$\frac{1}{1}$

HW # 4

Math 121A

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UCB extension

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$\boxed{1.6}$ for $u = e^x \cos y$

(a) verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

(b) verify that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(a) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\sin y\, e^x\right) = -\sin y\, e^x$

$\frac{\partial^2}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(e^x \cos y\right) = -\sin y\, e^x$ ✓

hence the same.

(b) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^x \cos y\right) = e^x \cos y$

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-e^x \sin y\right) = -e^x \cos y$ ✓

so add to give $e^x \cos y - e^x \cos y = 0$

$\boxed{1.7}$ if $z = x^2 + 2y^2$, $x = r\cos\theta$, $y = r\sin\theta$, find

$\left(\frac{\partial z}{\partial x}\right)_y$

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 2y^2\right) = 2x$

$\boxed{1.23}$ find $\frac{\partial^2 z}{\partial r \partial \theta} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial \theta}\right) = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial \theta}\left(x^2 + 2y^2\right)\right)$

$= \frac{\partial}{\partial r}\left(\frac{\partial}{\partial \theta}\left(r^2\cos^2\theta + 2r^2\sin^2\theta\right)\right)$

$= \frac{\partial}{\partial r}\left(r^2(-2\cos\theta\sin\theta) + 2r^2(2\sin\theta\cos\theta)\right)$

$= \frac{\partial}{\partial r}\left(-2r^2\cos\theta\sin\theta + 4r^2\sin\theta\cos\theta\right)$

$= -4r\cos\theta\sin\theta + 8r\sin\theta\cos\theta$

$= \boxed{4r\cos\theta\sin\theta}$

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$\boxed{2.6}$ Find maclaurin series for $e^{x+y}$.

$e^{x+y} = e^x e^y$
so can find expansion for exp function for each independent variable $x, y$ and multiply both series.
or I can let $x+y = z$, expand in $z$, then substitute back for $z = x+y$.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$= 1 + (x+y) + \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!} + \frac{(x+y)^4}{4!} \cdots$$

expand $(x+y)^a$ using binomial:

$$(x+y)^a = x^a \quad + x^{a-1}\binom{a}{1} y \quad + x^{a-2}\binom{a}{2} y^2 \quad + \cdots + y^a$$

$$\text{so } (x+y)^3 = x^3 + \binom{3}{1} x^2 y + \binom{3}{2} x y^2 + \binom{3}{3} x^0 y^3$$
$$= x^3 + \frac{3!}{2!} x^2 y + \frac{3!}{2!} x y^2 + y^3 = x^3 + 3x^2 y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$
$$= x^4 + \frac{4!}{3!} x^3 y + \frac{4!}{2!\,2!} x^2 y^2 + \frac{4!}{1!\,3!} x y^3 + y^4$$
$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$$

$$\text{so } e^z = 1 + (x+y) + \frac{1}{2}\left(x^2 + 2xy + y^2\right) + \frac{1}{6}\left(x^3 + 3x^2 y + 3xy^2 + y^3\right)$$
$$+ \frac{1}{12}\left(x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4\right) + \cdots$$

$$= 1 + (x+y) + \frac{x^2}{2} + xy + \frac{y^2}{2} + \frac{x^3}{6} + \frac{1}{2} x^2 y + \frac{1}{2} x y^2 + \frac{1}{6} y^3$$
$$+ \frac{1}{12} x^4 + \frac{1}{3} x^3 y + \frac{1}{2} x^2 y^2 + \frac{1}{3} x y^3 + \frac{1}{12} y^4 + \cdots \longrightarrow$$

$$= \boxed{1 + (x+y) + \left(xy + \frac{x^2}{2} + \frac{y^2}{2}\right) + \left(\frac{x^3}{6} + \frac{1}{2}x^2y + \frac{1}{2}xy^2 + \frac{1}{6}y^3\right) + \left(\frac{1}{12}x^4 + \frac{1}{3}x^3y + \frac{1}{2}x^2y^2 + \frac{1}{3}xy^3 + \frac{1}{12}y^4\right) + \cdots}$$

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$\boxed{1.24}$ Find $\frac{\partial^2 z}{\partial x \partial y}$ when $z = x^2 + 2y^2$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(4y) = 0$$

$\boxed{2.3}$ Find a two-variable maclaurin series for

$$\frac{\ln(1+x)}{1+y}$$

this is a function in 2 variables $x, y$.

expanding in maclaurin series of $f(x,y)$ is (about $z=0$).

expand $\ln(1+x)$ : $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $\quad -1 < x \leq 1$

expand $(1+y)^{-1}$ : $1 + (-1)y + \frac{(-1)(-1-1)}{2!}y^2 + \frac{(-1)(-2)(-3)}{3!}y^3$

$|y| < 1$

so we have.

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)\left(1 - y + y^2 - y^3 + y^4 \cdots\right)$$

multiply each term in the $x$ series by the $y$ series:

$$= \left(x - xy + xy^2 - xy^3 + \cdots\right) + \left(-\frac{x^2}{2} + \frac{x^2 y}{2} - \frac{x^2 y^2}{2} + \frac{x^2 y^3}{2} \cdots\right) + \left(\frac{x^3}{3} - \frac{x^3 y}{3} + \frac{x^3 y^2}{3} \cdots\right)$$

This can be rearranged as an increasing powers of $x$ or $y$ or by increasing powers of $xy$ together. book did not say. I assume the third option:

$$\boxed{x - \underbrace{\left(xy + \frac{x^2}{2}\right)}_{\text{exponents add to 2}} + \underbrace{\left(xy^2 + \frac{x^3}{3}\right)}_{\text{exponents add to 3}} + \underbrace{\left(-xy^3 - \frac{x^2y^2}{2} - \frac{x^3y}{3}\right)}_{\text{exponents add to 4}} + \cdots}$$

$\boxed{2.8}$ Find a 2 variable maclaurin series for $e^x \cos y$ and $e^x \sin y$ by finding series for $e^z = e^{x+iy}$ and taking real and imaginary parts.

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need to expand $e^z = e^{x+iy}$.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$ From Table page 24

$$= 1 + (x+iy) + \frac{(x+iy)^2}{2} + \frac{(x+iy)^3}{3!} + \cdots$$

$$= 1 + (x+iy) + \frac{1}{2}(x^2 + 2ixy - y^2) + \frac{1}{2\cdot 3}(x^3 + 3ix^2y - 3xy^2 - iy^3) + \cdots$$

Collect real part and imaginary part:

$$= \left(1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{1}{2\cdot 3}x^3 - \frac{3}{2\cdot 3}xy^2 + \cdots\right)$$
$$+ i\left(y + xy + \frac{3}{2\cdot 3}x^2y - \frac{1}{2\cdot 3}y^3 - \cdots\right)$$

$$= \left(1 + x + \frac{1}{2}(x^2 - y^2) + \frac{1}{2\cdot 3}(x^3 - 3xy^2) + \cdots\right)$$
$$+ i\left(y + xy + \frac{1}{2\cdot 3}(3x^2y - y^3) + \cdots\right)$$

Since $e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$
$$= e^x \cos y + i\,(e^x \sin y)$$

Then
$$\boxed{\begin{aligned} e^x \cos y &= \left(1 + x + \tfrac{1}{2}(x^2 - y^2) + \tfrac{1}{6}(x^3 - 3xy^2) + \cdots\right) \\ \text{and} \quad e^x \sin y &= \left(y + xy + \tfrac{1}{6}(3x^2y - y^3) + \cdots\right) \end{aligned}}$$

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$\boxed{4.2}$ use differentials to show that for large $n$ and small $a$

$$\sqrt{n+a} - \sqrt{n} \approx \frac{a}{2\sqrt{n}}$$

Consider $n = x$, so we have $\sqrt{x+dx} - \sqrt{x}$

This is a differential of $\sqrt{x}$ ✓

i.e $d(\sqrt{x}) = d(x^{1/2}) = \frac{1}{2} x^{-1/2}\, dx$

replace back $x$ by 'n', and $dx$ by 'a' we get

$$= \boxed{\frac{1}{2} \frac{a}{\sqrt{n}}} \checkmark$$

Find approx value of $\sqrt{10^{12} + 15} - \sqrt{10^{12}}$
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hence $a = 15$, $n = 10^{12}$

so result $= \frac{1}{2} \frac{a}{\sqrt{n}} = \frac{1}{2} \frac{15}{(10^{12})^{1/2}} = \frac{1}{2} \frac{15}{10^{6}}$

$$= \boxed{7.5 \times 10^{-6}}$$

✓

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$\boxed{4.8}$ About how much (in percent) does an error of 1% in a and b affect $a^2b^3$ ?

let $R = a^2b^3$

so we want to find $\frac{dR}{R}$

but $dR = \frac{\partial R}{\partial a} da + \frac{\partial R}{\partial b} db$ since $R$ is function of $(a,b)$.

so $dR = 2ab^3\, da + 3b^2a^2\, db$

$$\frac{dR}{R} = \frac{2ab^3}{R} da + \frac{3b^2a^2}{R} db = \frac{2ab^3}{a^2b^3} da + \frac{3b^2a^2}{a^2b^3} db$$

$$= \frac{2}{a} da + \frac{3}{b} db = 2\underbrace{\frac{da}{a}}_{1\%} + 3\underbrace{\frac{db}{b}}_{1\%}$$

$$= 2(0.01) + 3(0.01) = 0.05$$

so $R$ is affected $\boxed{5\%}$

**4.9** Show that the approximate error $\frac{df}{f}$ of a product $f = gh$ is the sum of the approximate relative errors of the factors.

$$df = \frac{\partial f}{\partial g} dg + \frac{\partial f}{\partial h} dh$$

$$\frac{df}{f} = \frac{\frac{\partial f}{\partial g} dg}{f} + \frac{\frac{\partial f}{\partial h} dh}{f} = \frac{\frac{\partial f}{\partial g} dg}{gh} + \frac{\frac{\partial f}{\partial h} dh}{gh}$$

$$\frac{df}{f} = \frac{\partial f}{\partial g} \frac{dg}{g} \frac{1}{h} + \frac{\partial f}{\partial h} \frac{dh}{h} \frac{1}{g} \quad (1)$$

but $\frac{\partial f}{\partial g} = h$

and $\frac{\partial f}{\partial h} = g$

So (1) becomes

$$\frac{df}{f} = h \frac{dg}{g} \frac{1}{h} + g \frac{dh}{h} \frac{1}{g}$$

$$\boxed{\frac{df}{f} = \frac{dg}{g} + \frac{dh}{h}}$$ which means the result required.

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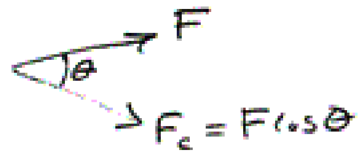
$\boxed{4.10}$ a force of 500 nt is measured with possible error of 1 nt. Its component in direction 60° away from its Line of action is required, where the angle is subject to an error of 0.5°. What is approx the Largest possible error in the component?




$$F_c = \text{Component} = F\cos\theta$$

$$\text{so} \quad dF_c = \frac{\partial F_c}{\partial F} dF + \frac{\partial F_c}{\partial \theta} d\theta$$

$$dF_c = \cos\theta \, dF - F\sin\theta \, d\theta$$

For $\theta = 60$, $d\theta = 0.5°$

and for $F = 500$, error $= 1$ i.e. $dF = 1$

$$\text{so} \quad dF_c = (\cos 60°)(\pm 1) - (500)\sin(60°)(\pm 0.5°)$$

$$dF_c = \underbrace{\tfrac{1}{2}(\pm 1)}_{A} - \underbrace{500\frac{\sqrt{3}}{2}(\pm 0.5°)}_{B}$$

convert to radian

$$\frac{180°}{0.5°} = \frac{\pi}{r} \Rightarrow \boxed{r = \frac{0.5}{180}\pi} = 8.726 \times 10^{-3}$$

There are 4 possible values (all +ve, all −ve, A +ve B −ve, A −ve B +ve)

(1) $dF_c = \frac{1}{2} - 500\frac{\sqrt{3}}{2}(+r) = -3.2787$ nt

(2) $dF_c = -\frac{1}{2} + 500\frac{\sqrt{3}}{2}(-r) = +3.2787$ nt

(3) $dF_c = \frac{1}{2} + 500\frac{\sqrt{3}}{2}(-r) = +4.2784$ nt

(4) $dF_c = -\frac{1}{2} - 500\frac{\sqrt{3}}{2}(+r) = -4.2784$ nt

So largest +ve error is $\boxed{4.278}$ nt, and largest −ve error is $\boxed{-4.278}$ nt, so error range is $2 \times 4.278$ or 8.556 nt

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$\boxed{4.13}$ without using calculator, estimate the change in length of space diagonal of a box whose dimensions are changed from $200 \times 200 \times 100$ to $201 \times 202 \times 99$


$L =$ length of space diagonal $= \sqrt{x^2+y^2+z^2}$

so $dL = \frac{\partial L}{\partial x}dx + \frac{\partial L}{\partial y}dy + \frac{\partial L}{\partial z}dz$

$\frac{\partial L}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}$ ; same for $y, z$.

so $\boxed{dL = \frac{x}{\sqrt{x^2+y^2+z^2}}dx + \frac{y}{\sqrt{x^2+y^2+z^2}}dy + \frac{z}{\sqrt{x^2+y^2+z^2}}dz}$

now $dx = 201 - 200 = 1$
$dy = 202 - 200 = 2$
$dz = 99 - 100 = -1$

but $\sqrt{x^2+y^2+z^2} = \sqrt{200^2+200^2+100^2} = \sqrt{4\times10^4+4\times10^4+10^4} = 100\sqrt{9} = 300$

so $dL = \frac{200}{300}\overset{dx}{(1)} + \frac{200}{300}\overset{dy}{(2)} + \frac{100}{300}\overset{dz}{(-1)}$

$dL = \frac{2}{3} + \frac{4}{3} - \frac{1}{3} = \boxed{\frac{5}{3}}$ so space diagonal has increased by $\frac{5}{3}$ or $\boxed{1.666}$

so new length $= 300 + \frac{5}{3} = 301.666$