Final Exam

Math 2520
Differential Equations and Linear Algebra

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Contents

1 Problem 1

Solve the following system of equations and write the solution as a parametric vector form

$$\begin{aligned} x + 2y - 3z &= 5 \\ 2x + y - 3z &= 13 \\ -x + y &= -8 \end{aligned}$$

Solution

In matrix form $Ax = b$, the above system is

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ -8 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 1 & -3 & 13 \\ -1 & 1 & 0 & -8 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ -1 & 1 & 0 & -8 \end{bmatrix}$$

$R_3 = R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 3 & -3 & -3 \end{bmatrix}$$

$R_3 = R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence original system (1) becomes

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} \tag{2}$$

The above shows that $z = t$ is a free variable and x, y are basic variables. Second row gives $-3y + 3t = 3$ or $-y + t = 1$ or $y = t - 1$. First row gives $x + 2y - 3t = 5$ or $x = 5 + 3t - 2(t - 1)$ or $x = 7 + t$. Hence the solution is

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 7 + t \\ t - 1 \\ t \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ t \\ t \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

The above is the solution in parametric vector form. For any value of the parameter t, a solution exist.

2 Problem 2

Compute the determinant using a cofactor expansion

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

<u>solution</u>

Expanding along the last row since it has most number of zeros gives (only the element $A(3,2) \neq 0$)

$$\begin{aligned} \det(A) &= (-1)^{3+2}(-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \\ &= 2\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \\ &= 2(-1) \\ &= -2 \end{aligned}$$

3 Problem 3

Let

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$$

$$u = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$$

a is u in NullSpace of A ? Justify your answer.

b Is u in columnspace of A ? Justify your answer.

c Determine the rank A and the Nullity of A. Show your work

solution

3.1 Part a

For an $m \times n$ matrix, the solution set corresponding to $A\vec{x} = \vec{0}$ is called the NullSpace(A). Therefore we need to first find this solution set by solving

$$\begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -3 & -4 & 0 \\ -4 & 6 & -2 & 0 \\ -3 & 7 & 6 & 0 \end{bmatrix}$$

$R_2 = R_2 + 4R_1$

$$\begin{bmatrix} 1 & -3 & -4 & 0 \\ 0 & -6 & -18 & 0 \\ -3 & 7 & 6 & 0 \end{bmatrix}$$

$R_3 = R_3 + 3R_1$

$$\begin{bmatrix} 1 & -3 & -4 & 0 \\ 0 & -6 & -18 & 0 \\ 0 & -2 & -6 & 0 \end{bmatrix}$$

$R_3 = R_3 - \frac{1}{3}R_2$

$$\begin{bmatrix} 1 & -3 & -4 & 0 \\ 0 & -6 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence (1) becomes

$$\begin{bmatrix} 1 & -3 & -4 \\ 0 & -6 & -18 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

The above shows that $z = t$ is free variable and x, y are basic variables. Second row gives $-6y - 18t = 0$ or $y = -3t$. First row gives $x - 3y - 4t = 0$ or $x = 3(-3t) + 4t$ or $x = -5t$. Hence the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5t \\ -3t \\ t \end{bmatrix}$$
$$= t\begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$$

Now we are ready to answer the question if $u = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$ is in the NullSpace(A). In other words, does there exist t which make $t\begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$. It is clear there is no such t. To show this, looking at the second row, it says $-3t = 3$ or $t = -1$. But third row says $t = -4$. Therefore there is no t which makes u in NullSpace(A). Hence u is not in NullSpace(A)

3.2 Part b

The columnspace of A is the set of all linear combinations of the columns of A. The basis for the columnspace are columns of A that correspond to the pivot columns are doing the above REF. From part A we found that column 1, 2 are the pivot columns. Hence the basis of columnspace of A are

$$\left\{ \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix} \right\}$$

Hence the columnspace of A is two dimensional subspace of $\mathbb{R}^3$. To find if u is in columnspace of A, we need to find if there exists a linear combination of these basis which gives u. Therefore we need to solve

$$c_1 \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$$

For c_1, c_2 to see if a solution exist. In matrix form the above becomes

$$\begin{bmatrix} 1 & -3 \\ -4 & 6 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} \tag{1}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -3 & 3 \\ -4 & 6 & 3 \\ -3 & 7 & -4 \end{bmatrix}$$

$R_2 = R_2 + 4R_1$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & -6 & 15 \\ -3 & 7 & -4 \end{bmatrix}$$

$R_3 = R_3 + 3R_1$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & -6 & 15 \\ 0 & -2 & 5 \end{bmatrix}$$

$R_3 = R_3 - \frac{1}{3}R_2$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & -6 & 15 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 = -\frac{R_2}{6}$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 = R_1 + 3R_2$

$$\begin{bmatrix} 1 & 0 & -\frac{9}{2} \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Hence (1) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} \\ -\frac{5}{2} \\ 0 \end{bmatrix}$$

Therefore the solution is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} \\ -\frac{5}{2} \end{bmatrix}$$

Therefore a solution exists. This means

$$-\frac{9}{2} \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$$

This means u is in the columnspace of A.

3.3 Part c

The rank of A is the dimension of the columnspace of A. Which is the same as the number of pivot columns found. In this case, it is 2 as found in part b above. Hence $rank(A) = 2$.

Nullity of A is the dimension of the nullspace of A. From part (a) we found that the nullspace of A is given by the one parameter vector $t \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$. Hence the dimension is 1. It is the number of the free variables. Therefore Nullity of $A = 1$. To verify this, we can use the rank-nullity theorem, which says for a matrix $m \times n$,

$$rank(A) + nullity(A) = n$$

Since $n = 3$ and since $rank(A) = 2$ then $nullity\,(A) = 1$.

4 Problem 4

a Using the definition, verify that the given transformation is linear transformation $T : C^2(I) \to C^0(I)$ defined by $T(y) = y'' + y$

b Find the kernel of T

solution

4.1 Part a

The transformation T is linear if

1. $T(u+v) = T(u) + T(v)$ for all u, v in $C^2(I)$
2. $T(cu) = cT(u)$ for all scalars c and u in $C^2(I)$

To show property 1:

$$T(u+v) = (u+v)'' + (u+v)$$

By linearity of second derivatives (and since u, v are in $C^2(I)$) the above becomes

$$\begin{aligned} T(u+v) &= (u'' + v'') + (u+v) \\ &= u'' + u + v'' + v \\ &= (u'' + u) + (v'' + v) \end{aligned}$$

But $(u'' + u) = T(u)$ and $(v'' + v) = T(v)$, Hence the above becomes

$$T(u+v) = T(u) + T(v)$$

To show property 2:

$$T(cu) = (cu)'' + (cu)$$

But since c is a scalar, we can move it outside the derivative and the above becomes

$$\begin{aligned} T(cu) &= c(u)'' + cu \\ &= c(u'' + u) \end{aligned}$$

But $u'' + u = T(u)$. Hence the above becomes

$$T(cu) = cT(u)$$

Both properties are satisfied. Hence T is linear transformation.

4.2 Part b

The kernel of $T : V \to W$ is $\ker(T) = \{u \in V : Tu = 0\}$. In this case $V = C^2(I)$ and $W = C^0(I)$. Hence we need to find all u, such that $T(u) = 0$. Which is the same as saying all u which satisfies $u'' + u = 0$. Hence $\ker(T)$ is the solution of this ode.

This is linear constant coefficient ode. The characteristic equation is $\lambda^2 + 1 = 0$. The solutions are $\lambda = \pm i$. Hence the basis functions are $\left\{e^{ix}, e^{-ix}\right\}$ (assuming the independent variable is x), or using Euler relation $\{\cos x, \sin x\}$. Therefore the solution is linear combination of these basis given by $u = c_1 \cos x + c_2 \sin x$ where c_1, c_2 are arbitrary constants. Hence

$$\ker(T) = \{u : u = c_1 \cos x + c_2 \sin x\}$$

5 Problem 5

Solve $\left(y+3x^2\right)dx + xdy = 0$

solution

Writing the ODE as

$$Mdx + Ndy = 0$$

Where $M = y + 3x^2, N = x$. Checking if the ODE is exact

$$\frac{\partial M}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = 1$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then it is exact in some region R. Let there exists constant function $\Phi\left(x,y\right) = c$ which satisfies

$$\frac{\partial \Phi}{\partial x} = M = y + 3x^2 \tag{1}$$

$$\frac{\partial \Phi}{\partial y} = N = x \tag{2}$$

For all $\left(x,y\right)$ in R. Integrating (1) w.r.t. x gives

$$\int \frac{\partial \Phi}{\partial x} dx = \int y + 3x^2 dx$$

$$\Phi = yx + x^3 + g(y) \tag{3}$$

Taking derivative of the above w.r.t. y gives

$$\frac{\partial \Phi}{\partial y} = x + g'(y) \tag{4}$$

Comparing (4) and (2) gives

$$x + g'(y) = x$$

$$g'(y) = 0$$

Hence $g(y) = c_1$ a constant. Substituting this in (3) gives

$$\Phi = yx + x^3 + c_1$$

But $\Phi = c$. Combining the constants c, c_1 into one constant, say C, the above becomes

$$C = yx + x^3$$

Solving for y gives

$$yx = C - x^3$$

For $x \neq 0$

$$y = \frac{C - x^3}{x}$$

6 Problem 6

Using the method of undermined coefficients, find the general solution of the given differential equation

$$y'' - y' - 2y = e^{-x} + 2\cos x \tag{1}$$

solution

The solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' - y' - 2y = 0$ and y_p is a particular solution to the ode. We start by finding y_h. Since this is linear second order with constant coefficient, then the characteristic equation method is used. The characteristic equation for $y'' - y' - 2y = 0$ is

$$\lambda^2 - \lambda - 2 = 0$$
$$(\lambda - 2)(\lambda + 1) = 0$$

Hence the roots are $\lambda_1 = 2, \lambda_2 = -1$. Therefore the basis set of solutions for y_h is the set

$$\left\{e^{2x}, e^{-x}\right\} \tag{2}$$

And y_h is linear combination of these basis. Therefore

$$y_h = c_1 e^{2x} + c_2 e^{-x} \tag{3}$$

Looking at RHS of (1) shows it is linear combination of basis $[e^{-x}, \cos x]$. For each basis in this list, we generate all possible derivatives. Which gives (ignoring sign changes and any leading constants as they will be parts of the unknowns to be found later on). This results in the following list

$$[\{e^{-x}\}, \{\cos x, \sin x\}] \tag{4}$$

Now we compare each basis in (2) with each basis in (4) to see if there is any duplication. We see that e^{-x} is in (4) as well in (2). We now multiply e^{-x} in (4) by an extra x and obtain new list

$$[\{xe^{-x}\}, \{\cos x, \sin x\}] \tag{4A}$$

We repeat this process again, checking if (2) still has any duplication in (4A). There are no duplication now. Hence the trial solution is linear combination of the basis in (4A). Which gives

$$y_p = Axe^{-x} + B\cos x + C\sin x \tag{5}$$

To determine A, B, C, we substitute y_p back in the ODE (1) and solve for these unknowns by comparing terms.

$$
\begin{aligned}
y_p' &= Ae^{-x} - Axe^{-x} - B\sin x + C\cos x && (6)\\
y_p'' &= -Ae^{-x} - Ae^{-x} + Axe^{-x} - B\cos x - C\sin x \\
&= -2Ae^{-x} + Axe^{-x} - B\cos x - C\sin x && (7)
\end{aligned}
$$

Substituting (5,6,7) into the ODE (1) gives

$$
\begin{aligned}
(-2Ae^{-x} + Axe^{-x} - B\cos x - C\sin x) - (Ae^{-x} - Axe^{-x} - B\sin x + C\cos x) - 2\left(Axe^{-x} + B\cos x + C\sin x\right) &= e^{-x} + 2\cos x\\
-2Ae^{-x} + Axe^{-x} - B\cos x - C\sin x - Ae^{-x} + Axe^{-x} + B\sin x - C\cos x - 2Axe^{-x} - 2B\cos x - 2C\sin x &= e^{-x} + 2\cos x
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
-3Ae^{-x} - 3B\cos x - 3C\sin x + B\sin x - C\cos x &= e^{-x} + 2\cos x\\
-3Ae^{-x} + \cos x\,(-3B - C) + \sin x\,(-3C + B) &= e^{-x} + 2\cos x
\end{aligned}
$$

Comparing terms on each side gives 3 equations to solve for A, B, C

$$
\begin{aligned}
-3A &= 1\\
-3B - C &= 2\\
-3C + B &= 0
\end{aligned}
$$

First equation gives $A = -\frac{1}{3}$. Multiplying second equation by -3 and adding the result to third equation gives

$$
\begin{aligned}
9B + 3C &= -6\\
-3C + B &= 0
\end{aligned}
$$

Adding gives

$$
\begin{aligned}
9B + B &= -6\\
10B &= -6\\
B &= -\frac{6}{10}\\
&= -\frac{3}{5}
\end{aligned}
$$

From $-3B - C = 2$ we now find $-3\left(-\frac{6}{10}\right) - C = 2$, or $C = -\frac{1}{5}$. Hence the particular solution (5) becomes

$$
y_p = -\frac{1}{3}xe^{-x} - \frac{3}{5}\cos x - \frac{1}{5}\sin x \tag{8}
$$

Substituting (8) and (3) in $y = y_h + y_p$ gives the final solution as

$$
y = c_1e^{2x} + c_2e^{-x} - \frac{1}{3}xe^{-x} - \frac{3}{5}\cos x - \frac{1}{5}\sin x
$$

7 Problem 7

Use the Laplace transform to solve the given initial-value problems. You can use the table of transformation

$$
\begin{aligned}
y'' + y &= e^{2t} \\
y(0) &= 0 \\
y'(0) &= 1
\end{aligned}
$$

solution

Taking the Laplace transform of both sides of $y'' + y = e^{2t}$ gives (using linearity)

$$\mathscr{L}\left(y''\right) + \mathscr{L}(y) = \mathscr{L}\left(e^{2t}\right) \tag{1}$$

Assuming $\mathscr{L}(y) = Y(s)$, and using the property that

$$\mathscr{L}\left(y''\right) = s^2 \mathscr{L}(y) - sy(0) - y'(0)$$

And from table 10.2.1 $\mathscr{L}\left(e^{2t}\right) = \frac{1}{s-2}, s > 2$, then the ode becomes

$$
\begin{aligned}
\left(s^2 \mathscr{L}(y) - sy(0) - y'(0)\right) + \mathscr{L}(y) &= \frac{1}{s-2} \\
\left(s^2 Y - s(0) - 1\right) + Y &= \frac{1}{s-2} \\
s^2 Y - 1 + Y &= \frac{1}{s-2} \\
Y\left(s^2 + 1\right) &= \frac{1}{s-2} + 1 \\
Y &= \frac{1}{(s-2)\left(s^2+1\right)} + \frac{1}{s^2+1} \qquad \text{(1A)}
\end{aligned}
$$

Using partial fractions on the first term in the RHS above gives

$$
\begin{aligned}
\frac{1}{(s-2)\left(s^2+1\right)} &= \frac{A}{s-2} + \frac{Bs + C}{s^2+1} \\
&= \frac{A\left(s^2+1\right) + (Bs + C)(s-2)}{(s-2)\left(s^2+1\right)} \\
&= \frac{As^2 + A + \left(Bs^2 - 2Bs + Cs - 2C\right)}{(s-2)\left(s^2+1\right)} \\
&= \frac{As^2 + A + Bs^2 - 2Bs + Cs - 2C}{(s-2)\left(s^2+1\right)} \\
&= \frac{s^2(A+B) + s(C-2B) + (A-2C)}{(s-2)\left(s^2+1\right)}
\end{aligned}
$$

Therefore

$$1 = s^2 (A + B) + s (C - 2B) + (A - 2C)$$

Comparing terms gives

$$\begin{aligned} A + B &= 0 \\ C - 2B &= 0 \\ A - 2C &= 1 \end{aligned}$$

In matrix form the above is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & 0 & -2 & 1 \end{bmatrix}$$

$R_3 = R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}$$

$R_3 = 2R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & -4 & 2 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -5 & 2 \end{bmatrix}$$

$R_3 = \frac{-1}{5} R_3, R_2 = -\frac{1}{2} R_2$ gives

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 & \frac{-2}{5} \end{bmatrix}$$

$R_2 = R_2 + \frac{1}{2}R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{-1}{5} \\ 0 & 0 & 1 & \frac{-2}{5} \end{bmatrix}$$

$R_1 = R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{5} \\ 0 & 1 & 0 & \frac{-1}{5} \\ 0 & 0 & 1 & \frac{-2}{5} \end{bmatrix}$$

Hence (2) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{-1}{5} \\ \frac{-2}{5} \end{bmatrix} \tag{3}$$

Therefore, since now in RREF form, the solution is

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{-1}{5} \\ \frac{-2}{5} \end{bmatrix}$$

Hence

$$\begin{aligned} \frac{1}{(s-2)\left(s^2+1\right)} &= \frac{A}{s-2} + \frac{Bs+C}{s^2+1} \\ &= \frac{1}{5}\frac{1}{s-2} + \frac{-\frac{1}{5}s - \frac{2}{5}}{s^2+1} \\ &= \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{s+2}{s^2+1} \\ &= \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} \end{aligned} \tag{4}$$

Substituting (4) back in (1A) gives

$$\begin{aligned} Y &= \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} + \frac{1}{s^2+1} \\ &= \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{s}{s^2+1} + \frac{3}{5}\frac{1}{s^2+1} \end{aligned} \tag{5}$$

Now we will use tables to do the inverse Laplace transform. From table 10.2.1

$$\begin{aligned} \mathscr{L}^{-1}\left(\frac{1}{s-2}\right) &= e^{2t} \qquad s>2 \\ \mathscr{L}^{-1}\left(\frac{s}{s^2+1}\right) &= \cos t \qquad s>0 \\ \mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right) &= \sin t \qquad s>0 \end{aligned}$$

Using this result in (5) and since $\mathscr{L}^{-1}(Y(s)) = y(t)$ then (5) becomes

$$y(t) = \frac{1}{5}e^{2t} - \frac{1}{5}\cos t + \frac{3}{5}\sin t$$

8 Problem 8

Find a series solution in powers of x of the differential equation $y'' + x^2 y' + y = 0$

solution

Let the solution be

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

Then

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} \end{aligned} \tag{2}$$

And

$$\begin{aligned} y''(x) &= \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2} \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned} \tag{3}$$

Substituting (1,2,3) into the given ODE gives

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n &= 0 \end{aligned} \tag{3A}$$

Now we make all powers of x the same by rewriting $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n+1} = \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n$. The way the above is done is by using the rule: When adding a value to the summation index n inside the sum, then we must at same time subtract the same value from the starting index n.

Hence (3A) now becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

To be able to compare coefficients of x, we expand up to $n = 1$ the sums in order to make all sums start from $n = 2$. This gives

$$\begin{aligned} (2)(1)a_2 + (1+2)(1+1) a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n + (a_0 + a_1 x) + \sum_{n=2}^{\infty} a_n x^n &= 0 \\ (2a_2 + a_0) + x(6a_3 + a_1) + \left(\sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} + \sum_{n=2}^{\infty} (n-1) a_{n-1} + \sum_{n=2}^{\infty} a_n \right) x^n &= 0 \end{aligned}$$

Now we are to compare coefficients on each power of x. The above gives the three equations

$$\begin{aligned} 2a_2 + a_0 &= 0 \\ 6a_3 + a_1 &= 0 \\ (n+2)(n+1)\,a_{n+2} + (n-1)\,a_{n-1} + a_n &= 0 \qquad n \geq 2 \end{aligned} \tag{4}$$

First equation above gives

$$a_2 = -\frac{1}{2}a_0$$

Second equation in (4) gives

$$a_3 = -\frac{a_1}{6}$$

And the third equation in (4) gives the <u>recursive equation</u> which allows us to find all a_n after these

$$(n+2)(n+1)\,a_{n+2} + (n-1)\,a_{n-1} + a_n = 0 \qquad n \geq 2$$

Or

$$a_{n+2} = \frac{-a_n - (n-1)\,a_{n-1}}{(n+2)(n+1)} \qquad n \geq 2 \tag{5}$$

Therefore, <u>for $n = 2$</u> the above gives

$$a_4 = \frac{-a_2 - a_1}{(2+2)(2+1)} = \frac{-a_2 - a_1}{12}$$

But $a_2 = -\frac{1}{2}a_0$, therefore the above becomes

$$a_4 = \frac{-\left(-\frac{1}{2}a_0\right) - a_1}{(4)(3)} = \frac{\frac{1}{2}a_0 - a_1}{12} = \frac{a_0 - 2a_1}{24} \tag{6}$$

And <u>for $n = 3$</u> (5) gives

$$a_5 = \frac{-a_3 - (3-1)\,a_2}{(3+2)(3+1)} = \frac{-a_3 - 2a_2}{20}$$

But $a_2 = -\frac{1}{2}a_0, a_3 = -\frac{a_1}{(2)(3)}$, the above becomes

$$a_5 = \frac{-\left(-\frac{a_1}{(2)(3)}\right) - 2\left(-\frac{1}{2}a_0\right)}{20} = \frac{\frac{a_1}{(2)(3)} + a_0}{20} = \frac{a_1 + 6a_0}{120} \tag{7}$$

And <u>for $n = 4$</u> (5) gives

$$a_6 = \frac{-a_4 - (4-1)\,a_3}{(4+2)(4+1)} = \frac{-a_4 - 3a_3}{30}$$

But $a_4 = \frac{a_0 - 2a_1}{24}$ and $a_3 = -\frac{a_1}{6}$. The above becomes

$$a_6 = \frac{-\left(\frac{a_0-2a_1}{24}\right) - 3\left(-\frac{a_1}{6}\right)}{30} = \frac{\frac{-a_0+2a_1}{24} + \frac{a_1}{2}}{30} = \frac{-a_0 + 2a_1 + 12a_1}{(30)(24)} = \frac{-a_0 + 14a_1}{(30)(24)} = \frac{-a_0 + 14a_1}{720}$$

And so on. Therefore, from (1)

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots \\ &= a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{a_1}{6} x^3 + \left(\frac{a_0 - 2a_1}{24}\right) x^4 + \left(\frac{a_1 + 6a_0}{120}\right) x^5 + \left(\frac{-a_0 + 14a_1}{720}\right) x^6 + \cdots \\ &= a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{a_1}{6} x^3 + \left(\frac{a_0}{24} - \frac{a_1}{12}\right) x^4 + \left(\frac{a_1}{120} + \frac{a_0}{20}\right) x^5 + \left(\frac{-a_0}{720} + \frac{7a_1}{360}\right) x^6 + \cdots \end{aligned}$$

Therefore

$$y(x) = a_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 + \frac{1}{20} x^5 - \frac{1}{720} x^6 + \cdots\right) + a_1 \left(x - \frac{1}{6} x^3 - \frac{1}{12} x^4 + \frac{1}{120} x^5 + \frac{7}{360} x^6 + \cdots\right) \tag{8}$$

The series solution above contains two unknowns a_0, a_1. There are the same as the constant of integrations. Since this is a second order ODE, then there will be two unknowns in the general solutions. These can be found from initial conditions. For example, assuming $y(0) = y_0, y'(0) = y'_0$. Then from (8) at $x = 0$, it gives $y(0) = a$. Taking one derivative of (8) gives

$$y'(x) = a_0 \left(-x + \frac{4}{24} x^3 + \cdots\right) + a_1 \left(1 - \frac{3}{6} x^2 + \cdots\right) \tag{9}$$

At $x = 0$ the above becomes $y'_0 = a_1$. Therefore (8) can be written as

$$y(x) = y(0) \left(1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 + \frac{1}{20} x^5 - \frac{1}{720} x^6 + \cdots\right) + y'(0) \left(x - \frac{1}{6} x^3 - \frac{1}{12} x^4 + \frac{1}{120} x^5 + \frac{7}{360} x^6 + \cdots\right) \tag{10}$$

9 Problem 9

a) Determine all the equilibrium points of the given system. b) Select two equilibrium points and classify them as saddle, node, spiral or center and whether they are stable or unstable.

$$x' = 2x - x^2 - xy$$
$$y' = 3y - 3xy - 2y^2$$

solution

9.1 Part a

equilibrium points are the solutions in x, y of

$$2x - x^2 - xy = 0 \tag{1}$$
$$3y - 3xy - 2y^2 = 0 \tag{2}$$

Which can be written as

$$x\left(2 - x - y\right) = 0 \tag{1}$$
$$y\left(3 - 3x - 2y\right) = 0 \tag{2}$$

From (1), we see that

$$x = 0 \tag{3}$$

is a solution and $2 - x - y = 0$ or

$$x = 2 - y \tag{4}$$

Is another solution. For each x value in (3,4), now we solve for y from (2). When $x = 0$ then (2) becomes

$$y\left(3 - 2y\right) = 0$$

Which has solutions $y = 0, y = \frac{3}{2}$. Therefore $\{0, 0\}$ and $\left\{0, \frac{3}{2}\right\}$ are two solutions found so far. And when $x = 2 - y$ then (2) becomes

$$y\left(3 - 3\left(2 - y\right) - 2y\right) = 0$$
$$y\left(3 - 6 + 3y - 2y\right) = 0$$
$$y\left(3 - 6 + y\right) = 0$$

Which has solutions $y = 0$ and $y = 3$. When $y = 0$ then $x = 2 - y$ gives $x = 2$. Therefore $\{2, 0\}$ is a solution, and when $y = 3$ then $x = 2 - y$ gives $x = 2 - 3 = -1$. Hence $\{-1, 3\}$ is another solution. Putting all these together gives the solutions as

$$\{0, 0\}, \left\{0, \frac{3}{2}\right\}, \{2, 0\}, \{-1, 3\}$$

9.2 Part b

The given system is matrix form is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{F}$$
$$= \begin{bmatrix} 2x - x^2 - xy \\ 3y - 3xy - 2y^2 \end{bmatrix}$$
$$= \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

The Jacobian matrix for the system is given by the gradient of **F**

$$J = \nabla \mathbf{F}$$
$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\partial(2x-x^2-xy)}{\partial x} & \frac{\partial(2x-x^2-xy)}{\partial y} \\ \frac{\partial(3y-3xy-2y^2)}{\partial x} & \frac{\partial(3y-3xy-2y^2)}{\partial y} \end{bmatrix}$$
$$= \begin{bmatrix} 2 - 2x - y & -x \\ -3y & 3 - 3x - 4y \end{bmatrix}$$

Selecting points $\{0,0\}$ and $\left\{0, \frac{3}{2}\right\}$ for analysis.

At Point $\{0,0\}$ the linearized system matrix A is the Jacobian matrix evaluated at this equilibrium point. Hence

$$A = \begin{bmatrix} 2 - 2x - y & -x \\ -3y & 3 - 3x - 4y \end{bmatrix}_{\substack{x=0 \\ y=0}}$$
$$= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

The eigenvalues are found by solving $\det(A - \lambda I) = 0$ or

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(3-\lambda) = 0$$

Hence $\lambda_1 = 2$ and $\lambda_2 = 3$. Since both eigenvalues are positive, then this is unstable critical point. It is a negative attractor also called a node.

At Point $\left\{0, \frac{3}{2}\right\}$ the linearized system matrix A is the Jacobian matrix evaluated at this equilibrium point. Hence

$$A = \begin{bmatrix} 2-2x-y & -x \\ -3y & 3-3x-4y \end{bmatrix}_{\substack{x=0 \\ y=\frac{3}{2}}}$$

$$= \begin{bmatrix} 2-\frac{3}{2} & 0 \\ -3\left(\frac{3}{2}\right) & 3-4\left(\frac{3}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{9}{2} & -3 \end{bmatrix}$$

The eigenvalues are found by solving $\det(A - \lambda I) = 0$ or

$$\begin{vmatrix} \frac{1}{2}-\lambda & 0 \\ -\frac{9}{2} & -3-\lambda \end{vmatrix} = 0$$

$$\left(\frac{1}{2}-\lambda\right)(-3-\lambda) = 0$$

Hence $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -3$. Since one eigenvalue is positive, and one eigenvalue is negative, then this is unstable critical point. It is a saddle point.