Quizz 3

Math 2520
Differential Equations and Linear Algebra

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Contents

1 Problem 1

Determine whether or not the given matrix A is diagonalizable. If it is find a diagonalizing matrix P and a diagonal matrix D such that $P^{-1}AP = D$

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Solution

The first step is to find the eigenvalues. This is found by solving

$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix} = 0$$
$$(3-\lambda)(-1-\lambda) = 0$$

Hence the eigenvalues are $\lambda_1 = 3, \lambda_2 = -1$. Since the eigenvalues are unique, then the matrix is diagonalizable. We need to determine the corresponding eigenvector in order to find P

$\lambda_1 = 3$

Solving

$$\begin{bmatrix} 3-\lambda_1 & 0 \\ 8 & -1-\lambda_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3-3 & 0 \\ 8 & -1-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence $v_1 = t$ is free variable and v_2 is base variable. Second row gives $8t - 4v_2 = 0$ or $v_2 = 2t$. Therefore the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Let $t = 1$, then

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\underline{\lambda_2 = -1}$

Solving

$$\begin{bmatrix} 3-\lambda_2 & 0 \\ 8 & -1-\lambda_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3+1 & 0 \\ 8 & -1+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$ gives

$$\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence $v_2 = t$ is free variable and v_1 is base variable. First row gives $4v_1 = 0$ or $v_1 = 0$. Therefore the eigenvector is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$, then

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Now that both eigenvectors are found, then

$$\begin{aligned} P &= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \end{aligned}$$

And D is the diagonal matrix of the eigenvalues arranged in same order as the corresponding eigenvectors. (Will verify below). Hence

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Therefore

$$P^{-1}AP = D$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \tag{1}$$

To verify the above, the LHS of (1) is evaluated directly, to confirm that D is indeed the result and it is diagonal of the eigenvalues. The first step is to find $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1}$. Since this is 2×2 then

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{\det(P)} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

But $\det(P) = 1$. The above becomes

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

Therefore the LHS of (1) becomes

$$\begin{aligned}
\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \times 3 & 0 \\ -2 \times 3 + 1 \times 8 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 0 \\ -6 + 8 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}
\end{aligned}$$

And now LHS of (1) becomes

$$\begin{aligned}
\begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} &= \begin{bmatrix} 3 & 0 \\ 2 \times 1 - 1 \times 2 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}
\end{aligned}$$

Hence

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Which confirms D is the matrix whose diagonal elements are the eigenvalues of A.

2 Problem 2

Find the general solution of the homogeneous differential equation $y''' + y' - 10y = 0$

Solution

This is a linear 3rd order constant coefficient ODE. Hence the method of characteristic equation will be used. Let the solution be $y = Ae^{\lambda x}$. Substituting this into the ODE gives

$$
\begin{aligned}
A\lambda^3 e^{\lambda x} + A\lambda e^{\lambda x} - 10Ae^{\lambda x} &= 0 \\
Ae^{\lambda x}\left(\lambda^3 + \lambda - 10\right) &= 0
\end{aligned}
$$

Which simplifies (for non-trivial y) to the characteristic equation which is a polynomial in λ

$$\lambda^3 + \lambda - 10 = 0$$

By inspection, we see that $\lambda = 2$ is a root. Therefore a factor of the equation is $(\lambda - 2)$. Now doing long division $\frac{\lambda^3+\lambda-10}{(\lambda-2)}$ gives $\lambda^2 + 2\lambda + 5$.

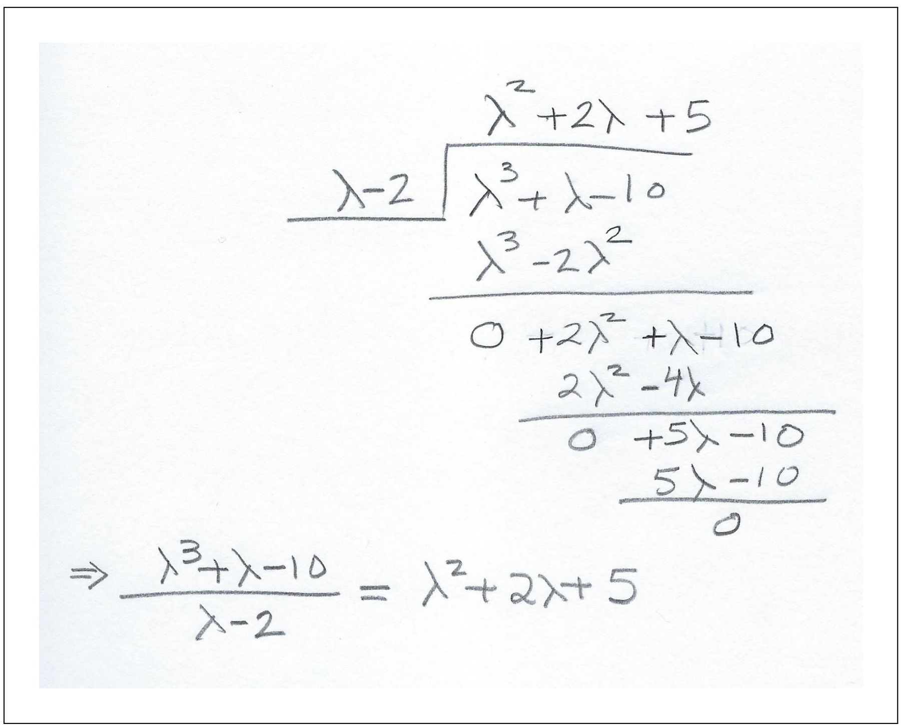

Figure 1: Polynmial long division to find remainder

Hence the above polynomial can be written as

$$(\lambda - 2)\left(\lambda^2 + 2\lambda + 5\right) = 0 \tag{1}$$

Now the roots for $\left(\lambda^2 + 2\lambda + 5\right) = 0$ are found using the quadratic formula.

$$
\begin{aligned}
\lambda &= -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} \\
&= -\frac{2}{2} \pm \frac{1}{2}\sqrt{4 - 4 \times 5} \\
&= -1 \pm \frac{1}{2}\sqrt{-16} \\
&= -1 \pm \frac{1}{2}(4i) \\
&= -1 \pm 2i
\end{aligned}
$$

Hence the roots of the characteristic equation are

$$
\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -1 + 2i \\
\lambda_3 &= -1 - 2i
\end{aligned}
$$

Therefore the basis solution are $\left\{e^{2x}, e^{(-1+2i)x}, e^{(-1-2i)x}\right\}$ and the general solution is a linear combination of these basis solutions which gives

$$
y = Ae^{2x} + Be^{(-1+2i)x} + Ce^{(-1-2i)x}
$$

Which can be simplified to

$$
y = Ae^{2x} + e^{-x}\left(Be^{2ix} + Ce^{-2ix}\right) \tag{2}
$$

By using Euler formula, the above can be simplified further as follows

$$
\begin{aligned}
Be^{2ix} + Ce^{-2ix} &= B(\cos(2x) + i\sin(2x)) + C(\cos(2x) - i\sin(2x)) \\
&= (B + C)\cos(2x) + \sin(2x)(i(B - C))
\end{aligned}
$$

Let $(B + C) = B_0$ a new constant and let $i(B - C) = C_0$ a new constant, the above becomes

$$
Be^{2ix} + Ce^{-2ix} = B_0\cos(2x) + C_0\sin(2x)
$$

Substituting the above back in (2) gives the general solution as

$$
y = Ae^{2x} + e^{-x}(B_0\cos(2x) + C_0\sin(2x))
$$

The constants A, B_0, C_0 can be found from initial conditions if given.

3 Problem 3

Using the method of undermined coefficients, compute the general solution of the given equation $y'' + 3y' + 2y = 2\sin(x)$

Solution

The solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' + 3y' + 2y = 0$ and y_p is a particular solution to the given ODE. The ode $y'' + 3y' + 2y = 0$ is linear second order constant coefficient ODE. Hence the method of characteristic equation will be used. Let the solution be $y_h = Ae^{\lambda x}$. Substituting this into $y'' + 3y' + 2y = 0$

$$\begin{aligned} A\lambda^2 e^{\lambda x} + A\lambda 3 e^{\lambda x} + 2Ae^{\lambda x} &= 0 \\ Ae^{\lambda x}\left(\lambda^2 + 3\lambda + 2\right) &= 0 \end{aligned}$$

And for non trivial solution the above simplifies to

$$\begin{aligned} \lambda^2 + 3\lambda + 2 &= 0 \\ (\lambda + 1)(\lambda + 2) &= 0 \end{aligned}$$

Hence the roots are $\lambda = -1, \lambda = -2$. Therefore the basis solutions for y_h are $\left\{e^{-x}, e^{-2x}\right\}$ and y_h is linear combination of these basis. Therefore

$$y_h = c_1 e^{-x} + c_2 e^{-2x} \tag{1}$$

Now y_p is found. Since the RHS is $\sin(x)$ then the trial solution is

$$y_p = A\cos(x) + B\sin(x) \tag{2}$$

This shows that the basis for y_p are $\{\sin x, \cos x\}$. There are no duplication between these basis and the basis for y_h, so no need to multiply by an extra x. Using (2) gives

$$y_p' = -A\sin(x) + B\cos(x) \tag{3}$$

$$y_p'' = -A\cos(x) - B\sin(x) \tag{4}$$

Substituting (2,3,4) back into the given ODE gives

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= 2\sin(x) \\ (-A\cos(x) - B\sin(x)) + 3(-A\sin(x) + B\cos(x)) + 2(A\cos(x) + B\sin(x)) &= 2\sin(x) \\ \cos(x)(-A + 3B + 2A) + \sin(x)(-B - 3A + 2B) &= 2\sin(x) \\ \cos(x)(3B + A) + \sin(x)(-3A + B) &= 2\sin(x) \end{aligned}$$

Comparing coefficients on both sides gives two equations to solve for A, B

$$\begin{aligned} 3B + A &= 0 \\ -3A + B &= 2 \end{aligned}$$

Multiplying the second equation by -3 gives

$$
\begin{aligned}
3B + A &= 0 \\
9A - 3B &= -6
\end{aligned}
$$

Adding the above two equations gives $10A = -6$. Hence $A = -\frac{3}{5}$ and therefore $3B = \frac{3}{5}$ or $B = \frac{1}{5}$. Substituting these values of A, B into (2) gives

$$
y_p = -\frac{3}{5}\cos(x) + \frac{1}{5}\sin(x)
$$

Hence the solution becomes

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left(-\frac{3}{5}\cos(x) + \frac{1}{5}\sin(x)\right) \\
&= c_1 e^{-x} + c_2 e^{-2x} - \frac{3}{5}\cos(x) + \frac{1}{5}\sin(x)
\end{aligned}
$$

4 Problem 4

Show that the given vector functions are linearly independent

$$\vec{x}_1(t) = \begin{bmatrix} e^t \\ 2e^t \end{bmatrix} \qquad \vec{x}_2(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

Solution

These functions are defined for all t. Hence domain is $t \in (-\infty, \infty)$. The Wronskian of these vectors is

$$\begin{aligned} W(t) &= \begin{vmatrix} e^t & \sin t \\ 2e^t & \cos t \end{vmatrix} \\ &= e^t \cos t - 2e^t \sin t \\ &= e^t (\cos t - 2\sin t) \end{aligned}$$

We just need find one value t_0 where $W(t_0) \neq 0$ to show linearly independence. At $t = 0$ the above becomes

$$W(t = 0) = 1$$

Therefore the given vector functions are linearly independent. An alternative method is to write

$$\begin{aligned} c_1\vec{x}_1(t) + c_2\vec{x}_2(t) &= \vec{0} \\ c_1 \begin{bmatrix} e^t \\ 2e^t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

If the above is true only for $c_1 = 0, c_2 = 0$ then $\vec{x}_1(t), \vec{x}_2(t)$ are linearly independent. The above can be written as

$$\begin{bmatrix} e^t & \sin t \\ 2e^t & \cos t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$$\begin{bmatrix} e^t & \sin t \\ 0 & \cos t - 2\sin t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Row two gives

$$c_2 (\cos t - 2\sin t) = 0$$

For this to be true for any t in the interval $t \in (-\infty, \infty)$, then only solution is $c_2 = 0$. First row now gives

$$c_1 e^t = 0$$

But e^t is never zero which means $c_1 = 0$.

Since the only solution to $c_1\vec{x}_1 + c_2\vec{x}_2 = \vec{0}$ is $c_1 = c_2 = 0$, then this shows that $\vec{x}_1, \vec{x}_2$ are linearly independent.