HW 2

Math 2520
Differential Equations and Linear Algebra

Summer 2021
Normandale college, Bloomington, Minnesota.

Nasser M. Abbasi

June 17, 2021 Compiled on June 17, 2021 at 2:18am

Contents

1 Problem 1

Let $A=\begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix}, B=\begin{bmatrix}7 & -5 & 1\\ 1 & -4 & -3\end{bmatrix}, C=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}$. Compute a) $A+2B$. b) A^T. c)$A+C$, d)CB

Solution

1.1 part a

$$
\begin{aligned}
A+2B &= \begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix} + 2\begin{bmatrix}7 & -5 & 1\\ 1 & -4 & -3\end{bmatrix}\\
&= \begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix} + \begin{bmatrix}14 & -10 & 2\\ 2 & -8 & -6\end{bmatrix}\\
&= \begin{bmatrix}2+14 & 0-10 & -1+2\\ 4+2 & -5-8 & 2-6\end{bmatrix}\\
&= \begin{bmatrix}16 & -10 & 1\\ 6 & -13 & -4\end{bmatrix}
\end{aligned}
$$

1.2 part b

The transpose operation exchanges rows with columns, therefore

$$
\begin{aligned}
A^T &= \begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix}^T\\
&= \begin{bmatrix}2 & 4\\ 0 & -5\\ -1 & 2\end{bmatrix}
\end{aligned}
$$

1.3 part c

$$
A+C = \begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix} + \begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}
$$

This is undefined because the size of the matrices must be the same in order to add them to together, because addition is done element by element.

1.4 part d

$$CB = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}$$

C has size 2×2 and B has size 2×3. Since inner dimensions agree, then the matrix product is defined. The result will be 2×3. Doing the matrix multiplication using the standard rows times columns method, gives

$$CB = \begin{bmatrix} 9 & -13 & -5 \\ -13 & 6 & -5 \end{bmatrix}$$

2 Problem 2

Determine the solution set to the given system. If using calculator, show three steps: 1. Write the augmented matrix and determine the size of the matrix. 2. Find the rref(augmented matrix) 3. Write the solution.

a)

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 1 \\
3x_1 + 5x_2 + x_3 &= 3 \\
2x_1 + 6x_2 + 7x_3 &= 1
\end{aligned}
$$

b)

$$
\begin{aligned}
x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\
-2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\
3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2
\end{aligned}
$$

If the system is inconsistent state why.

Solution

2.1 part a

In matrix form $Ax = b$ the system becomes

$$
\begin{aligned}
Ax &= b \\
\begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 1 \\ 2 & 6 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}
\end{aligned}
$$

The size of the A matrix is 3×3. Now the augmented matrix is setup in order to solve the system.

$$
\begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 5 & 1 & 3 \\ 2 & 6 & 7 & 1 \end{bmatrix}
$$

In the above, the vector b was appended to the right side of the A matrix. The augmented matrix has size 3×4. The augmented matrix is now converted to Echelon form using the allowed row operations. (Multiplication by constant, or adding multiples of another row to the row). In all the following, the notation $R_i = R_i + R_j$ means to replace row i with row i added to row j.

$R_2 = 3R_1 - R_2$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 2 & 6 & 7 & 1 \end{bmatrix}$$

$R_3 = 2R_1 - R_3$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -5 & 1 \end{bmatrix}$$

$R_3 = 2R_2 + R_3$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Now it is in row echelon form. Next step is to convert to row reduced echelon form. Multiplying last row by -1 gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_2 = R_2 - 2R_3$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_1 = R_1 - R_3$ gives

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_1 = R_1 - 2R_2$ gives

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

This is now in row reduced Echelon form. The basic variables are x_1, x_2, x_3. There are no free variables. Hence the original system now becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

The last row gives $x_3 = -1$. Second row gives $x_2 = 2$ and first row gives $x_1 = -2$. The solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

2.2 part b

In matrix form $Ax = b$ the system becomes

$$Ax = b$$

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ -2 & 4 & 5 & -5 \\ 3 & -6 & -6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

The A matrix size is 3×4. The augmented matrix is

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 1 \end{bmatrix}$$

$R_2 = R_2 + 2R_1$ gives

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & 1 & 5 \\ 3 & -6 & -6 & 8 & 1 \end{bmatrix}$$

$R_3 = R_3 - 3R_1$ gives

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & -3 & -1 & -2 \end{bmatrix}$$

$R_3 = R_3 + R_2$ gives

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

The system is inconsistent. Because the last row says $0 = 3$. Therefore there are no solutions.

3 Problem 3

Given the following matrix function $A(t) = \begin{bmatrix} -7 & t^2 \\ 1+t & \cos\left(\frac{\pi t}{2}\right) \end{bmatrix}$. (a) determine the derivative $\frac{dA}{dt}$. (b) Determine $\int_0^t A(\tau)\, d\tau$

Solution

3.1 Part a

$$\begin{aligned}
\frac{dA}{dt} &= \frac{d}{dt}\begin{bmatrix} -7 & t^2 \\ 1+t & \cos\left(\frac{\pi t}{2}\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{d}{dt}(-7) & \frac{d}{dt}\left(t^2\right) \\ \frac{d}{dt}(1+t) & \frac{d}{dt}\cos\left(\frac{\pi t}{2}\right) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2t \\ 1 & -\frac{\pi}{2}\sin\left(\frac{\pi t}{2}\right) \end{bmatrix}
\end{aligned}$$

3.2 Part b

$$\begin{aligned}
\int_0^t A(\tau)\, d\tau &= \int_0^t \begin{bmatrix} -7 & \tau^2 \\ 1+\tau & \cos\left(\frac{\pi\tau}{2}\right) \end{bmatrix} dt \\
&= \begin{bmatrix} -\int_0^t 7dt & \int_0^t \tau^2 dt \\ \int_0^t (1+\tau)\, dt & \int_0^t \cos\left(\frac{\pi\tau}{2}\right) dt \end{bmatrix} \\
&= \begin{bmatrix} -[7\tau]_0^t & \frac{1}{3}\left[\tau^3\right]_0^t \\ \left[\tau + \frac{\tau^2}{2}\right]_0^t & \frac{2}{\pi}\left[\sin\left(\frac{\pi\tau}{2}\right)\right]_0^t \end{bmatrix} \\
&= \begin{bmatrix} -7t & \frac{1}{3}t^3 \\ t + \frac{t^2}{2} & \frac{2}{\pi}\sin\left(\frac{\pi t}{2}\right) \end{bmatrix}
\end{aligned}$$

4 Problem 4

Write the system of equations with the given coefficient matrix and right-hand side vector.

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 1 & -2 & 6 \\ 3 & 1 & 4 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Solution

The problem is asking to write $Ax = b$. Let the variables be x_1, x_2, x_3, x_4. Hence the above becomes

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 1 & -2 & 6 \\ 3 & 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Carrying the matrix vector multiplication using standard method of a row times a column gives the system of equations

$$\begin{aligned} x_1 - x_2 + 2x_3 + 3x_4 &= 1 \\ x_1 + x_2 - 2x_3 + 6x_4 &= -1 \\ 3x_1 + x_2 + 4x_3 + 2x_4 &= 2 \end{aligned}$$

5 Problem 5

Reduce to row-echelon form and determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Solution

5.1 Part a

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{bmatrix}$$

$R_2 = R_2 - R_1$ gives

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & -4 & 0 \\ 3 & -2 & 6 \end{bmatrix}$$

$R_3 = 2R_3, R_1 = 3R_1$ gives

$$\begin{bmatrix} 6 & 3 & 12 \\ 0 & -4 & 0 \\ 6 & -4 & 12 \end{bmatrix}$$

$R_3 = R_3 - R_1$ gives

$$\begin{bmatrix} 6 & 3 & 12 \\ 0 & -4 & 0 \\ 0 & -7 & 0 \end{bmatrix}$$

$R_3 = R_3 + \frac{7}{4}R_2$ gives

$$\begin{bmatrix} 6 & 3 & 12 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The above is row echelon form. Pivots are $A(1,1)$ and $A(2,2)$. Number of pivots is 2. Hence Rank is 2

5.2 Part b

$$B=\begin{bmatrix}2 & -1 & 3\\ 3 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$$

$R_1 = 3R_1, R_2 = 2R_2$ gives

$$\begin{bmatrix}6 & -3 & 9\\ 6 & 2 & -4\\ 2 & -2 & 1\end{bmatrix}$$

$R_2 = R_2 - R_1$ gives

$$\begin{bmatrix}6 & -3 & 9\\ 0 & 5 & -13\\ 2 & -2 & 1\end{bmatrix}$$

$R_3 = 3R_3$ gives

$$\begin{bmatrix}6 & -3 & 9\\ 0 & 5 & -13\\ 6 & -6 & 3\end{bmatrix}$$

$R_3 = R_3 - R_1$ gives

$$\begin{bmatrix}6 & -3 & 9\\ 0 & 5 & -13\\ 0 & -3 & -6\end{bmatrix}$$

$R_3 = 5R_3, R_2 = 3R_2$ gives

$$\begin{bmatrix}6 & -3 & 9\\ 0 & 15 & -39\\ 0 & -15 & -30\end{bmatrix}$$

$R_3 = R_3 + R_2$ gives

$$\begin{bmatrix}6 & -3 & 9\\ 0 & 15 & -39\\ 0 & 0 & -69\end{bmatrix}$$

Pivots are $A(1,1), A(2,2), A(3,3)$. Three pivots. Hence <u>rank is</u> 3.

6 Problem 6

Reduce the matrix to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix and list the pivot columns

$$A = \begin{bmatrix} 1 & 3 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

Solution

$R_2 = R_2 - 2R_1$ gives

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & -2 & -2 & -8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

$R_3 = R_3 - 3R_1$ gives

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & -2 & -2 & -8 \\ 0 & -3 & -3 & -12 \end{bmatrix}$$

$R_2 = 3R_2, R_3 = 2R_3$ gives

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & -6 & -6 & -24 \\ 0 & -6 & -6 & -24 \end{bmatrix}$$

$R_3 = R_3 - R_2$ gives

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & -6 & -6 & -24 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 = \frac{R_2}{6}$ gives (to simplify)

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basic variables are x_1, x_2. Free variables are x_3, x_4. The above is row echelon form. Now we apply the reduced Echelon phase by zeroing all entries in the pivot columns above the pivots. In the above, the pivots are $A(1,1)$ and $A(2,2)$. First, all pivot entries are changed to 1

$R_2 = -R_2$ gives

$$\begin{bmatrix} 1 & 3 & 4 & 8 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 = R_1 - 3R_2$ gives

$$\begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above is the reduced echelon form. I do not know how to circle the pivots using Latex. But they are $A(1,1) = 1$ and $A(2,2) = 1$ in the above final matrix. I the original matrix they are $A(1,1) = 1$ and $A(2,2) = 4$.

The pivot columns are the first and second columns. In the final matrix, these are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

7 Problem 7

Determine A^{-1} if possible using Gauss-Jordan method

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & -4 \\ -1 & -2 & 5 \\ 2 & 6 & 7 \end{bmatrix}$$

Solution

7.1 Part a

The augmented matrix becomes, after adding the identity 3×3 matrix to the right side of the original matrix

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix}$$

Elimination is now applied to transform the left half of the above matrix to becomes the identity matrix. What then results on the right half will be A^{-1}.

$R_2 = R_2 + R_1$ gives

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & -1 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - 5R_1$ gives

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & -1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - 2R_2$ gives

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & -1 & 1 & 0 \\ 0 & 0 & 0 & -3 & -2 & 1 \end{bmatrix}$$

Since the last row in the left half is all zeros, it means it is not possible to obtain the identity matrix on the left side. Therefore this matrix has no inverse since it is singular. The rank is only 2. We stop here.

7.2 Part b

The augmented matrix becomes, after adding the identity matrix to the right side

$$\begin{bmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ -1 & -2 & 5 & 0 & 1 & 0 \\ 2 & 6 & 7 & 0 & 0 & 1 \end{bmatrix}$$

Now elimination is applied to transform the left half to the identity matrix. What results on the right side will be A^{-1}.

$R_2 = R_2 + R_1$ gives

$$\begin{bmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 2 & 6 & 7 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - 2R_1$ gives

$$\begin{bmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 15 & -2 & 0 & 1 \end{bmatrix}$$

Now reduced echelon phase starts.

$R_3 = \frac{R_3}{15}$ gives

$$\begin{bmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{2}{15} & 0 & \frac{1}{15} \end{bmatrix}$$

$R_2 = R_2 - R_3$ gives

$$\begin{bmatrix} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{17}{15} & 1 & -\frac{1}{15} \\ 0 & 0 & 1 & -\frac{2}{15} & 0 & \frac{1}{15} \end{bmatrix}$$

$R_1 = R_1 + 4R_3$ gives

$$\begin{bmatrix} 1 & 3 & 0 & 1+4\left(-\frac{2}{15}\right) & 0 & 4\left(\frac{1}{15}\right) \\ 0 & 1 & 0 & \frac{17}{15} & 1 & -\frac{1}{15} \\ 0 & 0 & 1 & -\frac{2}{15} & 0 & \frac{1}{15} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & \frac{7}{15} & 0 & \frac{4}{15} \\ 0 & 1 & 0 & \frac{17}{15} & 1 & -\frac{1}{15} \\ 0 & 0 & 1 & -\frac{2}{15} & 0 & \frac{1}{15} \end{bmatrix}$$

$R_1 = R_1 - 3R_2$ gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{15} - 3\left(\frac{17}{15}\right) & -3 & \frac{4}{15} - 3\left(-\frac{1}{15}\right) \\ 0 & 1 & 0 & \frac{17}{15} & 1 & -\frac{1}{15} \\ 0 & 0 & 1 & -\frac{2}{15} & 0 & \frac{1}{15} \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{44}{15} & -3 & \frac{7}{15} \\ 0 & 1 & 0 & \frac{17}{15} & 1 & -\frac{1}{15} \\ 0 & 0 & 1 & -\frac{2}{15} & 0 & \frac{1}{15} \end{array}\right]$$

Since the left half is the identity matrix, then the right half is the matrix inverse of the original matrix. Therefore

$$B^{-1} = \begin{bmatrix} -\frac{44}{15} & -3 & \frac{7}{15} \\ \frac{17}{15} & 1 & -\frac{1}{15} \\ -\frac{2}{15} & 0 & \frac{1}{15} \end{bmatrix}$$

8 Problem 8

Use A^{-1} to find the solution to the given system

$$\begin{aligned} 6x_1 + 20x_2 &= -8 \\ 2x_1 + 7x_2 &= 4 \end{aligned}$$

<u>Solution</u>

The first step is to find A^{-1}. In matrix form, the above system is

$$Ax = b$$
$$\begin{bmatrix} 6 & 20 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

Hence the augmented matrix to find A^{-1} is

$$\begin{bmatrix} 6 & 20 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix}$$

Now elimination is applied to transform the left half to the identity matrix. What results on the right side will be A^{-1}.

$R_2 = 3R_2$ gives

$$\begin{bmatrix} 6 & 20 & 1 & 0 \\ 6 & 21 & 0 & 3 \end{bmatrix}$$

$R_2 = R_2 - R_1$ gives

$$\begin{bmatrix} 6 & 20 & 1 & 0 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

$R_1 = \frac{R_1}{6}$ gives

$$\begin{bmatrix} 1 & \frac{20}{6} & \frac{1}{6} & 0 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

$R_1 = R_1 - \frac{20}{6} R_2$ gives

$$\begin{bmatrix} 1 & 0 & \frac{1}{6} + \left(\frac{20}{6}\right) & -10 \\ 0 & 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{7}{2} & -10 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

Since the left half is now the identity matrix, then the right half is the inverse. Therefore

$$A^{-1} = \begin{bmatrix} \frac{7}{2} & -10 \\ -1 & 3 \end{bmatrix}$$

Now that A^{-1} is found, then the solution is found as follows. By premultiplying both sides of the equation by A^{-1}

$$A^{-1}Ax = A^{-1}b$$

But $A^{-1}A = I$, the identity matrix. Hence the above simplifies to

$$x = A^{-1}b$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & -10 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

Applying the standard matrix vector multiplications on the right side gives the solution as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -68 \\ 20 \end{bmatrix}$$

9 Problem 9

Use the cofactor expansion theorem to evaluate the given determinant along the specified row or column.

$$A = \begin{vmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{vmatrix} \qquad \text{column 3}$$

$$B = \begin{vmatrix} 6 & -1 & 2 \\ -4 & 7 & 1 \\ 0 & 3 & 1 \end{vmatrix} \qquad \text{row 2}$$

Solution

9.1 Part a

The expansion along column 3 is given by

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{vmatrix} \\ &= (-1)^{1+3}(4)\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} + (-1)^{2+3}(4)\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} + (-1)^{3+3}(6)\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} \\ &= 4\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} - 4\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} + 6\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} \\ &= (4)(5) - (4)(-7) + (6)(-8) \\ &= 20 + 28 - 48 \\ &= 0 \end{aligned}$$

Since the determinant is zero, the matrix is singular.

9.2 Part b

The expansion along row 2 is given by (the sign of each cofactor is found using $(-1)^{i+j}$ where i is the row number and j is the column number).

$$\begin{aligned}
\det(B) &= \begin{vmatrix} 6 & -1 & 2 \\ -4 & 7 & 1 \\ 0 & 3 & 1 \end{vmatrix} \\
&= (-1)^{2+1}(-4)\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} + (-1)^{2+2}(7)\begin{vmatrix} 6 & 2 \\ 0 & 1 \end{vmatrix} + (-1)^{2+3}(1)\begin{vmatrix} 6 & -1 \\ 0 & 3 \end{vmatrix} \\
&= 4\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} + 7\begin{vmatrix} 6 & 2 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 6 & -1 \\ 0 & 3 \end{vmatrix} \\
&= (4)(-7) + (7)(6) - (18) \\
&= -28 + 42 - 18 \\
&= -4
\end{aligned}$$

10 Problem 10

Find A^{-1}

$$A = \begin{bmatrix} 3e^t & e^{2t} \\ 2e^t & 2e^{2t} \end{bmatrix}$$

Solution

The augment matrix is

$$\begin{bmatrix} 3e^t & e^{2t} & 1 & 0 \\ 2e^t & 2e^{2t} & 0 & 1 \end{bmatrix}$$

$R_1 = 2R_1, R_2 = 3R_2$ gives

$$\begin{bmatrix} 6e^t & 2e^{2t} & 2 & 0 \\ 6e^t & 6e^{2t} & 0 & 3 \end{bmatrix}$$

$R_2 = R_2 - R_1$ gives

$$\begin{bmatrix} 6e^t & 2e^{2t} & 2 & 0 \\ 0 & 4e^{2t} & -2 & 3 \end{bmatrix}$$

$R_1 = R_1 - \frac{1}{2}R_2$ gives

$$\begin{bmatrix} 6e^t & 0 & 3 & -\frac{3}{2} \\ 0 & 4e^{2t} & -2 & 3 \end{bmatrix}$$

$R_1 = \frac{R_1}{6}$ gives

$$\begin{bmatrix} e^t & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 4e^{2t} & -2 & 3 \end{bmatrix}$$

$R_2 = \frac{R_2}{4}$ gives

$$\begin{bmatrix} e^t & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & e^{2t} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

$R_1 = \frac{R_1}{e^t}$ (since $e^t \neq 0$) gives

$$\begin{bmatrix} 1 & 0 & \frac{1}{2}e^{-t} & -\frac{1}{4}e^{-t} \\ 0 & e^{2t} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

$R_2 = \frac{R_1}{e^{2t}}$ (since $e^{2t} \neq 0$) gives

$$\begin{bmatrix} 1 & 0 & \frac{1}{2}e^{-t} & -\frac{1}{4}e^{-t} \\ 0 & 1 & -\frac{1}{2}e^{-2t} & \frac{3}{4}e^{-2t} \end{bmatrix}$$

Since the left half is now the identity matrix, then the right half is the inverse. Therefore

$$A^{-1} = \begin{bmatrix} \frac{1}{2}e^{-t} & -\frac{1}{4}e^{-t} \\ -\frac{1}{2}e^{-2t} & \frac{3}{4}e^{-2t} \end{bmatrix}$$

11 Problem 11

Let A and B be 3×3 matrices with $\det(A) = 3$ and $\det(B) = -4$. Compute $\det\left(B^{-1}AB\right)^2$

Solution

The determinant of a product of matrices is the product of their determinants. Hence

$$\det\left(B^{-1}AB\right)^2 = \det\left(B^{-1}AB\right)\det\left(B^{-1}AB\right) \tag{1}$$

But

$$\det\left(B^{-1}AB\right) = \det\left(B^{-1}\right)\det(A)\det(B) \tag{2}$$

Also, the determinant of the inverse of an invertible matrix is the inverse of the determinant. Hence $\det\left(B^{-1}\right) = \frac{1}{\det(B)}$ and the above reduces to

$$\begin{aligned}\det\left(B^{-1}AB\right) &= \frac{1}{\det(B)}\det(A)\det(B)\\ &= \det(A)\\ &= 3\end{aligned}$$

Substituting the above into (1) gives

$$\begin{aligned}\det\left(B^{-1}AB\right)^2 &= (3)(3)\\ &= 9\end{aligned}$$

Note that knowing $\det(B) = -4$ was not really needed, as it cancels out. We just needed to know that $\det(B) \neq 0$. (in other words, that B is not singular).

12 Problem 12

Use Cramer's rule to solve the given linear system

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$

Solution

The system is matrix form is

$$Ax = b$$

$$\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

Hence, using Cramer's rule gives

$$x_1 = \frac{\begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix}}$$

$$x_2 = \frac{\begin{vmatrix} 3 & 6 \\ -5 & 8 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix}}$$

But $\begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix} = 12 - 10 = 2$. The above becomes

$$x_1 = \frac{\begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix}}{2} = \frac{24 + 16}{2} = \frac{40}{2} = 20$$

$$x_2 = \frac{\begin{vmatrix} 3 & 6 \\ -5 & 8 \end{vmatrix}}{2} = \frac{24 + 30}{2} = \frac{54}{2} = 27$$

Therefore the solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 27 \end{bmatrix}$$