Mechanical Vibration
EGME 431
California State University, Fullerton

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spring 2009 Compiled on October 14, 2025 at 5:11pm [public]

Contents

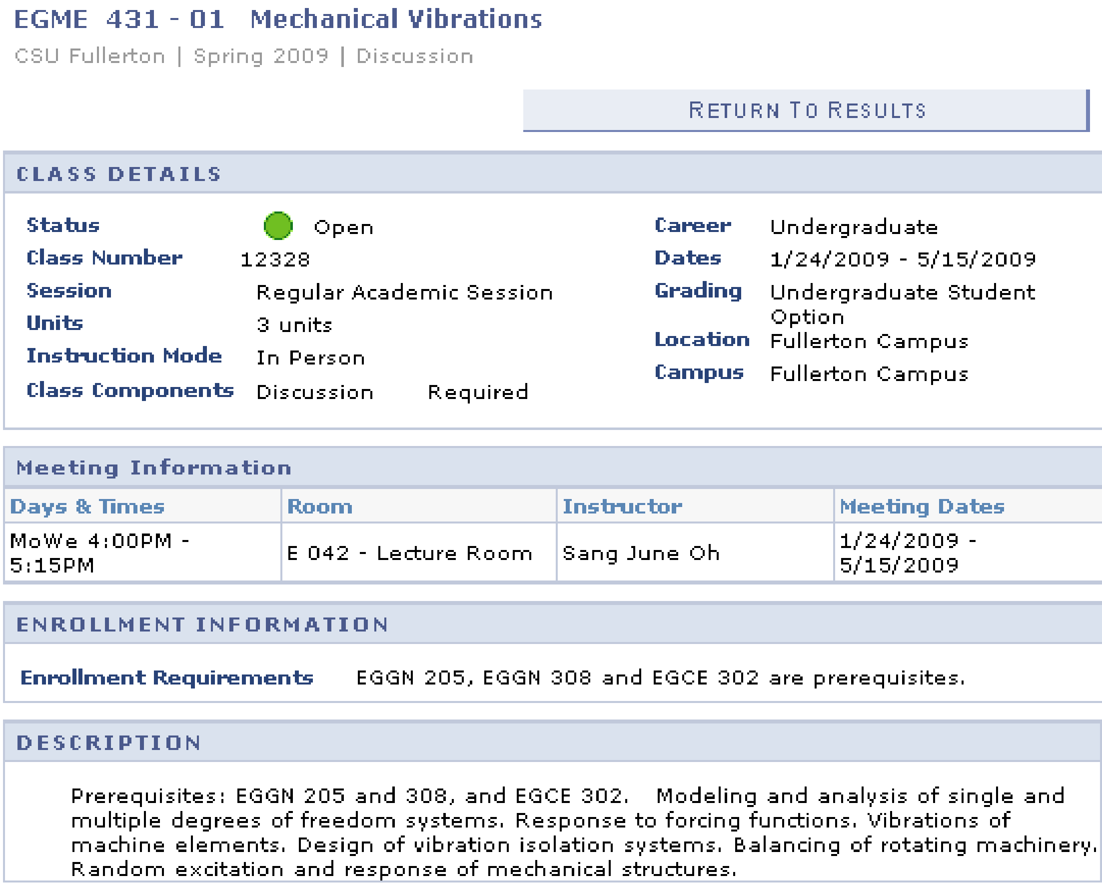
Chapter 1

Introduction

Local contents

I took this course in Spring 2009 at CSUF to learn more about Vibration and control of vibration. Not part of a degree program. In the Mechanical Eng. department.

1.1 course description



EGME 431 - 01 Mechanical Vibrations

CSU Fullerton | Spring 2009 | Discussion

RETURN TO RESULTS

CLASS DETAILS

Status	Open	Career	Undergraduate
Class Number	12328	Dates	1/24/2009 - 5/15/2009
Session	Regular Academic Session	Grading	Undergraduate Student Option
Units	3 units	Location	Fullerton Campus
Instruction Mode	In Person	Campus	Fullerton Campus
Class Components	Discussion Required		

Meeting Information

Days & Times	Room	Instructor	Meeting Dates
MoWe 4:00PM - 5:15PM	E 042 - Lecture Room	Sang June Oh	1/24/2009 - 5/15/2009

ENROLLMENT INFORMATION

Enrollment Requirements EGGN 205, EGGN 308 and EGCE 302 are prerequisites.

DESCRIPTION

Prerequisites: EGGN 205 and 308, and EGCE 302. Modeling and analysis of single and multiple degrees of freedom systems. Response to forcing functions. Vibrations of machine elements. Design of vibration isolation systems. Balancing of rotating machinery. Random excitation and response of mechanical structures.

Figure 1.1: class info

1.2 Textbook

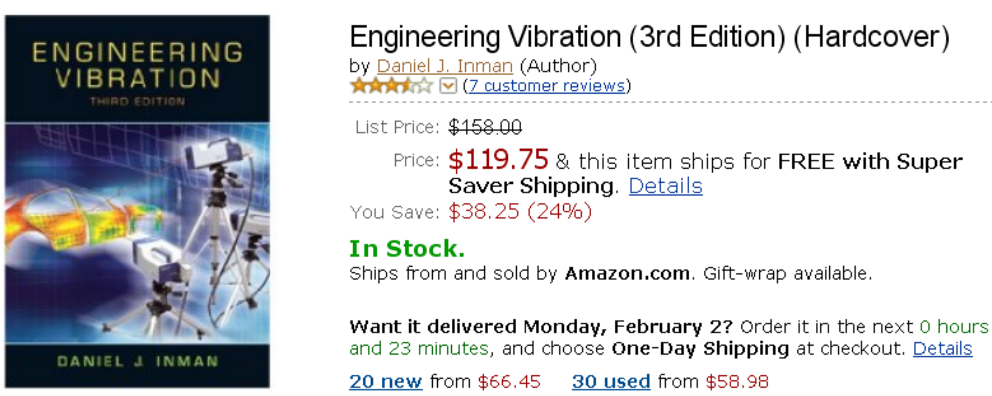


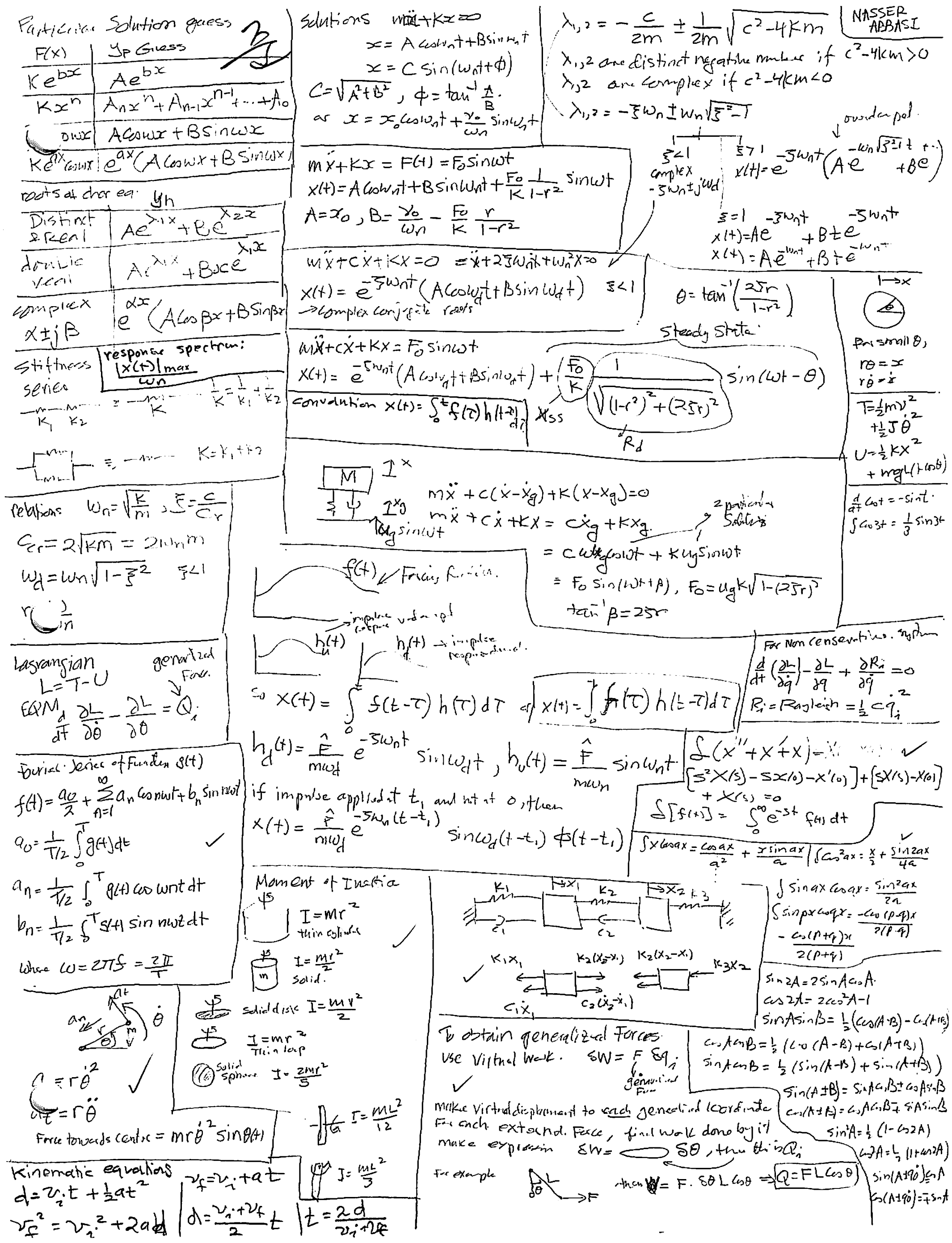
Figure 1.2: Text book

1.3 Instructor

Dr Sang June Oh

`sjoh@fullerton.edu` 714-278-2458

1.4 sheetsheet



Chapter 2

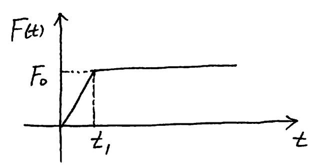
Quizz

EGME 431 QUIZ FOR HW #3

Given an undamped spring-mass system,

$$m\ddot{X} + KX = F(t)$$

Where



$$F(t) = \begin{cases} F_0\left(\frac{t}{t_1}\right) & t < t_1 \\ F_0 & t > t_1 \end{cases}$$

Find the response $X(t)$ for $t > t_1$

You may or may not use the following formulas.

$$\left\{ \begin{aligned} \omega_d &= \omega_n\sqrt{1-\zeta^2} \\ h(t) &= \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t \\ X(t) &= \int_0^t F(\tau)h(t-\tau)\,d\tau \end{aligned} \right\}$$

Chapter 3

HWs

Local contents

3.1 HW1

Local contents

3.1.1 Description of HW

My grade is 66/70.

1. Equation of motion of spring mass on an incline surface.
2. Equation of motion of pendulum attached to spring.
3. Solving 2nd order ODE's.
4. Equation of motion of mass attached to 2 pulley's.

3.1.2 Problem 1.6

Figure P1.5

1.6. Find the equation of motion for the system of Figure P1.6 and compute the formula for the natural frequency. In particular, using static equilibrium along with Newton's law, determine what effect gravity has on the equation of motion and the system's natural frequency. Assume the block slides without friction.

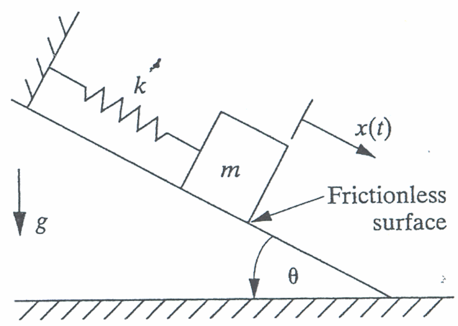


Figure P1.6

Figure 3.1: Problem

Taking displacement along the x-direction shown to be from the static equilibrium position, then applying $\sum F_x = m\ddot{x}$ along the shown x direction, we obtain

$$m\ddot{x} = -kx$$
$$\ddot{x} + \frac{k}{m}x = 0$$

which is the equation of motion. To obtain the natural frequency, we consider free vibration $\ddot{x} + \frac{k}{m}x = 0$, which implies that $\omega_n = \sqrt{\frac{k}{m}}$, hence we see that the natural frequency is independent of g.

We see that gravity has no effect on the spring mass system, this is because we use x to be from the static equilibrium position of the spring.

3.1.3 Problem 1.16

1.16. A machine part is modeled as a pendulum connected to a spring as illustrated in Figure P1.16. Ignore the mass of pendulum's rod and derive the equation of motion. Then, following the procedure used in Example 1.1.1, linearize the equation of motion, and compute the formula for the natural frequency. Assume that the rotation is small enough so that the spring only deflects horizontally.

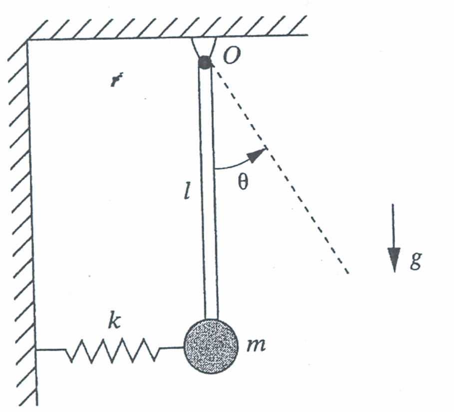
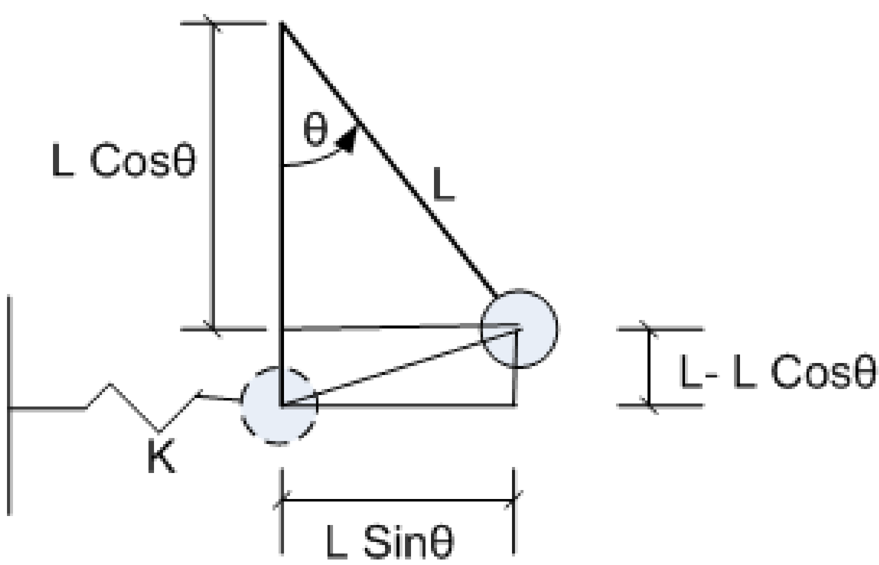


Figure P1.16

Figure 3.2: Problem

First we need to derive the equation of motion. Considering the following diagram



Using as generalized coordinates θ, we obtain

$$T = \frac{1}{2} m \left(L\dot{\theta}\right)^2$$
$$U = \frac{1}{2} k (L \sin\theta)^2 + mg(L - L\cos\theta)$$

Notice that in the calculation of U above, we assumed that the spring stretches by $L\sin\theta$ in the horizontal direction only, which we are allowed to do for small θ.

Now we can find Lagrangian

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2} m \left(L\dot{\theta}\right)^2 - \frac{1}{2} k L^2 \sin^2\theta - mgL(1 - \cos\theta) \end{aligned}$$

Hence the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$
$$\frac{d}{dt}\left(mL^2\dot{\theta}\right) - \left(-kL^2 \sin\theta\cos\theta - mgL\sin\theta\right) = 0$$
$$mL^2\ddot{\theta} + kL^2 \sin\theta\cos\theta + mgL\sin\theta = 0$$

The above is nonlinear equation. Linearize around $\theta = 0$ (equilibrium point) using Taylor series, and for small θ we obtain $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, hence the above becomes

$$mL\ddot{\theta} + kL\theta + mg\theta = 0$$
$$\ddot{\theta} + \left(\frac{mg + kL}{mL}\right)\theta = 0$$

Hence effective $\omega_n$can be found from

$$\omega_n^2 = \frac{mg + kL}{mL}$$

Hence

$$\omega_n = \sqrt{\frac{g}{L} + \frac{k}{m}}$$

Comparing the above to the natural frequency of pendulum with no spring attached which is $\omega_n = \sqrt{\frac{g}{L}}$, we can see the effect of adding a spring on the natural frequency: The more stiff the spring is, in other words, the larger k is, the larger ω_n will become, and the smaller the period of oscillation will be. We conclude that a pendulum with a spring attached to it will always oscillate with a period which is smaller than the same pendulum without the spring attached. This makes sense as a mass with spring alone has $\omega_n = \sqrt{\frac{k}{m}}$

3.1.4 Problem 1.32

1.32. Solve $\ddot{x} + 2\dot{x} + 2x = 0$ for $x_0 = 0$ mm, $v_0 = 1$ mm/s and sketch the response. You may wish to sketch $x(t) = e^{-t}$ and $x(t) = -e^{-t}$ first.

We need to solve $\ddot{x} + 2\dot{x} + 2x = 0$ for $x_0 = 0mm$ and $v_0 = 1mm/s$

The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$ which has roots $\lambda_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2\pm\sqrt{4-8}}{2} = -1 \pm j$

Hence the solution is

$$x_h = e^{-t}(A\cos t + B\sin t)$$

is the general solution. Now we use I.C. to find A, B. When $t = 0$

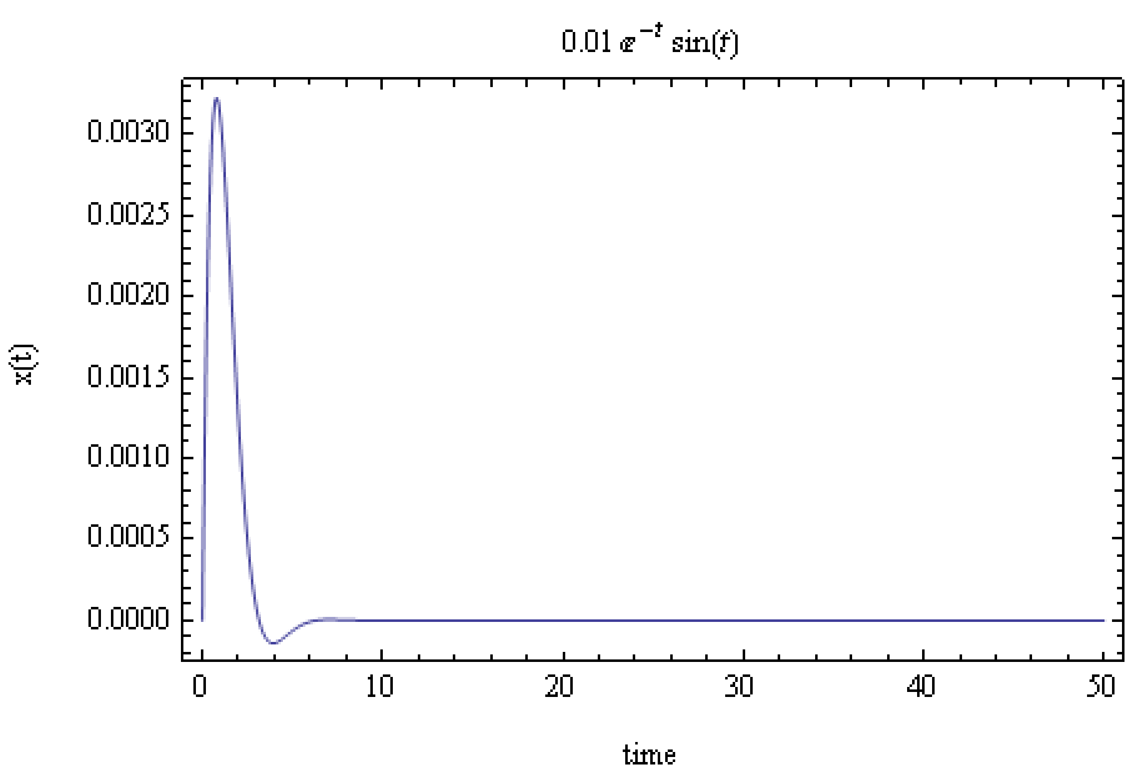
$$0 = A$$

Hence $x_h = Be^{-t}\sin t$, and $\dot{x}_h = Be^{-t}\cos t - Be^{-t}\sin t$ and at $t = 0$, we obtain $0.01 = B$

Then

$$x_h = 0.01e^{-t}\sin t$$

This is a plot of the solution for t up to 50 seconds



3.1.5 Problem 1.43

$x_0 = 100$ mm.

1.43. Solve $\ddot{x} - \dot{x} + x = 0$ with $x_0 = 1$ and $v_0 = 0$ for $x(t)$ and sketch the response.

1.44. A spring–mass–damper system has mass of 100 kg, stiffness of 3000 N/m, and damping coefficient of 300 kg/s. Calculate the undamped natural frequency, the damping ratio, and

We need to solve $\ddot{x} - \dot{x} + x = 0$ for $x_0 = 1$ and $v_0 = 0$

The characteristic equation is $\lambda^2 - \lambda + 1 = 0$ which has roots $\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

Hence the solution is

$$x_h = e^{\frac{1}{2}t}\left(A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right)$$

is the general solution. Now we use I.C. to find A, B. When $t = 0$

$$1 = A$$

Hence $x_h = e^{\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right)$, and

$$\dot{x}_h = \frac{1}{2}e^{\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right) + e^{\frac{1}{2}t}\left(-\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t + B\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t\right)$$

and at $t = 0$, we obtain

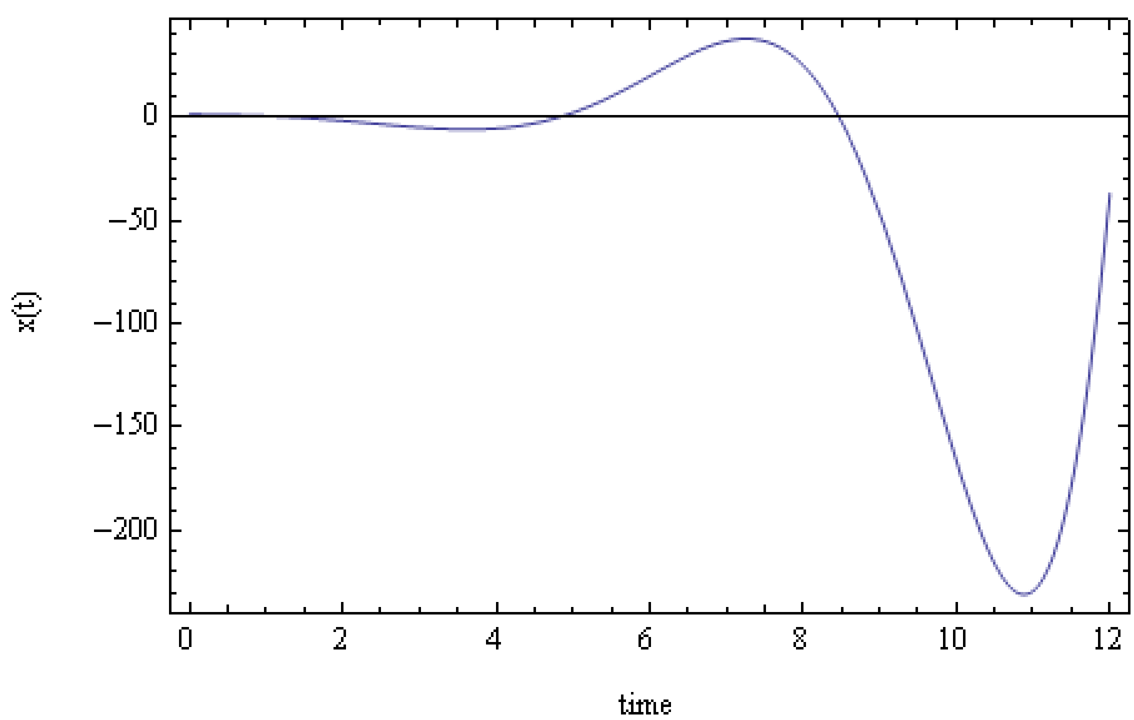
$$0 = \frac{1}{2} + B\frac{\sqrt{3}}{2}$$
$$B = \frac{-1}{\sqrt{3}}$$

Hence

$$x_h = e^{\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)$$

This is a plot of the solution for t up to 12 seconds

$$\frac{1}{3}e^{t/2}\left(3\cos\left(\frac{\sqrt{3}\,t}{2}\right) - \sqrt{3}\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$



3.1.6 Problem 1.62

How much effect does the rotational inertia have on the undamped natural frequency?

1.62. Use Lagrange's formulation to calculate the equation of motion and the natural frequency of the system of Figure P1.62. Model each of the brackets as a spring of stiffness k, and assume the inertial of the pulleys is negligible.

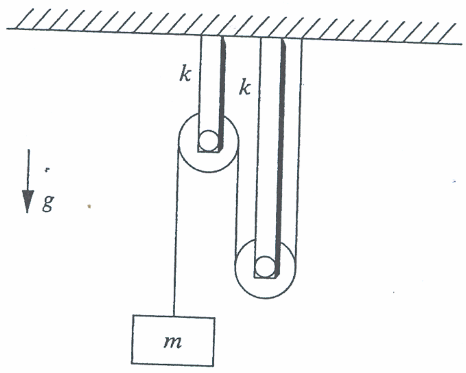


Figure P1.62

This is a single degree of freedom linear system. Assume x from static equilibrium, then (using parallel springs) we obtain

$$T = \frac{1}{2} m\dot{x}^2$$
$$U = \frac{1}{2} kx^2 + \frac{1}{2} kx^2 = kx^2$$

Hence $L = T - U = \frac{1}{2} m\dot{x}^2 - kx^2$ and the Lagrangian equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$
$$\frac{d}{dt}(m\dot{x}) - (-2kx) = 0$$

Hence equation of motion is

$$m\ddot{x} + 2kx = 0$$

And $\omega_n = \sqrt{\frac{2k}{m}}$

3.1.7 Problem 1.90

Section 1.8 (see also Problem 1.43)

1.90. Consider the system of Figure P1.90. (a) Write the equations of motion in terms of the angle, θ, the bar makes with the vertical. Assume linear deflections of the springs and linearize the equations of motion. (b) Discuss the stability of the linear system's solutions in terms of the physical constants, m, k, and l. Assume the mass of the rod acts at the center as indicated in the figure.

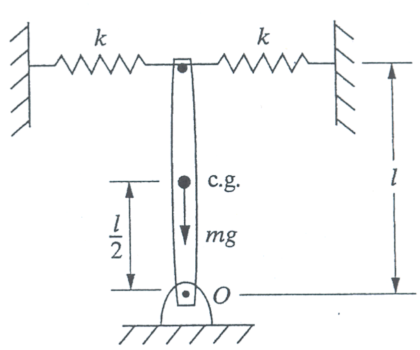
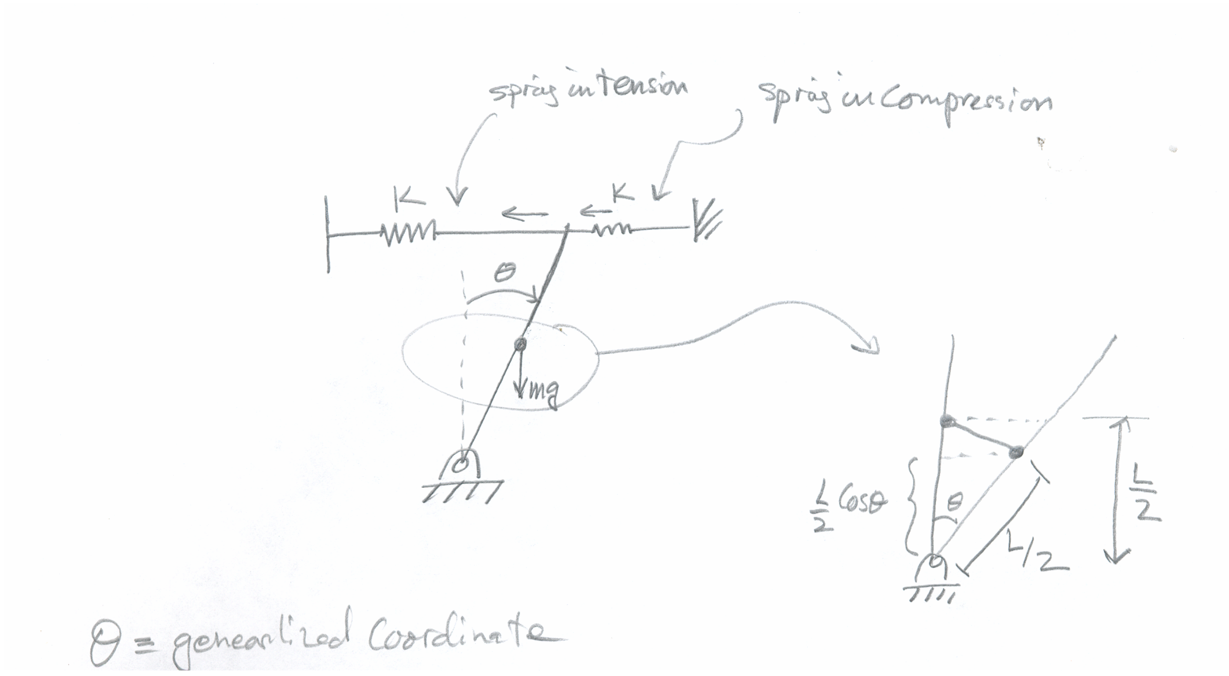


Figure P1.90

3.1.7.1 Part(a)



$$T = \frac{1}{2}m\left(\frac{l}{2}\dot{\theta}\right)^2$$
$$= \frac{1}{8}ml^2\dot{\theta}^2$$
$$U_{springs} = \frac{1}{2}k(l\sin\theta)^2 + \frac{1}{2}k(l\sin\theta)^2$$

Assuming small angle oscillation, $\sin\theta \simeq \theta$, hence

$$U_{springs} = kl^2\theta^2$$

and for the mass, since it losses potential, we have

$$U_{mass} = -mg\left(\frac{l}{2} - \frac{l}{2}\cos\theta\right)$$

Hence Lagrangian L is

$$\begin{aligned} L &= T - (U_{springs} + U_{mass}) \\ &= \frac{1}{8}ml^2\dot{\theta}^2 - \left(kl^2\theta^2 - mg\frac{l}{2}(1-\cos\theta)\right) \\ &= \frac{1}{8}ml^2\dot{\theta}^2 - kl^2\theta^2 + mg\frac{l}{2} - mg\frac{l}{2}\cos\theta \end{aligned}$$

Now find the Lagrangian equation

$$\begin{aligned} \frac{\partial L}{\partial \dot{\theta}} &= \frac{1}{4}ml^2\dot{\theta} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \frac{1}{4}ml^2\ddot{\theta} \\ \frac{\partial L}{\partial \theta} &= -2kl^2\theta + mg\frac{l}{2}\sin\theta \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= \frac{1}{4}ml^2\ddot{\theta} - \left(-2kl^2\theta + mg\frac{l}{2}\sin\theta\right) \\ &= \frac{1}{4}ml^2\ddot{\theta} + 2kl^2\theta - mg\frac{l}{2}\sin\theta \end{aligned}$$

And the equation of motion is

$$\begin{aligned} \frac{1}{4}ml^2\ddot{\theta} + 2kl^2\theta - mg\frac{l}{2}\sin\theta = 0 \\ \ddot{\theta} + \frac{8k}{m}\theta - 2\frac{g}{l}\sin\theta = 0 \end{aligned}$$

Linearize by setting $\sin\theta \simeq \theta$ we obtain equation of motion

$$\ddot{\theta} + \theta\left(\frac{8k}{m} - 2\frac{g}{l}\right) = 0 \tag{1}$$

Hence

$$\omega_n = \sqrt{2\left(4\frac{k}{m} - \frac{g}{l}\right)}$$

3.1.7.2 Part (b)

To discuss stability, we need to determine the location of the roots of the characteristic equation of the homogeneous EQM, hence from equation (1), we see that

$$\ddot{\theta} + \omega_n^2\theta = 0$$

And assuming solution $\theta(t) = e^{\lambda t}$ leads to the characteristic equation

$$\begin{aligned} \lambda^2 + \omega_n^2 &= 0 \\ \lambda^2 &= -\omega_n^2 \\ \lambda &= \pm\sqrt{-\omega_n^2} \\ &= \pm j\sqrt{\omega_n^2} \end{aligned}$$

Since $\omega_n^2 > 0$, then

$$\lambda = \pm j\omega_n$$

Since roots of the characteristic equation on the imaginary axis, this is a $\boxed{\text{marginally stable system}}$ regardless of the values of m, l, k.

Since we are looking at the linearized system, there is only one equilibrium point, and the system is either stable or not. Here we found it is marginally stable. The effect of changing k, l, m is to change the period of oscillation around the equilibrium point.

3.1.8 Key for HW1

HW # 1 Key

1.6 Find the equation of motion for the system of Figure P1.6, and find the natural frequency. In particular, using static equilibrium along with Newton's law, determine what effect gravity has on the equation of motion and the system's natural frequency. Assume the block slides without friction.

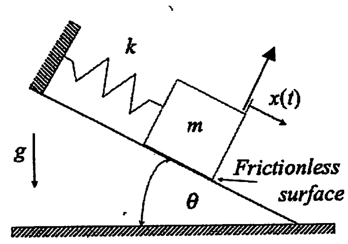
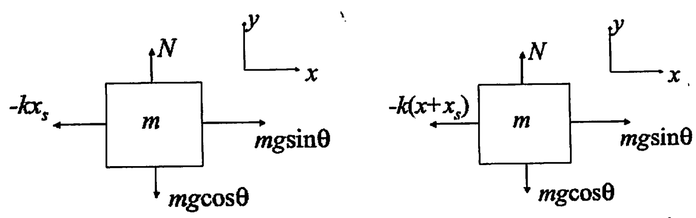


Figure P1.6

Solution:
Choosing a coordinate system along the plane with positive down the plane, the free-body diagram of the system for the static case is given and (a) and for the dynamic case in (b):



In the figures, N is the normal force and the components of gravity are determined by the angle θ as indicated. From the static equilibrium: $-kx_s + mg\sin\theta = 0$. Summing forces in (b) yields:

$$\sum F_i = m\ddot{x}(t) \Rightarrow m\ddot{x}(t) = -k(x + x_s) + mg\sin\theta$$

$$\Rightarrow m\ddot{x}(t) + kx = -kx_s + mg\sin\theta = 0$$

$$\Rightarrow \underline{m\ddot{x}(t) + kx = 0}$$

$$\Rightarrow \underline{\omega_n = \sqrt{\frac{k}{m}} \text{ rad/s}}$$

1.16 A machine part is modeled as a pendulum connected to a spring as illustrated in Figure P1.16. Ignore the mass of pendulum's rod and derive the equation of motion. Then following the procedure used in Example 1.1.1, linearize the equation of motion and compute the formula for the natural frequency. Assume that the rotation is small enough so that the spring only deflects horizontally.

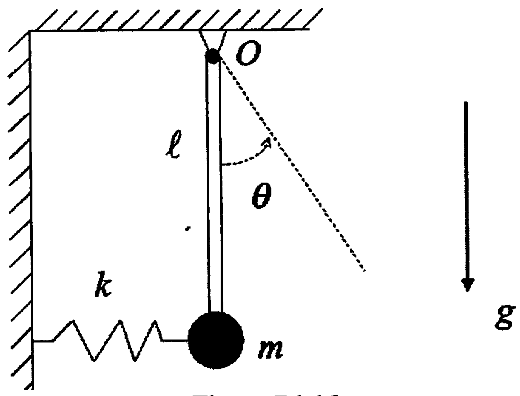
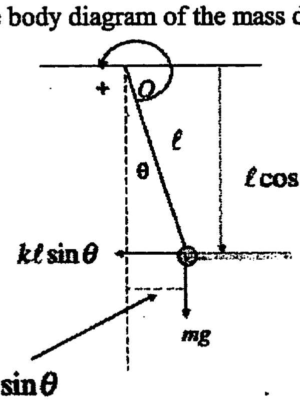


Figure P1.16

Solution: Consider the free body diagram of the mass displaced from equilibrium:



There are two forces acting on the system to consider, if we take moments about point O (then we can ignore any forces at O). This yields

$$\sum M_O = J_O\alpha \Rightarrow m\ell^2\ddot{\theta} = -mg\ell\sin\theta - k\ell\sin\theta \bullet \ell\cos\theta$$

$$\Rightarrow \underline{m\ell^2\ddot{\theta} + mg\ell\sin\theta + k\ell^2\sin\theta\cos\theta = 0}$$

Next consider the small θ approximations to that $\sin\theta \sim \theta$ and $\cos\theta=1$. Then the linearized equation of motion becomes:

$$\ddot{\theta}(t) + \left(\frac{mg + k\ell}{m\ell}\right)\theta(t) = 0$$

Thus the natural frequency is

$$\underline{\omega_n = \sqrt{\frac{mg + k\ell}{m\ell}} \text{ rad/s}}$$

1.32 Solve $\ddot{x}+2\dot{x}+2x=0$ for $x_0 = 0$ mm, $v_0 = 1$ mm/s and sketch the response. You may wish to sketch $x(t) = e^{-t}$ and $x(t) = -e^{-t}$ first.

Solution:

Given $\ddot{x}+2\dot{x}+x=0$ where $x_0 = 0$, $v_0 = 1$ mm/s

Let: $x = ae^{rt} \Rightarrow \dot{x} = are^{rt} \Rightarrow \ddot{x} = ar^2e^{rt}$

Substitute into the equation of motion to get

$$ar^2e^{rt} + 2are^{rt} + ae^{rt} = 0 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow r_{1,2} = -1 \pm i$$

So

$$x = c_1e^{(-1+i)t} + c_2e^{(-1-i)t} \Rightarrow \dot{x} = (-1+i)c_1e^{(-1+i)t} + (-1-i)c_2e^{(-1-i)t}$$

Initial conditions:

$$x_0 = x(0) = c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1 \qquad (1)$$
$$v_0 = \dot{x}(0) = (-1+i)c_1 + (-1-i)c_2 = 1 \qquad (2)$$

Substituting equation (1) into (2)

$$v_0 = (-1+i)c_1 - (-1-i)c_1 = 1$$
$$c_1 = -\frac{1}{2}i,\ c_2 = \frac{1}{2}i$$
$$x(t) = -\frac{1}{2}ie^{(-1+i)t} + \frac{1}{2}ie^{(-1-i)t} = -\frac{1}{2}ie^{-t}\left(e^{it} - e^{-it}\right)$$

Applying Euler's formula

$$x(t) = -\frac{1}{2}ie^{-t}(\cos t + i\sin t - (\cos t - i\sin t))$$
$$\underline{x(t) = e^{-t}\sin t}$$

Alternately use equations (1.36) and (1.38). The plot is similar to figure 1.11.

1.43 Solve $\ddot{x}-\dot{x}+x=0$ with $x_0 = 1$ and v_0 =0 for *x(t)* and sketch the response.

Solution: This is a problem with negative damping which can be used to tie into Section 1.8 on stability, or can be used to practice the method for deriving the solution using the method suggested following equation (1.13) and eluded to at the start of the section on damping. To this end let $x(t) = Ae^{\lambda t}$ the equation of motion to get:

$$(\lambda^2-\lambda+1)e^{\lambda t}=0$$

This yields the characteristic equation:

$$\lambda^2-\lambda+1=0 \Rightarrow \lambda=\frac{1}{2}\pm\frac{\sqrt{3}}{2}j, \text{ where } j=\sqrt{-1}$$

There are thus two solutions as expected and these combine to form

$$x(t)=e^{0.5t}(Ae^{\frac{\sqrt{3}}{2}jt}+Be^{-\frac{\sqrt{3}}{2}jt})$$

Using the Euler relationship for the term in parenthesis as given in Window 1.4, this can be written as

$$x(t)=e^{0.5t}(A_1\cos\frac{\sqrt{3}}{2}t+A_2\sin\frac{\sqrt{3}}{2}t)$$

Next apply the initial conditions to determine the two constants of integration:

$$x(0)=1=A_1(1)+A_2(0)\Rightarrow A_1=1$$

Differentiate the solution to get the velocity and then apply the initial velocity condition to get

$$\dot{x}(t)=$$

$$\frac{1}{2}e^0(A_1\cos\frac{\sqrt{3}}{2}0+A_2\sin\frac{\sqrt{3}}{2}0)+e^0\frac{\sqrt{3}}{2}(-A_1\sin\frac{\sqrt{3}}{2}0+A_2\cos\frac{\sqrt{3}}{2}0)=0$$

$$\Rightarrow A_1+\sqrt{3}(A_2)=0\Rightarrow A_2=-\frac{1}{\sqrt{3}},$$

$$\Rightarrow \underline{x(t)=e^{0.5t}(\cos\frac{\sqrt{3}}{2}t-\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t)}$$

This function oscillates with increasing amplitude as shown in the following plot which shows the increasing amplitude. This type of response is referred to as a flutter instability. This plot is from Mathcad.

1.62 Use Lagrange's formulation to calculate the equation of motion and the natural frequency of the system of Figure P1.62. Model each of the brackets as a spring of stiffness k, and assume the inertia of the pulleys is negligible.

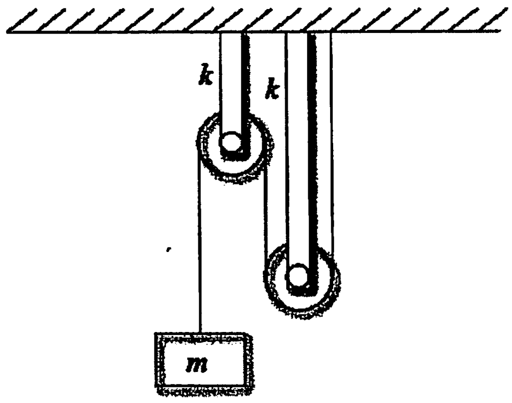


Figure P1.62

Solution: Let x denote the distance mass m moves, then each spring will deflects a distance $x/4$. Thus the potential energy of the springs is

$$U = 2\times\frac{1}{2}k\left(\frac{x}{4}\right)^2 = \frac{k}{16}x^2$$

The kinetic energy of the mass is

$$T = \frac{1}{2}m\dot{x}^2$$

Using the Lagrange formulation in the form of Equation (1.64):

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{x}}\left(\frac{1}{2}m\dot{x}^2\right)\right) + \frac{\partial}{\partial x}\left(\frac{kx^2}{16}\right) = 0 \Rightarrow \frac{d}{dt}(m\dot{x}) + \frac{k}{8}x = 0$$

$$\Rightarrow \underline{m\ddot{x} + \frac{k}{8}x = 0} \Rightarrow \underline{\omega_n = \frac{1}{2}\sqrt{\frac{k}{2m}}} \text{ rad/s}$$

1.64 Lagrange's formulation can also be used for non-conservative systems by adding the applied non-conservative term to the right side of equation (1.64) to get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial U}{\partial q_i}+\frac{\partial R_i}{\partial \dot{q}_i}=0$$

Here R_i is the *Rayleigh dissipation function* defined in the case of a viscous damper attached to ground by

$$R_i=\frac{1}{2}c\dot{q}_i^2$$

Use this extended Lagrange formulation to derive the equation of motion of the damped automobile suspension of Figure P1.64

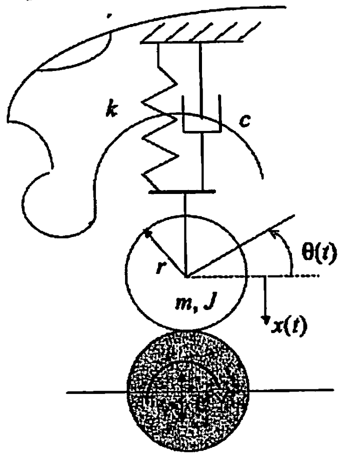


Figure P1.64

Solution: The kinetic energy is (see Example 1.4.1):

$$T=\frac{1}{2}(m+\frac{J}{r^2})\dot{x}^2$$

The potential energy is:

$$U=\frac{1}{2}kx^2$$

The Rayleigh dissipation function is

$$R=\frac{1}{2}c\dot{x}^2$$

The Lagrange formulation with damping becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial U}{\partial q_i}+\frac{\partial R_i}{\partial \dot{q}_i}=0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}(m+\frac{J}{r^2})\dot{x}^2\right)\right)+\frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right)+\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}c\dot{x}^2\right)=0$$

$$\Rightarrow (m+\frac{J}{r^2})\ddot{x}+c\dot{x}+kx=0$$

Problems and Solutions Section 1.8 (1.90 through 1.93)

1.90 Consider the system of Figure 1.90 and (a) write the equations of motion in terms of the angle, θ, the bar makes with the vertical. Assume linear deflections of the springs and linearize the equations of motion. Then (b) discuss the stability of the linear system's solutions in terms of the physical constants, m, k, and ℓ. Assume the mass of the rod acts at the center as indicated in the figure.

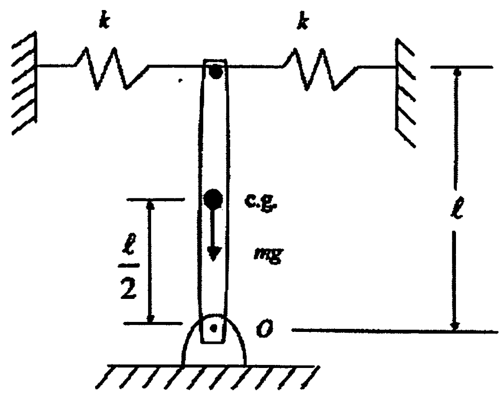


Figure P1.90

Solution: Note that from the geometry, the springs deflect a distance $kx = k(\ell\sin\theta)$ and the cg moves a distance $\ell/2 \cos\theta$. Thus the total potential energy is

$$U = 2\times\frac{1}{2}k(\ell\sin\theta)^2 - \frac{mg\ell}{2}\cos\theta$$

wrong

$mg\left(\frac{\ell}{2} - \frac{\ell}{2}\cos\theta\right) = -mg\frac{\ell}{2}\cos\theta$

and the total kinetic energy is

$$T = \frac{1}{2}J_O\dot\theta^2 = \frac{1}{2}\frac{m\ell^2}{3}\dot\theta^2$$

The Lagrange equation (1.64) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) + \frac{\partial U}{\partial\theta} = \frac{d}{dt}\left(\frac{m\ell^2}{3}\dot\theta\right) + 2k\ell\sin\theta\cos\theta - \frac{1}{2}mg\ell\sin\theta = 0$$

Using the linear, small angle approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ yields

$$\text{a)} \quad \frac{m\ell^2}{3}\ddot\theta + \left(2k\ell^2 - \frac{mg\ell}{2}\right)\theta = 0$$

Since the leading coefficient is positive the sign of the coefficient of θ determines the stability.

b)

if $2k\ell - \frac{mg}{2} > 0 \Rightarrow 4k > \frac{mg}{\ell} \Rightarrow$ the system is stable

if $4k = mg \Rightarrow \theta(t) = at + b \Rightarrow$ the system is unstable

if $2k\ell - \frac{mg}{2} < 0 \Rightarrow 4k < \frac{mg}{\ell} \Rightarrow$ the system is unstable

Note that physically this results states that the system's response is stable as long as the spring stiffness is large enough to over come the force of gravity.

3.2 HW2

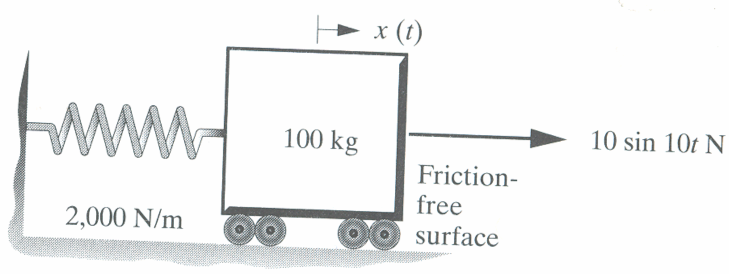
Local contents

3.2.1 Description of HW

1. Find solution to spring mass system with forcing function for different IC.
2. Compute IC for spring mass system to cause it to oscillate at some frequency.
3. Find EQM for spring-damper-mass system on incline surface.
4. Car and passengers and different speeds, deflection calculations.
5. The unbalanced mass rotating, the washing machine problem.

3.2.2 Problem 2.7

2.7. Consider the system in Figure P2.7, write the equation of motion, and calculate the response assuming (a) that the system is initially at rest, and (b) that the system has an initial displacement of 0.05 m.



Solution sketch: Obtain the Lagrangian, find EQM, solve in terms of general initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$, then solve parts (a) and (b) using this general solution.

This is one degree of freedom system. Using x as the generalized coordinates, we first obtain the Lagrangian L

$$L = T - U$$

Where

$$\begin{aligned}
T &= \frac{1}{2} m\dot{x}^2 \\
U &= \frac{1}{2} kx^2 \\
L &= \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \\
\frac{\partial L}{\partial \dot{x}} &= m\dot{x} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= m\ddot{x} \\
\frac{\partial L}{\partial x} &= -kx
\end{aligned}$$

Hence the EQM is (using the Lagrangian equation), and $F = 10$ and $\omega = 10$ rad/sec. (the forcing frequency)

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= F \sin \omega t \\
m\ddot{x} + kx &= F \sin \omega t \\
\ddot{x} + \frac{k}{m} x &= \frac{F}{m} \sin \omega t \\
\ddot{x} + \omega_n^2 x &= \frac{F}{m} \sin \omega t \qquad (1)
\end{aligned}$$

Where $\omega_n^2 = \frac{k}{m}$. The solution is

$$x(t) = x_h(t) + x_p(t) \qquad (2)$$

To obtain $x_h t(t)$

$$\ddot{x}_h(t) + \omega_n^2 x_h(t) = 0$$

Assume $x_h(t) = e^{\lambda t}$ and substitute in the above ODE we obtain the characteristic equation

$$\begin{aligned}
\lambda^2 + \omega_n^2 &= 0 \\
\lambda &= \pm j\omega_n
\end{aligned}$$

Hence

$$x_h(t) = A\cos\omega_n t + B\sin\omega_n t \tag{3}$$

Guess

$$\begin{aligned}
x_p(t) &= c_1\cos\omega t + c_2\sin\omega t\\
\dot{x}_p(t) &= -\omega c_1\sin\omega t + \omega c_2\cos\omega t\\
\ddot{x}_p(t) &= -\omega^2 c_1\cos\omega t - \omega^2 c_2\sin\omega t
\end{aligned}$$

Notice, the above guess is valid only under the $\boxed{\text{condition that } \omega \neq \omega_n}$ which is the case in this problem. Now, substitute the above 3 equations into (1) we obtain

$$\begin{aligned}
\left(-\omega^2 c_1\cos\omega t - \omega^2 c_2\sin\omega t\right) + \omega_n^2\left(c_1\cos\omega t + c_2\sin\omega t\right) &= \frac{F}{m}\sin\omega t\\
\sin\omega t\left(-\omega^2 c_2 + \omega_n^2 c_2\right) + \cos\omega t\left(-\omega^2 c_1 + \omega_n^2 c_1\right) &= \frac{F}{m}\sin\omega t
\end{aligned}$$

By comparing coefficients, we obtain

$$\begin{aligned}
c_2\left(\omega_n^2 - \omega^2\right) &= \frac{F}{m}\\
c_2 &= \frac{F/m}{\left(\omega_n^2 - \omega^2\right)}
\end{aligned}$$

and $c_1 = 0$, hence

$$\boxed{x_p(t) = \frac{F/m}{\left(\omega_n^2 - \omega^2\right)}\sin\omega t}$$

Then from (2) we obtain

$$\begin{aligned}
x(t) &= x_h(t) + x_p(t)\\
&= x_h(t) + \frac{F/m}{\left(\omega_n^2 - \omega^2\right)}\sin\omega t
\end{aligned}$$

Using (3) in the above

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{F/m}{\left(\omega_n^2 - \omega^2\right)}\sin\omega t \tag{4}$$

Now assume $x(0) = x_0$ and $\dot{x}(0) = v_0$ For the condition $x(0) = x_0$ we obtain

$$\boxed{x_0 = A}$$

For the condition $\dot{x}(0) = v_0$ we obtain

$$\begin{aligned}
\dot{x}(t) &= -A\omega_n\sin\omega_n t + B\omega_n\cos\omega_n t + \omega\frac{F/m}{\left(\omega_n^2 - \omega^2\right)}\cos\omega t\\
\dot{x}(0) &= v_0 = B\omega_n + \omega\frac{F/m}{\left(\omega_n^2 - \omega^2\right)}
\end{aligned}$$

Hence

$$B = \frac{v_0}{\omega_n} - \frac{\omega}{\omega_n}\frac{F/m}{\left(\omega_n^2 - \omega^2\right)}$$

Hence (4) can be written as

$$x(t) = x_0\cos\omega_n t + \left(\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n}\frac{F/m}{\left(\omega_n^2 - \omega^2\right)}\right)\sin\omega_n t + \frac{F/m}{\left(\omega_n^2 - \omega^2\right)}\sin\omega t$$

Let $\frac{\omega}{\omega_n} = r$, the above becomes

$$x(t) = x_0\cos\omega_n t + \left(\frac{v_0}{\omega_n} - \frac{\frac{F}{m}\,r}{\omega_n^2\left(1 - r^2\right)}\right)\sin\omega_n t + \frac{F/m}{\omega_n^2\left(1 - r^2\right)}\sin\omega t$$

But $\omega_n^2 = \frac{k}{m}$ hence

$$x(t) = x_0\cos\omega_n t + \left(\frac{v_0}{\omega_n} - \frac{\frac{F}{m}\,r}{\frac{k}{m}\left(1 - r^2\right)}\right)\sin\omega_n t + \frac{\frac{F}{m}}{\frac{k}{m}\left(1 - r^2\right)}\sin\omega t$$

Therefore, the general solution is

$$x(t) = x_0\cos\omega_n t + \left(\frac{v_0}{\omega_n} - \frac{F}{k}\frac{r}{\left(1 - r^2\right)}\right)\sin\omega_n t + \frac{F}{k}\frac{1}{\left(1 - r^2\right)}\sin\omega t \tag{5}$$

3.2.2.1 Part(a)

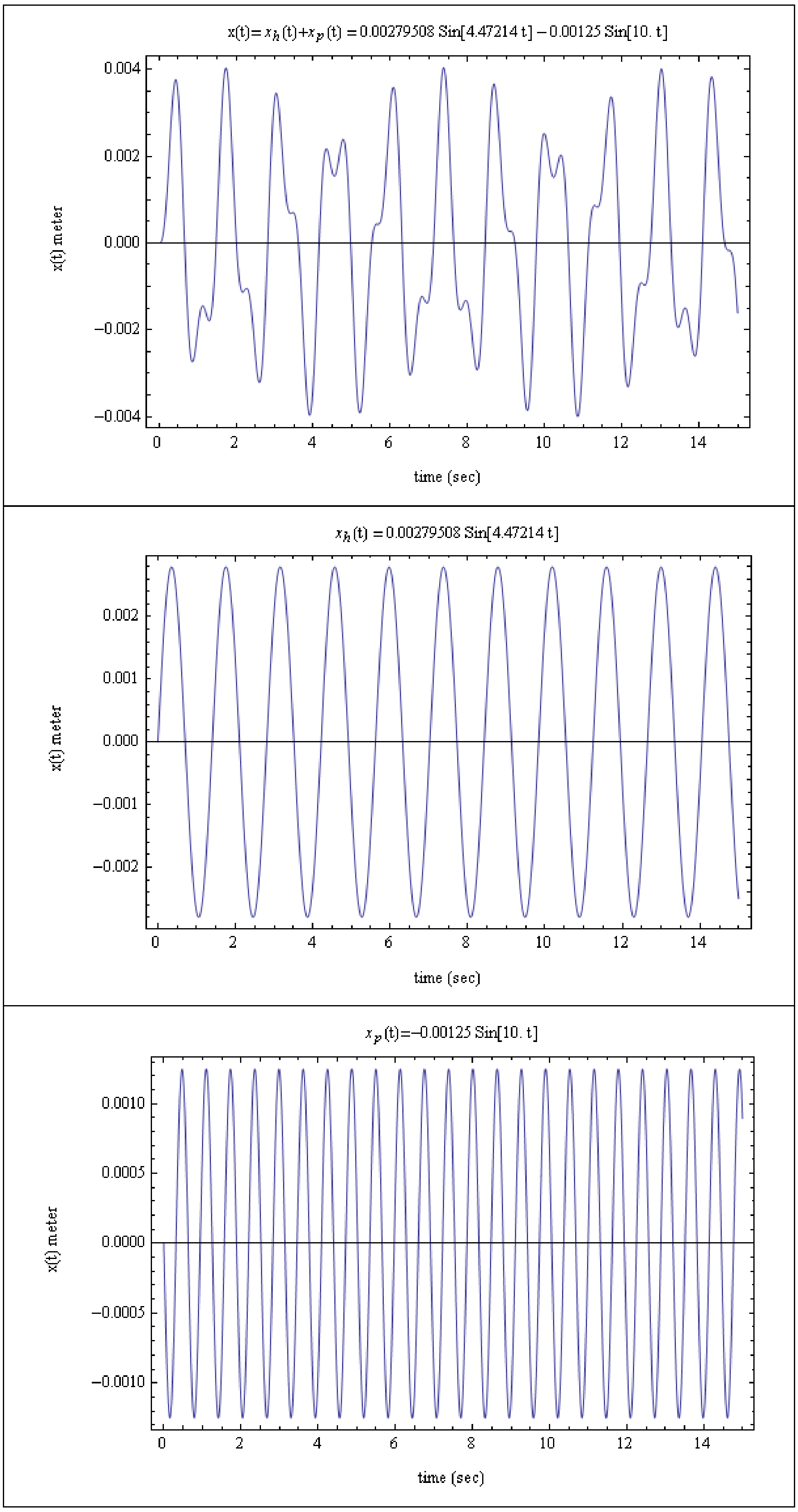
When $x_0 = 0, v_0 = 0$ we obtain from (5)

$$x(t) = \left(-\frac{F}{k}\frac{r}{(1-r^2)}\right)\sin\omega_n t + \frac{F}{k}\frac{1}{(1-r^2)}\sin\omega t \tag{6}$$

Substitute numerical values, and plot the solution. $F = 10, \omega = 10$ rad/sec., $k = 2000, m = 100, \omega_n = \sqrt{\frac{2000}{100}} = 4.472\,1, r = \frac{\omega}{\omega_n} = \frac{10}{4.472\,1} = 2.236\,1$, then equation (6) becomes

$$\begin{aligned} x(t) &= \left(-\frac{10}{2000}\frac{2.236\,1}{(1-2.236\,1^2)}\right)\sin 4.472\,1t + \frac{10}{2000}\frac{1}{(1-2.236\,1^2)}\sin 10t \\ &= 0.002795\sin 4.472\,1t - 0.001250\sin 10t \end{aligned}$$

In the following plot, we show the homogeneous solution and the particular solution separately, then show the general solution.


x(t)= x_h(t)+x_p(t) = 0.00279508 Sin[4.47214 t] – 0.00125 Sin[10. t]
x(t) meter
time (sec)
x_h(t) = 0.00279508 Sin[4.47214 t]
x(t) meter
time (sec)
x_p(t)=–0.00125 Sin[10. t]
x(t) meter
time (sec)

3.2.2.2 Part(b)

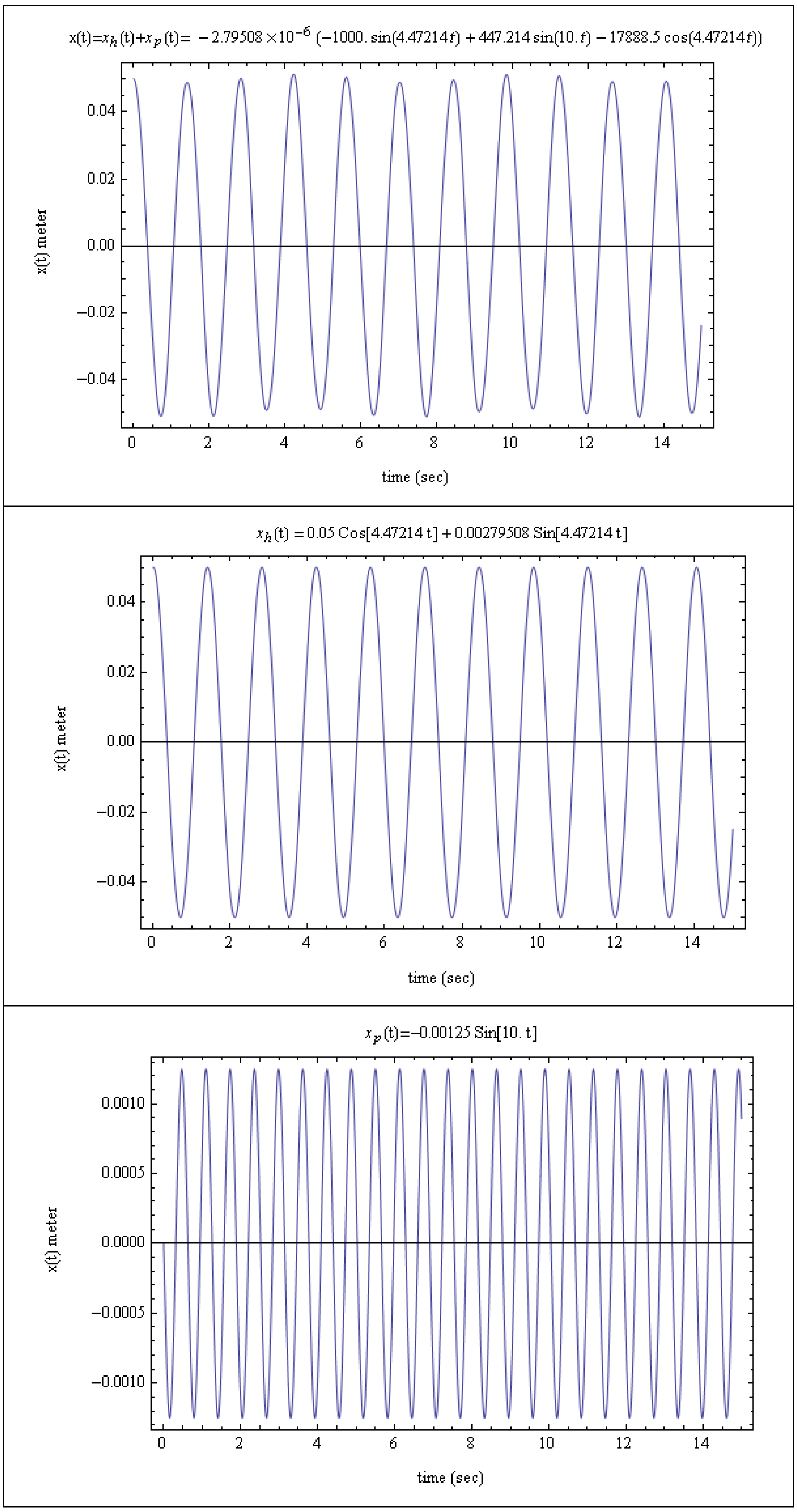
When $x_0 = 0.05$ and $v_0 = 0$ we obtain from (5)

$$x(t) = 0.05\cos\omega_n t + \left(-\frac{F}{k}\frac{r}{(1-r^2)}\right)\sin\omega_n t + \frac{F}{k}\frac{1}{(1-r^2)}\sin\omega t$$

Substitute numerical values found in part(a), then the solution becomes

$$\begin{aligned} x(t) &= 0.05\cos 4.472\,1t + \left(-\frac{10}{2000}\frac{2.236\,1}{(1-2.236\,1^2)}\right)\sin 4.472\,1t + \frac{10}{2000}\frac{1}{(1-2.236\,1^2)}\sin 10t \\ &= 0.05\cos 4.472\,1t + 0.002795\sin 4.472\,1t - 0.001250\sin 10t \end{aligned}$$

In the following plot, we show the homogeneous solution and the particular solution separately, then show the general solution.



3.2.3 Problem 2.10

> **2.10.** Compute the initial conditions such that the response of
>
> $$m\ddot{x} + kx = F_0 \cos \omega t$$
>
> oscillates at only one frequency (ω).

Following the approach taken in problem 2.7, the EQM is

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m} \cos \omega t$$

And $x(t) = x_h(t) + x_p(t)$ where $x_h(t) = A \cos \omega_n t + B \sin \omega_n t$. For $x_p(t)$, guess $x_p(t) = c_1 \cos \omega t + c_2 \sin \omega t$ and following the same steps in problem 2.7, we obtain

$$\sin \omega t(-\omega^2 c_2 + \omega_n^2 c_2) + \cos \omega t(-\omega^2 c_1 + \omega_n^2 c_1) = \frac{F_0}{m} \cos \omega t$$

Notice that the above guess is valid only under the $\boxed{\text{condition that } \omega \neq \omega_n}$. Compare coefficients, we find $c_2 = 0$ and

$$c_1 = \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2}$$

Hence

$$x_p(t) = \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2} \cos \omega t$$

Then, the general solution is

$$x(t) = A \cos \omega_n t + B \sin \omega_n t + \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2} \cos \omega t \tag{1}$$

Let, at $t = 0$, $x(0) = x_0$, and $\dot{x}(0) = v_0$,then from (1), we find

$$x_0 = A + \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2}$$
$$A = x_0 - \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2}$$

And since

$$\dot{x}(t) = -A\omega_n \sin \omega_n t + B\omega_n \cos \omega_n t - \omega \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2} \sin \omega t$$

Then

$$v_0 = B\omega_n$$
$$B = \frac{v_0}{\omega_n}$$

Therefore, the general solution is (from (1))

$$\boxed{x(t) = \left(x_0 - \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2}\right) \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t + \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2} \cos \omega t}$$

To make the response oscillate at frequency ω only, we can set $v_0 = 0$ to eliminate the $\sin \omega_n t$, and set $x_0 = \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2}$ to eliminate the $\cos \omega_n t$ term. Hence, the initial conditions are

$$v_0 = 0$$
$$x_0 = \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2}$$

3.2.4 Problem 2.29

2.29. Write the equation of motion for the system given in Figure P2.29 for the case that $F(t) = F\cos\omega t$ and the surface is friction free. Does the angle θ affect the magnitude of oscillation?

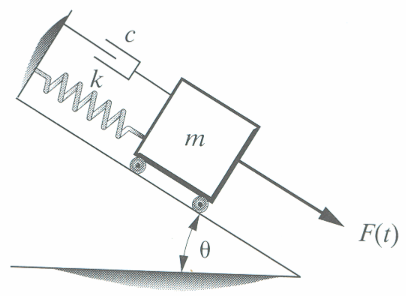
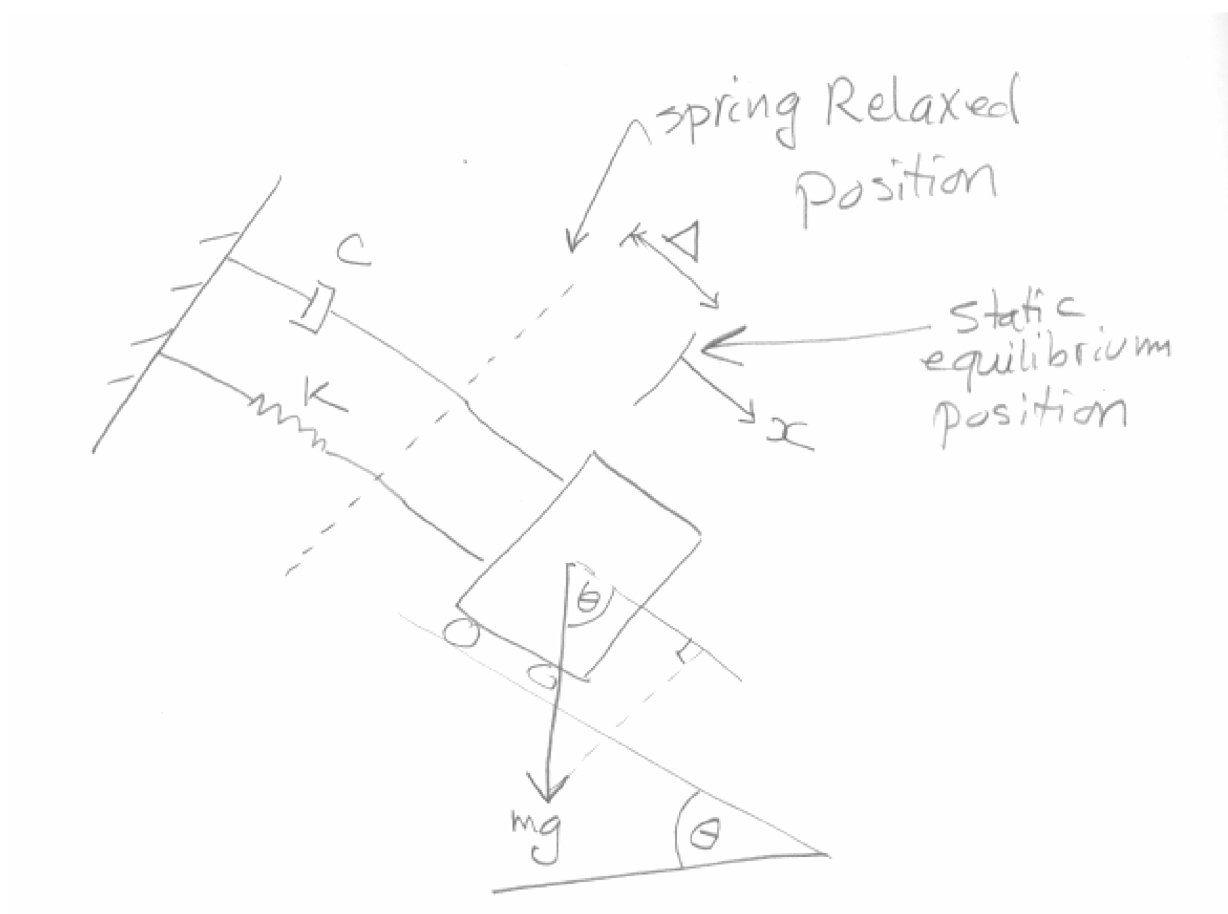


Figure P2.29

This is one degree of freedom system. Using x along the inclined surface as the generalized coordinates, we first obtain the Lagrangian L



We first note that $k\Delta = mg\cos\theta$ and the mass will lose potential as it slides down the surface. We measure everything from the relaxed position (not the static equilibrium.) This is done to show more clearly that the angle do not affect the solution.

$$L = T - U$$

Where

$$\begin{aligned} T &= \frac{1}{2}m\left(\frac{d}{dt}(x+\Delta)\right)^2 \\ &= \frac{1}{2}m\dot{x}^2 \\ U &= \frac{1}{2}k(x+\Delta)^2 - mg(x+\Delta)\cos\theta \end{aligned}$$

Hence

$$\begin{aligned}
L &= \frac{1}{2}m\dot{x}^2 - \left(\frac{1}{2}k(x+\Delta)^2 - mg(x+\Delta)\cos\theta\right)\\
&= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x+\Delta)^2 + mgx\cos\theta + mg\Delta\cos\theta\\
\frac{\partial L}{\partial \dot{x}} &= m\dot{x}\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= m\ddot{x}\\
\frac{\partial L}{\partial x} &= -k(x+\Delta) + mg\cos\theta\\
&= -kx - k\Delta + mg\cos\theta
\end{aligned}$$

But $k\Delta = mg\cos\theta$, hence the above reduces to

$$\frac{\partial L}{\partial x} = -kx$$

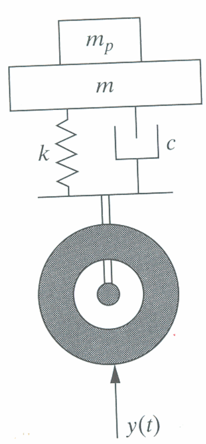
Hence the EQM is (using the Lagrangian equation)

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= F\cos\omega t\\
m\ddot{x} + kx &= F\cos\omega t\\
\ddot{x} + \omega_n^2 x &= \frac{F}{m}\cos\omega t \qquad (1)
\end{aligned}$$

Where $\omega_n^2 = \frac{k}{m}$. We see that the angle θ is not in the EQM. Hence the solution does not involve θ and the oscillation magnitude is not affected by the angle. Intuitively, the reason for this is because the angle effect is already counted for to reach the static equilibrium. Once the system is in static equilibrium, the angle no longer matters as far as the solution is concerned.

3.2.5 Problem 2.46

2.46. Consider Example 2.4.1 for car 1, illustrated in Figure P2.46, if three passengers totaling 200 kg are riding in the car. Calculate the effect of the mass of the passengers on the deflection at 20, 80, 100, and 150 km/h. What is the effect of the added passenger mass on car 2?



n with the mass of the occupants, m_p, included.

system for car 2

From example 2.4.1, we note the following table

TABLE 2.1 COMPARISON OF CAR VELOCITY, FREQUENCY, AND DISPLACEMENT FOR TWO DIFFERENT CARS

Speed (km/h)	ω_b	r_1	r_2	x_1 (cm)	x_2 (cm)
20	5.817	0.923	1.158	3.19	2.32
80	23.271	3.692	4.632	0.12	0.07
100	29.088	4.615	5.79	0.09	0.05
150	43.633	6.923	8.686	0.05	0.03

Also, from example 2.4.1, the mass of car 1 is $1007kg$ and the mass of car 2 is $1585kg$. Hence we write

$$m_1 = 1007kg$$
$$m_2 = 1585kg$$

To find the deflection of the car, we use equation 2.70 in the book, which is

$$X = Y\sqrt{\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}$$

Where X is the magnitude of the steady state deflection and Y is the magnitude of the base deflection, which is given as 0.01 meters in the example.

Hence, for each speed, we calculate ω_p and then we find $\omega_n = \sqrt{\frac{k}{m_1+m_p}}$ and then find $r = \frac{\omega_p}{\omega_n}$ and then find $\xi = \frac{c}{2\sqrt{k(m_1+m_p)}}$ and then using equation(1), we calculate X. This is done for each different speed (all for car m_1). Next, we do the same for car m_2. These calculation are shown in the following table. Note also that $c = 2000$ N s/m as given in the example and $k = 4\times 10^4$ N/m

car 1

$v\ (km/h)$	$\omega_p = 0.2909\ v$	$\omega_n = \sqrt{\frac{k}{m_1+m_p}}$	$r = \frac{\omega_p}{\omega_n}$	$\xi = \frac{c}{2\sqrt{k(m_1+m_p)}}$	$X = Y\sqrt{\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}\ (cm)$
20	5.818	5.7567	1.0106	0.14392	3.571
80	23. 272	5.7567	4.0426	0.14392	0.0997
100	29.09	5.7567	5.0532	0.14392	0.0718
150	43.635	5.7567	7.5799	0.14392	0.0425

car 2

$v\ (km/h)$	$\omega_p = 0.2909\ v$	$\omega_n = \sqrt{\frac{k}{m_2+m_p}}$	$r = \frac{\omega_p}{\omega_n}$	$\xi = \frac{c}{2\sqrt{k(m_2+m_p)}}$	$X = Y\sqrt{\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}\ (cm)$
20	5.818	4.7338	1.229	0.11835	1.7726
80	23.272	4.7338	4.9161	0.11835	0.066141
100	29.09	4.7338	6.1452	0.11835	0.04797
150	43.635	4.7338	9.2178	0.11835	0.02857

Observations: The heavier car (car 2) has smaller defection (X values) for all speeds. Adding passengers, causes ω_n to change. This results in making the deflection smaller when passengers are in the car as compared without them. Heaver cars and heavier passenger results in smaller deflection values. For the lighter car however, adding the passenger did not result in smaller deflection for all speeds. For speed $v = 20$, adding the passenger caused a larger defection (3.19 vs. 3.571). As car 1 speed became larger, the deflection became smaller for both cars.

So, in conclusion: lighter cars have larger deflections at bumps, and the faster the car, the smaller the deflection.

3.2.6 Problem 2.57

2.57. Consider a typical unbalanced machine problem as given in Figure P2.57 with a machine mass of 120 kg, a mount stiffness of 800 kN/m, and a damping value of 500 kg/s. The amplitude of the out-of-balance force is measured to be 374 N at a running speed of 3000 rev/min. (a) Determine the amplitude of motion due to the out of balance. (b) If the out-of-balance mass is estimated to be 1% of the total mass, estimate the value of e.

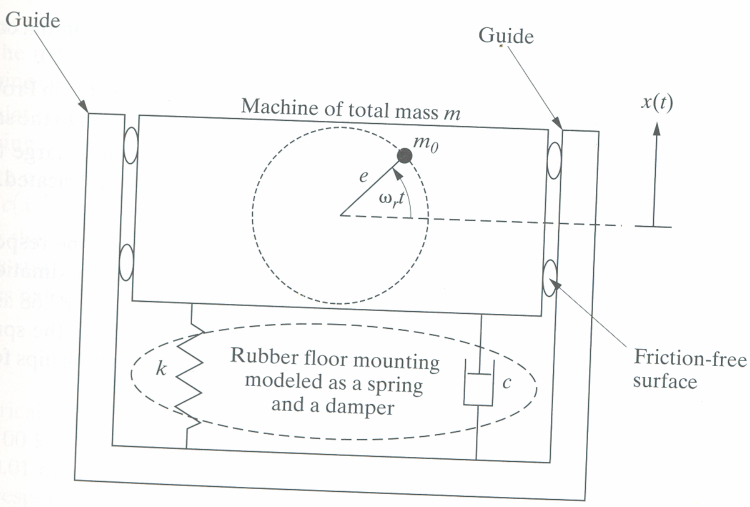


Figure P2.57 Typical unbalanced machine problem.

Given $m = 120\ kg, k = 800 \times 10^3$N/M, $c = 500$ kg/s, and mass m_0 has angular speed of $\omega_r = \frac{3000 \times 2\pi}{60} = 100\pi$ radians per seconds

3.2.6.1 Part(a)

The rotating mass will cause a downward force as the result of the centripetal force $m_0 e\omega_r^2 \sin(\omega_r t)$. Hence the reaction to this force on the machine will be in the upward direction. Hence

$$F_r = m_0 e\omega_r^2 \sin(\omega_r t)$$

Hence the machine equation of motion is

$$\begin{aligned} m\ddot{x} + c\dot{x} + kx &= m_0 e\omega_r^2 \sin(\omega_r t) \\ \ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x &= \frac{m_0}{m} e\omega_r^2 \sin(\omega_r t) \end{aligned} \tag{1}$$

By guessing $x_p = X\sin(\omega_r t - \theta)$ then we find that (The method of undetermined coefficients is used, derivation is show in text book at page 115)

$$X = \frac{m_o e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \tag{2}$$

This is the maximum magnitude of motion in steady state. In the above, $r = \frac{\omega_r}{\omega_n}$. Hence to find X we substitute the given values in the above expression. We first note that we are told that $m_0 e\omega_r^2 = 374N$, hence $m_o e = \frac{374}{\omega_r^2}$ but we found that $\omega_r = 100\pi$ rad/sec, hence

$$m_o e = \frac{374}{(100\ \pi)^2} = 0.003789\,4$$

And

$$r = \frac{\omega_r}{\omega_n} = \frac{100\pi}{\sqrt{\frac{k}{m}}} = \frac{100\ \pi}{\sqrt{\frac{800\times 10^3}{120}}} = 3.847\,6$$

And

$$\xi = \frac{c}{2\sqrt{km}} = \frac{500}{2\sqrt{800 \times 10^3 \times 120}} = 0.02551\,6$$

Substituting into (2) gives

$$\begin{aligned} X &= \left(\frac{0.003789\,4}{120}\right) \frac{3.847\,6^2}{\sqrt{(1 - 3.847\,6^2)^2 + (2 \times 0.02551\,6 \times 3.847\,6)^2}} \\ &= 3.386\,3 \times 10^{-5}\ meter \end{aligned}$$

3.2.6.2 Part(b)

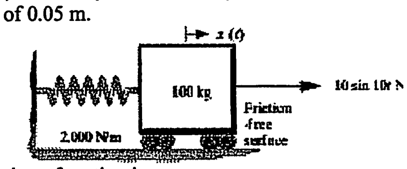
We are told that $m_o = 0.01 \times m$, hence $m_o = 0.01 \times 120 = 1$. And since we are told that $m_0 e\omega_r^2 = 374N$ then

$$e = \frac{374}{m_0\omega_r^2} = \frac{374}{1. \times (100\pi)^2} = 3.8 \times 10^{-3} \text{ meter}$$

3.2.7 Key for HW2

EGME 431 Handed 3/16/09 HW 2 SOLUTION

2.7 Consider the system in Figure P2.7, write the equation of motion and calculate the response assuming a) that the system is initially at rest, and b) that the system has an initial displacement of 0.05 m.



Solution: The equation of motion is

$$m\ddot{x} + kx = 10\sin 10t$$

Let us first determine the general solution for

$$\ddot{x} + \omega_n^2 x = f_0 \sin\omega t$$

Replacing the cosine function with a sine function in Eq. (2.4) and following the same argument, the general solution is:

$$x(t) = A_1 \sin\omega_n t + A_2 \cos\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\sin\omega t$$

Using the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = v_0$, a general expression for the response of a spring-mass system to a harmonic (sine) excitation is:

$$x(t) = \left(\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n}\cdot\frac{f_0}{\omega_n^2 - \omega^2}\right)\sin\omega_n t + x_0\cos\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\sin\omega t$$

Given: k=2000 N/m, m=100 kg, ω=10 rad/s,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{100}} = \sqrt{20}\text{ rad/s} = 4.472\text{ rad/s} \qquad f_0 = \frac{F_0}{m} = \frac{10}{100} = 0.1\text{ N/kg}$$

a) $x_0 = 0$ m, $v_0 = 0$ m/s

Using the general expression obtained above:

$$x(t) = \left(0 - \frac{10}{\sqrt{20}}\cdot\frac{0.1}{\sqrt{20}^2 - 10^2}\right)\sin\sqrt{20}t + 0 + \frac{0.1}{\sqrt{20}^2 - 10^2}\sin 10t$$

$$= 2.795\times 10^{-3}\sin 4.472t - 1.25\times 10^{-3}\sin 10t$$

b) $x_0 = 0.05$ m, $v_0 = 0$ m/s

$$x(t) = \left(0 - \frac{10}{\sqrt{20}}\cdot\frac{0.1}{\sqrt{20}^2 - 10^2}\right)\sin\sqrt{20}t + 0.05\cos\sqrt{20}t + \frac{0.1}{\sqrt{20}^2 - 10^2}\sin 10t$$

$$= 0.002795\sin 4.472t + 0.05\cos 4.472t - 0.00125\sin 10t$$

$$= 5.01\times 10^{-2}\sin(4.472t + 1.515) - 1.25\times 10^{-3}\sin 10t$$

2.10 Compute the initial conditions such that the response of:

$$m\ddot{x} + kx = F_0 \cos\omega t$$

oscillates at only one frequency (ω).

Solution: From Eq. (2.11):

$$x(t) = \frac{v_0}{\omega_n}\sin\omega_n t + (x_0 - \frac{f_0}{\omega_n^2 - \omega^2})\cos\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\cos\omega t$$

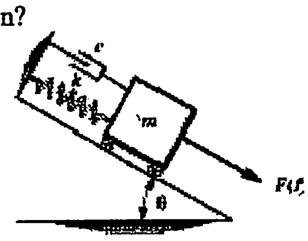
For the response of $m\ddot{x} + kx = F_0\cos\omega t$ to have only one frequency content, namely, of the frequency of the forcing function, ω, the coefficients of the first two terms are set equal to zero. This yields that the initial conditions have to be

$$x_0 = \frac{f_0}{\omega_n^2 - \omega^2} \quad \text{and} \quad v_0 = 0$$

Then the solution becomes

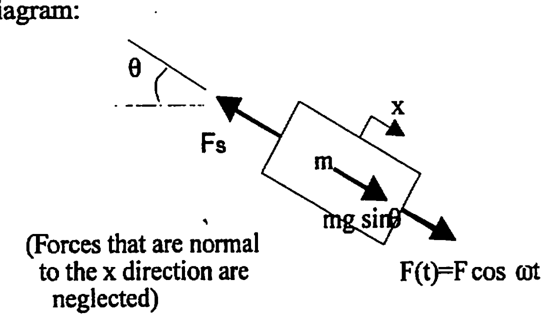
$$x(t) = \frac{f_0}{\omega_n^2 - \omega^2}\cos\omega t$$

2.29 Write the equation of motion for the system given in Figure P2.29 for the case that $F(t) = F\cos\omega t$ and the surface is friction free. Does the angle θ effect the magnitude of oscillation?



Solution:

Free body diagram:



Assuming x = 0 to be at the equilibrium:

$$\sum F_x = F + mg\sin\theta - F_s = m\ddot{x}$$

where $F_s = k(x + \frac{mg\sin\theta}{k})$ and $F(t) = F\cos\omega t$

Then the equation of motion is:

$$m\ddot{x} + kx = F\cos\omega t$$

Note that the equation of motion does not contain θ which means that the magnitude of the response is not affected by the angle of the incline.

2.46 Consider Example 2.4.1 for car 1 illustrated in Figure P2.46, if three passengers totaling 200 kg are riding in the car. Calculate the effect of the mass of the passengers on the deflection at 20, 80, 100, and 150 km/h. What is the effect of the added passenger mass on car 2?

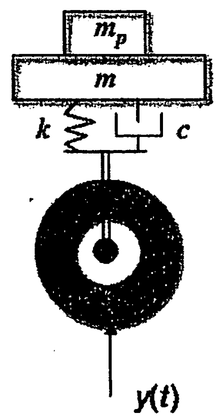


Figure P2.46 Model of a car suspension with the mass of the occupants, m_p, included.

Solution:

Add a mass of 200 kg to each car. From Example 2.4.1, the given values are: m_1 = 1207 kg, m_2 = 1785 kg, k = 4x10^4 N/m; c = 2,000 kg/s, ω_b = 0.29v.

Car 1: $\omega_1 = \sqrt{\frac{k}{m}} = \sqrt{\frac{4\times10^4}{1207}} = 5.76$ rad/s

$$\zeta_1 = \frac{c}{2\sqrt{km_1}} = \frac{2000}{2\sqrt{(4\times10^5)(1207)}} = 0.144$$

Car 2: $\omega_2 = \sqrt{\frac{k}{m}} = \sqrt{\frac{4\times10^4}{1785}} = 4.73$ rad/s

$$\zeta_2 = \frac{c}{2\sqrt{km_2}} = \frac{2000}{2\sqrt{(4\times10^5)(1785)}} = 0.118$$

Using Equation (2.71): $X = Y\left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}$ produces the following:

Speed (km/h)	ω_b (rad/s)	r_1	r_2	x_1 (cm)	x_2 (cm)
20	5.817	1.01	1.23	3.57	1.77
80	23.271	3.871	4.71	0.107	0.070
100	29.088	5.05	6.15	0.072	0.048
150	2.40	7.58	9.23	0.042	0.028

At lower speeds there is little effect from the passengers weight, but at higher speeds the added weight reduces the amplitude, particularly in the smaller car.

wrong
43.635

2.57 Consider a typical unbalanced machine problem as given in Figure P2.57 with a machine mass of 120 kg, a mount stiffness of 800 kN/m and a damping value of 500 kg/s. The out of balance force is measured to be 374 N at a running speed of 3000 rev/min. a) Determine the amplitude of motion due to the out of balance. b) If the out of balance mass is estimated to be 1% of the total mass, estimate the value of the *e*.

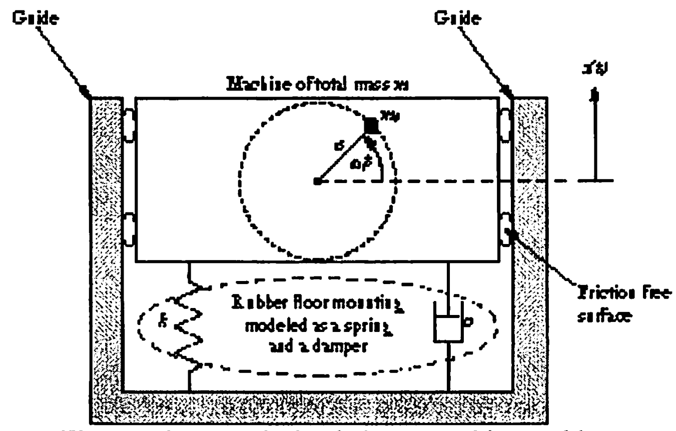


Figure P2.57 Typical unbalance machine problem.

Solution:

a) Using equation (2.84) with $m_0e = F_0/\omega_r^2$ yields:

$k := 800 \cdot 1000$ $\quad m := 120$ $\quad c := 500$ $\quad F0 := 374$

$\omega r := 100 \cdot \pi$ $\quad \omega n := \sqrt{\dfrac{k}{m}}$ $\quad \zeta := \dfrac{c}{2 \cdot \sqrt{k \cdot m}}$ $\quad k = 8 \cdot 10^5$

$r := \dfrac{\omega r}{\omega n}$ $\quad \omega n = 81.65$ $\quad r = 3.848$ $\quad \zeta = 0.026$

$$X := \frac{F0}{\omega r^2 \cdot m} \cdot \frac{r^2}{\sqrt{\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2}} \qquad X = 3.386 \cdot 10^{-5}$$

b) Use the fact that $F_0 = m_0 e \omega_r^2$ to get

$$e := \frac{F0}{\omega r^2 \cdot (0.01 \cdot m)} \qquad e = 3.158 \cdot 10^{-3}$$

in meters.

3.3 HW3

Local contents

3.3.1 Description of HW

1. Find solution to second order ODE with impulse as input.
2. Spring-damper-mass dropped from height h, find resulting EQM.
3. Find solution to second order ODE with 2 impulses as input, one delayed.
4. Find response of undamped system to half sin input force, using convolution.
5. As above, but the forcing function is triangle looking. Use convolution also.
6. Find Fourier series for sawtooth function.
7. Solving 2nd order ODE with impulse as input using Laplace transform.
8. Find shock response spectrum to half-sine input (hard).
9. Find $H(s)$, the transfer function for ODE.
10. Find the frequency response for the above $H(s)$, i.e. set $s = jw$ and plot.

3.3.2 Problem 3.2

Problem

Calculate the solution to $\ddot{x} + 2\dot{x} + 3x = \sin t + \delta(t - \pi)$ with IC $x(0) = 0, \dot{x}(0) = 1$ and plot the solution.

Answer

$$\ddot{x} + 2\dot{x} + 3x = \sin t + \delta(t - \pi)$$
$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \sin t + \delta(t - \pi)$$

Hence $\boxed{\omega_n = \sqrt{3}}$ and $2\xi\omega_n = 2$, hence $\boxed{\xi = \frac{1}{\sqrt{3}}} = 0.577\,35$, hence this is $\boxed{\text{underdamped system}}$.

Since $x = x_h + x_p$, then

$$x_h = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

We have 2 particular solutions. The first x_{p_1} is due to $\sin t$ and the second x_{p_2} is due to $\delta(t - \pi)$. When the forcing function is $\sin t$, we guess

$$x_{p_1} = c_1\cos t + c_2\sin t$$

and when the forcing function is $\delta(t - \pi)$ the response is

$$x_{p_2} = \frac{1}{\omega_d m}e^{-\xi\omega_n(t-\pi)}\sin\omega_d(t - \pi)\,\Phi(t - \pi)$$

From x_{p_1} we find $\dot{x}_{p_1}$ and $\ddot{x}_{p_1}$ and plug these into $\ddot{x} + 2\dot{x} + 3x = \sin t$ to find c_1 and c_2, next we find A, B by using the IC, and then at the end we add the solution x_{p_2}. Notice that x_{p_2} do not enter into the calculation of A, B since the impulse $\delta(t - \pi)$ is not effective at $t = 0$.

$$\dot{x}_{p_1} = -c_1\sin t + c_2\cos t$$
$$\ddot{x}_{p_1} = -c_1\cos t - c_2\sin t$$

Hence

$$\ddot{x}_{p_1} + 2\dot{x}_{p_1} + 3x_{p_1} = \sin t$$
$$(-c_1\cos t - c_2\sin t) + 2(-c_1\sin t + c_2\cos t) + 3(c_1\cos t + c_2\sin t) = \sin t$$
$$\sin t(-c_2 - 2c_1 + 3c_2) + \cos t(-c_1 + 2c_2 + 3c_1) = \sin t$$

Hence $(-2c_1 + 2c_2) = 1$ and $(2c_2 + 2c_1) = 0$. This results in

$$c_1 = -\frac{1}{4}$$
$$c_2 = \frac{1}{4}$$

Hence

$$\boxed{x_{p_1} = -\tfrac{1}{4}\cos t + \tfrac{1}{4}\sin t}$$

Therefore

$$x_h + x_{p_1} = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) - \frac{1}{4}\cos t + \frac{1}{4}\sin t$$

Now we use IC's to find A, B. At $t = 0$ we obtain

$$\boxed{A = \tfrac{1}{4}}$$

And

$$\dot{x}_h + \dot{x}_{p_1} = -\xi\omega_n e^{-\xi\omega_n t}\left(\frac{1}{4}\cos\omega_d t + B\sin\omega_d t\right) + e^{-\xi\omega_n t}\left(-\frac{1}{4}\omega_d \sin\omega_d t + \omega_d B\cos\omega_d t\right) + \frac{1}{4}\sin t + \frac{1}{4}\cos t$$

At $t = 0$ we have

$$1 = -\xi\omega_n\left(\frac{1}{4}\right) + (\omega_d B) + \frac{1}{4}$$

$$B = \frac{\left(1 + \frac{\xi\omega_n}{4} - \frac{1}{4}\right)}{\omega_d}$$

But $\omega_d = \omega_n\sqrt{1-\xi^2} = \sqrt{3}\sqrt{1-\left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{3}\sqrt{\frac{2}{3}}$, Hence $\boxed{\omega_d = \sqrt{2}}$ then the above becomes

$$\boxed{B = \frac{1}{\sqrt{2}}}$$

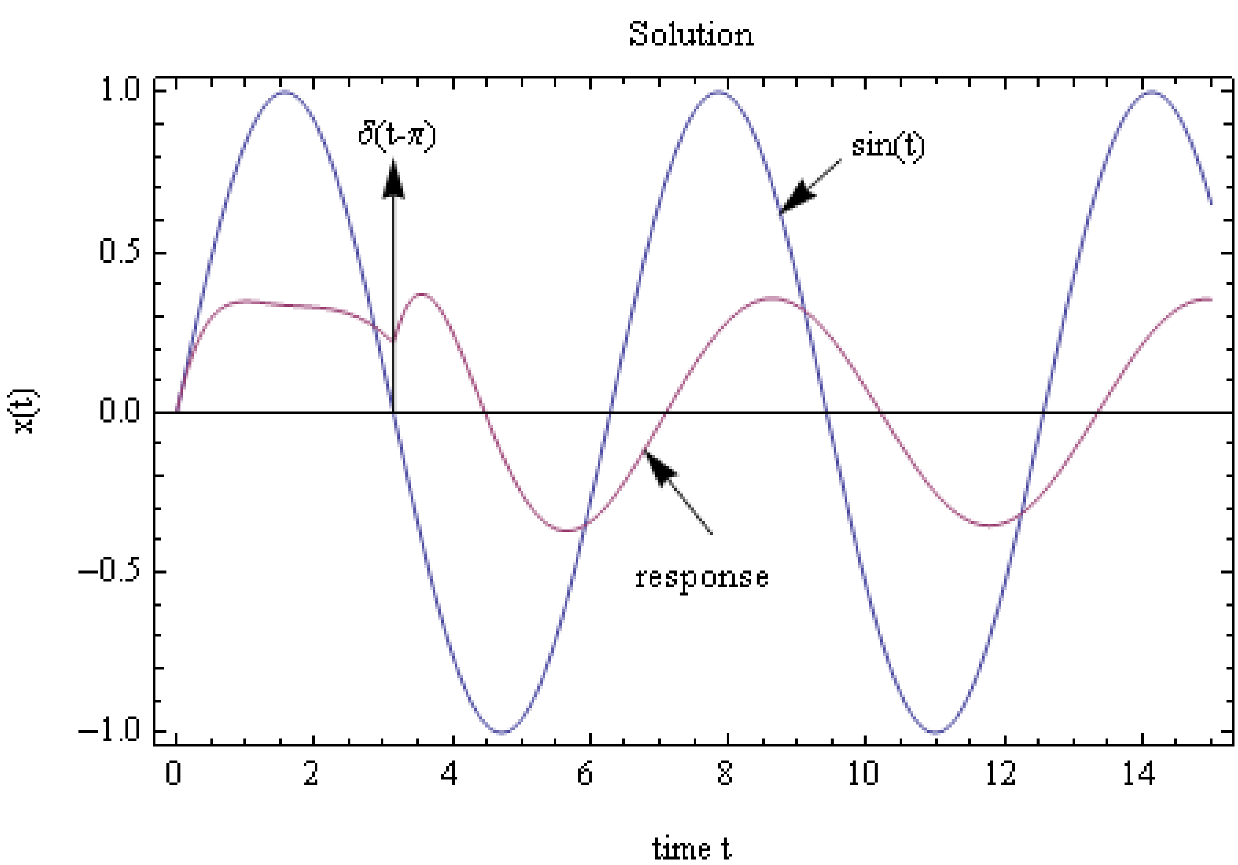
Hence the final solution is

$$x(t) = x_h + x_{p_1} + x_{p_2} = e^{-\xi\omega_n t}\left(\frac{1}{4}\cos\omega_d t + \frac{1}{\sqrt{2}}\sin\omega_d t\right) - \frac{1}{4}\cos t + \frac{1}{4}\sin t + \frac{1}{\omega_d m}e^{-\xi\omega_n(t-\pi)}\sin\omega_d(t-\pi)\,\Phi(t-\pi)$$

Substitute values for the parameters above we obtain

$$\boxed{x(t) = e^{-t}\left(\frac{1}{4}\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right) - \frac{1}{4}\cos t + \frac{1}{4}\sin t + \frac{1}{\sqrt{2}}e^{-(t-\pi)}\sin\sqrt{2}(t-\pi)\,\Phi(t-\pi)}$$

This is a plot of the solution superimposed on the forcing functions



3.3.3 Problem 3.8

hit by the cam?

3.8. The vibration of packages dropped from a height of h meters can be approximated by considering Figure P3.8 and modeling the point of contact as an impulse applied to the system at the time of contact. Calculate the vibration of the mass m after the system falls and hits the ground. Assume that the system is underdamped.

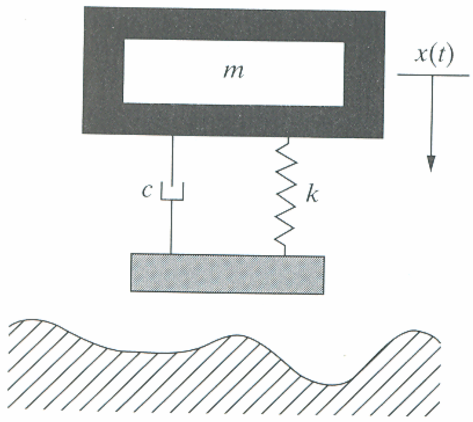


Figure P3.8 Vibration model of a package being dropped onto the ground.

The magnitude of the impulse resulting when the mass hits the ground is given by the change of momentum that occurs. Hence

$$\hat{F} = Ft = m(v_{final} - v_0)$$

But assuming the mass is dropped from rest, hence $v_0 = 0$, and $v_{final} = gt$ where $t = \sqrt{2\frac{h}{g}}$ where h is the height that mass falls. Hence

$$\begin{aligned}\hat{F} &= mv_{final}\\ &= m\sqrt{2gh}\end{aligned}$$

Hence the equation of motion is

$$\boxed{m\ddot{x}(t) + c\dot{x}(t) + kx(t) = m\sqrt{2gh}\delta(t)}$$

Since underdamped, $x(t) = h(t) = \frac{\hat{F}}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t$, hence the solution is

$$\begin{aligned}x(t) &= \frac{m\sqrt{2gh}}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t\\ &= \frac{\sqrt{2gh}}{\omega_d}e^{-\xi\omega_n t}\sin\omega_d t\end{aligned}$$

Taking $t = 0$ as time of impact.

3.3.4 Problem 3.11

Problem

Compute response of the system $3\ddot{x}(t) + 6\dot{x}(t) + 12x(t) = 3\delta(t) - \delta(t-1)$ with IC $x(0) = 0.01m$ and $v(0) = 1m/s$. Plot the response.

Answer

$$\begin{aligned}3\ddot{x}(t) + 6\dot{x}(t) + 12x(t) &= 3\delta(t) - \delta(t-1)\\ \ddot{x}(t) + 2\dot{x}(t) + 4x(t) &= \delta(t) - \frac{1}{3}\delta(t-1)\\ \ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x &= \delta(t) - \frac{1}{3}\delta(t-1)\end{aligned}$$

Where $m = 1, \omega_n^2 = 4$, hence $\omega_n = 2$ and $2\xi\omega_n = 2$, hence $\xi = \frac{1}{2}$. This is an underdamped system.

$\omega_d = \omega_n\sqrt{1-\xi^2} = 2\sqrt{1-\left(\frac{1}{2}\right)^2} = 2\sqrt{\frac{3}{4}}$, Hence $\omega_d = \sqrt{3}$

$$x_h = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

The response due to the forcing function $\delta(t)$ is given by

$$x_{p_1}(t) = \frac{1}{\omega_d m} e^{-\xi\omega_n t} \sin(\omega_d t)$$

The response due to the other forcing function $\delta(t-1)$ is given by

$$x_{p_2}(t) = -\frac{1}{3}\frac{1}{\omega_d m} e^{-\xi\omega_n (t-1)} \sin\omega_d (t-1)\,\Phi(t-1)$$

Now we determine A, B from IC's

$$\begin{aligned} x_h(0) + x_{p_1}(0) &= 0.01 \\ &= \left[e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{1}{\omega_d m} e^{-\xi\omega_n t}\sin(\omega_d t) \right]_{t=0} \end{aligned}$$

Hence $\boxed{A = 0.01}$ Now to find B

$$\begin{aligned} \dot{x}_h(t) + \dot{x}_{p_1}(t) = &-\xi\omega_n e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + e^{-\xi\omega_n t}(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t) \\ &+ \frac{-\xi\omega_n}{\omega_d m} e^{-\xi\omega_n t}\sin(\omega_d t) + \frac{\omega_d}{\omega_d m} e^{-\xi\omega_n t}\cos(\omega_d t) \end{aligned}$$

But $\dot{x}_h(0) + \dot{x}_{p_1}(0) = 1$, hence from the above, and noting that $m = 1$

$$\begin{aligned} 1 &= -A\xi\omega_n + B\omega_d + 1 \\ B &= \frac{A\xi\omega_n}{\omega_d} \\ &= \frac{0.01\left(\frac{1}{2}\right)2}{\sqrt{3}} \end{aligned}$$

Hence

$$B = \frac{1}{100\sqrt{3}}$$

Therefore

$$x_h = e^{-\xi\omega_n t}\left(\frac{1}{100}\cos\omega_d t + \frac{1}{100\sqrt{3}}\sin\omega_d t\right)$$

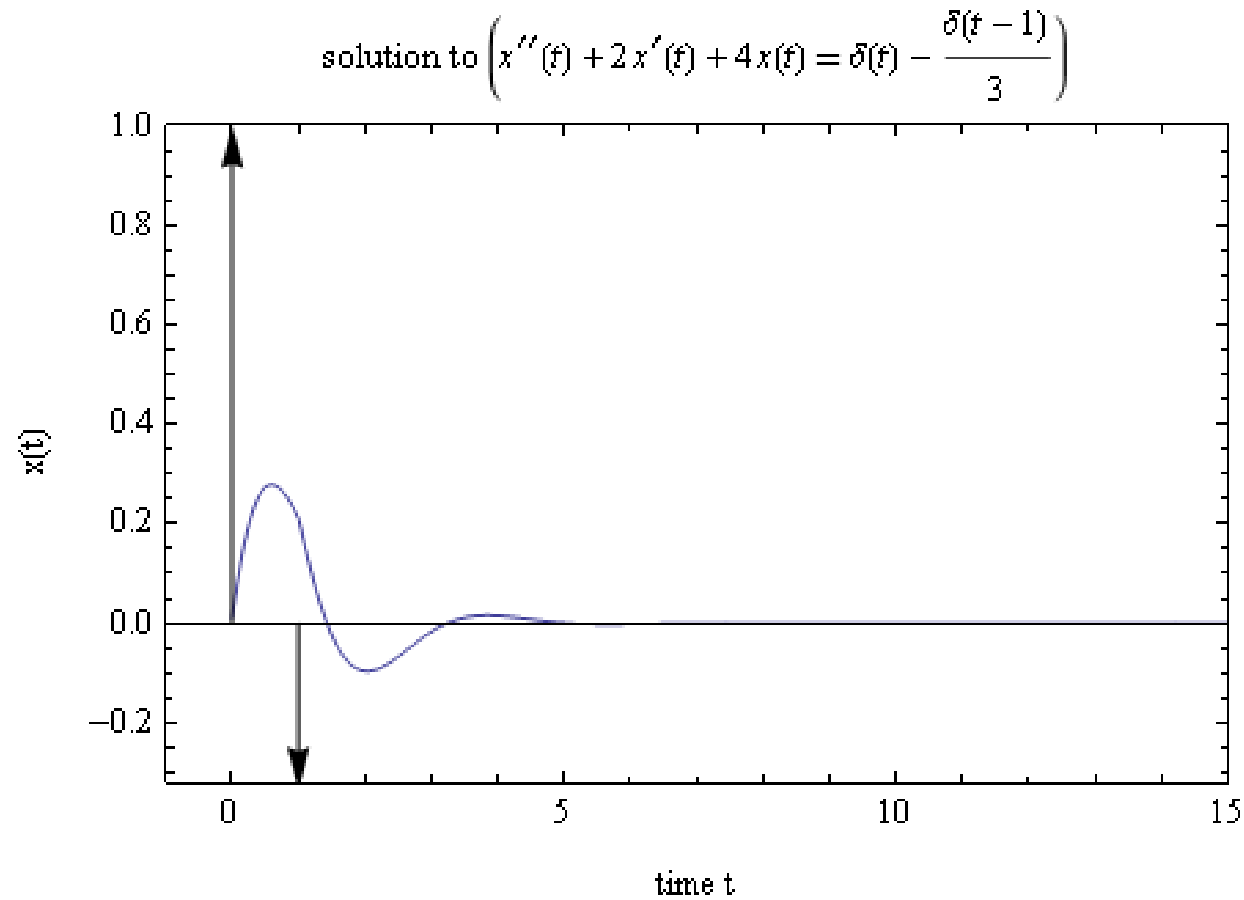
Now we can combine the above solution to obtain the final solution

$$\begin{aligned} x(t) &= x(h) + x_{p_1}(t) + x_{p_2}(t) \\ &= e^{-\xi\omega_n t}\left(\frac{1}{100}\cos\omega_d t + \frac{1}{100\sqrt{3}}\sin\omega_d t\right) \\ &+ \frac{1}{\omega_d m} e^{-\xi\omega_n t}\sin(\omega_d t) \\ &- \frac{1}{3}\frac{1}{\omega_d m} e^{-\xi\omega_n (t-1)}\sin\omega_d(t-1)\,\Phi(t-1) \end{aligned}$$

Substitute numerical values for the above parameters, we obtain

$$x(t) = \frac{e^{-t}}{100}\left(\cos\sqrt{3}t + \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right) + \frac{1}{\sqrt{3}}e^{-t}\sin\left(\sqrt{3}t\right) - \frac{1}{3}\frac{1}{\sqrt{3}}e^{-(t-1)}\sin\left(\sqrt{3}(t-1)\right)\Phi(t-1)$$

This is a plot of the response



3.3.5 Problem 3.16

3.16. Calculate the response of an underdamped system to the excitation given in Figure P3.16.

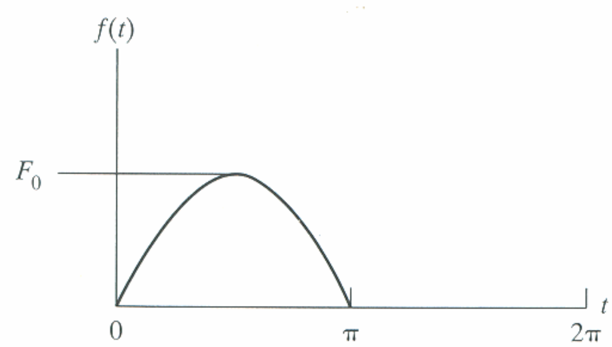


Figure P3.16 Plot of a pulse input of the form $f(t) = F_0 \sin t$.

Let the response by $x(t)$. Hence $x(t) = x_h(t) + x_p(t)$, where $x_p(t)$ is the particular solution, which is the response due the the above forcing function. Using convolution

$$x_p(t) = \int_0^t f(\tau)\, h(t-\tau)\, d\tau$$

Where $h(t)$ is the unit impulse response of a second order underdamped system which is

$$h(t) = \frac{1}{m\omega_d} e^{-\xi\omega_n t} \sin \omega_d t$$

hence

$$\begin{aligned} x_p(t) &= \frac{F_0}{m\omega_d} \int_0^t \sin(\tau)\, e^{-\xi\omega_n(t-\tau)} \sin(\omega_d(t-\tau))\, d\tau \\ &= \frac{F_0 e^{-\xi\omega_n t}}{m\omega_d} \int_0^t e^{\xi\omega_n\tau} \sin(\tau) \sin(\omega_d(t-\tau))\, d\tau \end{aligned}$$

Using $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ then

$$\sin(\tau)\sin(\omega_d(t-\tau)) = \frac{1}{2}[\cos(\tau - \omega_d(t-\tau)) - \cos(\tau + \omega_d(t-\tau))]$$

Then the integral becomes

$$x_p(t) = \frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\left(\int_0^t e^{\xi\omega_n\tau}\cos(\tau - \omega_d(t-\tau))\,d\tau - \int_0^t e^{\xi\omega_n\tau}\cos(\tau + \omega_d(t-\tau))\,d\tau\right)$$

Consider the first integral I_1 where

$$I_1 = \int_0^t e^{\xi\omega_n\tau}\cos(\tau - \omega_d(t-\tau))\,d\tau$$

Integrate by parts, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n\tau} \to v = \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}$ and let $u = \cos(\tau - \omega_d(t-\tau)) \to du = -(1+\omega_d)\sin(\tau - \omega_d(t-\tau))$, hence

$$\begin{aligned}I_1 &= \left[\cos(\tau - \omega_d(t-\tau))\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\right]_0^t \\ &\quad - \int_0^t \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}[-(1+\omega_d)\sin(\tau - \omega_d(t-\tau))]\,d\tau \\ &= \left[\cos(t - \omega_d(t-t))\frac{e^{\xi\omega_n t}}{\xi\omega_n} - \cos(0 - \omega_d(t-0))\frac{1}{\xi\omega_n}\right] \\ &\quad + \frac{(1+\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\sin(\tau - \omega_d(t-\tau))\,d\tau \\ &= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\sin(\tau - \omega_d(t-\tau))\,d\tau \qquad (1)\end{aligned}$$

Integrate by parts again the last integral above, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n\tau} \to v = \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}$ and let $u = \sin(\tau - \omega_d(t-\tau)) \to du = (1+\omega_d)\cos(\tau - \omega_d(t-\tau))$, hence

$$\begin{aligned}\int_0^t e^{\xi\omega_n\tau}\sin(\tau - \omega_d(t-\tau))\,d\tau &= \left[\sin(\tau - \omega_d(t-\tau))\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\right]_0^t \\ &\quad - \int_0^t \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}(1+\omega_d)\cos(\tau - \omega_d(t-\tau))\,d\tau \\ &= \frac{1}{\xi\omega_n}\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right] - \\ &\quad \frac{(1+\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\cos(\tau - \omega_d(t-\tau))\,d\tau \qquad (2)\end{aligned}$$

Substitute (2) into (1) we obtain

$$\begin{aligned}I_1 &= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \\ &\quad \frac{(1+\omega_d)}{\xi\omega_n}\left(\frac{1}{\xi\omega_n}\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right] - \frac{(1+\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\cos(\tau - \omega_d(t-\tau))\,d\tau\right) \\ &= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right] \\ &\quad - \frac{(1+\omega_d)^2}{(\xi\omega_n)^2}\int_0^t e^{\xi\omega_n\tau}\cos(\tau - \omega_d(t-\tau))\,d\tau \\ &= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right] - \frac{(1+\omega_d)^2}{(\xi\omega_n)^2}I_1\end{aligned}$$

Hence

$$
\begin{aligned}
I_1 + \frac{(1+\omega_d)^2}{(\xi\omega_n)^2} I_1 &= \frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right] \\
I_1\left(\frac{(\xi\omega_n)^2 + (1+\omega_d)^2}{(\xi\omega_n)^2}\right) &= \frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right] \\
I_1 &= \left(\frac{(\xi\omega_n)^2}{(\xi\omega_n)^2 + (1+\omega_d)^2}\right) \\
&\quad \left(\frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right]\right) \\
&= \frac{\xi\omega_n\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1+\omega_d)\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1+\omega_d)^2}
\end{aligned}
$$

Now consider the second integral I_2 where

$$
I_2 = \int_0^t e^{\xi\omega_n \tau} \cos(\tau + \omega_d(t-\tau))\, d\tau
$$

Integrate by parts, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n \tau} \to v = \frac{e^{\xi\omega_n \tau}}{\xi\omega_n}$ and let $u = \cos(\tau + \omega_d(t-\tau)) \to du = -(1-\omega_d)\sin(\tau + \omega_d(t-\tau))$, hence

$$
I_2 = \left[\cos(\tau + \omega_d(t-\tau))\frac{e^{\xi\omega_n \tau}}{\xi\omega_n}\right]_0^t - \int_0^t \frac{e^{\xi\omega_n \tau}}{\xi\omega_n}\left[-(1-\omega_d)\sin(\tau + \omega_d(t-\tau))\right] d\tau
$$

$$
\begin{aligned}
&= \left[\cos(t + \omega_d(t-t))\frac{e^{\xi\omega_n t}}{\xi\omega_n} - \cos(0 + \omega_d(t-0))\frac{1}{\xi\omega_n}\right] \\
&+ \frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n \tau}\sin(\tau + \omega_d(t-\tau))\, d\tau
\end{aligned}
$$

$$
= \frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n \tau}\sin(\tau + \omega_d(t-\tau))\, d\tau \tag{3}
$$

Integrate by parts again the last integral above, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n \tau} \to v = \frac{e^{\xi\omega_n \tau}}{\xi\omega_n}$ and let $u = \sin(\tau + \omega_d(t-\tau)) \to du = (1-\omega_d)\cos(\tau + \omega_d(t-\tau))$, hence

$$
\begin{aligned}
\int_0^t e^{\xi\omega_n \tau}\sin(\tau + \omega_d(t-\tau))\, d\tau &= \left[\sin(\tau + \omega_d(t-\tau))\frac{e^{\xi\omega_n \tau}}{\xi\omega_n}\right]_0^t - \int_0^t \frac{e^{\xi\omega_n \tau}}{\xi\omega_n}(1-\omega_d)\cos(\tau + \omega_d(t-\tau))\, d\tau \\
&= \frac{1}{\xi\omega_n}\left[\sin(t)\, e^{\xi\omega_n t} - \sin(\omega_d t)\right] - \frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n \tau}\cos(\tau + \omega_d(t-\tau))\, d\tau
\end{aligned} \tag{4}
$$

Substitute (4) into (3) we obtain

$$
\begin{aligned}
I_2 &= \frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \\
&\frac{(1-\omega_d)}{\xi\omega_n}\left(\frac{1}{\xi\omega_n}\left[\sin(t)\, e^{\xi\omega_n t} - \sin(\omega_d t)\right] - \frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n \tau}\cos(\tau + \omega_d(t-\tau))\, d\tau\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\, e^{\xi\omega_n t} - \sin(\omega_d t)\right] \\
&- \frac{(1-\omega_d)^2}{(\xi\omega_n)^2}\int_0^t e^{\xi\omega_n \tau}\cos(\tau + \omega_d(t-\tau))\, d\tau
\end{aligned}
$$

$$
= \frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right] - \frac{(1-\omega_d)^2}{(\xi\omega_n)^2} I_2
$$

Hence

$$I_2 + \frac{(1-\omega_d)^2}{(\xi\omega_n)^2} I_2 = \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]$$

$$I_2\left(\frac{(\xi\omega_n)^2 + (1-\omega_d)^2}{(\xi\omega_n)^2}\right) = \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]$$

$$I_2 = \left(\frac{(\xi\omega_n)^2}{(\xi\omega_n)^2 + (1-\omega_d)^2}\right)\left(\frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]\right)$$

$$= \frac{\xi\omega_n\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1-\omega_d)\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1-\omega_d)^2}$$

Using the above expressions for I_1, I_2, we find (and multiplying the solution by $(\Phi(t) - \Phi(t-\pi))$ since the force is only active from $t = 0$ to $t = \pi$, we obtain

$$x_p(t) = \frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}(I_1 - I_2)\left(\Phi(t) - \Phi(t-\pi)\right)$$

$$\begin{aligned} &= (\Phi(t) - \Phi(t-\pi)) * \\ &\frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\frac{\xi\omega_n\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1+\omega_d)\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1+\omega_d)^2} \\ &- \frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\frac{\xi\omega_n\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1-\omega_d)\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1-\omega_d)^2} \end{aligned} \tag{5}$$

Hence $x_p(t) = (\Phi(t) - \Phi(t-\pi))$

$$\left[\frac{F_0}{2m\omega_d}e^{-\xi\omega_n t}\left(\frac{\xi\omega_n\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1+\omega_d)\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1+\omega_d)^2} - \frac{\xi\omega_n\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1-\omega_d)\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1-\omega_d)^2}\right)\right]$$

But

$$\begin{aligned} (\xi\omega_n)^2 + (1+\omega_d)^2 &= \xi^2\omega_n^2 + 1 + \omega_d^2 + 2\omega_d \\ &= \xi^2\omega_n^2 + 1 + \omega_n^2(1-\xi^2) + 2\omega_d \\ &= 1 + 2\omega_d + \omega_n^2 \end{aligned}$$

and

$$(\xi\omega_n)^2 + (1-\omega_d)^2 = 1 - 2\omega_d + \omega_n^2$$

Hence $x_p(t)$ can now be written as

$$\begin{aligned} x_p(t) = &\frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\frac{\xi\omega_n\cos(t)\,e^{\xi\omega_n t} - \xi\omega_n\cos(\omega_d t) + (1+\omega_d)\sin(t)\,e^{\xi\omega_n t} + (1+\omega_d)\sin(\omega_d t)}{1 + 2\omega_d + \omega_n^2} \\ &- \frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\frac{\xi\omega_n\cos(t)\,e^{\xi\omega_n t} - \xi\omega_n\cos(\omega_d t) + (1-\omega_d)\sin(t)\,e^{\xi\omega_n t} + (1-\omega_d)\sin(\omega_d t)}{1 - 2\omega_d + \omega_n^2} \end{aligned}$$

And

$$x_h(t) = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

Hence the overall solution is

$$x(t) = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + x_p(t)$$

The above solution is a bit long due to integration by parts. I will not solve the same problem using Laplace transformation method. The differential equation is

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2 x(t) = f(t)$$

Take Laplace transform, we obtain (assuming $x(0) = x_0$ and $\dot{x}(0) = v_0$)

$$\begin{aligned} \left(s^2X - sx(0) - \dot{x}(0)\right) + 2\xi\omega_n(sX - x(0)) + \omega_n^2 X &= F(s) \\ \left(s^2X - sx_0 - v_0\right) + 2\xi\omega_n(sX - x_0) + \omega_n^2 X &= F(s) \end{aligned} \tag{7}$$

Now we find Laplace transform of $f(t)$

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\, dt \\
&= \int_0^\pi e^{-st} F_0 \sin t \; dt \\
&= F_0 \left[\int_0^\pi e^{-st} \sin t \; dt \right]
\end{aligned}
$$

Integration by parts gives

$$
F(s) = F_0 \left[\frac{1 + e^{-\pi s}}{1 + s^2} \right] \tag{8}
$$

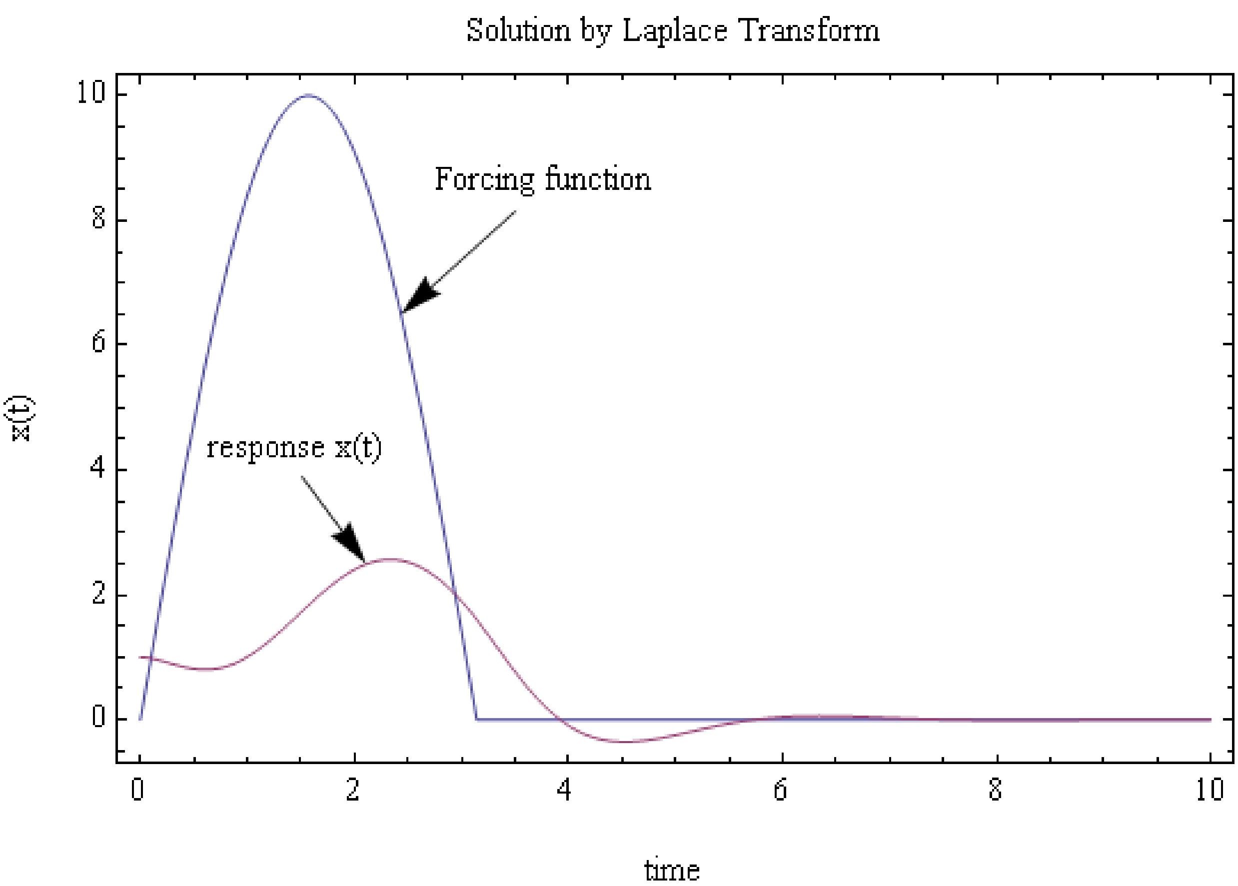
Substitute (8) into (7) we obtain

$$
\begin{aligned}
\left(s^2 X - s x_0 - v_0\right) + 2\xi\omega_n (sX - x_0) + \omega_n^2 X = F_0 \left[\frac{1 + e^{-\pi s}}{1 + s^2} \right] \\
X\left(s^2 + 2\xi\omega_n s + \omega_n^2\right) - s x_0 - v_0 - 2\xi\omega_n x_0 &= \frac{F_0(1 + e^{-\pi s})}{1 + s^2} \\
X\left(s^2 + 2\xi\omega_n s + \omega_n^2\right) &= \frac{F_0(1 + e^{-\pi s})}{1 + s^2} + s x_0 + v_0 + 2\xi\omega_n x_0 \\
&= \frac{F_0(1 + e^{-\pi s}) + \left(1 + s^2\right) s x_0 + v_0 \left(1 + s^2\right) + 2\xi\omega_n x_0 \left(1 + s^2\right)}{1 + s^2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
X &= \frac{F_0(1 + e^{-\pi s}) + \left(1 + s^2\right) s x_0 + v_0 \left(1 + s^2\right) + 2\xi\omega_n x_0 \left(1 + s^2\right)}{\left(1 + s^2\right)\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)} \\
&= \frac{F_0 + v_0 + \frac{F_0}{e^{\pi s}} + s x_0 + s^2 v_0 + s^3 x_0 + 2\xi\omega_n x_0 + 2 s^2 \xi\omega_n x_0}{\left(1 + s^2\right)\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)}
\end{aligned}
$$

Now we can use inverse Laplace transform on the above. It is easier to do partial fraction decomposition and use tables. I used CAS to do this and this is the result. I plot the solution $x(t)$. I used the following values to be able to obtain a plot $\xi = 0.5, \omega_n = 2, F_0 = 10, x_0 = 1, v_0 = 0$



3.3.6 Problem 3.21

3.21. A machine resting on an elastic support can be modeled as a single-degree-of-freedom spring–mass system arranged in the vertical direction. The ground is subject to a motion $y(t)$ of the form illustrated in Figure P3.21. The machine has a mass of 5000 kg and the support has stiffness 1.5×10^3 N/m. Calculate the resulting vibration of the machine.

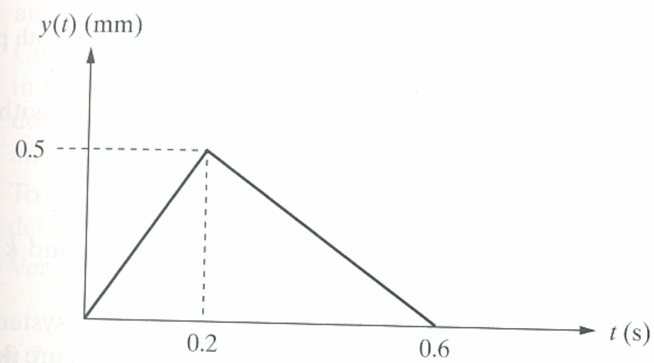


Figure P3.21 Triangular pulse input.

The acceleration $\ddot{x}$ of the mass is measured w.r.t. to the inertial frame, but the spring length is measured relative to the ground which is moving with displacement $y(t)$, hence the equation of motion of the mass m is given by

$$m\ddot{x}(t) + k(x(t) - y(y)) = 0$$

Therefore

$$m\ddot{x}(t) + k\;x(t) = k\;y(t) \tag{1}$$

Where $y(t)$ is given as

$$y(t) = \begin{cases} 2.5t & 0 \le t \le 0.2 \\ 0.75 - 1.25t & 0.2 < t \le 0.6 \\ 0 & 0.6 < t \end{cases}$$

The solution to (1) is given by $x(t) = x_h(t) + x_p(t)$ where $x_p(t)$ can be found using convolution, and $x_h(t)$ is as usual given by

$$x_h = A\cos\omega_n + B\sin\omega_n$$

Let us first find $x_p(t)$. Note that the impulse response $h(t)$ to undamped system is given by

$$h(t) = \frac{1}{m\omega_n}\sin\omega_n t$$

Hence for $0 \le t \le 0.2$,

$$\begin{aligned} x_{p(0\cdots0.2)}(t) &= \int_0^t f(\tau)\,(kh(t-\tau))\,d\tau \\ &= \int_0^t 2.5\tau\left(\frac{k}{m\omega_n}\sin\omega_n(t-\tau)\right)d\tau \\ &= \frac{2.5k}{m\omega_n}\int_0^t \tau\sin\omega_n(t-\tau)\;d\tau \\ &= 2.5\omega_n\int_0^t \tau\sin\omega_n(t-\tau)\;d\tau \end{aligned} \tag{2}$$

Integration by parts, $\int u dv = uv - \int v du$ where $u = \tau$, $dv = \sin\omega_n(t-\tau)$, hence $v = \frac{-\cos(\omega_n(t-\tau))}{-\omega_n}$, therefore (2) becomes

$$\begin{aligned} x_{p(0\cdots0.2)}(t) &= 2.5\omega_n\left(\left[\tau\frac{\cos\omega_n(t-\tau)}{\omega_n}\right]_0^t - \int_0^t \frac{\cos(\omega_n(t-\tau))}{\omega_n}\,d\tau\right) \\ &= 2.5\omega_n\left(\frac{t}{\omega_n} + \frac{1}{\omega_n^2}[\sin(\omega_n(t-\tau))]_0^t\right) \\ &= 2.5\omega_n\left(\frac{t}{\omega_n} + \frac{1}{\omega_n^2}[\sin\omega_n(t-t) - \sin\omega_n(t)]\right) \\ &= 2.5\left(t - \frac{\sin\omega_n t}{\omega_n}\right) \end{aligned}$$

For $0.2 < t \leq 0.6$

$$\begin{aligned}
x_{p(0.2\cdots 0.6)}(t) &= \omega_n \int_0^{0.2} 2.5\tau \sin\omega_n(t-\tau)\ d\tau + \int_{0.2}^{t} f(\tau)\left(kh(t-\tau)\right) d\tau \\
&= 2.5\omega_n \int_0^{0.2} \tau \sin\omega_n(t-\tau)\ d\tau + \int_{0.2}^{t} (0.75-1.25\tau)\left(\frac{k}{m\omega_n}\sin\omega_n(t-\tau)\right) d\tau \\
&= 2.5\omega_n \int_0^{0.2} \tau \sin\omega_n(t-\tau)\ d\tau + \\
&\int_{0.2}^{t} 0.75\frac{k}{m\omega_n}\sin\omega_n(t-\tau)\, d\tau \\
&- \int_{0.2}^{t} 1.25\tau\frac{k}{m\omega_n}\sin\omega_n(t-\tau)\, d\tau
\end{aligned} \tag{3}$$

For the first integral in (3), we obtain

$$\begin{aligned}
I_1 &= 2.5\omega_n \int_0^{0.2} \tau \sin\omega_n(t-\tau)\ d\tau \\
&= 2.5\omega_n\left(\left[\tau\frac{\cos\omega_n(t-\tau)}{\omega_n}\right]_0^{0.2} - \int_0^{0.2}\frac{\cos\left(\omega_n(t-\tau)\right)}{\omega_n}\ d\tau\right) \\
&= 2.5\omega_n\left(0.2\frac{\cos\omega_n(t-0.2)}{\omega_n} + \frac{1}{\omega_n^2}\left[\sin\left(\omega_n(t-\tau)\right)\right]_0^{0.2}\right) \\
&= 2.5\omega_n\left(0.2\frac{\cos\omega_n(t-0.2)}{\omega_n} + \frac{1}{\omega_n^2}\left(\sin\omega_n(t-0.2) - \sin\omega_n t\right)\right) \\
&= 0.5\cos\omega_n(t-0.2) + \frac{2.5}{\omega_n}\sin\omega_n(t-0.2) - \frac{2.5}{\omega_n}\sin\omega_n t
\end{aligned}$$

For the second integral in (3) we obtain

$$\begin{aligned}
I_2 &= 0.75\omega_n \int_{0.2}^{t} \sin\omega_n(t-\tau)\, d\tau \\
&= 0.75\left[\cos\omega_n(t-\tau)\right]_{0.2}^{t} \\
&= 0.75\left(1-\cos\omega_n(t-0.2)\right)
\end{aligned}$$

For the third integral in (3) we obtain

$$\begin{aligned}
I_3 &= \int_{0.2}^{t} 1.25\tau\frac{k}{m\omega_n}\sin\omega_n(t-\tau)\, d\tau \\
&= 1.25\omega_n \int_{0.2}^{t} \tau\sin\omega_n(t-\tau)\, d\tau
\end{aligned}$$

Integration by parts gives

$$\begin{aligned}
I_3 &= 1.25\omega_n\left(\left[\tau\frac{\cos\omega_n(t-\tau)}{\omega_n}\right]_{0.2}^{t} - \int_{0.2}^{t}\frac{\cos\left(\omega_n(t-\tau)\right)}{\omega_n}\ d\tau\right) \\
&= 1.25\omega_n\left(\frac{t}{\omega_n} - 0.2\frac{\cos\omega_n(t-0.2)}{\omega_n} + \frac{1}{\omega_n^2}\left[\sin\omega_n(t-\tau)\right]_{0.2}^{t}\right) \\
&= 1.25\omega_n\left(\frac{t}{\omega_n} - 0.2\frac{\cos\omega_n(t-0.2)}{\omega_n} + \frac{1}{\omega_n^2}\left[-\sin\omega_n(t-0.2)\right]\right) \\
&= 1.25\left(t - 0.2\cos\omega_n(t-0.2) - \frac{1}{\omega_n}\sin\omega_n(t-0.2)\right)
\end{aligned}$$

Hence

$$\begin{aligned}
x_{p(0.2\cdots 0.6)}(t) &= I_1 + I_2 - I_3 \\
&= 0.5\cos\omega_n(t-0.2) + \frac{2.5}{\omega_n}\sin\omega_n(t-0.2) - \frac{2.5}{\omega_n}\sin\omega_n t + \\
&0.75\left(1-\cos\omega_n(t-0.2)\right) \\
&- 1.25\left(t - 0.2\cos\omega_n(t-0.2) - \frac{1}{\omega_n}\sin\omega_n(t-0.2)\right) \\
&= 0.5\cos\omega_n(t-0.2) + \frac{2.5}{\omega_n}\sin\omega_n(t-0.2) - \frac{2.5}{\omega_n}\sin\omega_n t + \\
&0.75 - 0.75\cos\omega_n(t-0.2) \\
&- 1.25t + 0.25\cos\omega_n(t-0.2) + \frac{1.25}{\omega_n}\sin\omega_n(t-0.2) \\
&= 0.75 - 1.25t + \frac{3.75}{\omega_n}\sin\omega_n(t-0.2) - \frac{2.5}{\omega_n}\sin\omega_n t
\end{aligned}$$

For $t > 0.6$

$$
\begin{aligned}
x_{p(0.6\cdots t)}(t) &= 2.5\omega_n \int_0^{0.2} \tau \sin \omega_n (t-\tau)\ d\tau + \int_{0.2}^{0.6} (0.75 - 1.25\tau) \left(\frac{k}{m\omega_n} \sin \omega_n (t-\tau) \right) d\tau \\
&= 2.5\omega_n \int_0^{0.2} \tau \sin \omega_n (t-\tau)\ d\tau + \\
&\int_{0.2}^{0.6} 0.75 \frac{k}{m\omega_n} \sin \omega_n (t-\tau)\, d\tau \\
&- \int_{0.2}^{0.6} 1.25\tau \frac{k}{m\omega_n} \sin \omega_n (t-\tau)\, d\tau
\end{aligned} \tag{4}
$$

For the first integral in (4), we obtain

$$
I_1 = 0.5 \cos \omega_n (t - 0.2) + \frac{2.5}{\omega_n} \sin \omega_n (t-0.2) - \frac{2.5}{\omega_n} \sin \omega_n t
$$

For the second integral in (4) we obtain

$$
\begin{aligned}
I_2 &= 0.75\omega_n \int_{0.2}^{0.6} \sin \omega_n (t-\tau)\, d\tau \\
&= 0.75 [\cos \omega_n (t-\tau)]_{0.2}^{0.6} \\
&= 0.75 (\cos \omega_n (t-0.6) - \cos \omega_n (t-0.2)) \\
&= 0.75 \cos \omega_n (t-0.6) - 0.75 \cos \omega_n (t-0.2)
\end{aligned}
$$

For the third integral in (4) we obtain

$$
\begin{aligned}
I_3 &= \int_{0.2}^{0.6} 1.25\tau \frac{k}{m\omega_n} \sin \omega_n (t-\tau)\, d\tau \\
&= 1.25\omega_n \int_{0.2}^{0.6} \tau \sin \omega_n (t-\tau)\, d\tau
\end{aligned}
$$

Integration by parts gives

$$
\begin{aligned}
I_3 &= 1.25\omega_n \left(\left[\tau \frac{\cos \omega_n (t-\tau)}{\omega_n} \right]_{0.2}^{0.6} - \int_{0.2}^{0.6} \frac{\cos\left(\omega_n (t-\tau)\right)}{\omega_n}\ d\tau \right) \\
&= 1.25\omega_n \left(0.6 \frac{\cos \omega_n (t-0.6)}{\omega_n} - 0.2 \frac{\cos \omega_n (t-0.2)}{\omega_n} - \frac{1}{\omega_n^2} (\sin \omega_n (t-0.6) - \sin \omega_n (t-0.2)) \right) \\
&= 0.75 \cos \omega_n (t-0.6) - 0.25 \cos \omega_n (t-0.2) - \frac{1.25}{\omega_n} \sin \omega_n (t-0.6) + \frac{1.25}{\omega_n} \sin \omega_n (t-0.2)
\end{aligned}
$$

Hence

$$
\begin{aligned}
x_{p(0.6\cdots t)}(t) &= I_1 + I_2 - I_3 \\
&= 0.5 \cos \omega_n (t-0.2) + \frac{2.5}{\omega_n} \sin \omega_n (t-0.2) - \frac{2.5}{\omega_n} \sin \omega_n t \\
&+ 0.75 \cos \omega_n (t-0.6) - 0.75 \cos \omega_n (t-0.2) \\
&- 0.75 \cos \omega_n (t-0.6) + 0.25 \cos \omega_n (t-0.2) + \frac{1.25}{\omega_n} \sin \omega_n (t-0.6) - \frac{1.25}{\omega_n} \sin \omega_n (t-0.2) \\
&= \frac{3.75}{\omega_n} \sin \omega_n (t-0.2) - \frac{2.5}{\omega_n} \sin \omega_n t - \frac{1.25}{\omega_n} \sin \omega_n (t-0.6)
\end{aligned}
$$

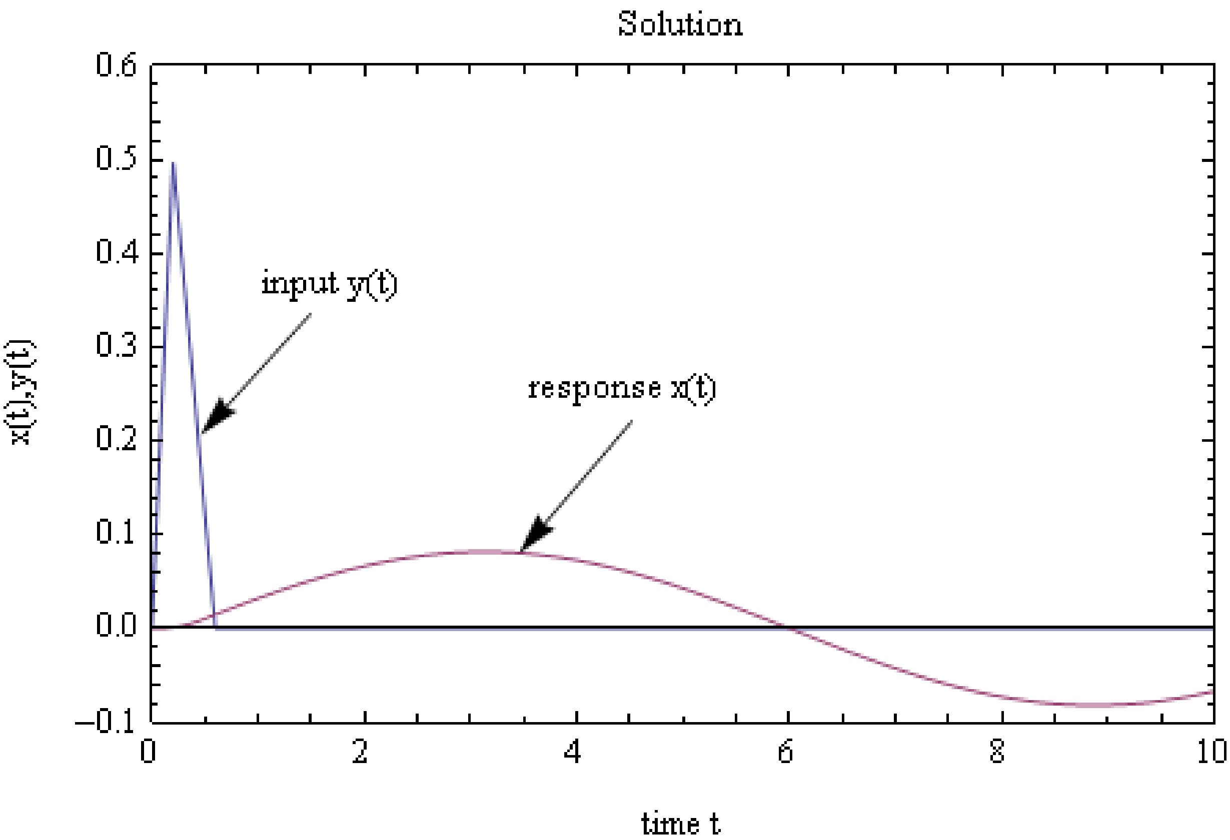
Hence, the overall response is, assuming zero initial conditions, is given by

$$
x(t) = \begin{cases}
2.5\left(t - \frac{\sin \omega_n t}{\omega_n}\right) & 0 \le t \le 0.2 \\
0.75 - 1.25t + \frac{3.75}{\omega_n} \sin \omega_n (t-0.2) - \frac{2.5}{\omega_n} \sin \omega_n t & 0.2 < t \le 0.6 \\
\frac{3.75}{\omega_n} \sin \omega_n (t-0.2) - \frac{2.5}{\omega_n} \sin \omega_n t - \frac{1.25}{\omega_n} \sin \omega_n (t-0.6) & t > 0.6
\end{cases}
$$

Noting that $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1500}{5000}} = 0.54772$, the above becomes

$$
x(t) = \begin{cases}
2.5t - 4.5644 \sin \omega_n t & 0 \le t \le 0.2 \\
0.75 - 1.25t + 6.8466 \sin \omega_n (t-0.2) - 4.5644 \sin \omega_n t & 0.2 < t \le 0.6 \\
6.8466 \sin \omega_n (t-0.2) - 4.5644 \sin \omega_n t - 2.2822 \sin \omega_n (t-0.6) & t > 0.6
\end{cases}
$$

This is a plot of the solution superimposed on top of the forcing function



3.3.7 Problem 3.29

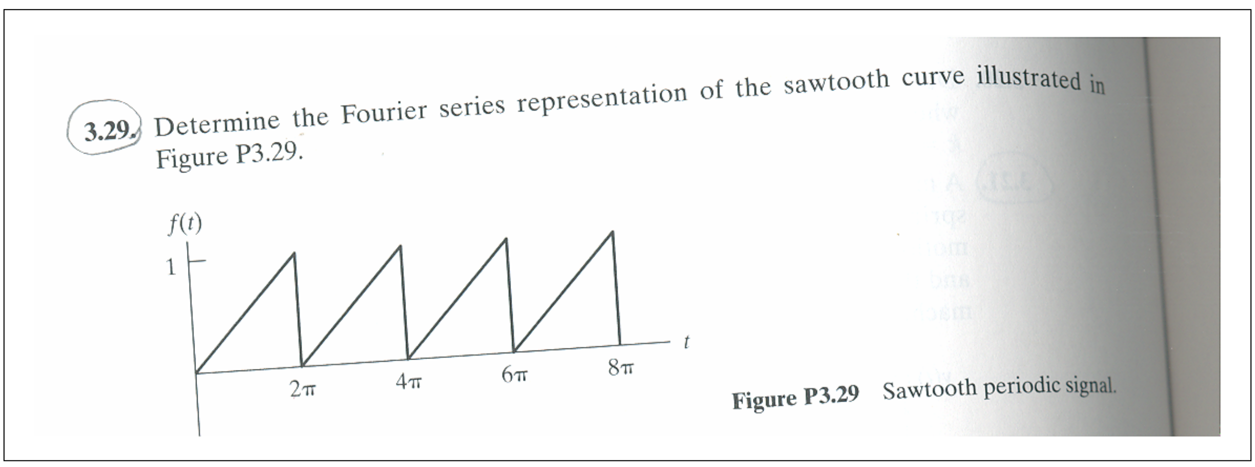

3.29. Determine the Fourier series representation of the sawtooth curve illustrated in Figure P3.29.

Figure P3.29 Sawtooth periodic signal.

Let $f(t)$ be the function shown above. Let $\tilde{f}(t)$ be its approximation using Fourier series. Hence

$$\tilde{f}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} nt\right) + b_n \sin\left(\frac{2\pi}{T} nt\right)$$

Where T is the period of $f(t)$ and

$$a_0 = \frac{2}{T}\int_0^T f(t)\, dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi}{T} nt\right) dt \qquad n = 1, 2, \cdots$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi}{T} nt\right) dt \qquad n = 1, 2, \cdots$$

For $f(t)$ we see that $T=2\pi$ and $f(t)=\frac{t}{T}$ for $0\leq t\leq T$, hence

$$\begin{aligned}
a_0 &= \frac{2}{T}\int_0^T \frac{t}{T}dt\\
&= \frac{2}{2\pi}\int_0^T \frac{t}{2\pi}dt\\
&= \frac{1}{2\pi^2}\left[\frac{t^2}{2}\right]_0^{2\pi}\\
&= \frac{1}{4\pi^2}\left[4\pi^2\right]\\
&= 1
\end{aligned}$$

And

$$\begin{aligned}
a_n &= \frac{2}{T}\int_0^{2\pi} \frac{t}{T}\cos(nt)\,dt \qquad n=1,2,\cdots\\
&= \frac{2}{2\pi}\int_0^{2\pi} \frac{1}{2\pi}t\cos(nt)\,dt\\
&= \frac{1}{2\pi^2}\int_0^{2\pi} t\cos(nt)\,dt\\
&= \frac{1}{2\pi^2}\left(\left[t\frac{\sin nt}{n}\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\sin nt\,dt\right)\\
&= \frac{1}{2\pi^2}\left(0+\frac{1}{n}\left[\frac{\cos nt}{n}\right]_0^{2\pi}\right)\\
&= \frac{1}{2\pi^2}\left(\frac{1}{n^2}[\cos 2n\pi - 1]\right)\\
&= \frac{1}{2n^2\pi^2}(\cos 2n\pi - 1)\\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
b_n &= \frac{2}{T}\int_0^{2\pi} \frac{t}{T}\sin(nt)\,dt \qquad n=1,2,\cdots\\
&= \frac{2}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\sin(nt)\,dt\\
&= \frac{1}{2\pi^2}\left(\left[-\frac{t\cos nt}{n}\right]_0^{2\pi} + \int_0^{2\pi}\frac{\cos nt}{n}dt\right)\\
&= \frac{1}{2\pi^2}\left(\left[\frac{-2\pi\cos 2\pi n}{n}\right] - \frac{1}{n}\left[\frac{\sin nt}{n}\right]_0^{2\pi}\right)\\
&= \frac{1}{2\pi^2}\left(\frac{-2\pi\cos 2\pi n}{n}\right)\\
&= \frac{-\cos 2\pi n}{n\pi}\\
&= \frac{-1}{n\pi}
\end{aligned}$$

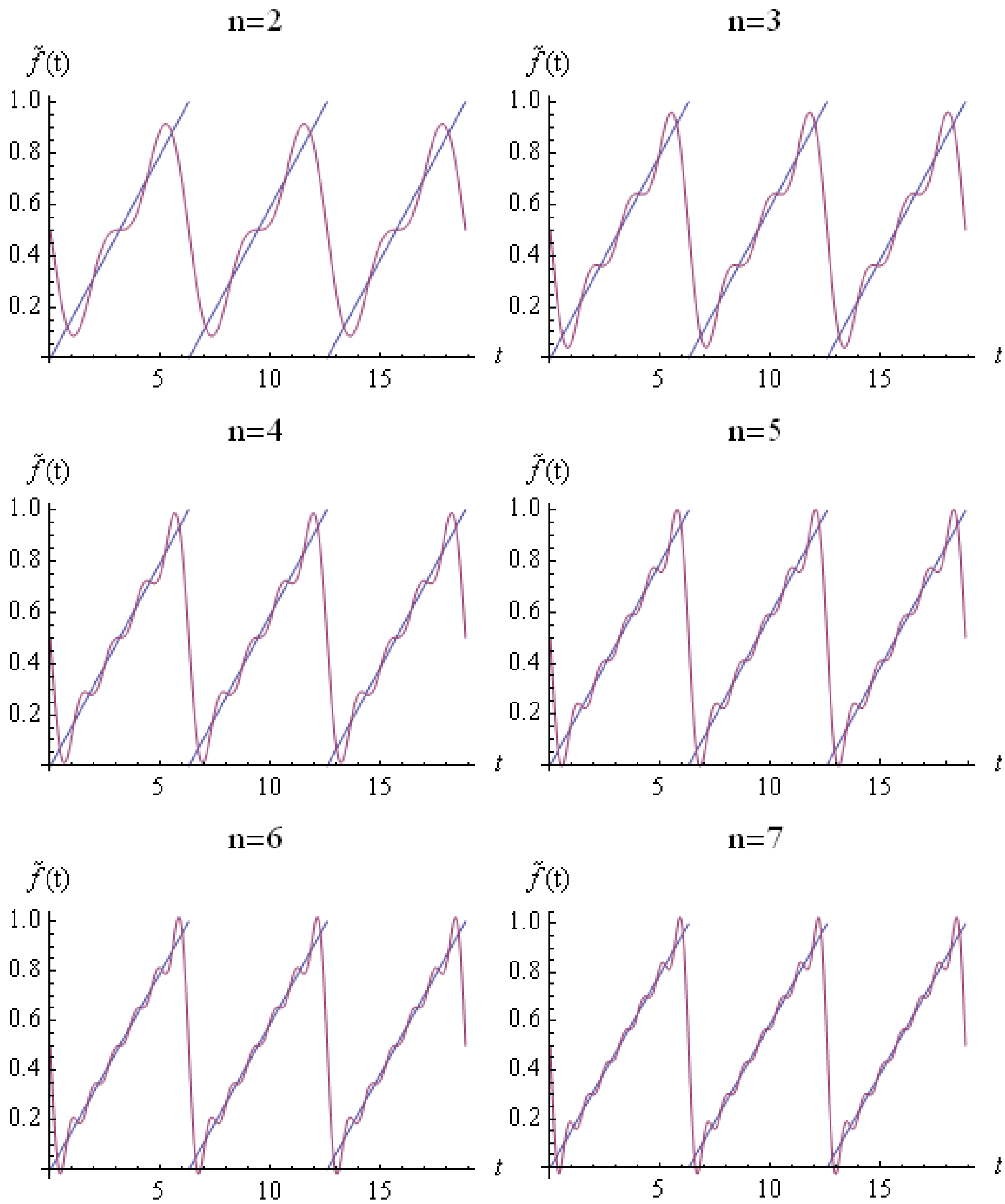
Hence

$$\begin{aligned}
\tilde{f}(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{2\pi}{T}nt\right) + b_n\sin\left(\frac{2\pi}{T}nt\right)\\
&= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{-1}{n\pi}\sin(nt)
\end{aligned}$$

These are few terms in the series

$$\tilde{f}(t) = \frac{1}{2} - \frac{1}{\pi}\sin t - \frac{1}{2\pi}\sin 2t - \frac{1}{3\pi}\sin 3t - \cdots$$

This is a plot of the above for increasing number of n



3.3.8 Problem 3.38

Problem

Solve the following system using Laplace transform $100\ddot{x}(t) + 2000x(t) = 50\delta(t)$ where the units are in Newtons and the initial conditions are both zero.

Answer

Divide the equation by 50 we obtain

$$2\ddot{x}(t) + 40x(t) = \delta(t)$$

Let $m = 2, k = 40$,hence the equation becomes

$$m\ddot{x}(t) + kx(t) = \delta(t)$$

Applying Laplace transform

$$m\left(s^2X(s) - sx_0 - v_0\right) + kX(s) = 1$$

But due to zero initial conditions, the above simplifies to

$$ms^2X(s) + kX(s) = 1$$

$$X(s)\left[ms^2 + k\right] = 1$$

$$X(s) = \frac{1}{ms^2 + k}$$

From tables, the inverse Laplace transform of $\frac{\alpha}{s^2+\alpha^2}$ is $\sin \alpha t$, but

$$\frac{1}{ms^2+k} = \frac{\frac{1}{m}}{s^2+\frac{k}{m}} = \frac{1}{m}\frac{1}{\sqrt{\frac{k}{m}}}\left(\frac{\sqrt{\frac{k}{m}}}{s^2+\frac{k}{m}}\right)$$

Hence, letting $\boxed{\alpha = \sqrt{\frac{k}{m}}}$ we see that inverse laplace transform of $\frac{1}{ms^2+k}$ is the same as the inverse laplace transform of $\frac{1}{m}\frac{1}{\alpha}\left(\frac{\alpha}{s^2+\alpha^2}\right)$ which is $\boxed{\frac{1}{m}\frac{1}{\alpha}\sin \alpha t}$

But $\alpha = \omega_n$, hence

$$x(t) = \frac{1}{m\omega_n}\sin \omega_n t$$

or

$$\begin{aligned} x(t) &= \frac{1}{2\sqrt{\frac{40}{2}}}\sin\sqrt{\frac{40}{2}}t \\ &= 0.1118\sin(4.4721t) \end{aligned}$$

3.3.9 Problem 3.44

Problem

Calculate the response spectrum of an undamped system to the forcing function

$$F(t) = \begin{cases} F_0 \sin \frac{\pi t}{t_1} & 0 \leq t \leq t_1 \\ 0 & t > t_1 \end{cases}$$ assuming zero initial conditions.

Answer

Solution sketch: Find the response $x(t)$ of the system to the above input. Then find t where this response is maximum, call this $x_{\max}$, then plot $\left(x_{\max}\frac{k}{F_0}\right)$ vs. $\frac{t\omega_n}{2\pi}$

The system EQM is

$$x''(t) + \omega_n^2 x(t) = \frac{F(t)}{m}$$

For $0 < t \leq t_1$,

$$\begin{aligned} x_1(t) &= x_h(t) + x_p(t) \\ &= A\cos\omega_n t + B\sin\omega_n t + x_p(t) \end{aligned}$$

Guess $\boxed{x_p(t) = c_1\cos\omega t + c_2\sin\omega t}$, hence $x_p'(t) = -\omega c_1 \sin\omega t + \omega c_2\cos\omega t$ and $x_p''(t) = -\omega^2 c_1\cos\omega t - \omega^2 c_2\sin\omega t$, hence substitute these into the EQM and compare, we obtain

$$\left(-\omega^2 c_1\cos\omega t - \omega^2 c_2\sin\omega t\right) + \omega_n^2\left(c_1\cos\omega t + c_2\sin\omega t\right) = \frac{F_0}{m}\sin\frac{\pi t}{t_1}$$

The input is half sin where $\omega t = \frac{\pi t}{t_1}$, hence $\boxed{\omega = \frac{\pi}{t_1}}$,hence the above becomes

$$\left(-\omega^2 c_1 + \omega_n^2 c_1\right)\cos\omega t + \left(-\omega^2 c_2 + \omega_n^2 c_2\right)\sin\omega t = \frac{F_0}{m}\sin\omega t$$

Hence $\boxed{c_1 = 0}$ and $c_2\left(-\omega^2+\omega_n^2\right) = \frac{F_0}{m}$ or $\boxed{c_2 = \frac{\frac{F_0}{m}}{\omega_n^2-\omega^2}}$, Then the solution becomes

$$x_1(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{\frac{F_0}{m}}{\omega_n^2-\omega^2}\sin\omega t$$

And since $x(0) = 0$ then $A = 0$ and take derivative we obtain

$$x_1'(t) = \omega_n B\cos\omega_n t + \omega\frac{\frac{F_0}{m}}{\omega_n^2-\omega^2}\cos\omega t$$

And since $x'(0) = 0$ then the above results in

$$\begin{aligned} 0 &= \omega_n B + \omega\frac{\frac{F_0}{m}}{\omega_n^2-\omega^2} \\ B &= \frac{\omega}{\omega_n}\frac{\frac{F_0}{m}}{\omega^2-\omega_n^2} \end{aligned}$$

Hence the solution becomes

$$
\begin{aligned}
x_1(t) &= \frac{\omega}{\omega_n}\frac{\frac{F_0}{m}}{\omega^2-\omega_n^2}\sin\omega_n t + \frac{\frac{F_0}{m}}{\omega_n^2-\omega^2}\sin\omega t\\
&= \frac{\frac{F_0}{m}}{\omega_n^2-\omega^2}\left(\sin\omega t - \frac{\omega}{\omega_n}\sin\omega_n t\right)\\
&= \frac{\frac{F_0}{m}}{\omega_n^2\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega t - \frac{\omega}{\omega_n}\sin\omega_n t\right)\\
&= \frac{\frac{F_0}{m}}{\frac{k}{m}\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega t - \frac{\omega}{\omega_n}\sin\omega_n t\right)
\end{aligned}
$$

Hence

$$
x_1(t) = \frac{\frac{F_0}{k}}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega t - \frac{\omega}{\omega_n}\sin\omega_n t\right) \qquad 0<t\leq t_1 \tag{1}
$$

Now we need to find where the maximum is. Take derivative, and set it to zero, we obtain

$$
x_1'(t) = \frac{F_0}{k}\frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}(\omega\cos\omega t - \omega\cos\omega_n t) = 0
$$

For $\omega\neq\omega_n$, we need to solve

$$
\cos\omega t - \cos\omega_n t = 0
$$

Using $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$, then the above becomes

$$
\begin{aligned}
-2\sin\left(\frac{(\omega+\omega_n)t}{2}\right)\sin\left(\frac{(\omega-\omega_n)t}{2}\right) &= 0\\
\sin\left(\frac{(\omega+\omega_n)t}{2}\right)\sin\left(\frac{(\omega-\omega_n)t}{2}\right) &= 0
\end{aligned}
$$

Hence, either $\frac{(\omega+\omega_n)t_p}{2} = n\pi$ or $\frac{(\omega-\omega_n)t_p}{2} = n\pi$for $n=\pm1,\pm2,\cdots$ or the time t_p which makes the maximum $x(t)$ is one of the following

$$
t_p = \begin{cases} \frac{2n\pi}{\omega+\omega_n} \\ \frac{2n\pi}{\omega-\omega_n} \end{cases} \qquad n=\pm1,\pm2,\cdots
$$

We now need to find which one of the above 2 solution gives a larger maximum. Using the first solution $t_p = \frac{2n\pi}{\omega+\omega_n}$, then (1) becomes

$$
\begin{aligned}
x_{\max}(t_{p_1}) &= \frac{F_0}{k\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega\left(\frac{2n\pi}{\omega+\omega_n}\right) - \frac{\omega}{\omega_n}\sin\omega_n\left(\frac{2n\pi}{\omega+\omega_n}\right)\right)\\
&= \frac{F_0}{k\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\left(\frac{2n\pi\frac{\omega}{\omega_n}}{1+\frac{\omega}{\omega_n}}\right) - \frac{\omega}{\omega_n}\sin\left(\frac{2n\pi}{1+\frac{\omega}{\omega_n}}\right)\right)
\end{aligned}
$$

And at $t_p = \frac{2n\pi}{\omega-\omega_n}$, then (1) becomes

$$
\begin{aligned}
x_{\max}(t_{p_2}) &= \frac{F_0}{k\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega\left(\frac{2n\pi}{\omega-\omega_n}\right) - \frac{\omega}{\omega_n}\sin\omega_n\left(\frac{2n\pi}{\omega-\omega_n}\right)\right)\\
&= \frac{F_0}{k\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\left(\frac{2n\pi\frac{\omega}{\omega_n}}{\frac{\omega}{\omega_n}-1}\right) - \frac{\omega}{\omega_n}\sin\left(\frac{2n\pi}{\frac{\omega}{\omega_n}-1}\right)\right)
\end{aligned}
$$

Need now to find which of the above is larger. Let us take the difference and see if the result is positive or negative (is there an easier way?)

$$
\begin{aligned}
x(t_{p_1}) - x(t_{p_2}) &= \frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\left(\frac{2n\pi\frac{\omega}{\omega_n}}{1+\frac{\omega}{\omega_n}}\right) - \sin\left(\frac{2n\pi\frac{\omega}{\omega_n}}{\frac{\omega}{\omega_n}-1}\right) - \frac{\omega}{\omega_n}\sin\left(\frac{2n\pi}{1+\frac{\omega}{\omega_n}}\right) + \frac{\omega}{\omega_n}\sin\left(\frac{2n\pi}{\frac{\omega}{\omega_n}-1}\right)\right)\\
&= \frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\left(\frac{2n\pi\frac{\omega}{\omega_n}}{1+\frac{\omega}{\omega_n}}\right) - \sin\left(\frac{2n\pi\frac{\omega}{\omega_n}}{\frac{\omega}{\omega_n}-1}\right) - \frac{\omega}{\omega_n}\sin\left(\frac{2n\pi}{1+\frac{\omega}{\omega_n}}\right) + \frac{\omega}{\omega_n}\sin\left(\frac{2n\pi}{\frac{\omega}{\omega_n}-1}\right)\right)
\end{aligned}
$$

Not sure how to continue. Now let us look at $t>t_1$. The solution here is

$$
x_2(t) = A\cos\omega_n t + B\sin\omega_n t
$$

But with IC given by $x_1(t_1)$ and $x_1'(t_1)$, hence from (1)

$$x_1(t_1) = \frac{\frac{F_0}{k}}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega t_1 - \frac{\omega}{\omega_n}\sin\omega_n t_1\right)$$

and

$$x_1'(t_1) = \frac{F_0}{k}\frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}(\omega\cos\omega t_1 - \omega\cos\omega_n t_1)$$

Hence

$$x_2(t_1) = A\cos\omega_n t_1 + B\sin\omega_n t_1 = \frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega t_1 - \frac{\omega}{\omega_n}\sin\omega_n t_1\right) \tag{3}$$

And

$$x_2'(t_1) = -\omega_n A\sin\omega_n t + B\omega_n\cos\omega_n t = \frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}(\omega\cos\omega t_1 - \omega\cos\omega_n t_1) \tag{4}$$

We need to solve (3) and (4) for A and B. Combining (3) and (4) we obtain

$$\begin{bmatrix}\cos\omega_n t_1 & \sin\omega_n t_1\\ -\omega_n\sin\omega_n t & \omega_n\cos\omega_n t\end{bmatrix}\begin{bmatrix}A\\B\end{bmatrix} = \begin{bmatrix}\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega t_1 - \frac{\omega}{\omega_n}\sin\omega_n t_1\right)\\ \frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}(\omega\cos\omega t_1 - \omega\cos\omega_n t_1)\end{bmatrix}$$

This is in the form $Ax = b$, solve for x, we obtain

$$\begin{bmatrix}A\\B\end{bmatrix} = \frac{1}{\omega_n}\begin{bmatrix}\omega_n\cos\omega_n t_1 & -\sin\omega_n t_1\\ \omega_n\sin\omega_n t & \cos\omega_n t\end{bmatrix}\begin{bmatrix}\left(\sin\omega t_1 - \frac{\omega}{\omega_n}\sin\omega_n t_1\right)\\ (\omega\cos\omega t_1 - \omega\cos\omega_n t_1)\end{bmatrix}\left(\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\right)$$

Hence

$$\begin{aligned}
A &= \frac{1}{\omega_n}\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left[\omega_n\cos\omega_n t_1\left(\sin\omega t_1 - \frac{\omega}{\omega_n}\sin\omega_n t_1\right) - \sin\omega_n t_1(\omega\cos\omega t_1 - \omega\cos\omega_n t_1)\right]\\
&= \frac{1}{\omega_n}\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}[\omega_n\cos\omega_n t_1\sin\omega t_1 - \omega\cos\omega_n t_1\sin\omega_n t_1 - \omega\sin\omega_n t_1\cos\omega t_1 + \omega\sin\omega_n t_1\cos\omega_n t_1]\\
&= \frac{1}{\omega_n}\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}[\omega_n\cos\omega_n t_1\sin\omega t_1 - \omega\sin\omega_n t_1\cos\omega t_1]
\end{aligned}$$

And

$$\begin{aligned}
B &= \frac{1}{\omega_n}\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left[\omega_n\sin\omega_n t_1\left(\sin\omega t_1 - \frac{\omega}{\omega_n}\sin\omega_n t_1\right) + \cos\omega_n t_1(\omega\cos\omega t_1 - \omega\cos\omega_n t_1)\right]\\
&= \frac{1}{\omega_n}\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}[\omega_n\sin\omega_n t_1\sin\omega t_1 - \omega\sin\omega_n t_1\sin\omega_n t_1 + \omega\cos\omega_n t_1\cos\omega t_1 - \omega\cos\omega_n t_1\cos\omega_n t_1]\\
&= \frac{1}{\omega_n}\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}[\omega_n\sin\omega_n t_1\sin\omega t_1 - \omega\sin\omega_n t_1\sin\omega_n t_1 + \omega\cos\omega_n t_1\cos\omega t_1 - \omega\cos\omega_n t_1\cos\omega_n t_1]
\end{aligned}$$

Ask about the above, why can't I get the answer shown in notes?

3.3.10 Solving 3.44 using convolution

To find the response $x(t)$ use convolution. Since this is an undamped system, then the impulse response is

$$h(t) = \frac{1}{m\omega_n}\sin\omega_n t$$

Hence, for $0 \le t \le t_1$

$$\begin{aligned}
x(t) &= \int_0^t F(\tau)\,h(t-\tau)\,d\tau\\
&= \int_0^t \left(F_0\sin\frac{\pi\tau}{t_1}\right)\frac{1}{m\omega_n}\sin\omega_n(t-\tau)\,d\tau\\
&= \frac{F_0}{m\omega_n}\int_0^t \sin\frac{\pi\tau}{t_1}\sin\omega_n(t-\tau)\,d\tau
\end{aligned}$$

Using $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$, then

$$\sin \overbrace{\frac{\pi\tau}{t_1}}^{A} \sin \overbrace{\omega_n(t-\tau)}^{B} = \frac{1}{2}\cos\left(\frac{\pi\tau}{t_1} - \omega_n(t-\tau)\right) - \frac{1}{2}\cos\left(\frac{\pi\tau}{t_1} + \omega_n(t-\tau)\right)$$

Hence the convolution integral becomes

$$\begin{aligned}
x(t) &= \frac{F_0}{m\omega_n}\int_0^t \frac{1}{2}\cos\left(\frac{\pi\tau}{t_1} - \omega_n(t-\tau)\right) - \frac{1}{2}\cos\left(\frac{\pi\tau}{t_1} + \omega_n(t-\tau)\right) d\tau \\
&= \frac{F_0}{2m\omega_n}\left[\int_0^t \cos\left(\frac{\pi\tau}{t_1} - \omega_n(t-\tau)\right) d\tau - \int_0^t \cos\left(\frac{\pi\tau}{t_1} + \omega_n(t-\tau)\right)\right] \\
&= \frac{F_0}{2m\omega_n}\left\{\left[\frac{\sin\left(\frac{\pi\tau}{t_1} - \omega_n(t-\tau)\right)}{\frac{\pi}{t_1} + \omega_n}\right]_0^t - \left[\frac{\sin\left(\frac{\pi\tau}{t_1} + \omega_n(t-\tau)\right)}{\frac{\pi}{t_1} - \omega_n}\right]_0^t\right\} \\
&= \frac{F_0}{2m\omega_n}\left\{\left(\frac{\sin\left(\frac{\pi t}{t_1}\right)}{\frac{\pi}{t_1} + \omega_n} - \frac{\sin(-\omega_n t)}{\frac{\pi}{t_1} + \omega_n}\right) - \left(\frac{\sin\left(\frac{\pi t}{t_1}\right)}{\frac{\pi}{t_1} - \omega_n} - \frac{\sin(\omega_n t)}{\frac{\pi}{t_1} - \omega_n}\right)\right\} \\
&= \frac{F_0}{2m\omega_n}\left\{\frac{\sin\frac{\pi t}{t_1}}{\frac{\pi}{t_1} + \omega_n} + \frac{\sin\omega_n t}{\frac{\pi}{t_1} + \omega_n} - \frac{\sin\frac{\pi t}{t_1}}{\frac{\pi}{t_1} - \omega_n} + \frac{\sin\omega_n t}{\frac{\pi}{t_1} - \omega_n}\right\}
\end{aligned}$$

And for $t > t_1$

$$\begin{aligned}
x(t) &= \int_0^{t_1} F(\tau)\, h(t-\tau)\, d\tau \ + \int_{t_1}^t 0 \times h(t-\tau)\, d\tau \\
&= \int_0^{t_1}\left(F_0 \sin\frac{\pi\tau}{t_1}\right)\frac{1}{m\omega_n}\sin\omega_n(t-\tau)\, d\tau \\
&= \frac{F_0}{m\omega_n}\int_0^{t_1}\sin\frac{\pi\tau}{t_1}\sin\omega_n(t-\tau)\, d\tau
\end{aligned}$$

As was done earlier, perform integration by parts, we obtain

$$\begin{aligned}
x(t) &= \frac{F_0}{2m\omega_n}\left\{\left[\frac{\sin\left(\frac{\pi\tau}{t_1} - \omega_n(t-\tau)\right)}{\frac{\pi}{t_1} + \omega_n}\right]_0^{t_1} - \left[\frac{\sin\left(\frac{\pi\tau}{t_1} + \omega_n(t-\tau)\right)}{\frac{\pi}{t_1} - \omega_n}\right]_0^{t_1}\right\} \\
&= \frac{F_0}{2m\omega_n}\left\{\frac{\sin(\pi - \omega_n(t-t_1))}{\frac{\pi}{t_1} + \omega_n} - \frac{\sin(-\omega_n t)}{\frac{\pi}{t_1} + \omega_n} - \frac{\sin(\pi + \omega_n(t-t_1))}{\frac{\pi}{t_1} - \omega_n} + \frac{\sin(\omega_n t)}{\frac{\pi}{t_1} - \omega_n}\right\} \\
&= \frac{F_0}{2m\omega_n}\left\{\frac{\sin(\pi - \omega_n(t-t_1))}{\frac{\pi}{t_1} + \omega_n} + \frac{\sin(\omega_n t)}{\frac{\pi}{t_1} + \omega_n} - \frac{\sin(\pi + \omega_n(t-t_1))}{\frac{\pi}{t_1} - \omega_n} + \frac{\sin(\omega_n t)}{\frac{\pi}{t_1} - \omega_n}\right\}
\end{aligned}$$

But $\sin(\pi - \alpha) = \sin\alpha$ and $\sin(\pi + \alpha) = -\sin\alpha$, hence the above becomes

$$x(t) = \frac{F_0}{2m\omega_n}\left\{\frac{\sin\omega_n(t-t_1)}{\frac{\pi}{t_1} + \omega_n} + \frac{\sin(\omega_n t)}{\frac{\pi}{t_1} + \omega_n} + \frac{\sin\omega_n(t-t_1)}{\frac{\pi}{t_1} - \omega_n} + \frac{\sin(\omega_n t)}{\frac{\pi}{t_1} - \omega_n}\right\}$$

Therefore, the final solution is

$$x(t) = \begin{cases} \frac{F_0}{2m\omega_n}\left\{\frac{\sin\frac{\pi t}{t_1}}{\frac{\pi}{t_1}+\omega_n} + \frac{\sin\omega_n t}{\frac{\pi}{t_1}+\omega_n} - \frac{\sin\frac{\pi t}{t_1}}{\frac{\pi}{t_1}-\omega_n} + \frac{\sin\omega_n t}{\frac{\pi}{t_1}-\omega_n}\right\} & 0 \le t \le t_1 \\ \frac{F_0}{2m\omega_n}\left\{\frac{\sin\omega_n(t-t_1)}{\frac{\pi}{t_1}+\omega_n} + \frac{\sin(\omega_n t)}{\frac{\pi}{t_1}+\omega_n} + \frac{\sin\omega_n(t-t_1)}{\frac{\pi}{t_1}-\omega_n} + \frac{\sin(\omega_n t)}{\frac{\pi}{t_1}-\omega_n}\right\} & t > t_1 \end{cases} \tag{1}$$

We can simplify the above more as follows

$$x(t) = \begin{cases} \frac{F_0}{2m\omega_n}\left\{\sin\frac{\pi t}{t_1}\left(\frac{1}{\frac{\pi}{t_1}+\omega_n} - \frac{1}{\frac{\pi}{t_1}-\omega_n}\right) + \sin\omega_n t\left(\frac{1}{\frac{\pi}{t_1}+\omega_n} + \frac{1}{\frac{\pi}{t_1}-\omega_n}\right)\right\} & 0 \le t \le t_1 \\ \frac{F_0}{2m\omega_n}\left\{\sin\omega_n(t-t_1)\left(\frac{1}{\frac{\pi}{t_1}+\omega_n} + \frac{1}{\frac{\pi}{t_1}-\omega_n}\right) + \sin(\omega_n t)\left(\frac{1}{\frac{\pi}{t_1}+\omega_n} + \frac{1}{\frac{\pi}{t_1}-\omega_n}\right)\right\} & t > t_1 \end{cases} \tag{1}$$

Hence

$$x(t) = \begin{cases} \frac{F_0}{2m\omega_n}\left\{\sin\frac{\pi t}{t_1}\left(\frac{\left(\frac{\pi}{t_1}-\omega_n\right)-\left(\frac{\pi}{t_1}+\omega_n\right)}{\left(\frac{\pi}{t_1}+\omega_n\right)\left(\frac{\pi}{t_1}-\omega_n\right)}\right) + \sin\omega_n t\left(\frac{\left(\frac{\pi}{t_1}-\omega_n\right)+\left(\frac{\pi}{t_1}+\omega_n\right)}{\left(\frac{\pi}{t_1}+\omega_n\right)\left(\frac{\pi}{t_1}-\omega_n\right)}\right)\right\} & 0 \le t \le t_1 \\ \frac{F_0}{2m\omega_n}\left\{\sin\omega_n(t-t_1)\left(\frac{\left(\frac{\pi}{t_1}-\omega_n\right)+\left(\frac{\pi}{t_1}+\omega_n\right)}{\left(\frac{\pi}{t_1}+\omega_n\right)\left(\frac{\pi}{t_1}-\omega_n\right)}\right) + \sin(\omega_n t)\left(\frac{\left(\frac{\pi}{t_1}-\omega_n\right)+\left(\frac{\pi}{t_1}+\omega_n\right)}{\left(\frac{\pi}{t_1}+\omega_n\right)\left(\frac{\pi}{t_1}-\omega_n\right)}\right)\right\} & t > t_1 \end{cases} \tag{1}$$

Hence

$$x(t) = \begin{cases} \frac{F_0}{2m\omega_n}\left\{\sin\frac{\pi t}{t_1}\left(\frac{-2\omega_n}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right) + \sin\omega_n t\left(\frac{2\frac{\pi}{t_1}}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\right\} & 0 \le t \le t_1 \\ \frac{F_0}{2m\omega_n}\left\{\sin\omega_n(t-t_1)\left(\frac{2\frac{\pi}{t_1}}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right) + \sin(\omega_n t)\left(\frac{2\frac{\pi}{t_1}}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\right\} & t > t_1 \end{cases} \tag{1}$$

or

$$x(t)=\begin{cases}\left(\frac{1}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{2m\omega_n}\left\{-2\omega_n\sin\frac{\pi t}{t_1}+2\frac{\pi}{t_1}\sin\omega_n t\right\} & 0\leq t\leq t_1\\ \left(\frac{\omega}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\sin\omega_n(t-t_1)-\sin\left(\omega_n t\right)\right\} & t>t_1\end{cases}\tag{1}$$

Hence

$$x(t)=\begin{cases}\left(\frac{1}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{-\omega_n\sin\frac{\pi t}{t_1}+\frac{\pi}{t_1}\sin\omega_n t\right\} & 0\leq t\leq t_1\\ \left(\frac{\omega}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\sin\omega_n(t-t_1)-\sin\omega_n t\right\} & t>t_1\end{cases}\tag{1}$$

To find where $x_{\max}$ is, we need to find $x_{\max}$. Take the derivative, we obtain

$$\dot{x}(t)=\begin{cases}\left(\frac{1}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{-\omega_n\frac{\pi}{t_1}\cos\frac{\pi t}{t_1}+\frac{\pi}{t_1}\omega_n\cos\omega_n t\right\} & 0\leq t\leq t_1\\ \left(\frac{\omega}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\omega_n\cos\omega_n(t-t_1)-\omega_n\cos\omega_n t\right\} & t>t_1\end{cases}$$

Now let $\dot{x}(t)=0$ for $t>t_1$ to find t_{peak}.

$$\left(\frac{\omega}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\omega_n\cos\omega_n(t_p-t_1)-\omega_n\cos\omega_n t_p\right\}=0$$

$$\cos\omega_n(t_p-t_1)-\cos\omega_n t_p=0\tag{2}$$

But

$$\cos\omega_n(t_p-t_1)=\cos\omega_n t_p\cos\omega_n t_1+\sin\omega_n t_p\sin\omega_n t_1$$

Substitute the above into (2) we obtain

$$\left(\cos\omega_n t_p\cos\omega_n t_1+\sin\omega_n t_p\sin\omega_n t_1\right)-\cos\omega_n t_p=0$$

Divide by $\cos\omega_n t_p$

$$\begin{aligned}\cos\omega_n t_1+\tan\omega_n t_p\sin\omega_n t_1-1&=0\\ \tan\omega_n t_p&=\frac{(1-\cos\omega_n t_1)}{\sin\omega_n t_1}\\ \omega_n t_p&=\tan^{-1}\left(\frac{1-\cos\omega_n t_1}{\sin\omega_n t_1}\right)\end{aligned}$$

Hence, the hypotenuse is $\sqrt{(1-\cos\omega_n t_1)^2+\sin^2\omega_n t_1}=\sqrt{2\left(1-\cos\omega_n t_1\right)}$ and so $\sin\omega_n t_p=-\sqrt{\frac{1}{2}\left(1-\cos\omega_n t_1\right)}$ and $\cos\omega_n t_p=\frac{-\sin\omega_n t_1}{\sqrt{2(1-\cos\omega_n t_1)}}$ and using these into (1) we find $x_{\max}$ when $t>t_1$ as

$$x_{\max}=\left(\frac{\omega}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\sin\omega_n(t_p-t_1)-\sin\omega_n t_p\right\}$$

But $\sin\omega_n(t_p-t_1)=\sin\omega_n t_p\cos\omega_n t_1-\cos\omega_n t_p\sin\omega_n t_1$, hence

$$\begin{aligned}x_{\max}&=\left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\sin\omega_n t_p\cos\omega_n t_1-\cos\omega_n t_p\sin\omega_n t_1-\sin\omega_n t_p\right\}\\ &=\left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{-\sqrt{\frac{1}{2}\left(1-\cos\omega_n t_1\right)}\cos\omega_n t_1+\frac{\sin\omega_n t_1}{\sqrt{2\left(1-\cos\omega_n t_1\right)}}\sin\omega_n t_1+\sqrt{\frac{1}{2}\left(1-\cos\omega_n t_1\right)}\right\}\\ &=\left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\frac{-(1-\cos\omega_n t_1)\cos\omega_n t_1+\sin^2\omega_n t_1+(1-\cos\omega_n t_1)}{\sqrt{2\left(1-\cos\omega_n t_1\right)}}\right\}\\ &=\left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\frac{-\cos\omega_n t_1+\cos^2\omega_n t_1+\sin^2\omega_n t_1+1-\cos\omega_n t_1}{\sqrt{2\left(1-\cos\omega_n t_1\right)}}\right\}\\ &=\left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\frac{-\cos\omega_n t_1+1+1-\cos\omega_n t_1}{\sqrt{2\left(1-\cos\omega_n t_1\right)}}\right\}\\ &=\left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{2F_0}{m\omega_n}\left\{\frac{1-\cos\omega_n t_1}{\sqrt{2\left(1-\cos\omega_n t_1\right)}}\right\}\\ &=\left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{F_0}{2m\omega_n}\left\{\sqrt{1-\cos\omega_n t_1}\right\}\end{aligned}$$

Hence

$$
\begin{aligned}
x_{\max} &= \left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{F_0}{2m\omega_n}\sqrt{1-\cos\omega_n t_1}\\
&= \left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{\omega_n F_0}{2m\omega_n^2}\sqrt{1-\cos\omega_n t_1}\\
&= \left(\frac{\frac{\omega}{\omega_n}}{\left(\left(\frac{\omega}{\omega_n}\right)^2-1\right)}\right)\frac{F_0}{2k}\sqrt{1-\cos\omega_n t_1}
\end{aligned}
$$

Hence

$$
\boxed{x_{\max}\frac{k}{F_0} = \left(\frac{r}{r^2-1}\right)\frac{1}{2}\sqrt{1-\cos\omega_n t_1}}
$$

Where $r=\frac{\omega}{\omega_n}$

A plot of $x_{\max}\frac{k}{F_0}$ vs. r gives the response spectrum

3.3.11 Problem 3.49

Problem Calculate the compliance transfer function for a system described by $ax''''+bx'''+cx''+dx'+ex=f(t)$ where $f(t)$ is the input and $x(t)$ is the displacement.

Answer

Take Laplace transform (assuming zero IC) we obtain

$$
as^4X(s)+bs^3X(s)+cs^2X(s)+dsX(s)+eX(s)=F(s)
$$

Hence

$$
X(s)\left[as^4+bs^3+cs^2+ds+e\right]=F(s)
$$

Hence

$$
\boxed{H(s)=\frac{X(s)}{F(s)}=\frac{1}{as^4+bs^3+cs^2+ds+e}}
$$

3.3.12 Problem 3.50

Problem

Calculate the frequency response function for the system of problem 3.49 for $a=1,b=4,c=11,d=16,e=8$

Answer

$$
\begin{aligned}
H(s) &= \frac{1}{as^4+bs^3+cs^2+ds+e}\\
&= \frac{1}{s^4+4s^3+11s^2+16s+11}
\end{aligned}
$$

Let $s=j\omega$

$$
\begin{aligned}
H(j\omega) &= \frac{1}{(j\omega)^4+4(j\omega)^3+11(j\omega)^2+16(j\omega)+11}\\
&= \frac{1}{\omega^4-4j\omega^3-11\omega^2+16j\omega+11}\\
&= \frac{1}{(\omega^4-11\omega^2+11)+j(16\omega-4\omega^3)}
\end{aligned}
$$

Hence

$$
\begin{aligned}
|H(j\omega)| &= \frac{1}{\sqrt{(\omega^4-11\omega^2+11)^2+(16\omega-4\omega^3)^2}}\\
&= \frac{1}{\sqrt{\omega^8-6\omega^6+15\omega^4+14\omega^2+121}}
\end{aligned}
$$

and

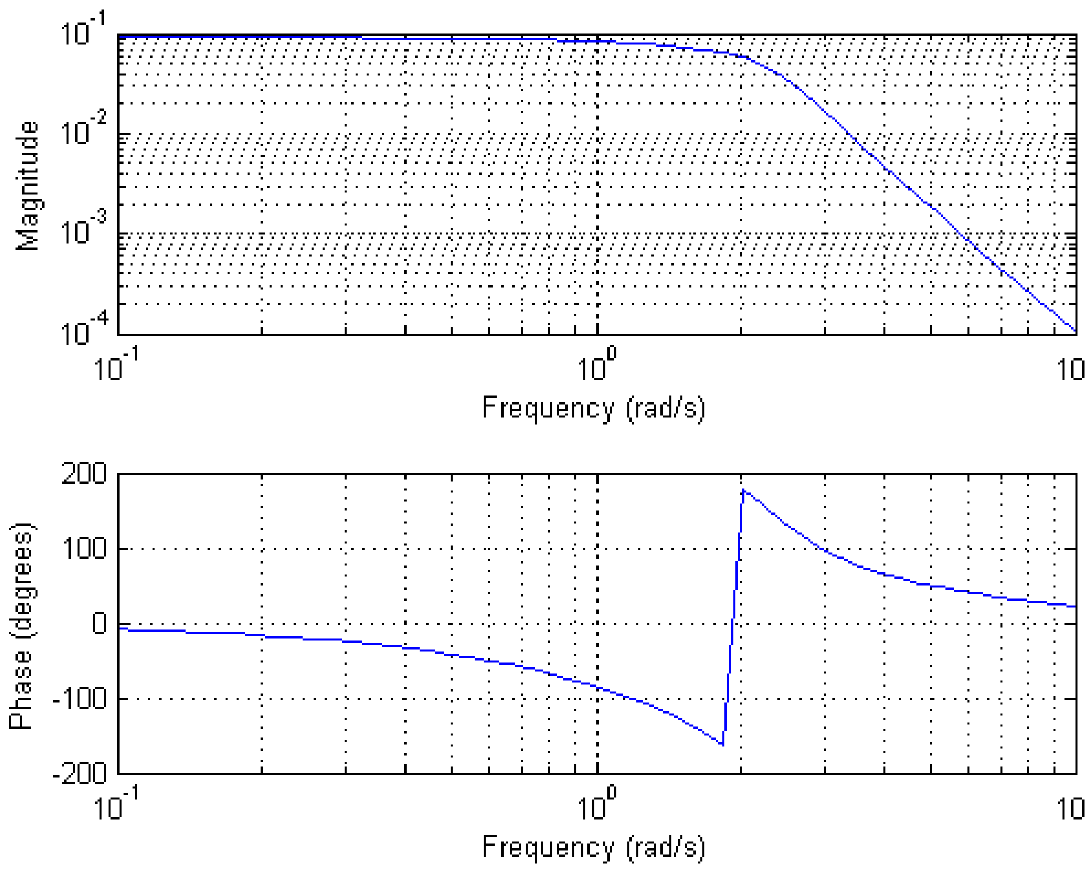
$$
Phase(H(j\omega))=-\tan^{-1}\left(\frac{16\omega-4\omega^3}{\omega^4-11\omega^2+11}\right)
$$

This is a plot of the magnitude and phase

```
EDU>> num=1;
EDU>> den=[1 4 11 16 11]
EDU>> sys=tf(num,den)

Transfer function:
               1
--------------------------------
s^4 + 4 s^3 + 11 s^2 + 16 s + 11

EDU>> w = logspace(-1,1);
EDU>> freqs(num,den,w)
```



3.3.13 Key for HW3

Key to HW#3
431, Vibration CSUF spring 2009

3- 3

3.2 Calculate the solution to

$$\ddot{x} + 2\dot{x} + 3x = \sin t + \delta(t - \pi)$$
$$x(0) = 0 \quad \dot{x}(0) = 1$$

and plot the response.

Solution: Given: $\ddot{x} + 2\dot{x} + 3x = \sin t + \delta(t - \pi)$, $x(0) = 0$, $\dot{x}(0) = 0$

$$\omega_n = \sqrt{\frac{k}{m}} = 1.732 \text{ rad/s}, \ \zeta = \frac{c}{2\sqrt{km}} = 0.5774, \ \omega_d = \omega_n\sqrt{1-\zeta^2} = 1.414 \text{ rad/s}$$

Total Solution:

$$x(t) = x_h + x_{p1} \qquad 0 < t < \pi$$
$$x(t) = x_h + x_{p1} + x_{p2} \qquad t > \pi$$

Homogeneous: Eq. (1.36)

$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) = Ae^{-t}\sin(1.414t + \phi)$$

Particular: #1 (Chapter 2)

$x_{p1}(t) = X\sin(\omega t - \theta)$, where $\omega = 1$ rad/s. Note that $f_0 = \frac{F_0}{m} = 1$

$$\Rightarrow X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} = 0.3536, \text{ and } \theta = \tan^{-1}\left[\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right] = 0.785 \text{ rad}$$

$$\Rightarrow x_{p1}(t) = 0.3536\sin(t - 0.7854)$$

Particular: #2 Equation 3.9

$$x_{p2}(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n(t-\pi)}\sin\omega_d(t-\tau) = \frac{1}{(1)(1.414)}e^{-(t-\pi)}\sin 1.414(t-\pi)$$

$$\Rightarrow x_{p2}(t) = 0.7071e^{-(t-\pi)}\sin 1.414(t-\pi)$$

The total solution for $0 < t < \pi$ becomes:

$$x(t) = Ae^{-t}\sin(1.414t + \phi) + 0.3536\sin(t - 0.7854)$$
$$\dot{x}(t) = -Ae^{-t}\sin(1.414t + \phi) + 1.414Ae^{-t}\cos(1.414t + \phi) + 0.3536\cos(t - 0.7854)$$
$$x(0) = 0 = A\sin\phi - 0.25 \Rightarrow A = \frac{0.25}{\sin\phi}$$
$$\dot{x}(0) = 1 = -A\sin\phi + 1.414A\cos\phi + 0.25 \Rightarrow 0.75 = 0.25 - 1.414(0.25)\frac{1}{\tan\phi}$$
$$\Rightarrow \phi = 0.34 \text{ and } A = 0.75$$

Thus for the first time interval, the response is

$$x(t) = 0.75e^{-t}\sin(1.414t + 0.34) + 0.3536\sin(t - 0.7854) \qquad 0 < t < \pi$$

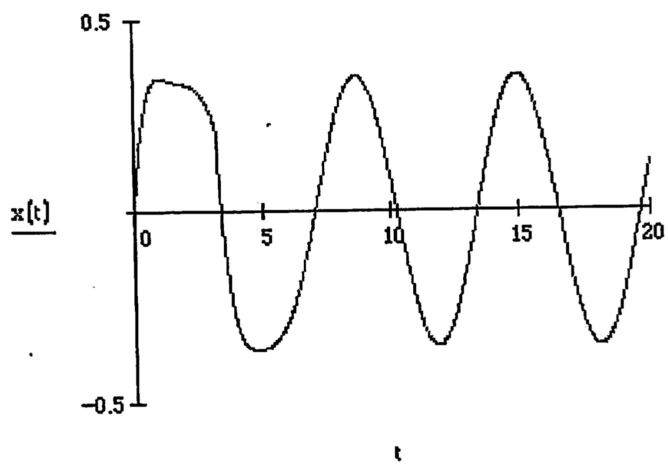
Next consider the application of the impulse at $t = \pi$:

3- 4

$x(t) = x_h + x_{p1} + x_{p2}$

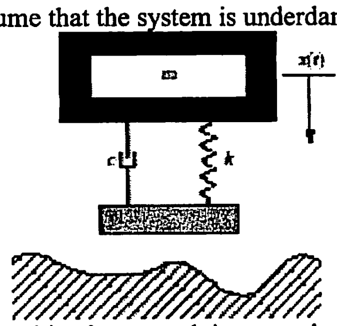
$x(t) = -0.433e^{-t}\sin(1.414t + 0.6155) + 0.3536\sin(t - 0.7854) - 0.7071e^{-(t-\pi)}\sin(1.414t - \pi) \quad t > \pi$

The response is plotted in the following (from Mathcad):



3- 10

3.8 The vibration packages dropped from a height of h meters can be approximated by considering Figure P3.8 and modeling the point of contact as an impulse applied to the system at the time of contact. Calculate the vibration of the mass m after the system falls and hits the ground. Assume that the system is underdamped.



Solution: When the system hits the ground, it responds as if an impulse force acted on it.

From Equation (3.6): $x(t) = \frac{\hat{F}e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$ where $\frac{\hat{F}}{m} = v_0$

Calculate v_0:

For falling mass: $x = \frac{1}{2}at^2$

So, $v_0 = gt^*$, where t^* is the time of impact from height h

$$h = \frac{1}{2}gt^{*2} \Rightarrow t^* = \sqrt{\frac{2h}{g}}$$

$$v_0 = \sqrt{2gh}$$

Let $t = 0$ when the end of the spring hits the ground

The response is $$x(t) = \frac{\sqrt{2gh}}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$$

Where ω_n, ω_d, and ζ are calculated from m, c, k. Of course the problem could be solved as a free response problem with $x_0 = 0$, $v_0 = \sqrt{2gh}$ or an impulse response with impact model as the unit velocity given.

3- 13

3.11 Calculate the response of the system

$$3\ddot{x}(t)+6\dot{x}(t)+12x(t)=3\delta(t)-\delta(t-1)$$

subject to the initial conditions $x(0)$ =0.01 m and $v(0)$ = 1 m/s. The units are in Newtons. Plot the response.

Solution: First compute the natural frequency and damping ratio:

$$\omega_n=\sqrt{\frac{12}{3}}=2\text{ rad/s},\ \zeta=\frac{6}{2\cdot 2\cdot 3}=0.5,\ \omega_d=2\sqrt{1-0.5^2}=1.73\text{ rad/s}$$

so that the system is underdamped. Next compute the responses to the two impulses:

$$x_1(t)=\frac{\hat{F}}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t=\frac{3}{3(1.73)}e^{-(t-1)}\sin 1.73(t-1)=0.577e^{-t}\sin 1.73t, t>0$$

$$x_2(t)=\frac{\hat{F}}{m\omega_d}e^{-\zeta\omega_n(t-1)}\sin\omega_d(t-1)=\frac{1}{3(1.73)}e^{-t}\sin 1.73t=0.193e^{-(t-1)}\sin 1.73(t-1), t>1$$

Now compute the response to the initial conditions from Equation (1.36)

$$x_h(t)=Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)$$

$$A=\sqrt{\frac{(v_0+\zeta\omega_n x_0)^2+(x_0\omega_d)^2}{\omega_d^2}},\quad \phi=\tan^{-1}\left[\frac{x_0\omega_d}{v_0+\zeta\omega_n x_0}\right]=0.071\text{ rad}$$

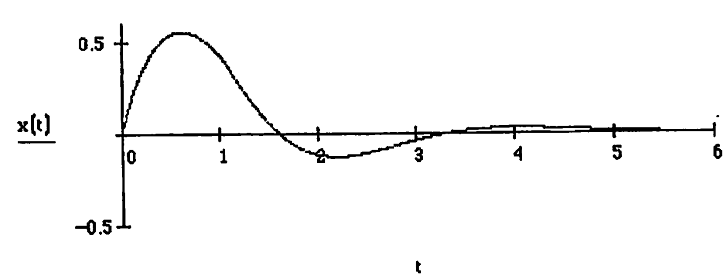
$$\Rightarrow x_h(t)=0.5775e^{-t}\sin(t+0.017)$$

Using the Heaviside function the total response is

$$x(t)=0.577e^{-t}\sin 1.73t+0.583e^{-t}\sin(t+0.017)+0.193e^{-(t-1)}\sin 1.73(t-1)\Phi(t-1)$$

This is plotted below in Mathcad:

$$x(t):=\left(\frac{e^{-\zeta\cdot\omega n\cdot t}}{\omega d}\cdot\sin(\omega d\cdot t)+A\cdot e^{-\zeta\cdot\omega n\cdot t}\cdot\sin(\omega d\cdot t+\phi)\right)+\left[\frac{e^{-\zeta\cdot\omega n\cdot(t-1)}}{-3\cdot\omega d}\cdot\sin[\omega d\cdot(t-1)]\right]\cdot\Phi(t-1)$$



Note the slight bump in the response at t = 1 when the second impact occurs.

3- 18

3.16 Calculate the response of an underdamped system to the excitation given in Figure P3.16.
Plot of a pulse input of the form $f(t) = F_0 \sin t$.

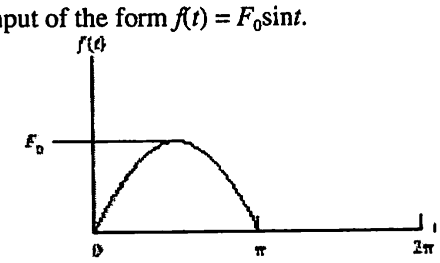


Figure P3.16

Solution:

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left[F(\tau) e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) \right] d\tau$$

$$F(t) = F_0 \sin(t) \qquad t < \pi \quad (\text{From Figure P3.16})$$

$$\text{For } t \le \pi, \qquad x(t) = \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left(\sin\tau e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) \right) d\tau$$

$$x(t) = \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \times$$

$$\left[\frac{1}{2\left[1 + 2\omega_d + \omega_n^2\right]} \left\{ e^{\zeta\omega_n t} \left[(\omega_d - 1)\sin t - \zeta\omega_n \cos t \right] - (\omega_d - 1)\sin\omega_d t - \zeta\omega_n \cos\omega_d t \right\} \right.$$
$$\left. + \frac{1}{2\left[1 + 2\omega_d + \omega_n^2\right]} \left\{ e^{\zeta\omega_n t} \left[(\omega_d - 1)\sin t - \zeta\omega_n \cos t \right] + (\omega_d - 1)\sin\omega_d t - \zeta\omega_n \cos\omega_d t \right\} \right]$$

$$\text{For } \tau > \pi, : \int_0^t f(\tau) h(t-\tau) d\tau = \int_0^\pi f(\tau) h(t-\tau) d\tau + \int_\pi^t (0) h(t-\tau) d\tau$$

3- 19

$$x(t)=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\int_0^{\pi}\left(\sin\tau e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)\right)d\tau$$

$$=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\times$$

$$\left[\frac{1}{2\left[1+2\omega_d+\omega_n^2\right]}\left\{\begin{array}{c}e^{\zeta\omega_n t}\left[(\omega_d-1)\sin\left[\omega_d(t-\pi)\right]-\zeta\omega_n\cos\left[\omega_d(t-\pi)\right]\right]\\ -(\omega_d-1)\sin\omega_d t-\zeta\omega_n\cos\omega_d t\end{array}\right\}\right.$$

$$\left.+\frac{1}{2\left[1+2\omega_d+\omega_n^2\right]}\left\{\begin{array}{c}e^{\zeta\omega_n t}\left[(\omega_d+1)\sin\left[\omega_d(t-\tau)\right]+\zeta\omega\cos\left[\omega_d(t-\pi)\right]\right]\\ +(\omega_d-1)\sin\omega_d t-\zeta\omega_n\cos\omega_d t\end{array}\right\}\right]$$

Alternately, one could take a Laplace Transform approach and assume the under-damped system is a mass-spring-damper system of the form

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F(t)$$

The forcing function given can be written as

$$F(t)=F_0\left(H(t)-H(t-\pi)\right)\sin(t)$$

Normalizing the equation of motion yields

$$\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2x(t)=f_0\left(H(t)-H(t-\pi)\right)\sin(t)$$

where $f_0=\frac{F_0}{m}$ and m, c and k are such that $0<\zeta<1$.

Assuming initial conditions, transforming the equation of motion into the Laplace domain yields

$$X(s)=\frac{f_0\left(1+e^{-\pi s}\right)}{\left(s^2+1\right)\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$$

The above expression can be converted to partial fractions

$$X(s)=f_0\left(1+e^{-\pi s}\right)\left(\frac{As+B}{s^2+1}\right)+f_0\left(1+e^{-\pi s}\right)\left(\frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$

where A, B, C, and D are found to be

3- 20

$$A = \frac{-2\zeta\omega_n}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

$$B = \frac{\omega_n^2 - 1}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

$$C = \frac{2\zeta\omega_n}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

$$D = \frac{\left(1-\omega_n^2\right) + \left(2\zeta\omega_n\right)^2}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

Notice that $X(s)$ can be written more attractively as

$$X(s) = f_0\left(\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right) + f_0 e^{-\pi s}\left(\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$
$$= f_0\left(G(s) + e^{-\pi s}G(s)\right)$$

Performing the inverse Laplace Transform yields

$$x(t) = f_0\left(g(t) + H(t-\pi)g(t-\pi)\right)$$

where $g(t)$ is given below

$$g(t) = A\cos(t) + B\sin(t) + Ce^{-\zeta\omega_n t}\cos(\omega_d t) + \left(\frac{D - C\zeta\omega_n}{\omega_d}\right)e^{-\zeta\omega_n t}\sin(\omega_d t)$$

ω_d is the damped natural frequency, $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

Let m=1 kg, c=2 kg/sec, k=3 N/m, and F_0=2 N. The system is solved numerically. Both exact and numerical solutions are plotted below

3- 21

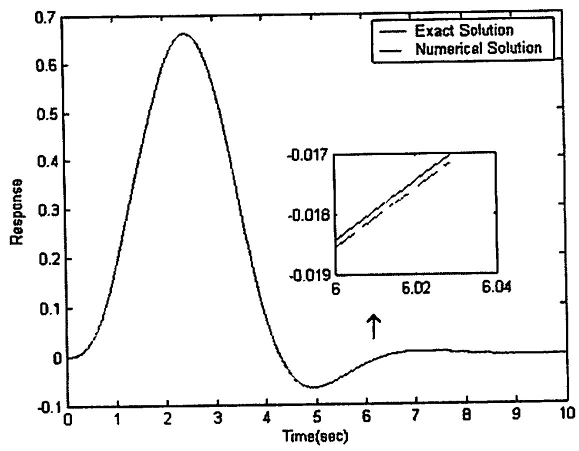


Figure 1 Analytical vs. Numerical Solutions

Below is the code used to solve this problem

```
% Establish a time vector
t=[0:0.001:10];

% Define the mass, spring stiffness and damping coefficient
m=1;
c=2;
k=3;

% Define the amplitude of the forcing function
F0=2;

% Calculate the natural frequency, damping ratio and normalized force amplitude
zeta=c/(2*sqrt(k*m));
wn=sqrt(k/m);
f0=F0/m;

% Calculate the damped natural frequency
wd=wn*sqrt(1-zeta^2);

% Below is the common denominator of A, B, C and D (partial fractions
% coefficients)
dummy=(1-wn^2)^2+(2*zeta*wn)^2;

% Hence, A, B, C, and D are given by
A=-2*zeta*wn/dummy;
B=(wn^2-1)/dummy;
C=2*zeta*wn/dummy;
```

3- 22

```
D=((1-wn^2)+(2*zeta*wn)^2)/dummy;

% EXACT SOLUTION
%
**************************************************************************
*
%
**************************************************************************
*
for i=1:length(t)
   % Start by defining the function g(t)
   g(i)=A*cos(t(i))+B*sin(t(i))+C*exp(-zeta*wn*t(i))*cos(wd*t(i))+((D-
C*zeta*wn)/wd)*exp(-zeta*wn*t(i))*sin(wd*t(i));
   % Before t=pi, the response will be only g(t)
   if t(i)<pi
      xe(i)=f0*g(i);
      % d is the index of delay that will correspond to t=pi
      d=i;
   else
      % After t=pi, the response is g(t) plus a delayed g(t). The amount
      % of delay is pi seconds, and it is d increments
      xe(i)=f0*(g(i)+g(i-d));
   end;
end;

% NUMERICAL SOLUTION
%
**************************************************************************
*
%
**************************************************************************
*

% Start by defining the forcing function
for i=1:length(t)
   if t(i)<pi
      f(i)=f0*sin(t(i));
   else
      f(i)=0;
   end;
end;

% Define the transfer functions of the system
% This is given below
%        1
% --------------------------
```

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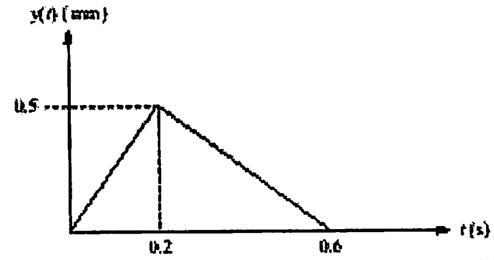
```
% s^2+2*zeta*wn+wn^2

% Define the numerator and denominator
num=[1];
den=[1 2*zeta*wn wn^2];
% Establish the transfer function
sys=tf(num,den);

% Obtain the solution using lsim
xn=lsim(sys,f,t);

% Plot the results
figure;
set(gcf,'Color','White');
plot(t,xe,t,xn,'--');
xlabel('Time(sec)');
ylabel('Response');
legend('Forcing Function','Exact Solution','Numerical Solution');
text(6,0.05,'\uparrow','FontSize',18);
axes('Position',[0.55 0.3/0.8 0.25 0.25])
plot(t(6001:6030),xe(6001:6030),t(6001:6030),xn(6001:6030),'--');
```

3- 30

3. 21 A machine resting on an elastic support can be modeled as a single-degree-of-freedom, spring-mass system arranged in the vertical direction. The ground is subject to a motion $y(t)$ of the form illustrated in Figure P3.221. The machine has a mass of 5000 kg and the support has stiffness 1.5×10^3 N/m. Calculate the resulting vibration of the machine.



Solution: Given $m = 5000$ kg, $k = 1.5 \times 10^3$ N/m, $\omega_n = \sqrt{k/m} = 0.548$ rad/s and that the ground motion is given by:

$$y(t) = \begin{cases} 2.5t & 0 \le t \le 0.2 \\ 0.75 - 1.25t & 0.2 \le t \le 0.6 \\ 0 & t \ge 0.6 \end{cases}$$

The equation of motion is $m\ddot{x} + k(x - y) = 0$ or $m\ddot{x} + kx = ky = F(t)$ The impulse response function computed from equation (3.12) for an undamped system is

$$h(t-\tau) = \frac{1}{m\omega_n}\sin\omega_n(t-\tau)$$

This gives the solution by integrating a yh across each time step:

$$x(t) = \frac{1}{m\omega_n}\int_0^t ky(\tau)\sin\omega_n(t-\tau)d\tau = \omega_n\int_0^t y(\tau)\sin\omega_n(t-\tau)d\tau$$

For the interval $0 \le t \le 0.2$:

$$x(t) = \omega_n\int_0^t 2.5\tau\sin\omega_n(t-\tau)d\tau$$

$$\Rightarrow \underline{x(t) = 2.5t - 4.56\sin 0.548t \text{ mm} \quad 0 \le t \le 0.2}$$

For the interval $0.2 \le t \le 0.6$:

$$x(t) = \omega_n\int_0^{0.2} 2.5\tau\sin\omega_n(t-\tau)d\tau + \omega_n\int_{0.2}^t (0.75 - 1.25\tau)\sin\omega_n(t-\tau)d\tau$$

$$= 0.75 - 0.5\cos 0.548(t-0.2) - 1.25t + 2.28\sin 0.548(t-0.2)$$

Combining this with the solution from the first interval yields:

$$\underline{x(t) = 0.75 + 1.25t - 0.5\cos 0.548(t-0.2)}$$

$$\underline{+6.48\sin 0.548(t-0.2) - 4.56\sin 0.548(t-0.2) \text{ mm} \quad 0.2 \le t \le 0.6}$$

Finally for the interval $t \ge 0.6$:

3- 31

$$x(t) = \omega_n \int_0^{0.2} 2.5t \sin \omega_n (t-\tau) d\tau + \omega_n \int_{0.2}^{0.6} (0.75 - 1.25t) \sin \omega_n (t-\tau) d\tau + \omega_n \int_0^t (0) \sin \omega_n (t-\tau) d\tau$$
$$= -0.5 \cos 0.548(t-0.2) - 2.28 \sin 0.548(t-0.6) + 2.28 \sin 0.548(t-0.2)$$

Combining this with the total solution from the previous time interval yields:

$$x(t) = -0.5 \cos 0.548(t-0.2) + 6.84 \sin 0.548(t-0.2) - 2.28 \sin 0.548(t-0.6) - 4.56 \sin 0.548t \quad \text{mm } t \geq 0.6$$

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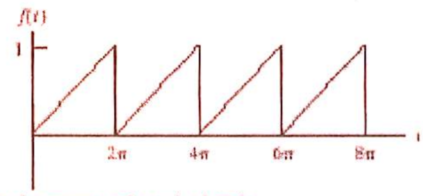
3.29 Determine the Fourier series representation of the sawtooth curve illustrated in Figure P3.29.



Solution: The sawtooth curve of period T is

$$F(t)=\frac{1}{2\pi}t \qquad 0\le t\le 2\pi$$

Determine coefficients a_0, a_n, b_n :

$$a_0=\frac{2}{T}\int_0^T F(t)dt=\frac{2}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)dt=\left(\frac{1}{2\pi^2}\right)\frac{1}{2}t^2\Big|_0^{2\pi}$$
$$=\frac{1}{4\pi^2}\left[4\pi^2-0\right]=1$$

$$a_n=\frac{2}{T}\int_0^T F(t)\cos n\omega_T t dt,\text{ where } \omega_T=\frac{2\pi}{T}=\frac{2\pi}{2\pi}=1$$
$$=\frac{2}{2\pi}\left[\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)\cos nt dt\right]=\frac{1}{2\pi^2}\left[\int_0^{2\pi} t\cos nt dt\right]$$
$$=\frac{1}{2\pi^2}\left[\frac{1}{n^2}\cos nt+\frac{1}{n}t\sin nt\right]_0^{2\pi}=\frac{1}{2\pi^2}\left[\frac{1}{n^2}(1-1)+\frac{1}{n}(0-0)\right]=0$$

$$b_n=\frac{2}{T}\int_0^T F(t)\sin n\omega_T t dt=\frac{2}{2\pi}\left[\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)\sin nt dt\right]=\frac{1}{2\pi^2}\left[\int_0^{2\pi} t\sin nt dt\right]$$
$$=\frac{1}{2\pi^2}\left[\frac{1}{n^2}\sin nt-\frac{1}{n}t\cos nt\right]_0^{2\pi}=\frac{1}{2\pi^2}\left[\frac{1}{n^2}(0-0)-\frac{1}{n}(2\pi-0)\right]$$
$$=\frac{1}{2\pi^2}\left(\frac{-2\pi}{n}\right)=\frac{-1}{\pi n}$$

Fourier Series

$$F(t)=\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{-1}{\pi n}\right)\sin nt$$
$$F(t)=\frac{1}{2}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin nt$$

3.38 Solve the following system for the response $x(t)$ using Laplace transforms:

$$100\ddot{x}(t) + 2000x(t) = 50\delta(t)$$

where the units are in Newtons and the initial conditions are both zero.

Solution:

First divide by the mass to get

$$\ddot{x} + 20x(t) = 0.5\delta(t)$$

Take the Laplace Transform to get

$$(s^2 + 20)X(s) = 0.5$$

So

$$X(s) = \frac{0.5}{s^2 + 20}$$

Taking the inverse Laplace Transform using entry 5 of Table 3.1 yields

$$X(s) = \frac{0.5}{\sqrt{20}} \cdot \frac{\omega}{s^2 + \omega^2} \quad \text{where } \omega = \sqrt{20}$$

$$\Rightarrow x(t) = \frac{1}{4\sqrt{5}} \sin\sqrt{20}t$$

3- 59

3.44 Calculate the response spectrum of an undamped system to the forcing function

$$F(t)=\begin{cases} F_0 \sin\dfrac{\pi t}{t_1} & 0 \le t \le t_1 \\ 0 & t > t_1 \end{cases}$$

assuming the initial conditions are zero.

Solution: Let $\omega = \pi / t_1$. The solution is the homogeneous solution $x_h(t)$ and the particular solution $x_p(t)$ or $x(t) = x_h(t) + x_p(t)$. Thus

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \left(\frac{F_0}{k - m\omega^2}\right)\sin\omega t$$

where A and B are constants and ω_n is the natural frequency of the system:
Using the initial conditions $x(0) = \dot{x}(0) = 0$ the constants A and B are

$$A = 0, \quad B = \frac{-F_0\omega}{\omega_n\left(k - m\omega^2\right)}$$

so that $x(t) = \dfrac{F_0 / k}{1-(\omega/\omega_n)^2}\left\{\sin\omega t - \dfrac{\omega}{\omega_n}\sin\omega_n t\right\}, \ 0 \le t \le t_1$

Which can be written as (where $\delta = F_0 / k$ the static deflection)

$$\frac{x(t)}{\delta} = \frac{1}{1-\left(\dfrac{\tau}{2t_1}\right)^2}\left\{\sin\frac{\pi t}{t_1} - \frac{\tau}{2t_1}\sin\frac{2\pi t}{\tau}\right\}, \ 0 \le t \le t_1$$

and where $\tau = 2\pi / \omega_n$. After t_1 the solution is a free response

$$x(t) = A'\cos\omega_n t + B'\sin\omega_n t, \ t > t_1$$

where the constants A' and B' can be found by using the values of $x(t = t_1)$ and $\dot{x}(t = t_1), t > t_0$. This gives

$$x(t = t_1) = a\left[-\frac{\tau}{2t_1}\sin\frac{2\pi t_1}{\tau}\right] = A'\cos\omega_n t_1 + B'\sin\omega_n t_1$$

$$\dot{x}(t = t_1) = a\left\{-\frac{\pi}{t_1} - \frac{\pi}{t_1}\cos\frac{2\pi t_1}{\tau}\right\} = -\omega_n A'\sin\omega_n t_1 + \omega_n B'\cos\omega_n t$$

where

$$a = \frac{\delta}{1-\left(\dfrac{\tau}{2t_1}\right)^2}$$

These are solved to yield

3- 60

$$A' = \frac{a\pi}{\omega_n t_1}\sin\omega_n t_1, \quad B' = -\frac{a\pi}{\omega_n t_1}\left[1+\cos\omega_n t_1\right]$$

So that after t_1 the solution is

$$\frac{x(t)}{\delta} = \frac{(\tau/t_1)}{2\left\{1-(\tau/2t_1)^2\right\}}\left[\sin 2\pi\left(\frac{t_1}{\tau}-\frac{t}{\tau}\right)-\sin 2\pi\frac{t}{\tau}\right], t \geq t_1$$

3- 66

3.50 Calculate the frequency response function for the compliance of Problem 3.49.

Solution: From problem 3.49,

$$H(s) = \frac{1}{as^4 + bs^3 + cs^2 + ds + e}$$

Substitute $s = j\omega$ to get the frequency response function:

$$H(j\omega) = \frac{1}{a(j\omega)^4 + b(j\omega)^3 + c(j\omega)^2 + d(j\omega) + e}$$

$$H(j\omega) = \frac{a\omega^4 - c\omega^2 + e - j(-b\omega^3 + d\omega)}{(a\omega^4 - c\omega^2 + e)^2 + (-b\omega^3 + d\omega)^2}$$

3.49 Calculate the compliance transfer function for a system described by

$$a\ddot{x} + b\ddot{x} + c\ddot{x} + d\dot{x} + ex = f(t)$$

where $f(t)$ is the input force and $x(t)$ is a displacement.

Solution:

The compliance transfer function is $\frac{X(s)}{F(s)}$.

Taking the Laplace Transform yields

$$(as^4 + bs^3 + cs^2 + ds + e)X(s) = F(s)$$

So, $\frac{X(s)}{F(s)} = \frac{1}{as^4 + bs^3 + cs^2 + ds + e}$