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HW # 2

Math 121A

NASSER ABBASI

(UCB extension)

ch 1

$\boxed{14.5}$ show that $1-\cos x = \frac{x^2}{2}$ with an error less than 0.003 for $|x| < \frac{1}{2}$

$$1-\cos x = 1 - \Big[\overbrace{1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}^{\text{Power series expansion for } \cos x}\Big]$$

$$= \underbrace{\frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots}$$

alternating series with $|a_{n+1}| < |a_n|$ since $|x| < 1$

~~so apply~~ apply $\boxed{14.3}$ rule

$|S - (a_1 + a_2 + \cdots)| \leq |a_{n+1}|$ i.e. convergent

here $n=1$ since we want to keep first term $\frac{x^2}{2}$.

So $|S - a_1| \leq |a_2|$

this is $(1-\cos x)$ | this is $\frac{x^2}{2}$ | this is $\frac{x^4}{4!}$

i.e. this says that error is less than the first neglected term. see theory (14.3).

So $\frac{x^4}{4!}$ is max error. which is largest at $x = \frac{1}{2}$.

hence error max value is $\frac{(\frac{1}{2})^4}{1\times2\times3\times4} = \frac{\frac{1}{16}}{2\cdot3\cdot4} = \frac{1}{16\cdot2\cdot3\cdot4} = \boxed{0.0026}$

which is less than 0.003.

Ch 1

$\boxed{14.6}$ show that $\ln(1-x) = -x$ with an error less than ~~b~~ 0.0056 for $|x| < 0.1$

expand $\ln(1-x)$ in power series

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

this is not an alternating series, so use theorem 14.4. page 31

second term 'N+1'

$$\left| \ln(1-x) \cancel{+} x \right| < \left| \frac{\frac{x^2}{2}}{1-|x|} \right|$$

First term 'N'

so max $|\text{error}| \leq \left| \frac{\left(\frac{0.1}{2}\right)^2}{1-0.1} \right| \leq 0.00277$

so this is < 0.0056. QED.

$\boxed{14.9}$ Find sum of $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots$

Series $=\sum_{1}^{\infty}\frac{1}{n(n+1)}$

$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{(n+1)}$, hence Series $=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots\right)-\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)$

take one term from each series, we get

$$=\frac{1}{1}+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\frac{1}{5}-\frac{1}{5}\cdots$$

$$=1$$

hence $\boxed{S \text{ for } \sum_{1}^{\infty}\frac{1}{n(n+1)}=1}$

Now to find the Reminder, rewrite as

$$1+\frac{1}{2}\underbrace{-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\frac{1}{5}-\frac{1}{5}}_{\text{alternating Series}}\cdots$$

$\Rightarrow \underbrace{\sum_{n=2}^{\infty}(-1)^n\frac{1}{(n)}}$

So apply theorem 14.3, which say that for alternating series, the reminder after n terms is $\leq |(n+1)^{th}$ term.$|$

here, the $|a_{n+1}|$ term is $\frac{1}{(n+1)}$ from

~~for $n=200$, $|a_n|=\frac{1}{200}=0.005$~~

~~Reminder $=|a_{n+1}|=\frac{1}{201}=0.00497512$~~

For $n=200$, $a_n=\frac{1}{n(n+1)}=\frac{1}{200(201)}=0.0000248$ ← This a_n from original Series

while Reminder $=\frac{1}{201}=0.004975$ which $>a_n$ hence not reliable estimate as much smaller.

$\boxed{15.5}$ use power series to evaluate the function at the given point.

$$\frac{d^4}{dx^4} \ln(1+x^3) \qquad \text{at } x=0.2$$

find power series for $\ln(1+x^3)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

replace x by x^3

$$\ln(1+x^3) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots$$

~~and~~ so $f' = 3x^2 - \dfrac{6x^5}{2} + \dfrac{9x^8}{3} - \dfrac{12x^{11}}{4} + \cdots$

$$f^2 = 6x - \frac{30x^4}{2} + \frac{72x^7}{3} - \frac{132x^{10}}{4} + \cdots$$

$$f^3 = 6 - \frac{120x^3}{2} + \frac{504x^6}{3} - \frac{1320x^9}{4} + \cdots$$

$$f^4 = 0 - \frac{360x^2}{2} + \frac{3024x^5}{3} - \frac{11880x^8}{4} + \cdots$$

$$f^4(x=0.2) = -(0.2)^2\frac{360}{2} + (0.2)^5\frac{3024}{3} - (0.2)^8\frac{11880}{4} + \cdots$$

$$\boxed{= -6.8850}$$

easier using series method, since eliminates need to do complicated differentiation many times.

ch 1
$\boxed{15.11}$ use power Series to evaluate $\int_0^1 \cos x^2 \, dx$.

expand $\cos x^2$ in power Series

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

replace x by x^2

$$\cos x^2 = 1 - \frac{x^4}{2} + \frac{x^8}{4!} - \frac{x^{12}}{6!}$$

integrate term by term

$$\int_0^1 \cos x^2 \, dx = \int_0^1 1 \, dx - \int_0^1 \frac{x^4}{2} dx + \int_0^1 \frac{x^8}{4!} - \int_0^1 \frac{x^{12}}{6!} \cdots$$

$$= [x]_0^1 - \frac{1}{2}\left[\frac{x^5}{5}\right]_0^1 + \frac{1}{4!}\left[\frac{x^9}{9}\right]_0^1 - \frac{1}{6!}\left[\frac{x^{13}}{13}\right]_0^1 \cdots$$

$$= 1 - \frac{1}{2}\,\frac{1}{5} + \frac{1}{4!}\,\frac{1}{9} - \frac{1}{6!}\,\frac{1}{13} \cdots$$

$$= 1 - \frac{1}{10} + \frac{1}{4!\,9} - \frac{1}{6!\,13} = 0.9045$$

this is easier than integrating $\cos x^2$ directly, which does not have simple integral.

$\boxed{15.16}$ use power series to evaluate $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

expand $\tan x$ in power series

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots}{1 - \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

$$\text{So } \lim_{x\to 0} \frac{\left(\frac{x - \frac{x^3}{3!} + \cdots}{1 - \frac{x^2}{2!} - \cdots}\right) - x}{x^3} = \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \cdots\right) - x\left(1 - \frac{x^2}{2!} + \cdots\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\overbrace{\left(x - \frac{x^3}{3!} + \cdots\right)}^{A} - \overbrace{\left(x - \frac{x^3}{2!} + \cdots\right)}^{B}}{x^3}$$

first term x in B cancel first term x in B, we get

$$\lim_{x\to 0} \frac{\left(-\frac{x^3}{3!} + \cdots\right) - \left(-\frac{x^3}{2!} + \cdots\right)}{x^3}$$

divid numerator and denominator by x^3

$$\lim_{x\to 0} \frac{\left(-\frac{1}{3!} + \underbrace{\cdots\cdots}_{\text{terms with } x \text{ in numerator}}\right) - \left(-\frac{1}{2!} + \cdots\right)}{1}$$

hence when $x=0$, we are left with $-\frac{1}{3!} + \frac{1}{2!} = \frac{1}{2} - \frac{1}{6}$

$$= \boxed{\frac{1}{3}}$$

ch 1

$\boxed{15.18}$ evaluate using power series

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$$

$$\lim_{x\to 0}\left(\frac{(e^x-1)-x}{xe^x-x}\right)$$

but $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

so $$\lim_{x\to 0}\left(\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)-1-x}{\left(x+x^2+\frac{x^3}{2!}+\frac{x^4}{3!}+\cdots\right)-x}\right)=\frac{\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)}{\left(x^2+\frac{x^3}{2!}+\frac{x^4}{3!}+\cdots\right)}$$

divide by x^2

$$\lim_{x\to 0}\left(\frac{\frac{1}{2}+\frac{x}{3!}+\cdots}{1+\frac{x}{2!}+\cdots}\right)=\boxed{\frac{1}{2}}$$

ch 1

$\boxed{16.3}$ show that $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ is convergent.

prem. Test: $\lim_{n\to\infty} a_n = 0$ hence can be convergent.

try integral test: $\int^{\infty} \frac{1}{n^{3/2}}\, dn = -\frac{2}{\sqrt{n}}\Big|^{\infty} = 0$

$\boxed{\text{hence convergent}}$

what is wrong with the following proof that it is divergent?

$$\frac{1}{\sqrt{8}} + \frac{1}{\sqrt{27}} + \cdots > \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{36}} + \cdots$$

$$> \frac{1}{3} + \frac{1}{6} + \cdots$$

$$> \frac{1}{3}\underbrace{\left(1 + \frac{1}{2} + \cdots\right)}_{\text{harmonic. diverges}}$$

<u>answer</u> the problem with this proof is that not enough terms were considered.

The question is

$$\frac{1}{n^{3/2}} \overset{?}{>} \frac{1}{3n} \qquad \text{for } \underline{\text{all}} \text{ terms.}$$

it looks to be true for first few, but try for $n=10$ we get

$$\frac{1}{\sqrt{10^3}} \overset{?}{>} \frac{1}{30} \quad \text{or} \quad \frac{1}{1000} \overset{?}{>} \frac{1}{900}$$

here we see that it is <u>not</u> true. $\frac{1}{1000} < \frac{1}{900}$

hence the "proof" for divergence is faulty as it did not consider enough terms.

ch 1

$\boxed{16.18}$ Find Maclaurin Series for $\arctan x = \int_0^x \frac{du}{1+u^2}$

MacLaurin series is taylor series expanded about the origin.

expand $\frac{1}{1+u^2}$ in ~~power series~~ Taylor series

could have used formula $(1+x)^p$ $|x|<1$

$$f(u) = \frac{1}{1+u^2} \Rightarrow f(0) = 1$$

$$f'(u) = \frac{-2u}{(1+u^2)^2} \Rightarrow f'(0) = 0$$

$$f''(u) = \frac{8u^2}{(1+u^2)^3} - \frac{2}{(1+u^2)^2} \Rightarrow f''(0) = -2$$

$$f'''(u) = \frac{-48u^3}{(1+u^2)^4} + \frac{24u}{(1+u^2)^3} \Rightarrow f'''(0) = 0$$

$$f''''(u) = \frac{384u^4}{(1+u^2)^5} - \frac{288u^2}{(1+u^2)^4} + \frac{24}{(1+u^2)^3} \Rightarrow f''''(0) = 24$$

so $f(u) = f(\bar{u}) + f'(\bar{u})(u-\bar{u}) + \frac{f''(\bar{u})(u-\bar{u})^2}{2!} + \frac{f'''(\bar{u})(u-\bar{u})^3}{3!} + \frac{f''''(\bar{u})(u-\bar{u})^4}{4!} + \cdots$

$$= 1 + 0 - \frac{2u^2}{2!} + 0 + \frac{24u^4}{24} - \cdots$$

$$\boxed{f(u) = 1 - u^2 + u^4 - u^6 + u^8 - \cdots}$$

so $\arctan x = \int_0^x f(u)\,du = \int_0^x 1 - \int_0^x u^2 + \int_0^x u^4 - \int_0^x u^6 \cdots$

$$\boxed{= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots}$$

$\boxed{16.22}$ use series you know to show that

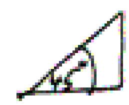
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

from problem 18, we found that $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots$

so <u>for $x=1$</u>, we get the same series.

hence $\boxed{\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots}$

so, the angle whose $\tan = 1$ is $45°$

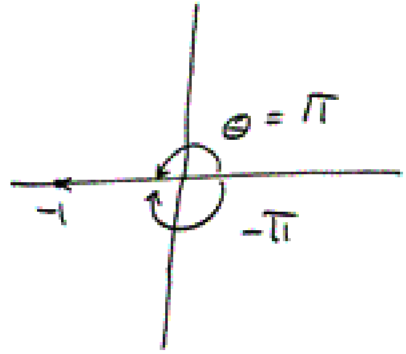


~~[illegible]~~ but ~~as~~ $45° = \frac{\pi}{4}$ as required to show.

Ch. 2

$\boxed{4.7}$ Plot the following numbers in complex plane. For each, give the numerical value of Re(z) and Im(z), its mod, and one value of the angle θ. Lable each plotted point in five ways as in figure 3.3. find and plot the complex conjugate.

$z = -1$ or $z = -\cancel{1} + 0i$

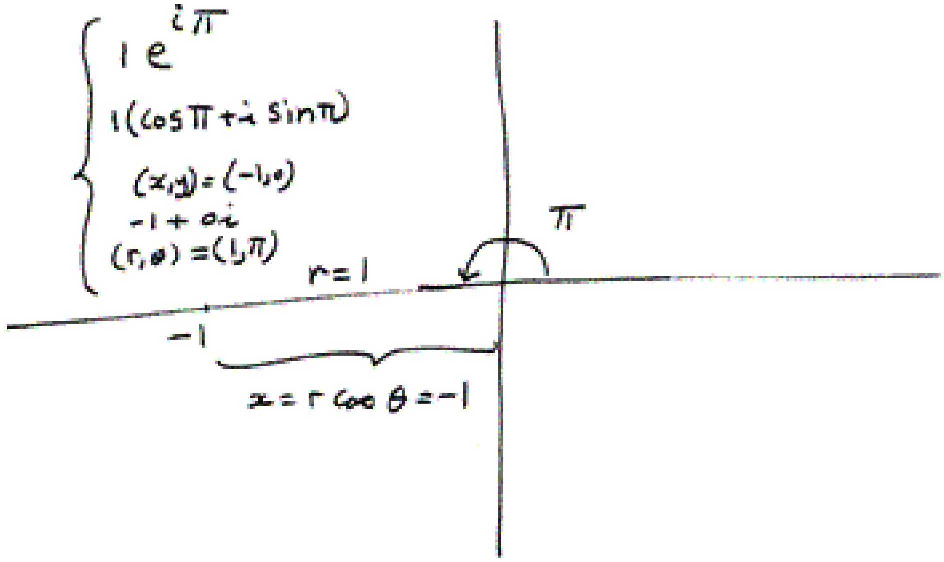


$Re(z) = -1$

$Im(z) = 0$

$|z| = 1$

$\theta = 180°$



$\bar{z} = -1$ because $\bar{z} = r(\cos(-\theta) + i\sin(-\theta))$

$\bar{z} = 1(\cos(-\pi) + i\sin(-\pi))$

$\bar{z} = 1(\cos(-\pi))$

but $\cos(-\pi) = \cos(\pi)$

so $\bar{z} = -1$

Ch 2

$\boxed{4.14}$ Plot $z = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

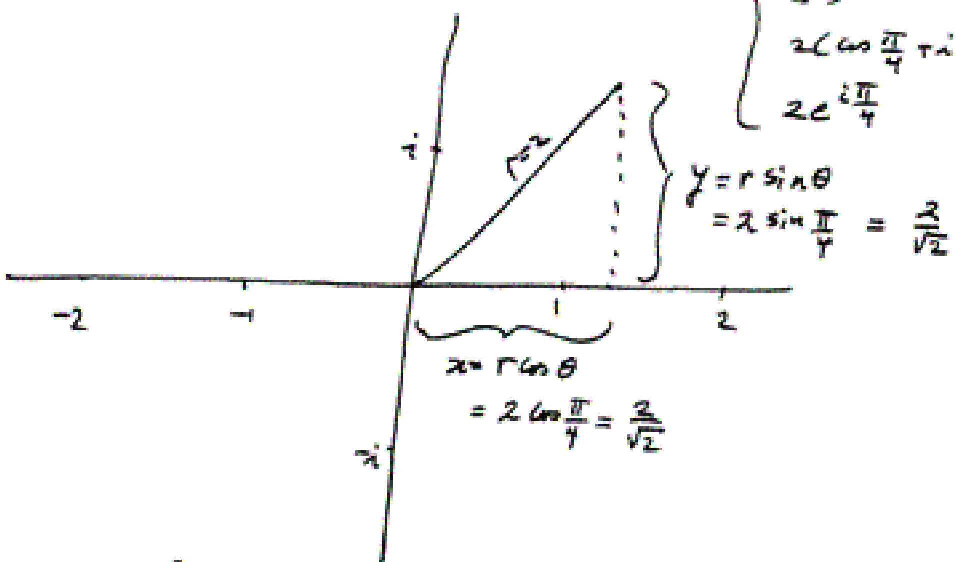
$Re(z) = 2\cos\frac{\pi}{4} = 2\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$

$Im(z) = 2\sin\frac{\pi}{4} = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

$|z| = \sqrt{Re^2(z) + Im^2(z)} = \sqrt{\frac{4}{2} + \frac{4}{2}} = \sqrt{4} = 2$

$\theta = \frac{\pi}{4}$

$$\left\{\begin{array}{l} (x,y) = \frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} + i\frac{2}{\sqrt{2}} \\ (r,\theta) = (2, \frac{\pi}{4}) \\ 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) \\ 2e^{i\frac{\pi}{4}} \end{array}\right.$$



$\boxed{5.6}$ Plot $\left(\frac{1+i}{1-i}\right)^2$

$= \left(\frac{1+i}{1-i}\frac{1-i}{1-i}\right)^2 = \left(\frac{2i}{2}\right)^2 = i^2 = -1$ ✓

this is the same as problem $\boxed{4.7}$ which I solved already. nothing more to do. ✓

$\boxed{5.13}$ Plot $5\left(\cos\frac{2\pi}{5} + i \sin\frac{2\pi}{5}\right)$

$r = 5$

$\theta = \frac{2\pi}{5} = 72^\circ$

$Re(z) = 5 \cos\frac{2\pi}{5} = 1.545$

$Im(z) = 5 \sin\frac{2\pi}{5} = 4.755$

$|z| = \sqrt{Re^2(z) + Im^2(z)} = 5$

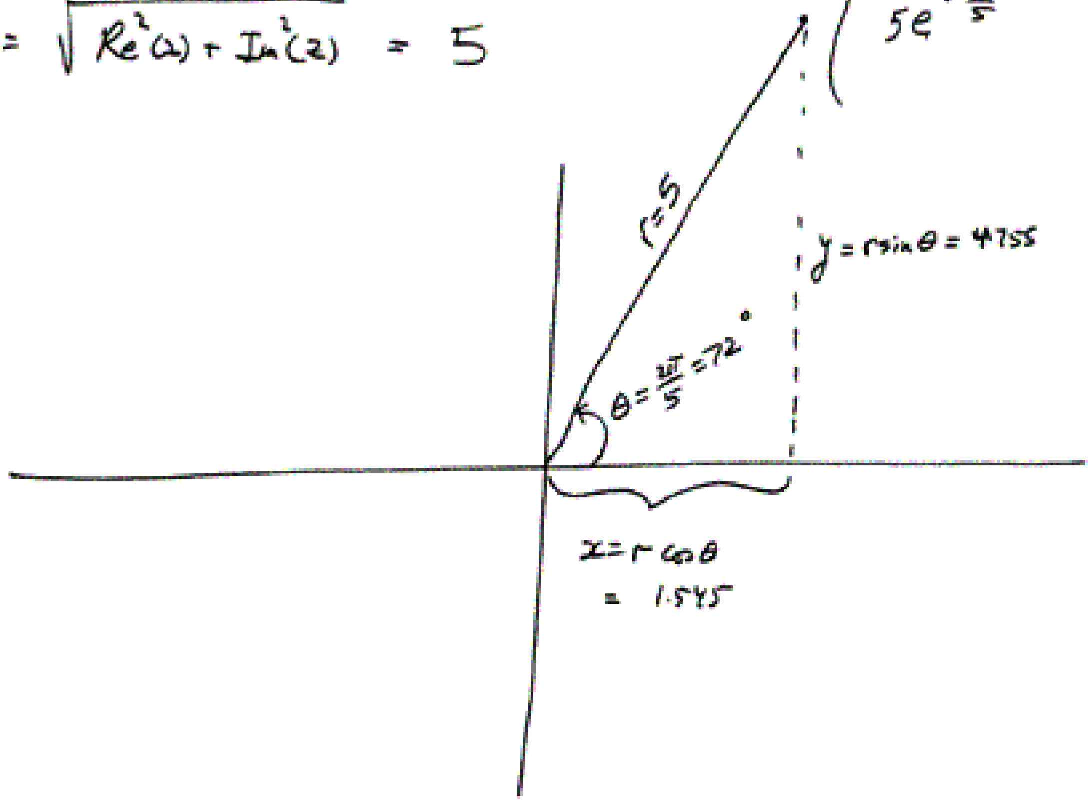
$(x,y) = (1.545, 4.755)$

$1.545 + i\, 4.755$

$(r,\theta) = (5, \frac{2\pi}{5})$

$5\left(\cos\frac{2\pi}{5} + i \sin\frac{2\pi}{5}\right)$

$5e^{i\frac{2\pi}{5}}$



$\boxed{5.20}$ Find in rectangular form $(a + ib)$. $1/z^2$

when $z = 2 - 3i$

$$\frac{1}{(2-3i)^2} = \frac{1}{4 - 12i - 9} = \frac{1}{-5-12i}$$ multiply by complex conjugate

$$= \frac{1}{(-5-12i)}\frac{(-5+12i)}{(-5+12i)} = \frac{-5+12i}{25 - 12^2 i^2} = \frac{-5+12i}{25+144} = \frac{-5+12i}{169} = \boxed{\frac{-5}{169} + \frac{12}{169} i}$$

when $z = x + iy$

$$\frac{1}{(x+iy)^2} = \frac{1}{x^2 + 2ixy - y^2} = \frac{1}{(x^2-y^2+i(2xy))} \frac{(x^2-y^2-i(2xy))}{(x^2-y^2-i\,2xy)}$$

$$= \frac{x^2-y^2-i\,2xy}{(x^2-y^2)^2+(2xy)^2} = \frac{x^2-y^2}{(x^2-y^2)^2+(2xy)^2} - i\frac{2xy}{(x^2-y^2)^2+(2xy)^2}$$

$\boxed{5.32}$ find mod of $(2-3i)^4$

$$|z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$$

$$z = 2-3i$$

$$|z| = \sqrt{z\bar{z}} = \sqrt{(2-3i)(2+3i)} = \sqrt{4-9i^2} = \sqrt{13}$$

$$\text{but } \underbrace{|z^4| = |z|^4}_{\text{this means}} \Rightarrow |z|^4 = \left(\sqrt{13}\right)^4 = (13)(13) = \boxed{169}$$

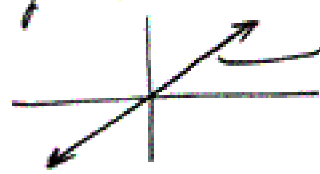
this means $\quad |(x+iy)^n| = |(x+iy)|^n$

This can be better seen from $(re^{i\theta})^4 = r^4 e^{i4\theta}$, r^4 → length.

$\boxed{5.39}$ Solve for all possible values of the real numbers x, y.

$$x+iy = y+ix$$

$$\left.\begin{array}{l} x = y \\ y = x \end{array}\right\}$$ any real value of x, y will work as long as $x=y$



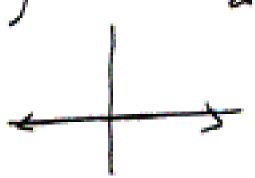
$\boxed{5.45}$

$$(x+iy)^2 = (x-iy)^2$$

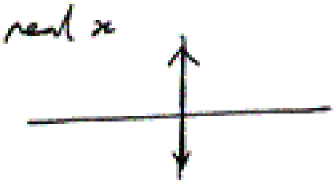
$$x^2 + i^2y^2 + 2xyi = x^2 + i^2y^2 - 2xyi$$

$$x^2 - y^2 + i\,2xy = x^2 - y^2 - i\,2xy$$

so $$\left.\begin{array}{l} x^2-y^2 = x^2-y^2 \\ -xy = xy \end{array}\right\} \Rightarrow$$ $x=0$, any real y
or $y=0$, any real x



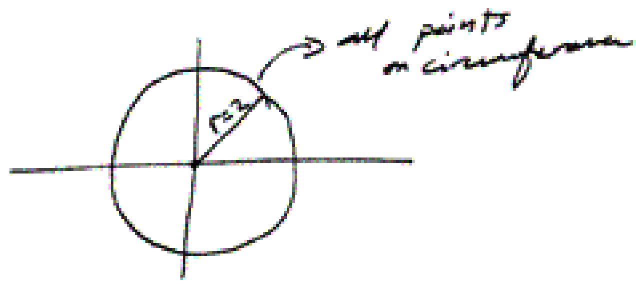
or



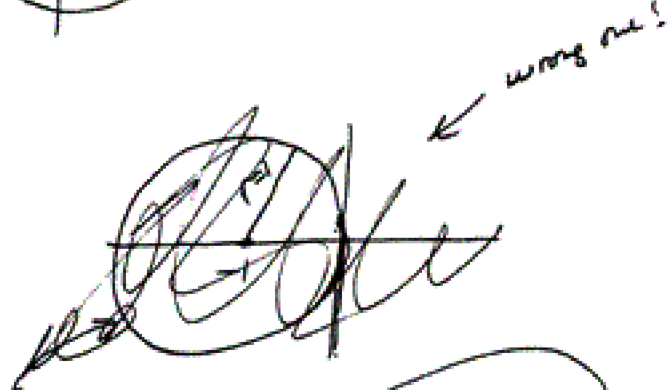
$\boxed{5.51}$ describe geometrically the set of complex points satisfying

$|z| = 2.$

this is a complex number whose length is 2.
then this describes all the points on the circumference of a circle whose radius $= 2$ centered at origin.

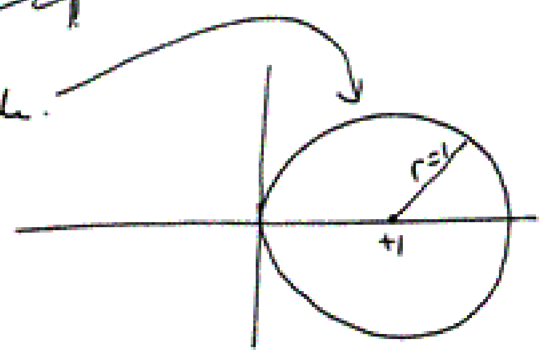


$\boxed{5.53}$ $|z-1| = 1$



this is a circle centered at
$z = 1 + 0i$
and radius $= 1$

so all points on circumference of this circle.



Can also solve this as follows
let $z = x+iy$. then ✓

$$|x+iy-1| = 1 \Rightarrow |(x-1)+iy| = 1$$

$$\Rightarrow (x-1)^2 + y^2 = 1^2$$

$$\Rightarrow x^2 - 2x + 1 + y^2 = 1 \Rightarrow x^2 + y^2 - 2x = 0$$

$$\Rightarrow \underbrace{(x-1)^2 + (y-0)^2 = 1^2}_{\text{general form of a circle equation, where}}$$

$(x-x_0)^2 + (y-y_0)^2 = r^2$ ✓, centre at (x_0, y_0) and radius $= r$

$\boxed{5.55}$ describe geometrically

$$z - \bar{z} = 5i$$

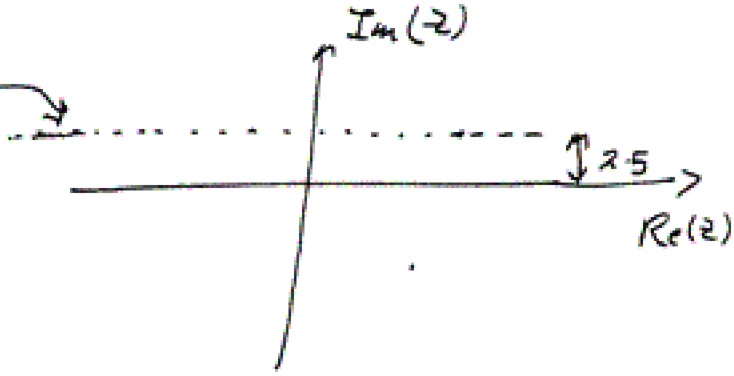
let $z = x + iy$.

then $(x+iy) - (x - iy) = 5i$

i.e. $2iy = 5i \Rightarrow \boxed{y = 2.5}$

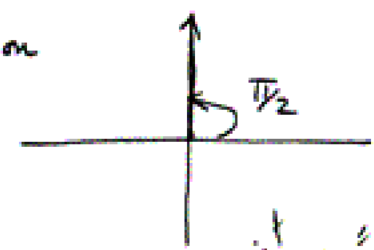
and any x real value will work.

so any complex number on this line



$\boxed{5.56}$ angle of $z = \frac{\pi}{2}$

so any complex number on The the positive $\mathrm{Im}(z)$ axis will satisfy this equation.



$\boxed{5.60}$ $|z - 1 + i| = 2$

$|z - (1 - i)| = 2$

This is a circle, radius $= 2$ Centered at $(1, -1)$

(my drawing is not perfect !)

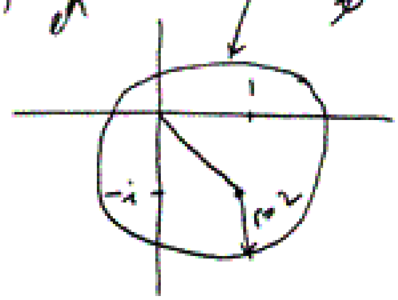
Can check it

$|(x+iy) - 1 + i| = 2$

$(x-1)^2 + (y+1)^2 = 4$

equation of circle $(x - x_0)^2 + (y - y_0)^2 = r^2$

any point on this circle.



ch 2

$\boxed{5.62}$ Describe geometrically

$$|z+1| + |z-1| = 8$$

$z = x + iy$. then $z+1 = (x+1)+iy$ and $z-1 = (x-1)+iy$.

Then $\sqrt{(x+1)^2+y^2} + \sqrt{(x-1)^2+y^2} = 8$ (1)

need to solve the above to find an equation in x, y.

$$\sqrt{(x+1)^2+y^2} = 8 - \sqrt{(x-1)^2+y^2}$$

square both sides to remove the $\sqrt{\ }$

$$(x+1)^2+y^2 = \left(8 - \sqrt{(x-1)^2+y^2}\right)^2$$

$$(x+1)^2+y^2 = 64 - 16\sqrt{(x-1)^2+y^2} + (x-1)^2+y^2)$$

$$x^2+2x+1+y^2 = 64 - 16\sqrt{(x-1)^2+y^2} + x^2-2x+1+y^2$$

$$4x = 64 - 16\sqrt{(x-1)^2+y^2}$$

or $16\sqrt{(x-1)^2+y^2} = 64-4x \Rightarrow 4\sqrt{(x-1)^2+y^2} = 16-x$

square again

$$16\left((x-1)^2+y^2\right) = (16-x)^2$$

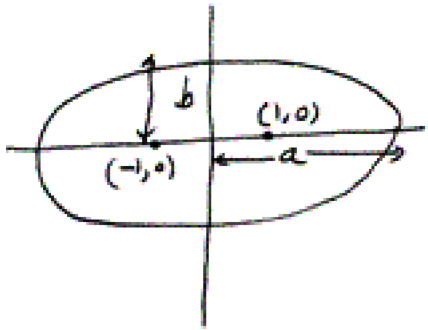
$$16(x^2-2x+1+y^2) = 256-32x+x^2$$

$$16+16x^2-32x+16y^2 = 256-32x+x^2$$

$$15x^2+16y^2 = 240 \Rightarrow \text{but } 240 = 15 \times 16$$

so $\boxed{\frac{x^2}{16}+\frac{y^2}{15}=1}$ this is an equation of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\Longrightarrow$



so $a = \sqrt{16} = 4$

$b = \sqrt{15}$

the foci of an ellipse are at $x = \pm\sqrt{a^2 - b^2}$

$$\cancel{= \pm\sqrt{256-225}} = \pm\sqrt{16-15} = \pm 1$$

So foci is $(1,0)$ and $(-1,0)$.

ch 2

$\boxed{6.4}$ test for convergence $\sum \left(\frac{1-i}{1+i}\right)^n$

to test for convergence of complex series, use the ratio test.

$$\rho_n = \left| \frac{\left(\frac{1-i}{1+i}\right)^{n+1}}{\left(\frac{1-i}{1+i}\right)^n} \right| = \left| \frac{(1-i)^{n+1}}{(1+i)^{n+1}} \frac{(1+i)^n}{(1-i)^n} \right|$$

$$= \left| \frac{(1-i)}{1+i} \right| = \left| \frac{(1-i)(1-i)}{(1+i)(1-i)} \right| = \left| \frac{1-2i \overset{-1}{+1}}{1+1} \right|$$

$$= \left| \frac{\cancel{2-2i}}{\cancel{2}} \right| = \cancel{|1-i|} \quad \frac{-2i}{2} = |-i|$$

so $\rho = 1$
not convergent
since must be < 1

$$\rho = \lim_{n \to \infty} \rho_n = |1-i| = \sqrt{1^2+1^2} = \sqrt{2} > 1$$

since $\rho > 1$, hence series $\boxed{\text{not convergent}}$

$\boxed{6.10}$ $\sum \left(\frac{1+i}{1-i\sqrt{3}}\right)^n$

$$\rho_n = \left| \frac{\left(\frac{1+i}{1-i\sqrt{3}}\right)^{n+1}}{\left(\frac{1+i}{1-i\sqrt{3}}\right)^n} \right| = \left| \frac{(1+i)^{n+1}}{(1-i\sqrt{3})^{n+1}} \frac{(1-i\sqrt{3})^n}{(1+i)^n} \right| = \left| \frac{(1+i)}{1-i\sqrt{3}} \right|$$

$$= \left| \frac{(1+i)(1+i\sqrt{3})}{(1-i\sqrt{3})(1+i\sqrt{3})} \right| = \left| \frac{1+i\sqrt{3}+i+i^2\sqrt{3}}{1-i^2\sqrt{3}^2} \right| = \left| \frac{1-\sqrt{3}+i(1+\sqrt{3})}{1+3} \right|$$

$$\rho = \lim_{n\to\infty} \rho_n = \left| \frac{1-\sqrt{3}+i(1+\sqrt{3})}{4} \right| = \sqrt{\left(\frac{1-\sqrt{3}}{4}\right)^2 + \left(\frac{1+\sqrt{3}}{4}\right)^2} = \frac{1}{4}\sqrt{(1-\sqrt{3})^2+(1+\sqrt{3})^2}$$

$$= \frac{1}{4}\sqrt{(1+3-2\sqrt{3})+(1+3+2\sqrt{3})} = \frac{1}{4}\sqrt{8} = \frac{1}{4}\sqrt{2^2 \cdot 2} = \frac{2}{4}\sqrt{2}$$

$$= \frac{1}{2}\sqrt{2} < 1$$ hence series is $\boxed{\text{convergent}}$

ch 2

$\boxed{6.12}$ test for convergence

$$\sum \frac{(3+2i)^n}{n!}$$

use ratio test.
$$\rho_n = \left| \frac{\frac{(3+2i)^{n+1}}{(n+1)!}}{\frac{(3+2i)^n}{n!}} \right| = \left| \frac{(3+2i)^{n+1} \quad n!}{(n+1)! \ (3+2i)^n} \right|$$

$$= \left| \frac{(3+2i) \quad n!}{n! \ (n+1)} \right| = \left| \frac{3+2i}{n+1} \right|$$

$$\rho = \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \left| \frac{3+2i}{n+1} \right| = 0$$

hence series is $\boxed{\text{convergent}}$

$\boxed{7.3}$ Find circle of convergence for

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

method of solution: find a_n, a_{n+1}. then find

$\rho_n = \left|\frac{a_{n+1}}{a_n}\right|$, then find $\lim_{n\to\infty} \rho_n$.

$$\text{Series} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}$$

$$\text{so } \rho_n = \left| \frac{\frac{z^{2n+1}}{(2n+2)!}}{\frac{z^{2n}}{(2n+1)!}} \right| = \left| \frac{z\,(2n+1)!}{(2n+2)!} \right| = \left| \frac{z}{2n+2} \right|$$

$$\rho = \lim_{n\to\infty} \left| \frac{z}{2n+2} \right| = 0$$

so this series converges for all values of z. so $R = \frac{1}{\rho} = \infty$

i.e. circle is the whole z plane.

$\boxed{7.12}$ Find circle of convergence $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2 z^n}{(2n)!}$

$$\rho_n = \left| \frac{\frac{(n+1\,!)^2 z^{n+1}}{(2(n+1))!}}{\frac{(n!)^2 z^n}{(2n)!}} \right| = \left| \frac{(n+1\,!)^2\, z\,(2n)!}{(2(n+1))!\,(n!)^2} \right| = \left| \frac{(n+1\,!)^2 z}{(2n+1)\,(n!)^2 \cdot (2n+2)} \right|$$

$$= \left| \frac{(n!\,(n+1))^2 z}{(2n+1)\,(n!)^2 \cdot (2n+2)} \right| = \left| \frac{(n+1)^2 z}{(2n+2)(2n+1)} \right| = \left| \frac{(n^2+2n+1)\, z}{(2n+2)(2n+1)} \right| = \cancel{\left| \frac{\left(1+\frac{2}{n}+\frac{1}{n^2}\right)z}{\frac{2}{n}+\frac{1}{n^2}} \right|}$$

$$\cancel{\rho = \lim_{n\to\infty} \rho_n = \left|\frac{z}{\infty}\right|} = \left| \frac{(n^2+2n+1)\, z}{4n^2+6n+2} \right| = \left| \frac{1+\frac{2}{n}+\frac{1}{n^2}\ z}{4+\frac{6}{n}+\frac{2}{n}} \right|$$

$\rho = \lim_{n\to\infty} \rho_n = \left|\frac{z}{4}\right|$ so $\left|\frac{z}{4}\right| < 1$ or $|z| < 4$. So circle with radius $R = \frac{1}{\rho} = 4$.

i.e. all z points inside circle with radius $= 4$, centered at $(0,0)$.

ch 2

$\boxed{8.1}$ show from using power series for e^z that

$$e^{z_1} e^{z_2} = e^{z_1+z_2}$$

<u>method of solution</u> express e^{z_1}, e^{z_2} in power series, Then do long multiplication, then collect all terms.

$$e^{z_1} e^{z_2} = \left(1 + \frac{z_1}{1!} + \frac{z_1^2}{2!} + \frac{z_1^3}{3!} + \cdots\right)\left(1 + \frac{z_2}{1!} + \frac{z_2^2}{2!} + \frac{z_2^3}{3!} + \cdots\right)$$

$$= 1 + z_2 + \frac{z_2^2}{2!} + \frac{z_2^3}{3!} + \cdots$$

$$z_1 + z_1 z_2 + \frac{z_1 z_2^2}{2!} + \frac{z_1 z_3^3}{3!} + \cdots$$

$$\frac{z_1^2}{2!} + \frac{z_1^2 z_2}{2!} + \frac{z_1^2 z_2^2}{2!\,2!} + \frac{z_1^2 z_2^3}{2!\,3!} + \cdots$$

$$\frac{z_1^3}{3!} + \frac{z_1^3 z_2}{3!} + \frac{z_1^3 z_2^2}{3!\,2!} + \frac{z_1^3 z_2^3}{3!\,3!} + \cdots$$

$\vdots$

etc.

Now add each column, we get

Third column

~~So~~ $1 + (z_2 + z_1) + \left(\frac{z_2^2}{2} + z_1 z_2 + \frac{z_1^2}{2}\right) +$

Fourth column.

First col

Second col

$$\left(\frac{z_2^3}{3!} + \frac{z_1 z_2^2}{2!} + \frac{z_1^2 z_2}{2!} + \frac{z_1^3}{3!}\right)$$

etc. but above can be reduced to:

$$= 1 + (z_2 + z_1) + \frac{(z_1+z_2)^2}{2} + \frac{(z_1+z_2)^3}{3!} + \cdots$$

$\Rightarrow$

in column 2 gives $(z_2 + z_1)$

column 3 gives $\frac{(z_1 + z_2)^2}{2}$

column 4 gives $\frac{(z_1 + z_2)^3}{3!}$

column n gives $\frac{(z_1 + z_2)^{n-1}}{(n-1)!}$

but this is the same as $e^{(z_1 + z_2)}$

hence $\boxed{e^{z_1} e^{z_2} = e^{z_1 + z_2}}$

$\boxed{8.2}$ show from power series that $\frac{d}{dz} e^z = e^z$

$$\frac{d}{dz}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right)$$

$$= \frac{d}{dz} 1 + \frac{d}{dz} z + \frac{d}{dz} \frac{z^2}{2!} + \cdots$$

$$= 0 + 1 + \frac{2z}{2!} + \frac{3z^2}{3 \times 2} + \frac{4z^3}{4 \times 3 \times 2} + \cdots$$

$$= 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots = e^z.$$