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Notes by Nasser M. Abbasi
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This is a simple way to remember how to calculate the FFT of a vector.
If n is the number of coordinates (or data points) in the vector x, then let

$$\mathrm{X} = fft(x)$$

X is a complex vector of the same number of coordinates (or data points) as x

Let $x = (a,b,c)$ be the vector (possibly complex) that we want to find the fft for.
We will do a dot product of the above vector with vectors whose coordinates are the roots of unity.

Recall that there are n roots ε such that $\varepsilon^n = 1$
But

$$1 = \cos(2\pi) + i\sin(2\pi) = e^{2\pi i}$$

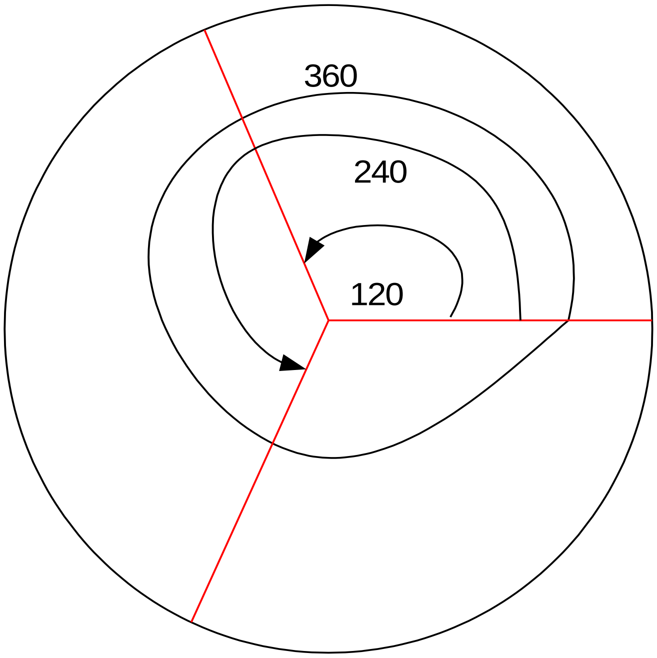
so the n roots of unity are

$$\varepsilon = 1^{\frac{1}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) = e^{\frac{2\pi i}{n}}$$

So, we divide the angle 2π by the number of roots, and each root will have the same magnitude of 1, but it will be at an angle of $k\left(\frac{2\pi}{n}\right)$ multiples where $k = 0,1,2,\ldots,n-1$.
This is because, with complex numbers, when we multiply one by the other, we add angles. Hence when we multiply a complex number by itself n times, we add n times the angle it had with the x-axis. Since we want to get 1 at the end (which has 360 angle), we divided 360 by n to get the above equation.

So, for $n=1$ there is one root, which is 1. for $n=2$ there are 2 roots, which are for $k=0,1$, which are 1 and $e^{\pi i}$ =-1 and so on.

To see this better, use the argand diagram. For example, this below are the 3 roots of unity. Since $n=3$, then we divide 360 degrees by the number of roots, and each unity root has an angle of $\frac{2\pi}{3}$ or 120 degrees away from the previous root.



3 roots of unity. Hence 360/3 = 120 degrees.

What does the roots of unity have to do with FFT?

Let me show how they are used.

In the case of $n=3$ (number of coordinates, or number of data points), we construct the 3 roots of unity.

Let $\omega = \exp^{\frac{2\pi i}{n}}$, then the roots of unity be written down as

$$\omega = (\omega^0, \omega^1, \ldots, \omega^{n-1})$$

but $n=3$, so we get

$$\omega = (\omega^0, \omega^1, \omega^2)$$

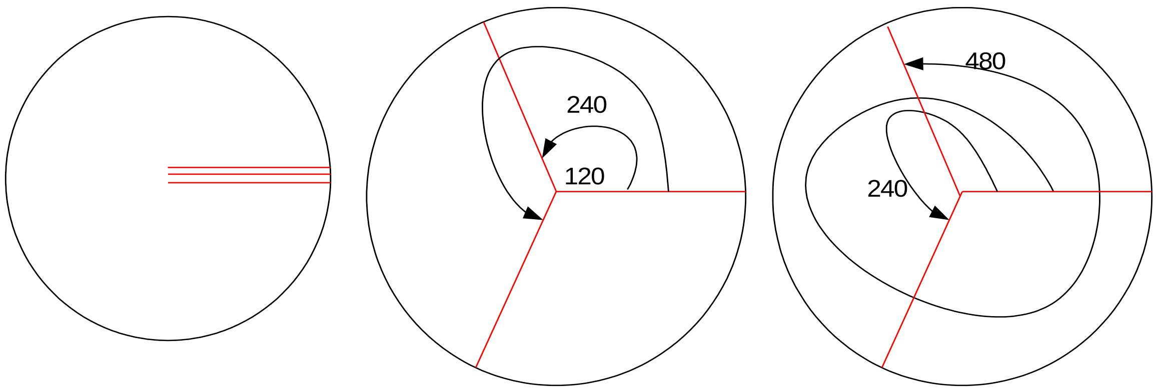
So, the exponent multipliers above, are the angle multipliers

Now, from this one set of roots of unity shown above, generate n sets by multiplying the exponents of ω inside the brackets by zero, then by one, then by two, then by three, etc… until $n-1$. When we multiply the exponent, this means we are rotating the root of unity vector around.

Since $n=3$ here, we will get the 3 different sets of roots of unity, all generated from the original $(\omega^0, \omega^1, \omega^2)$:

$(\omega^0, \omega^0, \omega^0)$
$(\omega^0, \omega^1, \omega^2)$
$(\omega^0, \omega^2, \omega^4)$

This is a graphical representation of the above 3 sets



Notice that the roots are the same, we just change the angle of rotation to get to the root each time.

Now, align the x vector on top of these roots of unity vectors, we get

$(a, \quad b, \quad c)$
$(\omega^0, \omega^0, \omega^0)$
$(\omega^0, \omega^1, \omega^2)$
$(\omega^0, \omega^2, \omega^4)$

Now to get the coordinates of X, do the dot product of x with each of the vectors below it one at a time. Remember that the dot product of two vectors is just one number (possibly complex) and not a set of numbers (or a vector).

So, the first coordinate of X will be

$$(a, \quad b, \quad c) \bullet (\omega^0, \omega^0, \omega^0)$$

And the second coordinate of X will be the dot product of x with the second vector of the roots of unity, that is

$$(a, \quad b, \quad c) \bullet (\omega^0, \omega^1, \omega^2)$$

And the third and final coordinate will be

$$(a, \quad b, \quad c) \bullet (\omega^0, \omega^2, \omega^4)$$

and the n^{th} coordinate is

$$(a, \quad b, \quad c) \bullet (\omega^0, \omega^{1*n}, \omega^{2*n})$$

Example

Let me show this with an simple example. Let

$$x = (1, 4, 5, 6)$$

be the data we want to find its FFT. Here $n = 4$, hence

$$\omega = (\omega^0, \omega^1, \omega^2, \omega^3)$$

so we need 4 vectors of roots of unity generated from the above by multiplying the exponents by 0,1,2 and 3 at a time, we get

$$(\omega^0, \omega^0, \omega^0, \omega^0)$$
$$(\omega^0, \omega^1, \omega^2, \omega^3)$$
$$(\omega^0, \omega^2, \omega^4, \omega^6)$$
$$(\omega^0, \omega^3, \omega^6, \omega^9)$$

Now do the dot product of x with each one of these vectors one at a time. Each time we do a dot product, we get one data point in the FFT domain generated.

Notice that

$$\omega^0 = e^{0\left(\frac{2\pi i}{4}\right)} = 1$$
$$\omega^1 = e^{1\left(\frac{2\pi i}{4}\right)} = e^{\frac{i\pi}{2}}$$
$$\omega^2 = e^{2\left(\frac{2\pi i}{4}\right)} = e^{i\pi}$$
$$\omega^3 = e^{3\left(\frac{2\pi i}{4}\right)} = e^{\frac{3\pi i}{2}}$$
$$\omega^4 = e^{4\left(\frac{2\pi i}{4}\right)} = e^{2\pi i}$$
$$\omega^6 = e^{6\left(\frac{2\pi i}{4}\right)} = e^{3\pi i}$$
$$\omega^9 = e^{9\left(\frac{2\pi i}{4}\right)} = e^{\frac{9\pi i}{2}}$$

notice that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

so we get

$$\omega^0 = 1$$
$$\omega^1 = i$$
$$\omega^2 = -1$$
$$\omega^3 = -i$$
$$\omega^4 = 1$$
$$\omega^6 = -1$$
$$\omega^9 = i$$

so, our 4 vectors of unity are now

$$(1,1,1,1)$$
$$(1,i,-1,-i)$$
$$(1,-1,1,-1)$$
$$(1,-i,-1,i)$$

Now do the dot product of x with each of the above vectors, and this will give us the FFT.

$$(1,4,5,6)\bullet(1,1,1,1) = 16$$
$$(1,4,5,6)\bullet(1,i,-1,-i) = -4-2i$$
$$(1,4,5,6)\bullet(1,-1,1,-1) = -4$$
$$(1,4,5,6)\bullet(1,-i,-1,i) = -4+2i$$

so,

$$FFT\left[x\right] = FFT\begin{bmatrix}1\\4\\5\\6\end{bmatrix} = \mathrm{X} = \begin{bmatrix}16\\-4-2i\\-4\\-4+2i\end{bmatrix}$$