Math 121 A

$\frac{3}{3}$ Very Good Work!

HW # 1

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chapter 1

use $S = \frac{a}{1-r}$ to find fraction equivalent to

$\boxed{1.5}$ $S = 0.58333\cdots$

the above can be written as

$$s = 0.58 + \underbrace{\frac{3}{1000} + \frac{3}{10000} + \cdots}_{a + ar + \cdots}$$

for $a = \frac{3}{1000}$

$r = \frac{1}{10}$

Since $r < 1$ then convergent and the sum is

$$\frac{a}{1-r} = \frac{\frac{3}{1000}}{1 - \frac{1}{10}} = \frac{\frac{3}{1000}}{\frac{9}{10}} = \frac{3}{900}$$

hence $0.58333\cdots = 0.58 + \frac{3}{900}$

$$= \frac{58}{100} + \frac{3}{900} = \frac{522+3}{900} = \boxed{\frac{525}{900}} = \frac{7}{12}$$

$\boxed{2.6}$ write series in form $a_1 + a_2 + \cdots$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

$$= \frac{(1!)^2}{(2)!} + \frac{(2!)^2}{4!} + \frac{(3!)^2}{6!} + \frac{(4!)^2}{8!} + \cdots$$

$$= \frac{2}{2} + \frac{2^2}{4\times3\times2} + \frac{(3\times2)^2}{(6\times5\times4\times3\times2)} + \frac{(4\times3\times2)^2}{8\times7\times6\times5\times4\times3\times2} + \cdots$$

$$= \boxed{1 + \frac{1}{6} + \frac{1}{20} + \frac{1}{70} + \cdots}$$

chap 1

$\boxed{2.7}$ write in abberviated $\sum$ Form

$$\frac{1}{3} + \frac{2}{5} + \frac{4}{7} + \frac{8}{9} + \frac{16}{11} + \cdots$$

denominator behaves as $2n+3$

numerator as 2^n

hence series $= \boxed{\sum_{n=0}^{\infty} \frac{2^n}{2n+3}}$

for example, $n=0 \Rightarrow \frac{2^0}{2(0)+3} = \frac{1}{3}$ ok

$n=1 \Rightarrow \frac{2^1}{2(1)+3} = \frac{2}{5}$ ok

$n=2 \Rightarrow \frac{2^2}{2(2)+3} = \frac{4}{7}$ ok

$n=3 \Rightarrow \frac{2^3}{2(3)+3} = \frac{8}{9}$ ok

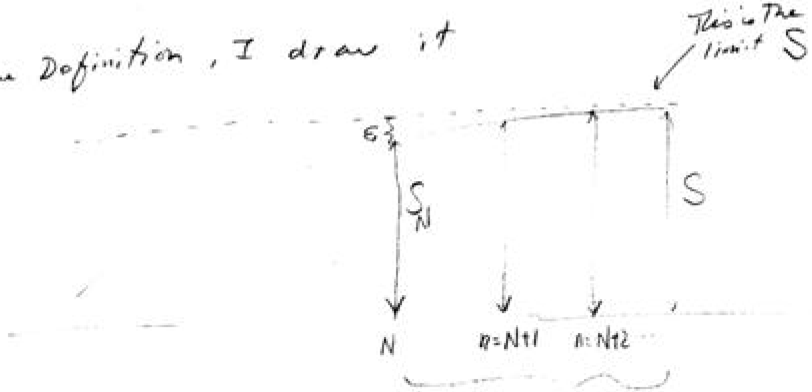
etc...

chapter 1

$\boxed{4.2}$ A careful math. definition of a Convergent Series with sum S is : Given any small positive number called ε, it is possible to find an integer N so that $|S - S_n| < \varepsilon$ for every $n \geq N$.
Select some ε and corresponding N for

$$\sum_{n=1}^{\infty} \frac{1}{5^n}$$

To visualize the Definition, I draw it



For all $n \geq N$, we have $|S - S_n| < \varepsilon$

series is Convergent by ratio test

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{5^{n+1}}}{\frac{1}{5^n}} = \frac{5^n}{5^{n+1}} = 5^{n-(n+1)} = 5^{-1} = \frac{1}{5} < 1$$

let $\varepsilon = 10^{-7}$

need to find the limiting sum S

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$$, this is a geometric series with $a = \frac{1}{5}$ and $r = \frac{1}{5}$

hence. $S = \dfrac{a}{1-r} = \dfrac{\frac{1}{5}}{1 - \frac{1}{5}} = \dfrac{\frac{1}{5}}{\frac{4}{5}} = \dfrac{1}{4}$ $\Longrightarrow$

and $S_n = \frac{a(1-r^n)}{1-r} = \frac{\frac{1}{5}\left(1-\frac{1}{5^n}\right)}{1-\frac{1}{5}} = \frac{\frac{1}{5}\left(1-\frac{1}{5^n}\right)}{\frac{4}{5}}$

$= \frac{1}{4}\left(1-\frac{1}{5^n}\right)$

Now I can find N.

$|S-S_n| = \frac{1}{4} - \frac{1}{4}\left(1-\frac{1}{5^n}\right) = \frac{1}{4}\left(1-\left(1-\frac{1}{5^n}\right)\right)$

$= \frac{1}{4}\left(\frac{1}{5^n}\right)$

so need $\frac{1}{4}\frac{1}{5^n} < 10^{-7}$, solve for n.

$\frac{1}{4}\frac{1}{5^n} < \frac{1}{10^7}$

i.e. $(4)\,5^n > 10^7$

i.e. $\log(4\,5^n) > \log 10^7$

i.e. $\log 4 + \log 5^n > 7$

i.e. $0.6 + n(\log 5) > 7$

i.e. $0.6 + n(0.698) > 7$

i.e. $0.698\,n > 6.4$

i.e. $n > 9.16$ i.e. $n = 10$

so $\boxed{N = 10}$ To test this $\Rightarrow$

for $n = 10$, $S_n = \dfrac{a(1-r^n)}{1-r} = \frac{1}{4}\left(1 - \frac{1}{5^n}\right)$

$= \frac{1}{4}\left(1 - \frac{1}{5^{10}}\right) = 0.2499999744$

so $|S - S_n| = 2.56 \times 10^{-8}$

which is smaller than $\varepsilon = 10^{-7}$

ok

Chapter 1

$\boxed{4.2}$

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

since by comparison test, this series is convergent

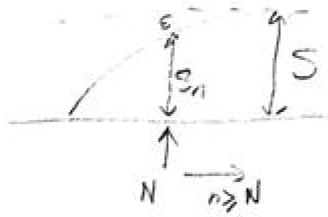
$\left(\frac{1}{n!} < \frac{1}{2^n} \text{ for } n>3 \right)$.

so $S - S_n$ for this series is smaller than

$S - S_n$ for the geometric series which is

$$\frac{a}{1-r} - \frac{a(1-r^n)}{1-r} = \frac{a - a(1-r^n)}{1-r} = \frac{a - a + ar^n}{1-r}$$

$$= \frac{ar^n}{1-r}$$



hence need $\left| \frac{ar^n}{1-r} \right| \leq \varepsilon$

let $\varepsilon = 10^{-7}$. for geometric series, $a = \frac{1}{2}$, $r = \frac{1}{2}$.

so need $\frac{\frac{1}{2} \frac{1}{2}^n}{\frac{1}{2}} < \frac{1}{10^7}$

i.e. $\left(\frac{1}{2}\right)^n \leq 10^{-7}$ or $\frac{1}{2^n} \leq \frac{1}{10^7}$

or $2^n > 10^7$ or $n \log 2 > 7$

or $n > \frac{7}{0.3}$ ~ $n > 23.2$

i.e. $\boxed{N = 24}$

chapter 11 use preliminary test to decide if divergent or more testing required.

$\boxed{5.1}$

$$\frac{1}{2} - \frac{4}{5} + \frac{9}{10} - \frac{16}{17} + \frac{25}{26} - \frac{36}{37} + \cdots$$

in preliminary test, find a_n. if $a_n \neq 0$ for $n \to \infty$ then divergent, else more testing needed to see if convergent.

The above series is $\boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^2+1}}$

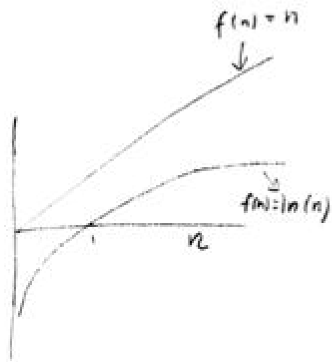
so $a_n = \frac{n^2}{n^2+1}$

$\lim_{n\to\infty} a_n = 1$ or $a_n = -1$ depending on sign

ie $a_n \neq 0$ as $n \to \infty$.

hence $\boxed{\text{divergent}}$

$\boxed{5.8}$ $\sum_{n=1}^{\infty} \frac{\ln n}{n}$



for large n
$\ln(n)$ grows much more slowly than n. see graph.

hence $\lim_{n\to\infty} a_n = \frac{\ln(n)}{n} \longrightarrow 0$

hence $\boxed{\text{more testing is needed}}$ to see if convergent

chapter 1

$\boxed{6.1}$ Show that $n! > 2^n$

$$S_1 = n! = 1 + 2 + 2\cdot 3 + 2\cdot 3\cdot 4 + 2\cdot 3\cdot 4\cdot 5 + \cdots$$
$$S_2 = 2^n = 1 + 2 + 2\cdot 2 + 2\cdot 2\cdot 2 + 2\cdot 2\cdot 2\cdot 2 + \cdots$$

looking at above 2 sequences, each term in S_1 is being multiplied by a factor larger than the factor that the corresponding term in S_2 is being multiplied with. (This is starting at $n=3$)

This implies the sum of terms of S_1 is larger than sum of terms of S_2. QED

$\boxed{6.2}$ Prove that harmonic series $\sum_1^\infty \frac{1}{n}$ is divergent by comparing it to series $1+\frac{1}{2}+(\frac{1}{4}+\frac{1}{4})+\cdots$

write out $\sum \frac{1}{n}$

$$1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{4\text{ terms}}+\underbrace{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}}_{8\text{ terms}}+\frac{1}{17}+\frac{1}{18}+\frac{1}{19}+\frac{1}{20}+\cdots$$

$$= 1+\frac{1}{2}+\frac{7}{12}+\frac{533}{840}+\frac{95549}{144144}+\cdots$$

$$= 1+\frac{1}{2}+0.58+0.634+0.662+\cdots$$

so we see that $\sum\frac{1}{n}$ can be rewritten as a series whose each term (after $n=2$) is larger than $\frac{1}{2}$ by collecting 4 terms, then 8 terms, then 16 terms, etc. and since there is ∞ number of terms, then we can keep doing this as we please so by comparison test to series $1+\frac{1}{2}+\frac{1}{2}+\cdots$ which is divergent, we conclude that $\sum\frac{1}{n}$ is divergent. QED

chapter 1

$\boxed{6.3}$ prove convergence of $\sum \frac{1}{n^2}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots$$

$$= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256}$$

combine, starting from $n=2$, in sequence $\sum \frac{1}{n^2}$, 2 terms, then 4 terms, then 8 terms, etc... to get

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) - 1 = \frac{13}{36} + \frac{26,581}{176,400} + \cdots + \cdots$$

less than $\frac{1}{2}$ + less than $\frac{1}{4}$ + less than $\frac{1}{8}$ + ...

hence, this is a series whose each term a_n, $n=1..\infty$, is smaller than corresponding term in geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$, which we know is convergent (since $r = \frac{1}{2} < 1$)

hence $\boxed{\sum \frac{1}{n^2} \text{ is convergent by comparison test}}$

<u>note</u> the term $a_1 = 1$ in $\frac{1}{n^2}$ was ignored.

this of course does not affect the convergence test.

chapter 1

$\boxed{6.5\,(a)}$ Test for convergence using comparison test

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} + \cdots$$

but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$

since $\sqrt{n} < n$, then $\frac{1}{\sqrt{n}} > \frac{1}{n}$

and since $\sum \frac{1}{n}$ diverges (see problem solution 6.2), then this implies that $\boxed{\sum^{\infty} \frac{1}{\sqrt{n}} \text{ diverges}}$, since each term in $\sum \frac{1}{\sqrt{n}}$ is larger than each corresponding term in $\sum \frac{1}{n}$.

$\boxed{6.5\,(b)}$ $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n} = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \frac{1}{\ln 5} + \frac{1}{\ln 6} + \cdots$

again, compare to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$

each term in $\sum \frac{1}{\ln n}$ is larger than each corresponding term in $\sum \frac{1}{n}$ which is divergent. hence $\displaystyle\sum_{2}^{\infty} \frac{1}{\ln n}$ $\boxed{\text{divergent}}$. This is because $\ln(n)$ is smaller than n for all positive n's.

n $\quad$ ln(n)

1 $\quad$ 2

Chapter 1

$\boxed{6.7}$ use integral test to find if series diverge or converges

$$S = \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} = \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \cdots$$

Since all terms are positive and $a_{n+1} < a_n$, then can use integral test.

$$I = \int^{\infty} \frac{1}{n \ln(n)}\,dn = \ln(\ln(n))\Big|^{\infty} = \ln(\ln(\infty)) = \ln(\infty) = \infty$$

since Integral diverges, then $\boxed{\sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \text{ diverges}}$

$\boxed{6.8}$ $S = \sum_{n=1}^{\infty} \frac{n}{n^2+4}$

$$I = \int^{\infty} \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln(x^2+4)\Big|^{\infty}$$

$$= \frac{1}{2}\ln(\infty) = \infty$$

hence series $\boxed{\text{diverges}}$

$\boxed{6.11}$ $\sum_{1}^{\infty} \frac{1}{n(1+\ln n)^{3/2}}$

$$I = \int^{\infty} \frac{1}{x(1+\ln(x))^{3/2}}\,dx = \frac{-2}{\sqrt{1+\ln(x)}}\Bigg|^{\infty}$$

as $x \to \infty$, $\frac{1}{\sqrt{1+\ln(x)}} \to 0$ hence converges

hence $\boxed{\text{Series converges}}$

chapter 1

$\boxed{6.12}$ So $\sum_{1}^{\infty} \frac{n}{(n^2+1)^2}$

$$I = \int^{\infty} \frac{x}{(x^2+1)^2} dx = -\frac{1}{2(1+x^2)} \Big|^{\infty}$$

when $x=\infty$, $I \longrightarrow 0$. hence $\boxed{\text{Series converges}}$

$\boxed{6.15}$ use integral test to prove the following so-called p-series test.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \begin{cases} \text{convergent if } p>1 \\ \text{divergent if } p\le 1 \end{cases}$$

$$I = \int^{\infty} \frac{1}{x^p} dx = \int^{\infty} x^{-p} dx = \frac{x^{-p+1}}{-p+1} \Big|^{\infty}$$

$$= \left(\frac{1}{1-p}\right)\left(\frac{1}{x^{1+p}}\right) \Big|^{\infty}$$

$\boxed{\text{when } p>1}$, then $\frac{1}{x^{1+p}} \longrightarrow 0$ as $x \longrightarrow \infty$ since x^{1+p} grows larger and larger. hence $I \longrightarrow 0$. hence $\boxed{\text{converges}}$

when $p<1$, then $1+p$ is negative, and so $\frac{1}{x^{1+p}} \longrightarrow \infty$ as x increases since denominator $\to 0$ now.

hence $\boxed{\text{divergent series}}$

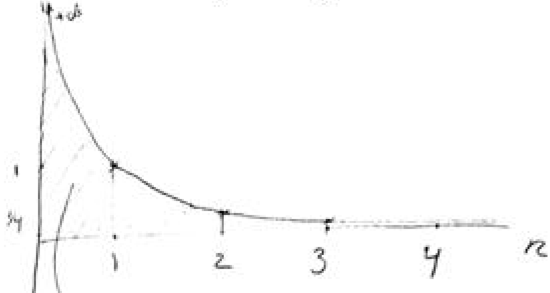
$\boxed{\text{when } p=1}$ $\frac{1}{1-p} = \frac{1}{0} = \infty$. so I diverges.

hence $\boxed{\text{sequence diverges}}$ also

chapter 1

$\boxed{6.16}$

$\sum \frac{1}{n^2}$ using integral test $\int_0^\infty \frac{1}{n^2} dn$ results in ∞. however this is wrong. the reason is that there is a pole at $n=0$, and so lower limit must start at a point to the right of 0.



$n=0 \quad f(n) = \frac{1}{n^2} = \infty$

$n=1 \quad f(n) = \frac{1}{n^2} = 1$

$n=2 \quad f(n) = \frac{1}{4}$

$n=3 \quad f(n) = \frac{1}{9}$

this area is the problem. needs not be considered in the integral. should start for $n=1$

Chapter 1

$\boxed{6.17}$

use integral test for divergence or convergence:

$$\sum_{n=0}^{\infty} e^{-n^2}$$

$$I = \int^{\infty} e^{-n^2}\,dn$$

can't evaluate integral. but area under $f(x) = e^{-x^2}$ is smaller than area under $f(x) = e^{-x}$, since e^{-x^2} approaches zero faster. So if I can show that $\int^{\infty} e^{-x}\,dx$ is finite, then this means $\int^{\infty} e^{-x^2}\,dx$ is finite as well.

$$I = \int^{\infty} e^{-n}\,dn = -e^{-n}\Big|^{\infty}$$

as $n \to \infty$ $e^{-n} \to 0$. hence integral converges.

hence $\int^{\infty} e^{-x^2}\,dx$ converges as well. hence

$$\sum_{n=0}^{\infty} e^{-n^2} \quad \boxed{\text{converges}}$$

chapter 1

$\boxed{6.18}$ use ratio test to find if series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}.$$

$$a_n = \frac{2^n}{n^2}, \qquad a_{n+1} = \frac{2^{n+1}}{(n+1)^2}$$

$$\text{so } \rho_n = \frac{|a_{n+1}|}{|a_n|} = \frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} = \frac{2^{n+1}\, n^2}{2^n\,(n+1)^2} = \underbrace{2\,\frac{n^2}{(n+1)^2}}_{\to 2}$$

since denominator has an extra 'n' factor, this converges x

$$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2/n^2}{(n+1)^2/n^2} = 1$$

$$\lim_{n\to\infty} \rho_n = \frac{2\, n^2}{n^2+2n+1} \Rightarrow 0 \text{ x}$$

larger than n^2

1/2

hence sequence $\boxed{\text{converges}}$ x

$\boxed{6.21}$ $\displaystyle\sum_{n=0}^{\infty} \frac{5^n (n!)^2}{(2n)!}$ $\qquad a_n = \dfrac{5^n (n!)^2}{(2n)!} \qquad a_{n+1} = \dfrac{5^{n+1}((n+1)!)^2}{(2(n+1))!}$

$$\text{so } \rho_n = \left|\frac{a_{n+1}}{a_n}\right| = \frac{5^{n+1}((n+1)!)^2\,(2n)!}{5^n (n!)^2\,(2(n+1))!} = \frac{5\,((n+1)!)^2\,(2n)!}{(n!)^2\,(2(n+1))!}$$

now $(n+1)! = 1\times 2\times\cdots\times n\times(n+1) = (n+1)\,n!$

$$\text{so } \rho_n = \frac{5\,((n+1)\,n!)^2\,(2n)!}{(n!)^2\,(2(n+1))!} = \frac{5\,(n+1)^2\,(2n)!}{(2(n+1))!}$$

now $(2(n+1))! = (2n+2)! = 1\times 2\times \cdots \times n\times n+1\times\cdots\times 2n\times(2n+1)\times(2n+2)$

$= (2n!)\,(2n+1)(2n+2)$

$$\text{so } \rho_n = \frac{5(n+1)^2\,(2n)!}{(2n)!\,(2n+1)(2n+2)} = \frac{5\,(n^2+2n+1)}{4n^2+6n+2} \Longrightarrow$$

divid by n^2 :

$$\rho_n = \frac{5\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{4+\frac{6}{n}+\frac{2}{n^2}}$$

$$\lim_{n\to\infty} \rho_n = \boxed{\frac{5}{4}}$$

> 1 hence $\boxed{\text{diverges}}$

$\boxed{6.27}$ use ratio test

$$\sum_{n=0}^{\infty} \frac{100^n}{n^{200}}$$

$$a_n = \frac{100^n}{n^{200}} \quad ; \quad a_{n+1} = \frac{100^{n+1}}{(n+1)^{200}}$$

$$\rho_n = \left|\frac{a_{n+1}}{a_n}\right| = \frac{100^{n+1}\; n^{200}}{100^n\;(n+1)^{200}} = 100\,\frac{n^{200}}{(n+1)^{200}}$$

now $\displaystyle\lim_{n\to\infty} \frac{n^{200}}{(n+1)^{200}} = 1$

so $\displaystyle\lim_{n\to\infty} \rho_n = (100)(1) = 100$ > 1

hence $\boxed{\text{diverges}}$

$\boxed{6.33}$ use special comparison test for convergence or divergence

$$\sum_{n=5}^{\infty} \frac{1}{2^n - n^2}$$

First need to find the comparison series. looking at

$2^n - n^2$ as $n \to \infty$ and looking at the log, we have

$\log 2^n = n \log 2$

$\log n^2 = 2 \log n$

since $\log n$ grows more slowly than n, then

2^n is the dominant term in denominator. So

compare with $\displaystyle\sum_{n=5}^{\infty} \frac{1}{2^n}$.

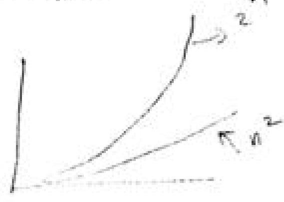
this is a convergent sequence, since geometric with $r = \frac{1}{2} < 1$

hence use test (a)

$$\frac{\frac{1}{2^n - n^2}}{\frac{1}{2^n}} = \frac{2^n}{2^n - n^2} = \frac{1}{1 - \frac{n^2}{2^n}}$$

$$\lim_{n\to\infty} \frac{1}{1 - \frac{n^2}{2^n}} = \cancel{0}\, 1 \quad \text{since } \lim_{n\to\infty} \frac{n^2}{2^n} \to 0$$

since n^2 grow slower than 2^n



since this is a finite limit, then

$$\sum_{n=5}^{\infty} \frac{1}{2^n - n^2} \quad \boxed{\text{Converges}}$$

$\boxed{6.34}$ use special comparison test to find if converges or diverges.

$$S = \sum_{n=1}^{\infty} \frac{n^2+3n+4}{n^4+7n^3+6n-3}$$

need to find a comparison sequence. looking at S.

as $n \to \infty$, numerator $\to n^2$. for denominator, as $n \to \infty$ it goes as n^4, so use $\frac{n^2}{n^4}$ or $\frac{1}{n^2}$ as comparison sequence. to find if $\sum \frac{1}{n^2}$ converges, use integral test

$$\int^{\infty} \frac{1}{n^2}\,dn = -\frac{1}{2n}\Big|^{\infty} \longrightarrow 0 \quad \text{as } n \to \infty.$$

so $\sum \frac{1}{n^2}$ converges. so use test (a).

$$\frac{\frac{n^2+3n+4}{n^4+7n^3+6n-3}}{\frac{1}{n^2}} = \frac{(n^2)(n^2+3n+4)}{n^4+7n^3+6n-3}$$

$$= \frac{n^4+3n^3+4n^2}{n^4+7n^3+6n-3} \text{. divide by } n^4 \longrightarrow \frac{1+\frac{3}{n}+\frac{4}{n^2}}{1+\frac{7}{n}+\frac{6}{n^3}-\frac{3}{n^4}}$$

as $n \to \infty$ above goes to 1

this is a finite limit. hence $\boxed{\text{convergent}}$

chapter 1

$\boxed{6.35}$ use special comparison test on

$$\sum_{n=3}^{\infty} \frac{(n-\ln n)^2}{5n^4-3n^2+1}$$

need to first find the comparison series.
in denominator, it goes as $5n^4$ for large n.
in numerator, for large n, $n > \ln n$. hence it goes as n^2.

so need to use $\frac{n^2}{5n^4}$ or $\frac{1}{n^2}$.

$\sum \frac{1}{n^2}$ is convergent by integral test. (see previous problem 6.34)

so use test (a)

$$\frac{\dfrac{(n-\ln n)^2}{5n^4-3n^2+1}}{\dfrac{1}{n^2}} = \frac{n^2\,(n-\ln n)^2}{5n^4-3n^2+1}$$

$$= \frac{n^2\,(n^2-2n\ln n-\ln^2 n)}{5n^4-3n^2+1} = \frac{n^4-2n^3\ln n-n^2\ln^2 n}{5n^4-3n^2+1}$$

$$= \frac{1-\frac{2\ln n}{n}-\frac{\ln^2 n}{n^2}}{5-\frac{3}{n}+\frac{1}{n^4}} = \frac{1-\frac{2\ln n}{n}-\left(\frac{\ln n}{n}\right)^2}{5-\frac{3}{n}+\frac{1}{n^4}}$$

as $n \to \infty$ $\frac{\ln n}{n} \to 0$ since $n > \ln n$. so limit $\longrightarrow \boxed{\frac{1}{5}}$

This is finite, hence $\boxed{\text{Converges}}$

$\boxed{7.2}$ test the following alternate series for convergence.

- $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$. First see if abs. convergent since if so, no need to do more testing since an alternate series that is abs. convergent is convergent

so look at $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ convert to log (?)

$\left.\begin{array}{l}\log 2^n = n \log 2 \\ \log n^2 = 2 \log n\end{array}\right\}$ since n grows faster than $\log n$, Then $\frac{\log 2^n}{\log n^2} \longrightarrow \infty$ as $n \to \infty$

so $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges. so series is not abs. convergent.

- it can still be convergent however.

An alternate series converges if $\frac{|a_{n+1}|}{|a_n|} \leq 1$ and

$\lim_{n \to \infty} a_n = 0$.

$|a_{n+1}| = \left| \frac{2^{n+1}}{(n+1)^2} \right|$; $|a_n| = \left| \frac{2^n}{n^2} \right|$

$\frac{|a_{n+1}|}{|a_n|} = \left| \frac{(2^{n+1})(n^2)}{(2^n)(n+1)^2} \right| = \left| \frac{(2)(n^2)}{n^2+2n+1} \right| = \left(\frac{\frac{2}{n^2} 2}{1+\frac{2}{n}+\frac{1}{n^2}} \right)$

as $n \to \infty$ above $\to \cancel{0}\ 2$. hence $\frac{|a_{n+1}|}{|a_n|} > 1$. hence

- $\boxed{\text{diverges}}$

chapter 1

$\boxed{7.3}$ use alternate series test

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

First do the absolute convergance test.

look at $\sum_{n=1}^{\infty} \frac{1}{n^2}$ this is a convergant series by integral test (see 6.34).

hence since abs. convergent, $\sum_{1}^{\infty} \frac{(-1)^n}{n^2}$ is $\boxed{\text{convergent}}$

$\boxed{7.5}$ $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

first look at $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ use comparison test with $\sum \frac{1}{n}$. since $\sum \frac{1}{n}$ diverges. $\frac{\frac{1}{\ln n}}{\frac{1}{n}} = \frac{n}{\ln n} \to \infty$ for large n, hence this is not absolutly convergent. then need to do more testing.

look at $\frac{|a_{n+1}|}{|a_n|} = \frac{\left|\frac{1}{\ln(n+1)}\right|}{\left|\frac{1}{\ln(n)}\right|} = \left|\frac{\ln(n)}{\ln(n+1)}\right|$ hence

$|a_{n+1}| \leq |a_n|$. so now look at $\lim_{n\to\infty} a_n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0$. hence $\boxed{\text{Converges Conditionally}}$

$\boxed{7.6}$ test alternate series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+5}$$

this is not absolutely convergent, since $\lim_{n\to\infty} \frac{n}{n+5} \to 1$

so use $\frac{|a_{n+1}|}{|a_n|}$ test.

$$\frac{\left|\frac{n+1}{(n+1)+5}\right|}{\left|\frac{n}{(n+5)}\right|} = \frac{(n+1)(n+5)}{(n+6)(n)} = \frac{n^2+6n+5}{n^2+6n}$$

so $\frac{|a_{n+1}|}{|a_n|} \geq 1$ i.e. $|a_{n+1}| \geq |a_n|$

hence $\boxed{\text{diverges}}$

$\boxed{9.1}$ test for convergence or divergence

$$\sum_{n=1}^{\infty} \frac{n-1}{(n+2)(n+3)}$$

preliminary test: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-1}{n^2+5n+6} = \lim_{n\to\infty} \frac{\frac{1}{n} - \frac{1}{n^2}}{1+\frac{5}{n}+\frac{6}{n^2}} = \frac{0}{1}$

$= 0$.

so must test more.

try ratio test

$$\rho_n = \frac{|a_{n+1}|}{|a_n|} = \frac{\frac{n}{(n+3)(n+4)}}{\frac{n-1}{(n+2)(n+3)}} = \frac{n(n+2)(n+3)}{(n-1)(n+3)(n+4)} = \frac{n^3+6n^2+8n}{n^3+6n^2+5n-12} = \frac{1+\frac{6}{n}+\frac{8}{n^2}}{1+\frac{6}{n}+\frac{5}{n^2}-\frac{12}{n^3}}$$

$\lim_{n\to\infty} \rho_n = \frac{1}{1} = 1$. hence use different test $\longrightarrow$

try comparison test

compare with $\sum \frac{1}{n}$ which diverges.

$$\frac{n-1}{(n+2)(n+3)} \overset{?}{\geq} \frac{1}{n}$$

i.e. $$\frac{\frac{n-1}{(n+2)(n+3)}}{\frac{1}{n}} \overset{?}{\geq} 1 \quad ; \quad \frac{(n-1)(n)}{n^2+5n+6} \overset{?}{\geq} 1$$

$$\frac{n^2-n}{n^2+5n+6} \overset{?}{\geq} 1$$ NO by looking at numerator and denominator.

so need to try against $\sum \frac{1}{n^2}$ for convergence (since $\sum \frac{1}{n^2}$ converges by integral test)

$$\frac{n-1}{(n+2)(n+3)} \overset{?}{\leq} \frac{1}{n^2}$$

$$\frac{\frac{n-1}{(n+2)(n+3)}}{\frac{1}{n^2}} \overset{?}{\leq} 1$$

$$\frac{(n-1)(n^2)}{n^2+5n+6} \overset{?}{\leq} 1 \qquad \frac{n^3-n^2}{n^2+5n+6} \overset{?}{\leq} 1 \qquad \frac{n-\frac{1}{n}}{\frac{1}{n}+\frac{5}{n^2}+\frac{6}{n^3}} \overset{?}{\leq} 1$$

as $n \to \infty$ this ratio is n. hence test is not usefull. need to try other test

try integral test ⟶

$$\int^{\infty} \frac{n-1}{(n+2)(n+3)}\,dn = -3\ln(2+n)+4\ln(3+n)\Big|^{\infty}$$

$$= 4\ln(\infty) - 3\ln(\infty) = \infty - \infty = 0$$

hence finite limit. hence $\boxed{\text{converges}}$

$\boxed{9.3}$ test for convergence or divergence

$$\sum_{1}^{\infty} \frac{1}{n^{\ln 3}}$$

premilist $\lim_{n\to\infty} \frac{1}{n^{\ln 3}} = 0$ hence must try other tests.

$\log 3 = 1.098 = C$ some constant.

try integral test

$$\int^{\infty} \frac{1}{n^{c}}\,dn = \frac{n^{1-C}}{1-C}\Big|^{\infty}$$

since $C = \ln 3 > 1$, then n^{1-C} goes to zero as $n\to\infty$

since $1-C$ is negative.

hence $I \longrightarrow 0$ as $n\to\infty$.

hence finite limit. hence $\boxed{\text{converges}}$

$\boxed{9.7}$ test for convergence or divergence

$$\sum_{n=0}^{\infty} \frac{(2n)!}{3^n (n!)^2}$$

prem. test. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2n)!}{3^n (n!)^2}$

$$= \lim_{n\to\infty} \frac{1\times 2\times\cdots\times n \times (n+1)\times(n+2)\times\cdots\times 2n}{3^n (1\times 2\times\cdots\times n)^2} = \frac{(n+1)(n+2)\cdots 2n}{3^n \; n!}$$

$$= \lim_{n\to\infty} \frac{n(1\times 2\times 3\times\cdots\times n)}{3^n \; n!} = \lim_{n\to\infty} \frac{n}{3^n} = 0$$

since 3^n grows faster than n.

hence need more testing.

ratio testing

$$a_{n+1} = \frac{(2(n+1))!}{3^{n+1}((n+1)!)^2}, \quad a_n = \frac{(2n)!}{3^n (n!)^2}$$

$$\rho_n = \frac{a_{n+1}}{a_n} = \frac{(2(n+1))! \; 3^n (n!)^2}{3^{n+1}((n+1)!)^2 \; (2n)!} = \frac{(2n+2)! \; (n!)^2}{3(1\times 2\times\cdots\times n\times(n+1))^2 \; (1\times 2\times\cdots\times n\times(n+1)\cdots 2n)}$$

$$\rho_n = \frac{1\times 2\times\cdots\times n\times(n+1)\times\cdots\times 2n\times(2n+1)\times(2n+2) \; (1\times 2\times\cdots\times n)^2}{3(1\times 2\times\cdots\times n\times(n+1))^2 \; (1\times 2\times\cdots\times n\times(n+1)\times\cdots\times 2n)}$$

$$= \frac{(n+2)\times(n+3)\times\cdots\times 2n\times(2n+1)(2n+2)}{3 \; (n+1)^2 \; ((n+2)(n+3)\cdots(2n))} = \frac{(2n+1)(2n+2)}{3(n+1)^2}$$

$$\rho_n = \frac{4n^2+6n+2}{3n^2+6n+3} \qquad \lim_{n\to\infty} \rho_n = \frac{4+\frac{6}{n}+\frac{2}{n^2}}{3+\frac{6}{n}+\frac{3}{n^2}} = \frac{4}{3} > 1$$ diverges

$\boxed{9.12}$

$$\sum_{n=2}^{\infty} \frac{1}{n^2-n}$$

prem. test

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2-n} = 0 \quad \text{since } n^2 > n \text{ for } n \geq 2.$$

so more testing needed.

try comparison test with $\sum_{n=2}^{\infty} \frac{1}{n^2}$ which is convergent by integral test.

is $\frac{1}{n^2-n} \overset{?}{\leq} \frac{1}{n^2}$ No so try another test.

try integral test

$$\int^{\infty} \frac{1}{n^2-n}\, dn = \ln(n-1) - \ln(n) \Big|^{\infty}$$

$$= \ln(\infty) - \ln(\infty) = \infty - \infty = 0$$

so finite limit. hence $\boxed{\text{Convergent}}$

$\boxed{9.21}$ $\sum_{n=1}^{\infty} a_n$ if $a_{n+1} = \frac{n}{2n+3} a_n$

$$a_1 = \frac{1}{2+3} a_0$$

$$a_2 = \frac{2}{4+3} a_1 = \frac{2}{4+3} \frac{1}{2+3} a_0$$

$$a_3 = \frac{3}{6+3} a_2 = \frac{3}{6+3} \frac{2}{4+3} \frac{1}{2+3} a_0$$

$$a_4 = \frac{4}{8+3} a_3 = \frac{4}{8+3} \frac{3}{6+3} \frac{2}{4+3} \frac{1}{2+3} a_0$$

so $\boxed{a_n = \frac{n!}{\prod_{n=1}^{\infty} (2n+3)}}$ $\Longrightarrow$

prom. test for $a_n = \dfrac{n!}{\prod_{n=1}^{\infty} (2n+3)}$

this goes to zero as $n \to \infty$. (because denominator is larger than numerator.

so need more testing.

chapter 1

$\boxed{10.1}$ find interval of convergence. be sure to investigate end points.

$$\sum_{n=0}^{\infty} (-1)^n x^n$$ use ratio test.

$$\rho = \frac{|a_{n+1}|}{|a_n|} = \left|\frac{x^{n+1}}{x^n}\right| = |x|$$

so $\boxed{\text{converges for } |x| < 1}$

$-1 \quad 0 \quad +1$

at $x = -1$, series is $\sum (-1)^n (-1)^n = \sum_{n=0}^{\infty} +1^{2n} = 1 + (1)^2 + (1)^4 + (1)^6 + \cdots$

$= 1 + 1 + 1 + 1 \cdots$

$\boxed{\text{This diverges at } x = -1}$

at $x = +1$

series is $\sum_{n=0}^{\infty} (-1)^n (+1)^n = \sum_{n=0}^{\infty} (-1)^{2n}$

$= -1^0 + (-1)^2 - (-1)^4 + (-1)^6 + \cdots$

$= 1 + 1 + 1 + 1 \cdots$ as above. so

$\boxed{\text{diverges at } x = +1}$

chapter 1

$\boxed{10.3}$ find interval of convergence.

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n(n+1)}$$

$$\rho_n = \frac{|a_{n+1}|}{|a_n|} = \left| \frac{\dfrac{x^{n+1}}{(n+1)(n+2)}}{\dfrac{x^n}{n(n+1)}} \right| = \left| \frac{x^{n+1}}{x^n} \frac{n(n+1)}{(n+1)(n+2)} \right|$$

$$\rho_n = \left| \frac{x \; n}{n+2} \right| = \left| \frac{x}{1+\frac{2}{n}} \right|$$

$$\lim_{n\to\infty} \rho_n = |x|$$ so $\boxed{\text{converges for } |x| < 1}$

at $x=+1$, series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (+1)^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n(n+1)}$ this removes -ve sign from all terms.

$$= \frac{-1^2}{2} + \frac{(-1)^4}{2\times 3} + \frac{(-1)^6}{3\times 4} + \cdots = \frac{1}{2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

now series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. to find if convergent, use integral test:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1)+Bn}{(n)(n+1)} \Rightarrow A(n+1)+Bn = 1 \Rightarrow A=1, B=-1$$

so $\displaystyle\int^{\infty} \frac{1}{n} - \frac{1}{n+1} \, dn = \ln(n) - \ln(1+n) \Big|^{\infty} = \infty - \infty = 0$

hence $\boxed{\text{convergent at } +1}$

at $x=-1$, series is $\displaystyle\sum \frac{(-1)^n (-1)^n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(+1)^{2n}}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

conjv[illegible]

same as above, so $\boxed{\text{convergent at } x = -1}$

chapter 1 find radius of convergence.

$\boxed{10.15}$

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n}$$

$$\rho_n = \frac{|a_{n+1}|}{|a_n|} = \left| \frac{\frac{(x-2)^{n+1}}{3^{n+1}}}{\frac{(x-2)^n}{3^n}} \right| = \left| \frac{(x-2)^{n+1} \; 3^n}{3^{n+1} (x-2)^n} \right|$$

$$\rho_n = \left| \frac{(x-2)}{3} \right| \quad \text{so} \lim_{n\to\infty} \rho_n = \left| \frac{x-2}{3} \right|$$

so convergent for $\left| \frac{x-2}{3} \right| < 1$ i.e. $\left| \frac{1}{3}(x-2) \right| < 1$

i.e. $|x-2| < 3$ i.e. $-3 < x-2 < 3$ or $\boxed{-1 < x < 5}$

at $\boxed{x=+5}$ $\sum_{n=1}^{\infty} \frac{-3^n}{3^n} = \sum_{n=1}^{\infty} -1 = -1-1-1-1\ldots$

so $\boxed{\text{divergent at } x=+5}$

at $x=-1$ $\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=1}^{\infty} -1 = -1-1-1\ldots$

so $\boxed{\text{divergent at } x=-1}$

?? ask about this. this should be $-1+1-1+1-1+1$ which is convergent to $\boxed{1 \text{ or } -1}$

chapter 1 find where series converge

$\boxed{10.19}$

$$\sum_0^\infty 8^{-n}(x^2-1)^n$$

let $y = x^2-1$ then $\sum_0^\infty 8^{-n} y^n$

$$\rho_n = \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{8^{-(n+1)} y^{n+1}}{8^{-n} y^n}\right| = \left|\frac{y}{8}\right|$$

so $\lim_{n\to\infty} \rho_n = \left|\frac{y}{8}\right|$ so converges for $\left|\frac{y}{8}\right| < 1$

or $|y| < 8$

i.e. $|x^2-1| < 8$?

or $|x^2| < 9$

or $\boxed{|x| < 3}$

test at -3

$$\sum_0^\infty 8^{-n}(9-1)^n = \sum_0^\infty 8^{-n} 8^n = \sum_0^\infty 8^0 = \sum_{n=0}^\infty 1$$

$= 1+1+1+\cdots$ $\boxed{\text{diverges at } -3}$

test at $+3$

$$\sum_0^\infty 8^{-n}(9-1)^n$$ this is the same as above

so $\boxed{\text{diverges at } +3}$

$\boxed{10.20}$ test for convergence or divergence :

$$\sum_0^\infty (-1)^n \frac{2^n}{n!} (x^2+1)^{2n}$$

let $y = x^2+1$ so series is $\sum_0^\infty (-1)^n \frac{2^n}{n!} y^{2n}$

$$\rho_n = \left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{2^{n+1} y^{2(n+1)}}{(n+1)!}}{\frac{2^n y^{2n}}{n!}} = \left|\frac{2^{n+1} y^{2(n+1)} n!}{(n+1)!\, 2^n y^{2n}}\right|$$

$$= \left|\frac{2y^2 \, n!}{\underbrace{1\times 2\times 3\times 4\times \cdots \times n}_{n!} \times n+1}\right| = \left|\frac{2y^2}{n+1}\right|$$

so $\lim_{n\to\infty} \rho_n = 0$

so converges for all values of y.

but $y = x^2+1$ i.e $x^2 = y-1$

or $x = \pm\sqrt{y-1}$

since converges for any value of y, x can attain any values as well.

$\boxed{\text{hence converges for any value of } X}$

Chapt. 1

$\boxed{13.2}$ expand in power series

$e^x \sin(x)$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$(e^x \sin x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!}$$

$$x^2 + 0 - \frac{x^4}{3!} + 0 + \frac{x^6}{5!} + 0 - \frac{x^8}{7!} + \cdots$$

$$\frac{x^3}{2!} + 0 - \frac{x^5}{2!\,3!} + 0 + \frac{x^7}{2!\,5!} + 0$$

$$+ \frac{x^4}{3!} + 0 - \frac{x^6}{3!\,3!} + 0 + 0$$

$$+ \frac{x^5}{4!} + 0 - \frac{x^7}{4!\,3!} + 0$$

$$= x + x^2 - \frac{x^3}{6} + \frac{x^3}{2!} + \frac{x^5}{5!} - \frac{x^5}{2!\,3!} + \frac{x^5}{4!} + \frac{x^6}{5!} - \frac{x^6}{3!\,3!}$$

$$= x + x^2 + \frac{-x^3 + 3x^3}{6} + \frac{x^5}{30} + \frac{x^6}{90} + \cdots$$

$$= \boxed{x + x^2 + \frac{x^3}{3} + \frac{x^5}{30} + \frac{x^6}{90} + \cdots}$$

Chapter 1

$\boxed{13.6}$ Find power series for $\dfrac{e^x}{1-x}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \qquad \text{for all } x$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \qquad -1 < x < 1$$

$$\frac{e^x}{1-x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right)\left(1 + x + x^2 + x^3 + x^4 + x^5 + \cdots\right)$$

$$\begin{aligned}
= 1 + x + x^2 + x^3 + x^4 + x^5 &+ \cdots \\
x + x^2 + x^3 + x^4 + x^5 &+ \cdots \\
\frac{x^2}{2!} + \frac{x^3}{2!} + \frac{x^4}{2!} + \frac{x^5}{2!} &+ \cdots \\
\frac{x^3}{3!} + \frac{x^4}{3!} + \frac{x^5}{3!} &+ \cdots \\
\frac{x^4}{4!} + \frac{x^5}{4!} &+ \cdots \\
\frac{x^5}{5!} &+ \cdots
\end{aligned}$$

$$1 + 2x + x^2 + \frac{x^2}{2} + 2x^3 + \frac{x^3}{2!} + \frac{x^3}{3!} + 2x^4 + \frac{x^4}{2!} + \frac{x^4}{3!} + \frac{x^4}{4!} +$$
$$2x^5 + \frac{x^5}{2!} + \frac{x^5}{3!} + \frac{x^5}{4!} + \frac{x^5}{5!} + \cdots$$

$$\boxed{= 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \frac{65}{24}x^4 + \frac{163}{60}x^5 + \cdots}$$

Chapter 1

$\boxed{13.8}$ Find power series for $\sec x = \frac{1}{\cos x}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

hence $(\sec x)(\cos x) = 1$

so assume $\sec x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

then $(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) = 1$

$$\Rightarrow \quad a_0 + 0 - \frac{a_0}{2}x^2 + 0 + \frac{a_0}{4!}x^4 + 0 + \cdots$$
$$a_1 x + 0 - \frac{a_1}{2}x^3 + 0 + \frac{a_1}{4!}x^5 + \cdots$$
$$a_2 x^2 + 0 - \frac{a_2}{2}x^4 + 0 + \cdots$$
$$a_3 x^3 + 0 - a_3\frac{x^5}{2} + \cdots$$
$$a_4 x^4 + 0 - \frac{a_4}{2}x^6 + \cdots$$

$$a_0 + a_1 x + x^2\left(a_2 - \frac{a_0}{2}\right) + x^3\left(a_3 - \frac{a_1}{2}\right) + x^4\left(\frac{a_0}{4!} - \frac{a_2}{2} + a_4\right) + \cdots = 1$$

by comparing terms.

hence $a_0 = 1$

$a_1 = 0$

$a_2 - \frac{a_0}{2} = 0 \implies a_2 = \frac{1}{2}$

$a_3 - \frac{a_1}{2} = 0 \implies a_3 = 0$

$\frac{a_0}{4!} - \frac{a_2}{2} + a_4 = 0 \implies a_4 = -\frac{a_0}{4!} + \frac{a_2}{2} = -\frac{1}{4!} + \frac{1}{4} = \frac{5}{24}$

hence $\boxed{\sec(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots}$

$\boxed{13.13}$ Find power series for

$\sin x^2$.

let $y = x^2$

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots$$

so $\sin x^2 = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots$

$$\boxed{\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots}$$

$\boxed{13.14}$

$\frac{\sin\sqrt{x}}{\sqrt{x}}$ $x > 0$

let $y = \sqrt{x}$, so $\frac{\sin\sqrt{x}}{\sqrt{x}} = \frac{\sin y}{y} = \frac{y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots}{y}$

$$= 1 - \frac{y^2}{3!} + \frac{y^4}{5!} - \frac{y^6}{7!} + \cdots$$

but $y = x^{1/2}$. so

$$\boxed{\frac{\sin\sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots}$$

chapter 1

$\boxed{13.16}$ Find power series for

$\sin[\ln(1+x)]$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad -1 < x \leq 1$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad$ for all x.

let $\ln(1+x) = y$.

$\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots$

$\sin(\ln(1+x)) = \underbrace{\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right]}_{\ln(1+x)} - \frac{1}{3!}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right]^3 + \frac{1}{5!}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right]^5 + \cdots$

then write it as

$= \ln(1+x) - \frac{1}{3!}\ln^3(1+x) + \frac{1}{5!}(\ln^5(1+x)) + \cdots$

$= \ln(1+x)\left[1 - \frac{1}{3!}\ln^2(1+x) + \frac{1}{5!}\ln^4(1+x) + \cdots\right]$

looking at only 3 terms in $\ln(1+x)$.

$= \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)\left[1 - \frac{1}{3!}\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)^2 + \frac{1}{5!}\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)^5 - \cdots\right]$

$\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)^2 = \left(\left(x - \frac{x^2}{2}\right) + \frac{x^3}{3}\right)^2 = \left(x - \frac{x^2}{2}\right)^2 + \frac{x^6}{3} + 2\,\frac{x^3}{3}\left(x - \frac{x^2}{2}\right)$

$= x^2 + \frac{x^4}{2} - 2\frac{x^3}{2} + \frac{x^6}{3} + \frac{2}{3}\left(x^4 - \frac{x^5}{2}\right) = x^2 + \frac{1}{2}x^4 - x^3 + \frac{1}{3}x^6 + \frac{2}{3}x^4 - \frac{1}{3}x^5$

$= x^2 - x^3 + \frac{7}{6}x^4 - \frac{1}{3}x^5 + \frac{1}{3}x^6$

so $\sin(\ln(1+x)) = \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)\left(1 - \frac{1}{3!}\left(x^2 - x^3 + \frac{7}{6}x^4 - \frac{1}{3}x^5 + \frac{1}{3}x^6\right)\right)$

$= \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)\left(1 - \frac{x^2}{6} + \frac{x^3}{6} - \frac{7}{36}x^4 + \frac{x^5}{18} + \frac{x^6}{18}\right) \Longrightarrow$

$$\sin(\ln(1+x)) = x + 0 - \frac{x^3}{6} + \frac{x^4}{6} - \frac{7}{36}x^5 + \frac{x^6}{18} + \frac{x^7}{18}$$

$$-\frac{x^2}{2} + 0 - \frac{x^4}{12} - \frac{x^5}{12} + \frac{7}{72}x^6 - \frac{x^7}{36} + \cdots$$

$$\frac{x^3}{3} + 0 - \frac{x^5}{18} + \frac{x^6}{18} - \frac{7x^7}{3\times 36} + \cdots$$

$$\boxed{x - \frac{x^2}{2} + \frac{1}{6}x^3 + \cdots}$$

these are the terms I can be sure about since I omitted $\frac{1}{5!}\ln^5(1+x)$ earlier as multiplication was becoming messy

is there a more direct method to do this ?

$\boxed{13.17}$ expand in power series $\int_0^x \cos t^2\, dt$.

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!}$$

so $$\cos t^2 = 1 - \frac{t^4}{2!} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \frac{t^{16}}{8!} - \dots$$

hence $$\int_0^x \cos t^2\, dt = \int_0^x 1 - \frac{t^4}{2} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \frac{t^{16}}{8!} \cdots \; dt$$

$$= t\Big|_0^x - \frac{1}{2}\left[\frac{t^5}{5}\right]_0^x + \frac{1}{4!}\left[\frac{t^9}{9}\right]_0^x - \frac{1}{6!}\left[\frac{t^{13}}{13}\right]_0^x + \cdots$$

$$= (x-0) - \frac{1}{2}\left(\frac{x^5}{5}\right) + \frac{1}{4!}\left(\frac{x^9}{9}\right) - \frac{1}{6!}\left(\frac{x^{13}}{13}\right) + \cdots$$

$$= x - \frac{x^5}{10} + \frac{x^9}{4!\,9} - \frac{x^{13}}{6!\,13} + \cdots$$

$$\boxed{= x - \frac{x^5}{10} + \frac{x^9}{216} - \frac{x^{13}}{9360} + \cdots}$$

Chapter 1

$\boxed{13.32}$ Find power series for $\ln(\cos x)$

$$\ln \cos x = -\int_0^x \tan u \, du.$$

but $\tan u = u + \dfrac{u^3}{3} + \dfrac{2u^5}{15} + \dfrac{17u^7}{315} + \cdots$

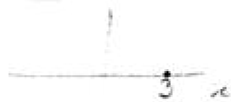
so $-\int_0^x \tan u \, du = -\int_0^x u + \frac{u^3}{3} + \frac{2}{15}u^5 + \frac{17}{315}u^7 + \cdots \quad du$

$$= -\left(\left[\frac{u^2}{2}\right]_0^x + \frac{1}{3}\left[\frac{u^4}{4}\right]_0^x + \frac{2}{15}\left[\frac{u^6}{6}\right]_0^x + \frac{17}{315}\left[\frac{u^8}{8}\right]_0^x + \cdots\right]$$

$$= -\left(\frac{x^2}{2} + \frac{1}{3}\frac{x^4}{4} + \frac{2}{15}\frac{x^6}{6} + \frac{17}{315}\frac{x^8}{8} + \cdots\right)$$

$$\boxed{= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \cdots}$$

$\boxed{13.37}$ expand e^x about $a=3$



i.e. expand $e^{(x-3+3)} = e^{(x-3)}e^3$

$$= e^3\left(1 + (x-3) + \frac{(x-3)^2}{2!} + \frac{(x-3)^3}{3!} + \frac{(x-4)^4}{4!} + \cdots\right)$$

$$= \boxed{e^3 + e^3(x-3) + \frac{e^3}{2}(x-3)^2 + \frac{e^3}{3!}(x-3)^3 + \frac{e^3}{4!}(x-4)^4 + \cdots}$$

ch 1

$\boxed{13.40}$

$f(x) = \sqrt{x}$ about point $a = 25$

$$\sqrt{x} = \sqrt{x-25+25} = \sqrt{25\left(1+\frac{x-25}{25}\right)} = 5\sqrt{1+\frac{x-25}{25}}$$

expand. $\left(1+\frac{x-25}{25}\right)^{1/2}$. let $z = \frac{x-25}{25}$

$$\text{so } (1+z)^{1/2} \equiv (1+z)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$$

$$\text{so } (1+z)^{1/2} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \dots$$

$$= 1 + \frac{x}{2} - \frac{\frac{1}{4}}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \dots$$

$$= 1 + \frac{x}{2} - \frac{\cancel{x}\,1}{(4)\,2!}x^2 + \frac{3}{2^3\,3!}x^3 + \dots$$

← These should be 'z' not 'x'

So answer is

$$\boxed{\sqrt{x} = 5\left(1 + \frac{\left(\frac{x-25}{25}\right)}{2} - \frac{\left(\frac{x-25}{25}\right)^2}{2^2\,2!} + \frac{3}{2^3\,3!}\left(\frac{x-25}{25}\right)^3 \dots\right)}$$

$$\boxed{\sqrt{x} = 5\left(1 + \tfrac{1}{2}\left(\frac{x-25}{25}\right) + \dots + \frac{n}{2^n\,n!}\left(\frac{x-25}{25}\right)^n + \dots\right.}$$

→ a different solution →

Chapter 1 **this is another way to solve $\boxed{13.40}$** 41

$\boxed{13.40}$ expand $f(x)=\sqrt{x}$ about point $a=25$

First find power series for $x^{1/2}$

$$f(x) = x^{1/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2}$$

$$f''(x) = -\frac{1}{4}x^{-3/2}$$

$$f'''(x) = \frac{3}{8}x^{-5/2}$$

$$f''''(x) = -\frac{15}{16}x^{-7/2}$$

So power series for $x^{1/2}$ at $x=25$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \frac{1}{3!}(x-a)^3 f'''(a) + \cdots$$

$$= \sqrt{25} + (x-25)\frac{1}{2}\frac{1}{\sqrt{25}} + \frac{1}{2!}(x-25)^2\left(-\frac{1}{4}\frac{1}{(\sqrt{25})^3}\right) + \frac{1}{3!}(x-25)^3\left(\frac{3}{8}\frac{1}{(\sqrt{25})^5}\right)$$

$$= 5 + \frac{(x-25)}{10} - \frac{(x-25)^2}{2\times 4\times 5^3} + \frac{(x-25)^3}{2\times 8\times 5^5} \cdots$$

$$\boxed{= 5 + \frac{1}{10}(x-25) - \frac{1}{1000}(x-25)^2 + \frac{(x-25)^3}{50000} \cdots}$$