HW 7
EE 409 (Linear Systems), CSUF spring 2010
Spring 2010
CSUF

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Contents

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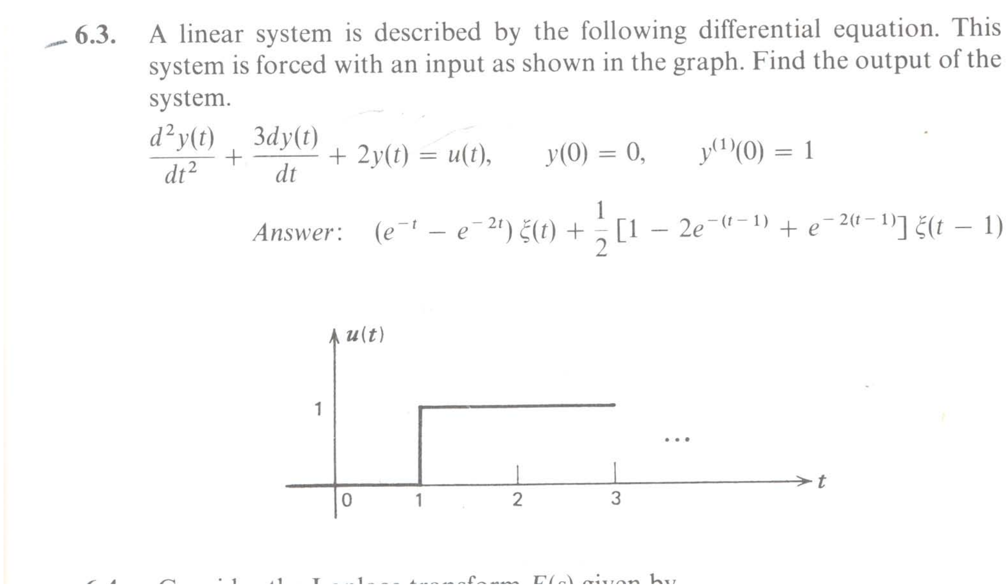
1 Problem 3.25



6.3. A linear system is described by the following differential equation. This system is forced with an input as shown in the graph. Find the output of the system.

$$\frac{d^2y(t)}{dt^2}+\frac{3dy(t)}{dt}+2y(t)=u(t),\qquad y(0)=0,\qquad y^{(1)}(0)=1$$

Answer: $(e^{-t}-e^{-2t})\,\xi(t)+\frac{1}{2}[1-2e^{-(t-1)}+e^{-2(t-1)}]\,\xi(t-1)$

Figure 1: Problem description

$$y''(t)+3y'(t)+2y(t)=u(t)$$

Using the Laplace approach. First we note that the input is a delayed step input, hence $u(t)=\xi(t-1)$ where $\xi(t)$ is the unit step function. Laplace transform of a delayed unit step is $\int_0^\infty \xi(t-1)\,e^{-st}dt=\int_1^\infty e^{-st}dt=\frac{[e^{-st}]_1^\infty}{-s}=\frac{e^{-s}}{s}$

Applying the Laplace transformation on the ODE gives

$$
\begin{aligned}
s^2 Y(s) - s y(0) - y'(0) + 3sY(s) - y(0) + 2Y(s) &= \frac{e^{-s}}{s} \\
s^2 Y(s) - 1 + 3sY(s) + 2Y(s) &= \frac{e^{-s}}{s} \\
Y(s)\left(s^2 + 3s + 2\right) - 1 &= \frac{e^{-s}}{s} \\
Y(s) &= \frac{1}{s^2 + 3s + 2} + \frac{e^{-s}}{s\left(s^2 + 3s + 2\right)} \qquad (1)
\end{aligned}
$$

Considering the first term on the RHS of (1), calling it $Y_1(s) = \frac{1}{s^2+3s+2}$, and using partial fractions gives

$$
\begin{aligned}
Y_1(s) &= \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \\
A &= \lim_{s->-1} \frac{1}{(s+2)} = 1 \\
B &= \lim_{s->-2} \frac{1}{(s+1)} = -1
\end{aligned}
$$

Hence

$$
Y_1(s) = \frac{1}{s+1} - \frac{1}{s+2}
$$

Considering the second term on the RHS, calling it $Y_2(s) = \frac{e^{-s}}{s(s^2+3s+2)}$, and using partial fractions gives

$$
\frac{Y_2(s)}{e^{-s}} = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}
$$

$$
\begin{aligned}
A &= \lim_{s->0} \frac{1}{(s+1)(s+2)} = \frac{1}{2} \\
B &= \lim_{s->-1} \frac{1}{s(s+2)} = -1 \\
C &= \lim_{s->-2} \frac{1}{s(s+1)} = \frac{1}{2}
\end{aligned}
$$

Hence

$$
\frac{Y_2(s)}{e^{-s}} = \frac{1}{2}\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2}
$$

Therefore

$$
\begin{aligned}
Y(s) &= Y_1(s) + Y_2(s) \\
&= \left(\frac{1}{s+1} - \frac{1}{s+2}\right) + \left(\frac{1}{2}\frac{e^{-s}}{s} - \frac{e^{-s}}{s+1} + \frac{1}{2}\frac{e^{-s}}{s+2}\right) \qquad (2)
\end{aligned}
$$

The effect of e^{-as} is to cause a time delay when finding the inverse Laplace transform.

$$e^{-as}F(s) \longrightarrow f(t-a)\,\xi(t-a)$$

Now, taking the inverse Laplace transform of (2) gives the solution

$$\begin{aligned} y(t) &= e^{-t}\xi(t) - e^{-2t}\xi(t) + \frac{1}{2}\xi(t-1) - e^{-(t-1)}\xi(t-1) + \frac{1}{2}e^{-2(t-1)}\xi(t-1) \\ &= \left(e^{-t} - e^{-2t}\right)\xi(t) + \frac{1}{2}\left(1 - 2e^{-(t-1)} + e^{-2(t-1)}\right)\xi(t-1) \end{aligned}$$