HW 5
EE 409 (Linear Systems), CSUF spring 2010
Spring 2010
CSUF

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Contents

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1 Problem 3.23 (a)

3.27. For the block diagram systems shown below, find

(a) The matrices (**A**, **B**, **C**, **D**) of the state-variable description.

(b) The matrix $\mathbf{e}^{\mathbf{A}t}$.

(c) The matrix $(j\omega\mathbf{I} - \mathbf{A})^{-1}$.

(d) The frequency-response function, with a sketch of the amplitude and phase responses.

(e) The impulse-response function, with a sketch.

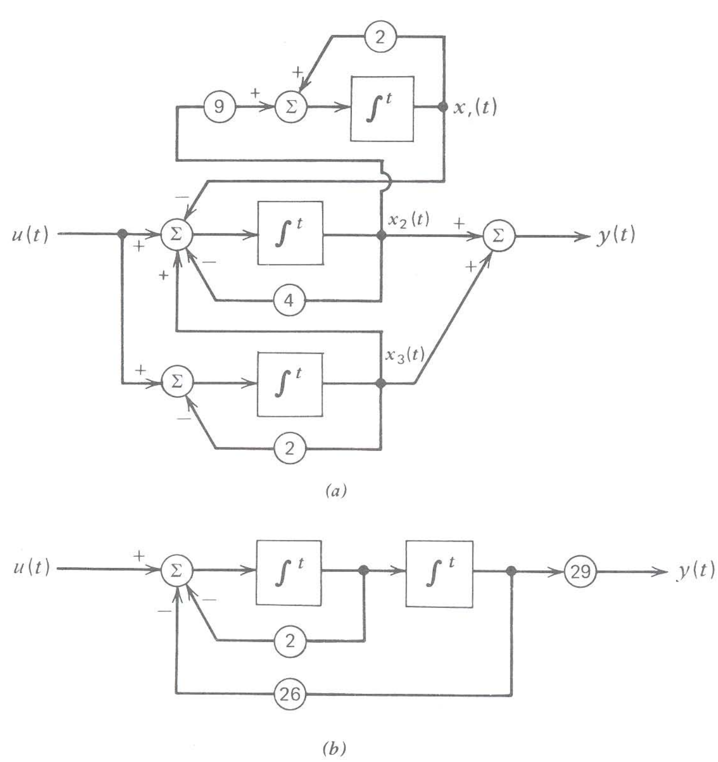

Figure 1: Problem description

1.1 Part(a)

Labeling the output from the branches as follows

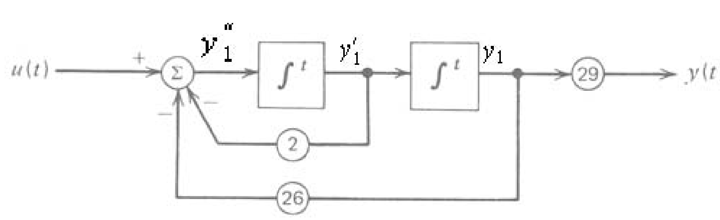

Figure 2: Problem description part(a) labeled

Then the differential equation becomes

$$y_1'' = u - 2y_1' - 26y_1$$

While the output equation become

$$y = 29y_1$$

Let $x_1 = y_1$

$$\left.\begin{array}{l} x_1 = y_1 \\ x_2 = y_1' \end{array}\right\} \rightarrow \left.\begin{array}{c} x_1' = y_1' = x_2 \\ x_2' = y_1'' = u - 2y_1' - 26y_1 = u - 2x_2 - 26x_1 \end{array}\right\}$$

Hence

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -26 & -2 \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{B} u(t)$$

$$y(t) = \overbrace{\begin{pmatrix} 29 & 0 \end{pmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overbrace{(0)}^{D} u(t)$$

1.2 Part b

To find e^{At} use the eigenvalue approach. Find find $|A - \lambda I|$

$$|A - \lambda I| = \left| \begin{pmatrix} 0 & 1 \\ -26 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \left| \begin{pmatrix} -\lambda & 1 \\ -26 & -2 - \lambda \end{pmatrix} \right| = -\lambda(-2 - \lambda) + 26$$

Now solve $-\lambda(-2 - \lambda) + 26 = 0$ or $\lambda^2 + 2\lambda + 26 = 0$, which has solutions

$$\lambda_1 = -1 + 5j$$
$$\lambda_2 = -1 - 5j$$

Hence we have the following 2 equations to solve for β_0 and β_1

$$e^{\lambda_1 t} = \beta_0 + \lambda_1 \beta_1$$
$$e^{\lambda_2 t} = \beta_0 + \lambda_2 \beta_1$$

Solving we find

$$\beta_0 = e^{-t}\left(\cos 5t + \frac{1}{5}\sin 5t\right)$$
$$\beta_1 = \frac{1}{5}e^{-t}\sin 5t$$

Hence

$$\begin{aligned} e^{At} &= \beta_0 + \beta_1 A \\ &= e^{-t}\left(\cos 5t + \frac{1}{5}\sin 5t\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{5}e^{-t}\sin 5t \begin{pmatrix} 0 & 1 \\ -26 & -2 \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} \cos 5t + \frac{1}{5}\sin 5t & \frac{1}{5}\sin 5t \\ \frac{-26}{5}\sin 5t & \cos 5t - \frac{1}{5}\sin 5t \end{pmatrix} \end{aligned}$$

1.3 Part c

To find matrix $(j\omega I - A)^{-1}$

$$\begin{aligned} j\omega I - A &= j\omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -26 & -2 \end{pmatrix} \\ &= \begin{pmatrix} j\omega & 0 \\ 0 & j\omega \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -26 & -2 \end{pmatrix} \\ &= \begin{pmatrix} j\omega & -1 \\ 26 & j\omega + 2 \end{pmatrix} \end{aligned}$$

Hence

$$\begin{pmatrix} j\omega & -1 \\ 26 & j\omega+2 \end{pmatrix}^{-1} = \frac{\begin{pmatrix} j\omega+2 & 1 \\ -26 & j\omega \end{pmatrix}}{(j\omega)(j\omega+2)+26} = \frac{\begin{pmatrix} j\omega+2 & 1 \\ -26 & j\omega \end{pmatrix}}{-\omega^2+2j\omega+26}$$
$$= \frac{1}{-\omega^2+2j\omega+26}\begin{pmatrix} j\omega+2 & 1 \\ -26 & j\omega \end{pmatrix}$$

1.4 Part d

To find the frequency response function. Assuming zero initial conditions, from equation 3.10.4 in the book

$$\begin{aligned} H(j\omega) &= C(j\omega I - A)^{-1}B \\ &= \begin{pmatrix} 29 & 0 \end{pmatrix} \frac{1}{-\omega^2+2j\omega+26}\begin{pmatrix} j\omega+2 & 1 \\ -26 & j\omega \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{-\omega^2+2j\omega+26}\begin{pmatrix} 29 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ j\omega \end{pmatrix} \\ &= \frac{29}{-\omega^2+2j\omega+26} \end{aligned}$$

Hence

$$|H(j\omega)| = \frac{29}{|-\omega^2+2j\omega+26|} = \frac{29}{\sqrt{(26-\omega^2)^2+4\omega^2}}$$

And phase is

$$\begin{aligned} \arg(H(j\omega)) &= \arg(29) - \arg\left(-\omega^2+2j\omega+26\right) \\ &= -\tan^{-1}\frac{2\omega}{26-\omega^2} \end{aligned}$$

1.5 Part e

The state solution is

$$x(t) = \int_0^t e^{A\tau}Bu(\tau)\,d\tau$$

and

$$y(t) = Cx(t) = \int_0^t Ce^{A\tau}Bu(\tau)\,d\tau$$

Hence, let $u(\tau) = \delta(t)$, then

$$\begin{aligned} h(t) &= Ce^{At}B \\ &= \begin{pmatrix} 29 & 0 \end{pmatrix} e^{-t}\begin{pmatrix} \cos 5t + \frac{1}{5}\sin 5t & \frac{1}{5}\sin 5t \\ \frac{-26}{5}\sin 5t & \cos 5t - \frac{1}{5}\sin 5t \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} 29 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{5}\sin 5t \\ \cos 5t - \frac{1}{5}\sin 5t \end{pmatrix} \\ &= e^{-t}\left(\frac{29}{5}\sin 5t\right)\xi(t) \end{aligned}$$