HW3, EGME 431 (Mechanical Vibration)

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1 Problem 3.2

Problem

Calculate the solution to $\ddot{x}+2\dot{x}+3x=\sin t+\delta(t-\pi)$ with IC $x(0)=0,\dot{x}(0)=1$ and plot the solution.

Answer

$$\ddot{x}+2\dot{x}+3x=\sin t+\delta(t-\pi)$$
$$\ddot{x}+2\xi\omega_n\dot{x}+\omega_n^2 x=\sin t+\delta(t-\pi)$$

Hence $\boxed{\omega_n=\sqrt{3}}$ and $2\xi\omega_n=2$, hence $\boxed{\xi=\frac{1}{\sqrt{3}}}=0.57735$, hence this is $\boxed{\text{underdamped system}}$.

Since $x=x_h+x_p$, then

$$x_h=e^{-\xi\omega_n t}(A\cos\omega_d t+B\sin\omega_d t)$$

We have 2 particular solutions. The first x_{p_1} is due to $\sin t$ and the second x_{p_2} is due to $\delta(t-\pi)$. When the forcing function is $\sin t$, we guess

$$x_{p_1}=c_1\cos t+c_2\sin t$$

and when the forcing function is $\delta(t-\pi)$ the response is

$$x_{p_2}=\frac{1}{\omega_d m}e^{-\xi\omega_n(t-\pi)}\sin\omega_d(t-\pi)\Phi(t-\pi)$$

From x_{p_1} we find $\dot{x}_{p_1}$ and $\ddot{x}_{p_1}$ and plug these into $\ddot{x}+2\dot{x}+3x=\sin t$ to find c_1 and c_2, next we find A,B by using the IC, and then at the end we add the solution x_{p_2}. Notice that x_{p_2} do not enter into the calculation of A,B since the impulse $\delta(t-\pi)$ is not effective at $t=0$.

$$\dot{x}_{p_1}=-c_1\sin t+c_2\cos t$$
$$\ddot{x}_{p_1}=-c_1\cos t-c_2\sin t$$

Hence

$$\ddot{x}_{p_1}+2\dot{x}_{p_1}+3x_{p_1}=\sin t$$
$$(-c_1\cos t-c_2\sin t)+2(-c_1\sin t+c_2\cos t)+3(c_1\cos t+c_2\sin t)=\sin t$$
$$\sin t(-c_2-2c_1+3c_2)+\cos t(-c_1+2c_2+3c_1)=\sin t$$

Hence $(-2c_1+2c_2)=1$ and $(2c_2+2c_1)=0$. This results in

$$c_1=-\frac{1}{4}$$
$$c_2=\frac{1}{4}$$

Hence

$$\boxed{x_{p_1}=-\tfrac{1}{4}\cos t+\tfrac{1}{4}\sin t}$$

Therefore

$$x_h + x_{p_1} = e^{-\xi\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) - \frac{1}{4}\cos t + \frac{1}{4}\sin t$$

Now we use IC's to find A, B. At $t = 0$ we obtain

$$\boxed{A = \tfrac{1}{4}}$$

And

$$\dot{x}_h + \dot{x}_{p_1} = -\xi\omega_n e^{-\xi\omega_n t}\left(\frac{1}{4}\cos\omega_d t + B\sin\omega_d t\right) + e^{-\xi\omega_n t}\left(-\frac{1}{4}\omega_d\sin\omega_d t + \omega_d B\cos\omega_d t\right) + \frac{1}{4}\sin t + \frac{1}{4}\cos t$$

At $t = 0$ we have

$$1 = -\xi\omega_n\left(\frac{1}{4}\right) + (\omega_d B) + \frac{1}{4}$$

$$B = \frac{\left(1 + \frac{\xi\omega_n}{4} - \frac{1}{4}\right)}{\omega_d}$$

But $\omega_d = \omega_n\sqrt{1-\xi^2} = \sqrt{3}\sqrt{1-\left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{3}\sqrt{\frac{2}{3}}$, Hence $\boxed{\omega_d = \sqrt{2}}$ then the above becomes

$$\boxed{B = \tfrac{1}{\sqrt{2}}}$$

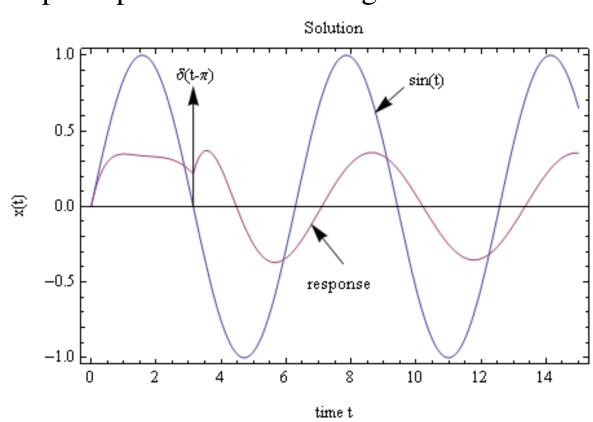
Hence the final solution is

$$\begin{aligned} x(t) &= x_h + x_{p_1} + x_{p_2} \\ &= e^{-\xi\omega_n t}\left(\frac{1}{4}\cos\omega_d t + \frac{1}{\sqrt{2}}\sin\omega_d t\right) - \frac{1}{4}\cos t + \frac{1}{4}\sin t + \frac{1}{\omega_d m}e^{-\xi\omega_n(t-\pi)}\sin\omega_d(t-\pi)\Phi(t-\pi) \end{aligned}$$

Substitute values for the parameters above we obtain

$$\boxed{x(t) = e^{-t}\left(\tfrac{1}{4}\cos\sqrt{2}t + \tfrac{1}{\sqrt{2}}\sin\sqrt{2}t\right) - \tfrac{1}{4}\cos t + \tfrac{1}{4}\sin t + \tfrac{1}{\sqrt{2}}e^{-(t-\pi)}\sin\sqrt{2}(t-\pi)\Phi(t-\pi)}$$

This is a plot of the solution superimposed on the forcing functions



2 Problem 3.8

hit by the cam?

3.8. The vibration of packages dropped from a height of h meters can be approximated by considering Figure P3.8 and modeling the point of contact as an impulse applied to the system at the time of contact. Calculate the vibration of the mass m after the system falls and hits the ground. Assume that the system is underdamped.

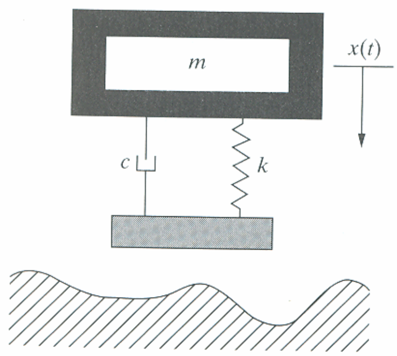


Figure P3.8 Vibration model of a package being dropped onto the ground.

3.9 Calcul...

The magnitude of the impulse resulting when the mass hits the ground is given by the change of momentum that occurs. Hence

$$\hat{F} = Ft = m(v_{final} - v_0)$$

But assuming the mass is dropped from rest, hence $v_0 = 0$, and $v_{final} = gt$ where $t = \sqrt{2\frac{h}{g}}$ where h is the height that mass falls. Hence

$$\begin{aligned} \hat{F} &= mv_{final} \\ &= m\sqrt{2gh} \end{aligned}$$

Hence the equation of motion is

$$\boxed{m\ddot{x}(t) + c\dot{x}(t) + kx(t) = m\sqrt{2gh}\delta(t)}$$

Since underdamped, $x(t) = h(t) = \frac{\hat{F}}{m\omega_d} e^{-\xi\omega_n t} \sin \omega_d t$, hence the solution is

$$\begin{aligned} x(t) &= \frac{m\sqrt{2gh}}{m\omega_d} e^{-\xi\omega_n t} \sin \omega_d t \\ &= \frac{\sqrt{2gh}}{\omega_d} e^{-\xi\omega_n t} \sin \omega_d t \end{aligned}$$

Taking $t = 0$ as time of impact.

3 Problem 3.11

Problem

Compute response of the system $3\ddot{x}(t)+6\dot{x}(t)+12x(t)=3\delta(t)-\delta(t-1)$ with IC $x(0)=0.01m$ and $v(0)=1m/s$. Plot the response.

Answer

$$3\ddot{x}(t)+6\dot{x}(t)+12x(t)=3\delta(t)-\delta(t-1)$$
$$\ddot{x}(t)+2\dot{x}(t)+4x(t)=\delta(t)-\frac{1}{3}\delta(t-1)$$
$$\ddot{x}+2\xi\omega_n\dot{x}+\omega_n^2 x=\delta(t)-\frac{1}{3}\delta(t-1)$$

Where $m=1,\omega_n^2=4$, hence $\boxed{\omega_n=2}$ and $2\xi\omega_n=2$, hence $\xi=\frac{1}{2}$. $\boxed{\text{This is an underdamped system}}$.

$\omega_d=\omega_n\sqrt{1-\xi^2}=2\sqrt{1-\left(\frac{1}{2}\right)^2}=2\sqrt{\frac{3}{4}}$, Hence $\boxed{\omega_d=\sqrt{3}}$

$$x_h=e^{-\xi\omega_n t}\left(A\cos\omega_d t+B\sin\omega_d t\right)$$

The response due to the forcing function $\delta(t)$ is given by

$$x_{p_1}(t)=\frac{1}{\omega_d m}e^{-\xi\omega_n t}\sin(\omega_d t)$$

The response due to the other forcing function $\delta(t-1)$ is given by

$$x_{p_2}(t)=-\frac{1}{3}\frac{1}{\omega_d m}e^{-\xi\omega_n(t-1)}\sin\omega_d(t-1)\Phi(t-1)$$

Now we determine A,B from IC's

$$\begin{aligned}x_h(0)+x_{p_1}(0)&=0.01\\&=\left[e^{-\xi\omega_n t}\left(A\cos\omega_d t+B\sin\omega_d t\right)+\frac{1}{\omega_d m}e^{-\xi\omega_n t}\sin(\omega_d t)\right]_{t=0}\end{aligned}$$

Hence $\boxed{A=0.01}$ Now to find B

$$\begin{aligned}\dot{x}_h(t)+\dot{x}_{p_1}(t)=&-\xi\omega_n e^{-\xi\omega_n t}\left(A\cos\omega_d t+B\sin\omega_d t\right)+e^{-\xi\omega_n t}\left(-A\omega_d\sin\omega_d t+B\omega_d\cos\omega_d t\right)\\&+\frac{-\xi\omega_n}{\omega_d m}e^{-\xi\omega_n t}\sin(\omega_d t)+\frac{\omega_d}{\omega_d m}e^{-\xi\omega_n t}\cos(\omega_d t)\end{aligned}$$

But $\dot{x}_h(0)+\dot{x}_{p_1}(0)=1$, hence from the above, and noting that $m=1$

$$\begin{aligned}1&=-A\xi\omega_n+B\omega_d+1\\B&=\frac{A\xi\omega_n}{\omega_d}\\&=\frac{0.01\left(\frac{1}{2}\right)2}{\sqrt{3}}\end{aligned}$$

Hence

$$\boxed{B = \frac{1}{100\sqrt{3}}}$$

Therefore

$$\boxed{x_h = e^{-\xi\omega_n t}\left(\frac{1}{100}\cos\omega_d t + \frac{1}{100\sqrt{3}}\sin\omega_d t\right)}$$

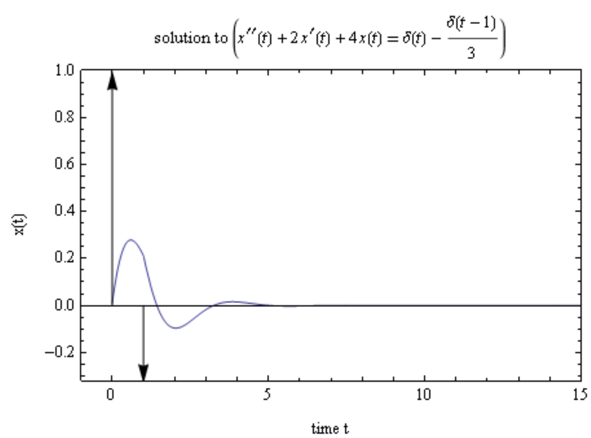
Now we can combine the above solution to obtain the final solution

$$\begin{aligned}
x(t) &= x(h) + x_{p_1}(t) + x_{p_2}(t) \\
&= e^{-\xi\omega_n t}\left(\frac{1}{100}\cos\omega_d t + \frac{1}{100\sqrt{3}}\sin\omega_d t\right) \\
&+ \frac{1}{\omega_d m} e^{-\xi\omega_n t}\sin(\omega_d t) \\
&- \frac{1}{3}\frac{1}{\omega_d m} e^{-\xi\omega_n (t-1)}\sin\omega_d (t-1)\,\Phi(t-1)
\end{aligned}$$

Substitute numerical values for the above parameters, we obtain

$$\boxed{x(t) = \frac{e^{-t}}{100}\left(\cos\sqrt{3}t + \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right) + \frac{1}{\sqrt{3}}e^{-t}\sin\left(\sqrt{3}t\right) - \frac{1}{3}\frac{1}{\sqrt{3}}e^{-(t-1)}\sin\left(\sqrt{3}(t-1)\right)\Phi(t-1)}$$

This is a plot of the response



4 Problem 3.16

3.16. Calculate the response of an underdamped system to the excitation given in Figure P3.16.

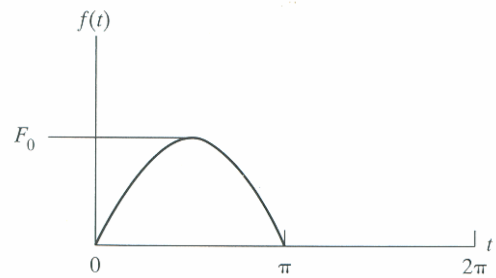


Figure P3.16 Plot of a pulse input of the form $f(t) = F_0 \sin t$.

Let the response by $x(t)$. Hence $x(t) = x_h(t) + x_p(t)$, where $x_p(t)$ is the particular solution, which is the response due the the above forcing function. Using convolution

$$x_p(t) = \int_0^t f(\tau)\, h(t-\tau)\, d\tau$$

Where $h(t)$ is the unit impulse response of a second order underdamped system which is

$$h(t) = \frac{1}{m\omega_d} e^{-\xi \omega_n t} \sin \omega_d t$$

hence

$$\begin{aligned} x_p(t) &= \frac{F_0}{m\omega_d} \int_0^t \sin(\tau)\, e^{-\xi\omega_n(t-\tau)} \sin(\omega_d(t-\tau))\, d\tau \\ &= \frac{F_0 e^{-\xi\omega_n t}}{m\omega_d} \int_0^t e^{\xi\omega_n \tau} \sin(\tau) \sin(\omega_d(t-\tau))\, d\tau \end{aligned}$$

Using $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ then

$$\sin(\tau)\sin(\omega_d(t-\tau)) = \frac{1}{2}[\cos(\tau - \omega_d(t-\tau)) - \cos(\tau + \omega_d(t-\tau))]$$

Then the integral becomes

$$x_p(t) = \frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d} \left(\int_0^t e^{\xi\omega_n\tau} \cos(\tau - \omega_d(t-\tau))\, d\tau - \int_0^t e^{\xi\omega_n\tau} \cos(\tau + \omega_d(t-\tau))\, d\tau \right)$$

Consider the first integral I_1 where

$$I_1 = \int_0^t e^{\xi\omega_n\tau}\cos\left(\tau - \omega_d\left(t-\tau\right)\right)d\tau$$

Integrate by parts, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n\tau} \rightarrow v = \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}$ and let $u = \cos\left(\tau - \omega_d\left(t-\tau\right)\right) \rightarrow du = -\left(1+\omega_d\right)\sin\left(\tau - \omega_d\left(t-\tau\right)\right)$, hence

$$\begin{aligned}
I_1 = &\left[\cos\left(\tau - \omega_d\left(t-\tau\right)\right)\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\right]_0^t \\
&- \int_0^t \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\left[-\left(1+\omega_d\right)\sin\left(\tau - \omega_d\left(t-\tau\right)\right)\right]d\tau \\
= &\left[\cos\left(t - \omega_d\left(t-t\right)\right)\frac{e^{\xi\omega_n t}}{\xi\omega_n} - \cos\left(0 - \omega_d\left(t-0\right)\right)\frac{1}{\xi\omega_n}\right] \\
&+ \frac{\left(1+\omega_d\right)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\sin\left(\tau - \omega_d\left(t-\tau\right)\right)d\tau \\
= &\frac{1}{\xi\omega_n}\left[\cos\left(t\right)e^{\xi\omega_n t} - \cos\left(\omega_d t\right)\right] + \frac{\left(1+\omega_d\right)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\sin\left(\tau - \omega_d\left(t-\tau\right)\right)d\tau \qquad (1)
\end{aligned}$$

Integrate by parts again the last integral above, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n\tau} \rightarrow v = \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}$ and let $u = \sin\left(\tau - \omega_d\left(t-\tau\right)\right) \rightarrow du = \left(1+\omega_d\right)\cos\left(\tau - \omega_d\left(t-\tau\right)\right)$, hence

$$\begin{aligned}
\int_0^t e^{\xi\omega_n\tau}\sin\left(\tau - \omega_d\left(t-\tau\right)\right)d\tau = &\left[\sin\left(\tau - \omega_d\left(t-\tau\right)\right)\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\right]_0^t \\
&- \int_0^t \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\left(1+\omega_d\right)\cos\left(\tau - \omega_d\left(t-\tau\right)\right)d\tau \\
= &\frac{1}{\xi\omega_n}\left[\sin\left(t\right)e^{\xi\omega_n t} + \sin\left(\omega_d t\right)\right] - \\
&\frac{\left(1+\omega_d\right)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\cos\left(\tau - \omega_d\left(t-\tau\right)\right)d\tau \qquad (2)
\end{aligned}$$

Substitute (2) into (1) we obtain

$$I_1 = \frac{1}{\xi\omega_n}\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{\xi\omega_n}\left(\frac{1}{\xi\omega_n}\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right] - \frac{(1+\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\cos(\tau - \omega_d(t-\tau))\,d\tau\right)$$

$$= \frac{1}{\xi\omega_n}\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right] - \frac{(1+\omega_d)^2}{(\xi\omega_n)^2}\int_0^t e^{\xi\omega_n\tau}\cos(\tau - \omega_d(t-\tau))\,d\tau$$

$$= \frac{1}{\xi\omega_n}\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right] - \frac{(1+\omega_d)^2}{(\xi\omega_n)^2}I_1$$

Hence

$$I_1 + \frac{(1+\omega_d)^2}{(\xi\omega_n)^2}I_1 = \frac{1}{\xi\omega_n}\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right]$$

$$I_1\left(\frac{(\xi\omega_n)^2 + (1+\omega_d)^2}{(\xi\omega_n)^2}\right) = \frac{1}{\xi\omega_n}\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right]$$

$$I_1 = \left(\frac{(\xi\omega_n)^2}{(\xi\omega_n)^2 + (1+\omega_d)^2}\right)\left(\frac{1}{\xi\omega_n}\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right]\right)$$

$$= \frac{\xi\omega_n\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1+\omega_d)\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1+\omega_d)^2}$$

Now consider the second integral I_2 where

$$I_2 = \int_0^t e^{\xi\omega_n\tau}\cos(\tau + \omega_d(t-\tau))\,d\tau$$

Integrate by parts, where $\int u\,dv = uv - \int v\,du$, Let $dv = e^{\xi\omega_n\tau} \rightarrow v = \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}$ and let $u = \cos(\tau + \omega_d(t-\tau)) \rightarrow$

$du = -(1-\omega_d)\sin(\tau+\omega_d(t-\tau))$, hence

$$I_2 = \left[\cos(\tau+\omega_d(t-\tau))\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\right]_0^t - \int_0^t \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\left[-(1-\omega_d)\sin(\tau+\omega_d(t-\tau))\right]d\tau$$

$$= \left[\cos(t+\omega_d(t-t))\frac{e^{\xi\omega_n t}}{\xi\omega_n} - \cos(0+\omega_d(t-0))\frac{1}{\xi\omega_n}\right] + \frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\sin(\tau+\omega_d(t-\tau))\,d\tau$$

$$= \frac{1}{\xi\omega_n}\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\sin(\tau+\omega_d(t-\tau))\,d\tau \tag{3}$$

Integrate by parts again the last integral above, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n\tau} \to v = \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}$ and let $u = \sin(\tau+\omega_d(t-\tau)) \to du = (1-\omega_d)\cos(\tau+\omega_d(t-\tau))$, hence

$$\int_0^t e^{\xi\omega_n\tau}\sin(\tau+\omega_d(t-\tau))\,d\tau = \left[\sin(\tau+\omega_d(t-\tau))\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\right]_0^t - \int_0^t \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}(1-\omega_d)\cos(\tau+\omega_d(t-\tau))\,d\tau$$
$$= \frac{1}{\xi\omega_n}\left[\sin(t)e^{\xi\omega_n t} - \sin(\omega_d t)\right] - \frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\cos(\tau+\omega_d(t-\tau))\,d\tau \tag{4}$$

Substitute (4) into (3) we obtain

$$I_2 = \frac{1}{\xi\omega_n}\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{\xi\omega_n}\left(\frac{1}{\xi\omega_n}\left[\sin(t)e^{\xi\omega_n t} - \sin(\omega_d t)\right] - \frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\cos(\tau+\omega_d(t-\tau))\,d\tau\right)$$

$$= \frac{1}{\xi\omega_n}\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)e^{\xi\omega_n t} - \sin(\omega_d t)\right] - \frac{(1-\omega_d)^2}{(\xi\omega_n)^2}\int_0^t e^{\xi\omega_n\tau}\cos(\tau+\omega_d(t-\tau))\,d\tau$$

$$= \frac{1}{\xi\omega_n}\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right] - \frac{(1-\omega_d)^2}{(\xi\omega_n)^2}I_2$$

Hence

$$I_2 + \frac{(1-\omega_d)^2}{(\xi\omega_n)^2} I_2 = \frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right]$$

$$I_2\left(\frac{(\xi\omega_n)^2 + (1-\omega_d)^2}{(\xi\omega_n)^2}\right) = \frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right]$$

$$I_2 = \left(\frac{(\xi\omega_n)^2}{(\xi\omega_n)^2 + (1-\omega_d)^2}\right)$$
$$\left(\frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right]\right)$$

$$= \frac{\xi\omega_n\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1-\omega_d)\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1-\omega_d)^2}$$

Using the above expressions for I_1, I_2, we find (and multiplying the solution by $(\Phi(t) - \Phi(t-\pi))$ since the force is only active from $t = 0$ to $t = \pi$, we obtain

$$x_p(t) = \frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}(I_1 - I_2)(\Phi(t) - \Phi(t-\pi))$$

$$\begin{aligned} &= (\Phi(t) - \Phi(t-\pi)) * \\ &\frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\frac{\xi\omega_n\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1+\omega_d)\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1+\omega_d)^2} \\ &- \frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\frac{\xi\omega_n\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1-\omega_d)\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1-\omega_d)^2} \end{aligned} \tag{5}$$

Hence $x_p(t) = (\Phi(t) - \Phi(t-\pi))$
$\left[\frac{F_0}{2m\omega_d} e^{-\xi\omega_n t}\left(\frac{\xi\omega_n\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1+\omega_d)\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1+\omega_d)^2} - \frac{\xi\omega_n\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1-\omega_d)\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1-\omega_d)^2}\right.\right.$

But

$$\begin{aligned} (\xi\omega_n)^2 + (1+\omega_d)^2 &= \xi^2\omega_n^2 + 1 + \omega_d^2 + 2\omega_d \\ &= \xi^2\omega_n^2 + 1 + \omega_n^2\left(1-\xi^2\right) + 2\omega_d \\ &= 1 + 2\omega_d + \omega_n^2 \end{aligned}$$

and

$$(\xi\omega_n)^2 + (1-\omega_d)^2 = 1 - 2\omega_d + \omega_n^2$$

Hence $x_p(t)$ can now be written as

$$\begin{aligned} x_p(t) = &\frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\frac{\xi\omega_n\cos(t)\, e^{\xi\omega_n t} - \xi\omega_n\cos(\omega_d t) + (1+\omega_d)\sin(t)\, e^{\xi\omega_n t} + (1+\omega_d)\sin(\omega_d t)}{1 + 2\omega_d + \omega_n^2} \\ &- \frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\frac{\xi\omega_n\cos(t)\, e^{\xi\omega_n t} - \xi\omega_n\cos(\omega_d t) + (1-\omega_d)\sin(t)\, e^{\xi\omega_n t} + (1-\omega_d)\sin(\omega_d t)}{1 - 2\omega_d + \omega_n^2} \end{aligned}$$

And

$$x_h(t) = e^{-\xi\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right)$$

Hence the overall solution is

$$x(t) = e^{-\xi\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + x_p(t)$$

The above solution is a bit long due to integration by parts. I will not solve the same problem using Laplace transformation method. The differential equation is

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2 x(t) = f(t)$$

Take Laplace transform, we obtain (assuming $x(0) = x_0$ and $\dot{x}(0) = v_0$)

$$\begin{aligned}\left(s^2X - sx(0) - \dot{x}(0)\right) + 2\xi\omega_n\left(sX - x(0)\right) + \omega_n^2 X &= F(s)\\ \left(s^2X - sx_0 - v_0\right) + 2\xi\omega_n\left(sX - x_0\right) + \omega_n^2 X &= F(s) \qquad (7)\end{aligned}$$

Now we find Laplace transform of $f(t)$

$$\begin{aligned}F(s) &= \int_0^\infty e^{-st} f(t)\,dt\\ &= \int_0^\pi e^{-st} F_0 \sin t\ dt\\ &= F_0\left[\int_0^\pi e^{-st}\sin t\ dt\right]\end{aligned}$$

Integration by parts gives

$$F(s) = F_0\left[\frac{1+e^{-\pi s}}{1+s^2}\right] \qquad (8)$$

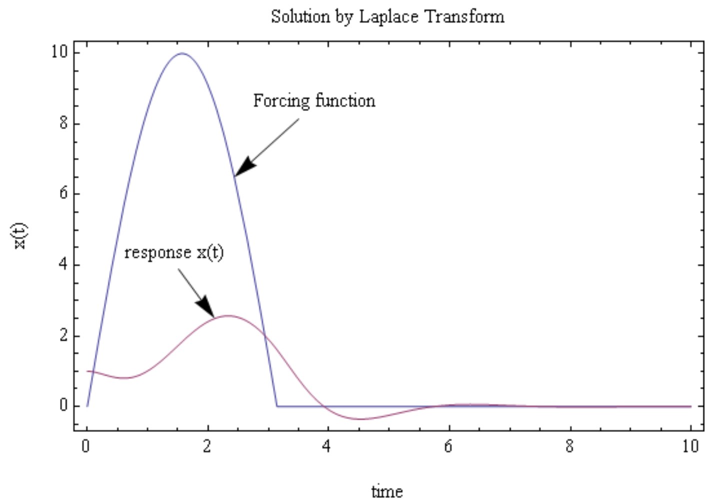
Substitute (8) into (7) we obtain

$$\begin{aligned}\left(s^2X - sx_0 - v_0\right) + 2\xi\omega_n\left(sX - x_0\right) + \omega_n^2 X &= F_0\left[\frac{1+e^{-\pi s}}{1+s^2}\right]\\ X\left(s^2 + 2\xi\omega_n s + \omega_n^2\right) - sx_0 - v_0 - 2\xi\omega_n x_0 &= \frac{F_0\left(1+e^{-\pi s}\right)}{1+s^2}\\ X\left(s^2 + 2\xi\omega_n s + \omega_n^2\right) &= \frac{F_0\left(1+e^{-\pi s}\right)}{1+s^2} + sx_0 + v_0 + 2\xi\omega_n x_0\\ &= \frac{F_0\left(1+e^{-\pi s}\right) + \left(1+s^2\right)sx_0 + v_0\left(1+s^2\right) + 2\xi\omega_n x_0\left(1+s^2\right)}{1+s^2}\end{aligned}$$

Hence

$$\begin{aligned}X &= \frac{F_0\left(1+e^{-\pi s}\right) + \left(1+s^2\right)sx_0 + v_0\left(1+s^2\right) + 2\xi\omega_n x_0\left(1+s^2\right)}{\left(1+s^2\right)\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)}\\ &= \frac{F_0 + v_0 + \frac{F_0}{e^{\pi s}} + sx_0 + s^2v_0 + s^3x_0 + 2\xi\omega_n x_0 + 2s^2\xi\omega_n x_0}{\left(1+s^2\right)\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)}\end{aligned}$$

Now we can use inverse Laplace transform on the above. It is easier to do partial fraction decomposition and use tables. I used CAS to do this and this is the result. I plot the solution $x(t)$. I used the following values to be able to obtain a plot $\xi = 0.5, \omega_n = 2, F_0 = 10, x_0 = 1, v_0 = 0$



5 Problem 3.21

3.21. A machine resting on an elastic support can be modeled as a single-degree-of-freedom spring–mass system arranged in the vertical direction. The ground is subject to a motion $y(t)$ of the form illustrated in Figure P3.21. The machine has a mass of 5000 kg and the support has stiffness 1.5×10^3 N/m. Calculate the resulting vibration of the machine.

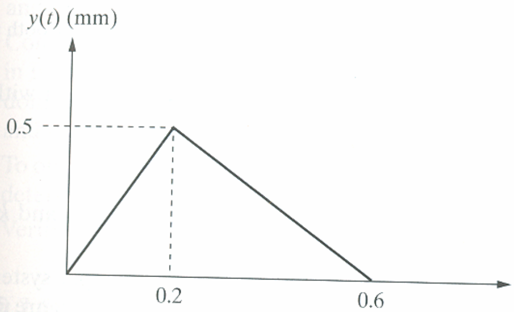


Figure P3.21 Triangular pulse input.

The acceleration $\ddot{x}$ of the mass is measured w.r.t. to the inertial frame, but the spring length is measured relative to the ground which is moving with displacement $y(t)$, hence the equation of motion of the mass m is given by

$$m\ddot{x}(t) + k(x(t) - y(y)) = 0$$

Therefore

$$m\ddot{x}(t) + k\,x(t) = k\,y(t) \tag{1}$$

Where $y(t)$ is given as

$$y(t) = \begin{cases} 2.5t & 0 \le t \le 0.2 \\ 0.75 - 1.25t & 0.2 < t \le 0.6 \\ 0 & 0.6 < t \end{cases}$$

The solution to (1) is given by $x(t) = x_h(t) + x_p(t)$ where $x_p(t)$ can be found using convolution, and $x_h(t)$ is as usual given by

$$x_h = A\cos\omega_n + B\sin\omega_n$$

Let us first find $x_p(t)$. Note that the impulse response $h(t)$ to undamped system is given by

$$h(t) = \frac{1}{m\omega_n}\sin\omega_n t$$

Hence for $0 \leq t \leq 0.2$,

$$\begin{aligned} x_{p(0\cdots 0.2)}(t) &= \int_0^t f(\tau)(kh(t-\tau))\,d\tau \\ &= \int_0^t 2.5\tau\left(\frac{k}{m\omega_n}\sin\omega_n(t-\tau)\right)d\tau \\ &= \frac{2.5k}{m\omega_n}\int_0^t \tau\sin\omega_n(t-\tau)\ d\tau \\ &= 2.5\omega_n\int_0^t \tau\sin\omega_n(t-\tau)\ d\tau \end{aligned} \tag{2}$$

Integration by parts, $\int udv = uv - \int vdu$ where $u = \tau$, $dv = \sin\omega_n(t-\tau)$, hence $v = \frac{-\cos(\omega_n(t-\tau))}{-\omega_n}$, therefore (2) becomes

$$\begin{aligned} x_{p(0\cdots 0.2)}(t) &= 2.5\omega_n\left(\left[\tau\frac{\cos\omega_n(t-\tau)}{\omega_n}\right]_0^t - \int_0^t \frac{\cos(\omega_n(t-\tau))}{\omega_n}\ d\tau\right) \\ &= 2.5\omega_n\left(\frac{t}{\omega_n} + \frac{1}{\omega_n^2}\left[\sin(\omega_n(t-\tau))\right]_0^t\right) \\ &= 2.5\omega_n\left(\frac{t}{\omega_n} + \frac{1}{\omega_n^2}\left[\sin\omega_n(t-t) - \sin\omega_n(t)\right]\right) \\ &= 2.5\left(t - \frac{\sin\omega_n t}{\omega_n}\right) \end{aligned}$$

For $0.2 < t \leq 0.6$

$$\begin{aligned} x_{p(0.2\cdots 0.6)}(t) &= \omega_n\int_0^{0.2} 2.5\tau\sin\omega_n(t-\tau)\ d\tau + \int_{0.2}^t f(\tau)(kh(t-\tau))\,d\tau \\ &= 2.5\omega_n\int_0^{0.2} \tau\sin\omega_n(t-\tau)\ d\tau + \int_{0.2}^t (0.75-1.25\tau)\left(\frac{k}{m\omega_n}\sin\omega_n(t-\tau)\right)d\tau \\ &= 2.5\omega_n\int_0^{0.2} \tau\sin\omega_n(t-\tau)\ d\tau + \\ &\int_{0.2}^t 0.75\frac{k}{m\omega_n}\sin\omega_n(t-\tau)\,d\tau \\ &-\int_{0.2}^t 1.25\tau\frac{k}{m\omega_n}\sin\omega_n(t-\tau)\,d\tau \end{aligned} \tag{3}$$

For the first integral in (3), we obtain

$$\begin{aligned} I_1 &= 2.5\omega_n\int_0^{0.2} \tau\sin\omega_n(t-\tau)\ d\tau \\ &= 2.5\omega_n\left(\left[\tau\frac{\cos\omega_n(t-\tau)}{\omega_n}\right]_0^{0.2} - \int_0^{0.2} \frac{\cos(\omega_n(t-\tau))}{\omega_n}\ d\tau\right) \\ &= 2.5\omega_n\left(0.2\frac{\cos\omega_n(t-0.2)}{\omega_n} + \frac{1}{\omega_n^2}\left[\sin(\omega_n(t-\tau))\right]_0^{0.2}\right) \\ &= 2.5\omega_n\left(0.2\frac{\cos\omega_n(t-0.2)}{\omega_n} + \frac{1}{\omega_n^2}\left(\sin\omega_n(t-0.2) - \sin\omega_n t\right)\right) \\ &= 0.5\cos\omega_n(t-0.2) + \frac{2.5}{\omega_n}\sin\omega_n(t-0.2) - \frac{2.5}{\omega_n}\sin\omega_n t \end{aligned}$$

For the second integral in (3) we obtain

$$
\begin{aligned}
I_2 &= 0.75\omega_n \int_{0.2}^{t} \sin\omega_n (t-\tau)\, d\tau \\
&= 0.75 \left[\cos\omega_n (t-\tau)\right]_{0.2}^{t} \\
&= 0.75 \left(1-\cos\omega_n (t-0.2)\right)
\end{aligned}
$$

For the third integral in (3) we obtain

$$
\begin{aligned}
I_3 &= \int_{0.2}^{t} 1.25\tau \frac{k}{m\omega_n} \sin\omega_n (t-\tau)\, d\tau \\
&= 1.25\omega_n \int_{0.2}^{t} \tau \sin\omega_n (t-\tau)\, d\tau
\end{aligned}
$$

Integration by parts gives

$$
\begin{aligned}
I_3 &= 1.25\omega_n \left(\left[\tau \frac{\cos\omega_n (t-\tau)}{\omega_n}\right]_{0.2}^{t} - \int_{0.2}^{t} \frac{\cos\left(\omega_n (t-\tau)\right)}{\omega_n}\, d\tau \right) \\
&= 1.25\omega_n \left(\frac{t}{\omega_n} - 0.2\frac{\cos\omega_n (t-0.2)}{\omega_n} + \frac{1}{\omega_n^2}\left[\sin\omega_n (t-\tau)\right]_{0.2}^{t} \right) \\
&= 1.25\omega_n \left(\frac{t}{\omega_n} - 0.2\frac{\cos\omega_n (t-0.2)}{\omega_n} + \frac{1}{\omega_n^2}\left[-\sin\omega_n (t-0.2)\right] \right) \\
&= 1.25 \left(t - 0.2\cos\omega_n (t-0.2) - \frac{1}{\omega_n}\sin\omega_n (t-0.2) \right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
x_{p(0.2\cdots 0.6)}(t) &= I_1 + I_2 - I_3 \\
&= 0.5\cos\omega_n (t-0.2) + \frac{2.5}{\omega_n}\sin\omega_n (t-0.2) - \frac{2.5}{\omega_n}\sin\omega_n t + \\
&\quad 0.75\left(1-\cos\omega_n (t-0.2)\right) \\
&\quad - 1.25\left(t - 0.2\cos\omega_n (t-0.2) - \frac{1}{\omega_n}\sin\omega_n (t-0.2) \right) \\
\\
&= 0.5\cos\omega_n (t-0.2) + \frac{2.5}{\omega_n}\sin\omega_n (t-0.2) - \frac{2.5}{\omega_n}\sin\omega_n t + \\
&\quad 0.75 - 0.75\cos\omega_n (t-0.2) \\
&\quad - 1.25t + 0.25\cos\omega_n (t-0.2) + \frac{1.25}{\omega_n}\sin\omega_n (t-0.2) \\
\\
&= 0.75 - 1.25t + \frac{3.75}{\omega_n}\sin\omega_n (t-0.2) - \frac{2.5}{\omega_n}\sin\omega_n t
\end{aligned}
$$

For $t > 0.6$

$$\begin{aligned}x_{p(0.6\cdots t)}(t) &= 2.5\omega_n \int_0^{0.2} \tau \sin \omega_n (t-\tau)\ d\tau + \int_{0.2}^{0.6} (0.75 - 1.25\tau) \left(\frac{k}{m\omega_n} \sin \omega_n (t-\tau)\right) d\tau \\ &= 2.5\omega_n \int_0^{0.2} \tau \sin \omega_n (t-\tau)\ d\tau + \\ &\int_{0.2}^{0.6} 0.75 \frac{k}{m\omega_n} \sin \omega_n (t-\tau)\, d\tau \\ &- \int_{0.2}^{0.6} 1.25\tau \frac{k}{m\omega_n} \sin \omega_n (t-\tau)\, d\tau \end{aligned} \tag{4}$$

For the first integral in (4), we obtain

$$I_1 = 0.5 \cos \omega_n (t - 0.2) + \frac{2.5}{\omega_n} \sin \omega_n (t-0.2) - \frac{2.5}{\omega_n} \sin \omega_n t$$

For the second integral in (4) we obtain

$$\begin{aligned} I_2 &= 0.75\omega_n \int_{0.2}^{0.6} \sin \omega_n (t-\tau)\, d\tau \\ &= 0.75 \left[\cos \omega_n (t-\tau)\right]_{0.2}^{0.6} \\ &= 0.75 \left(\cos \omega_n (t-0.6) - \cos \omega_n (t-0.2)\right) \\ &= 0.75 \cos \omega_n (t-0.6) - 0.75 \cos \omega_n (t-0.2) \end{aligned}$$

For the third integral in (4) we obtain

$$\begin{aligned} I_3 &= \int_{0.2}^{0.6} 1.25\tau \frac{k}{m\omega_n} \sin \omega_n (t-\tau)\, d\tau \\ &= 1.25\omega_n \int_{0.2}^{0.6} \tau \sin \omega_n (t-\tau)\, d\tau \end{aligned}$$

Integration by parts gives

$$\begin{aligned} I_3 &= 1.25\omega_n \left(\left[\tau \frac{\cos \omega_n (t-\tau)}{\omega_n}\right]_{0.2}^{0.6} - \int_{0.2}^{0.6} \frac{\cos(\omega_n (t-\tau))}{\omega_n}\ d\tau\right) \\ &= 1.25\omega_n \left(0.6 \frac{\cos \omega_n (t-0.6)}{\omega_n} - 0.2 \frac{\cos \omega_n (t-0.2)}{\omega_n} - \frac{1}{\omega_n^2} \left(\sin \omega_n (t-0.6) - \sin \omega_n (t-0.2)\right)\right) \\ &= 0.75 \cos \omega_n (t-0.6) - 0.25 \cos \omega_n (t-0.2) - \frac{1.25}{\omega_n} \sin \omega_n (t-0.6) + \frac{1.25}{\omega_n} \sin \omega_n (t-0.2) \end{aligned}$$

Hence

$$\begin{aligned} x_{p(0.6\cdots t)}(t) &= I_1 + I_2 - I_3 \\ &= 0.5 \cos \omega_n (t-0.2) + \frac{2.5}{\omega_n} \sin \omega_n (t-0.2) - \frac{2.5}{\omega_n} \sin \omega_n t \\ &+ 0.75 \cos \omega_n (t-0.6) - 0.75 \cos \omega_n (t-0.2) \\ &- 0.75 \cos \omega_n (t-0.6) + 0.25 \cos \omega_n (t-0.2) + \frac{1.25}{\omega_n} \sin \omega_n (t-0.6) - \frac{1.25}{\omega_n} \sin \omega_n (t-0.2) \\ &= \frac{3.75}{\omega_n} \sin \omega_n (t-0.2) - \frac{2.5}{\omega_n} \sin \omega_n t - \frac{1.25}{\omega_n} \sin \omega_n (t-0.6) \end{aligned}$$

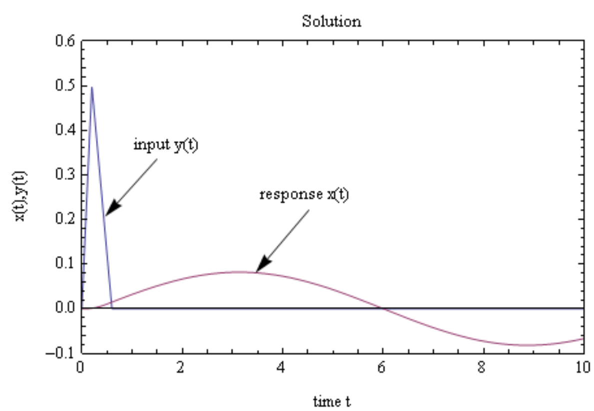
Hence, the overall response is, assuming zero initial conditions, is given by

$$x(t) = \begin{cases} 2.5\left(t - \frac{\sin \omega_n t}{\omega_n}\right) & 0 \le t \le 0.2 \\ 0.75 - 1.25t + \frac{3.75}{\omega_n}\sin \omega_n (t-0.2) - \frac{2.5}{\omega_n}\sin \omega_n t & 0.2 < t \le 0.6 \\ \frac{3.75}{\omega_n}\sin \omega_n (t-0.2) - \frac{2.5}{\omega_n}\sin \omega_n t - \frac{1.25}{\omega_n}\sin \omega_n (t-0.6) & t > 0.6 \end{cases}$$

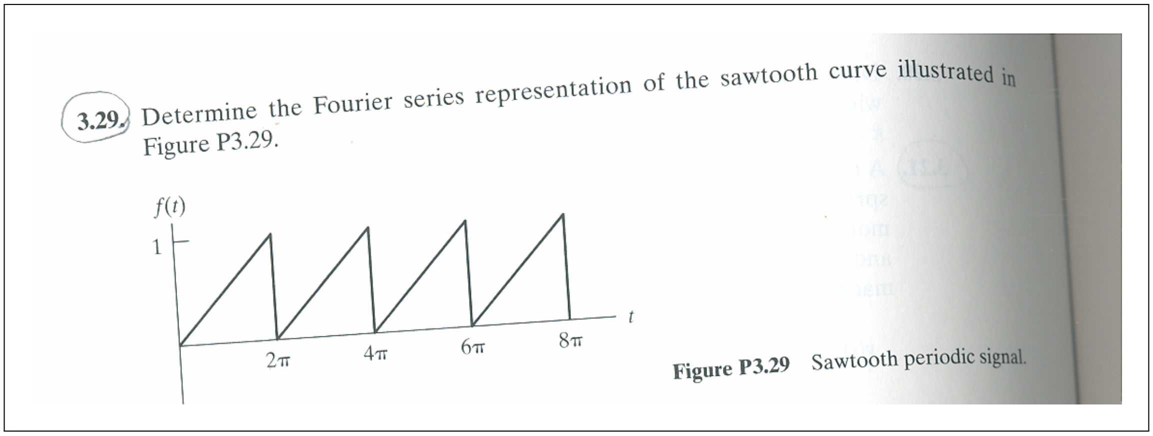
Noting that $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1500}{5000}} = 0.54772$, the above becomes

$$x(t) = \begin{cases} 2.5t - 4.5644 \sin \omega_n t & 0 \le t \le 0.2 \\ 0.75 - 1.25t + 6.8466 \sin \omega_n (t-0.2) - 4.5644 \sin \omega_n t & 0.2 < t \le 0.6 \\ 6.8466 \sin \omega_n (t-0.2) - 4.5644 \sin \omega_n t - 2.2822 \sin \omega_n (t-0.6) & t > 0.6 \end{cases}$$

This is a plot of the solution superimposed on top of the forcing function



6 Problem 3.29



Let $f(t)$ be the function shown above. Let $\tilde{f}(t)$ be its approximation using Fourier series. Hence

$$\tilde{f}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T}nt\right) + b_n \sin\left(\frac{2\pi}{T}nt\right)$$

Where T is the period of $f(t)$ and

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi}{T}nt\right)dt \qquad n = 1, 2, \cdots$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi}{T}nt\right)dt \qquad n = 1, 2, \cdots$$

For $f(t)$ we see that $T = 2\pi$ and $f(t) = \frac{t}{T}$ for $0 \le t \le T$, hence

$$\begin{aligned} a_0 &= \frac{2}{T}\int_0^T \frac{t}{T}dt \\ &= \frac{2}{2\pi}\int_0^T \frac{t}{2\pi}dt \\ &= \frac{1}{2\pi^2}\left[\frac{t^2}{2}\right]_0^{2\pi} \\ &= \frac{1}{4\pi^2}\left[4\pi^2\right] \\ &= 1 \end{aligned}$$

And

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_0^{2\pi} \frac{t}{T}\cos(nt)\,dt \qquad n = 1,2,\cdots \\
&= \frac{2}{2\pi}\int_0^{2\pi} \frac{1}{2\pi} t\cos(nt)\,dt \\
&= \frac{1}{2\pi^2}\int_0^{2\pi} t\cos(nt)\,dt \\
&= \frac{1}{2\pi^2}\left(\left[t\frac{\sin nt}{n}\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi} \sin nt\ dt\right) \\
&= \frac{1}{2\pi^2}\left(0 + \frac{1}{n}\left[\frac{\cos nt}{n}\right]_0^{2\pi}\right) \\
&= \frac{1}{2\pi^2}\left(\frac{1}{n^2}\left[\cos 2n\pi - 1\right]\right) \\
&= \frac{1}{2n^2\pi^2}\left(\cos 2n\pi - 1\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_0^{2\pi} \frac{t}{T}\sin(nt)\,dt \qquad n = 1,2,\cdots \\
&= \frac{2}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\sin(nt)\,dt \\
&= \frac{1}{2\pi^2}\left(\left[-\frac{t\cos nt}{n}\right]_0^{2\pi} + \int_0^{2\pi} \frac{\cos nt}{n} dt\right) \\
&= \frac{1}{2\pi^2}\left(\left[\frac{-2\pi\cos 2\pi n}{n}\right] - \frac{1}{n}\left[\frac{\sin nt}{n}\right]_0^{2\pi}\right) \\
&= \frac{1}{2\pi^2}\left(\frac{-2\pi\cos 2\pi n}{n}\right) \\
&= \frac{-\cos 2\pi n}{n\pi} \\
&= \frac{-1}{n\pi}
\end{aligned}
$$

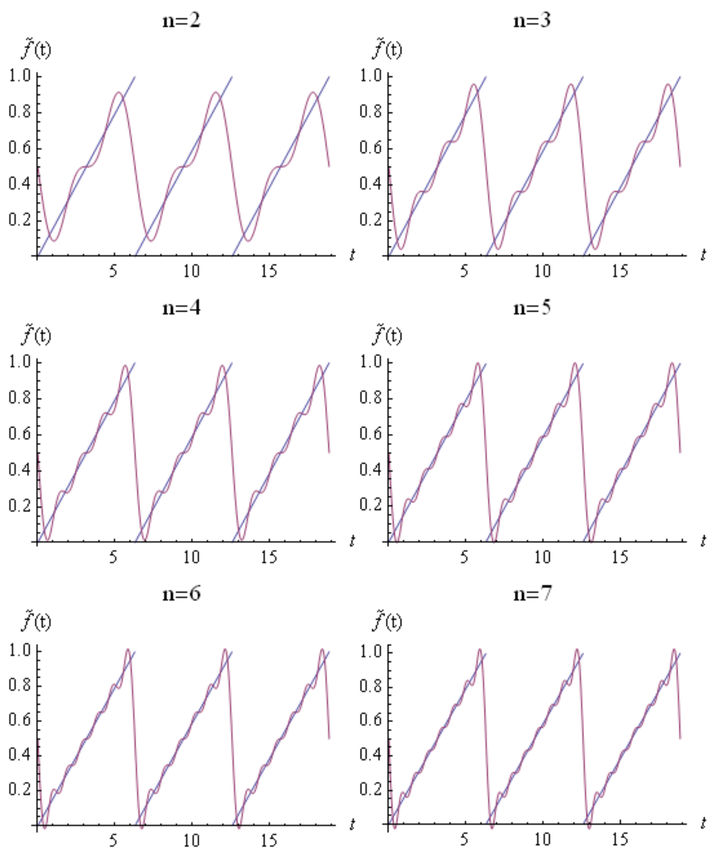
Hence

$$\begin{aligned}\tilde{f}(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} nt\right) + b_n \sin\left(\frac{2\pi}{T} nt\right) \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-1}{n\pi} \sin(nt)\end{aligned}$$

These are few terms in the series

$$\boxed{\tilde{f}(t) = \tfrac{1}{2} - \tfrac{1}{\pi}\sin t - \tfrac{1}{2\pi}\sin 2t - \tfrac{1}{3\pi}\sin 3t - \cdots}$$

This is a plot of the above for increasing number of n



7 Problem 3.38

Problem

Solve the following system using Laplace transform $100\ddot{x}(t)+2000x(t)=50\delta(t)$ where the units are in Newtons and the initial conditions are both zero.

Answer

Divide the equation by 50 we obtain

$$2\ddot{x}(t)+40x(t)=\delta(t)$$

Let $m=2,k=40$,hence the equation becomes

$$m\ddot{x}(t)+kx(t)=\delta(t)$$

Applying Laplace transform

$$m\left(s^2X(s)-sx_0-v_0\right)+kX(s)=1$$

But due to zero initial conditions, the above simplifies to

$$\begin{aligned}
ms^2X(s)+kX(s)&=1\\
X(s)\left[ms^2+k\right]&=1\\
X(s)&=\frac{1}{ms^2+k}
\end{aligned}$$

From tables, the inverse Laplace transform of $\frac{\alpha}{s^2+\alpha^2}$ is $\sin\alpha t$, but

$$\frac{1}{ms^2+k}=\frac{\frac{1}{m}}{s^2+\frac{k}{m}}=\frac{1}{m}\frac{1}{\sqrt{\frac{k}{m}}}\left(\frac{\sqrt{\frac{k}{m}}}{s^2+\frac{k}{m}}\right)$$

Hence, letting $\boxed{\alpha=\sqrt{\frac{k}{m}}}$ we see that inverse laplace transform of $\frac{1}{ms^2+k}$ is the same as the inverse laplace transform of $\frac{1}{m}\frac{1}{\alpha}\left(\frac{\alpha}{s^2+\alpha^2}\right)$ which is $\boxed{\frac{1}{m}\frac{1}{\alpha}\sin\alpha t}$

But $\alpha=\omega_n$, hence

$$\boxed{x(t)=\frac{1}{m\omega_n}\sin\omega_n t}$$

or

$$\begin{aligned}
x(t)&=\frac{1}{2\sqrt{\frac{40}{2}}}\sin\sqrt{\frac{40}{2}}t\\
&=0.1118\sin(4.4721t)
\end{aligned}$$

8 Problem 3.44

Problem

Calculate the response spectrum of an undamped system to the forcing function

$$F(t)=\begin{cases} F_0 \sin\frac{\pi t}{t_1} & 0\leq t\leq t_1 \\ 0 & t>t_1 \end{cases}$$ assuming zero initial conditions.

Answer

Solution sketch: Find the response $x(t)$ of the system to the above input. Then find t where this response is maximum, call this $x_{\max}$, then plot $\left(x_{\max}\frac{k}{F_0}\right)$ vs. $\frac{t\omega_n}{2\pi}$

The system EQM is

$$x''(t)+\omega_n^2 x(t)=\frac{F(t)}{m}$$

For $0<t\leq t_1$,

$$\begin{aligned} x_1(t) &= x_h(t)+x_p(t) \\ &= A\cos\omega_n t+B\sin\omega_n t+x_p(t) \end{aligned}$$

Guess $\boxed{x_p(t)=c_1\cos\omega t+c_2\sin\omega t}$, hence $x_p'(t)=-\omega c_1\sin\omega t+\omega c_2\cos\omega t$ and $x_p''(t)=-\omega^2 c_1\cos\omega t-\omega^2 c_2\sin\omega t$, hence substitute these into the EQM and compare, we obtain

$$\left(-\omega^2 c_1\cos\omega t-\omega^2 c_2\sin\omega t\right)+\omega_n^2\left(c_1\cos\omega t+c_2\sin\omega t\right)=\frac{F_0}{m}\sin\frac{\pi t}{t_1}$$

The input is half sin where $\omega t=\frac{\pi t}{t_1}$, hence $\boxed{\omega=\frac{\pi}{t_1}}$,hence the above becomes

$$\left(-\omega^2 c_1+\omega_n^2 c_1\right)\cos\omega t+\left(-\omega^2 c_2+\omega_n^2 c_2\right)\sin\omega t=\frac{F_0}{m}\sin\omega t$$

Hence $\boxed{c_1=0}$ and $c_2\left(-\omega^2+\omega_n^2\right)=\frac{F_0}{m}$ or $\boxed{c_2=\frac{\frac{F_0}{m}}{\omega_n^2-\omega^2}}$, Then the solution becomes

$$x_1(t)=A\cos\omega_n t+B\sin\omega_n t+\frac{\frac{F_0}{m}}{\omega_n^2-\omega^2}\sin\omega t$$

And since $x(0)=0$ then $A=0$ and take derivative we obtain

$$x_1'(t)=\omega_n B\cos\omega_n t+\omega\frac{\frac{F_0}{m}}{\omega_n^2-\omega^2}\cos\omega t$$

And since $x'(0)=0$ then the above results in

$$\begin{aligned} 0 &= \omega_n B+\omega\frac{\frac{F_0}{m}}{\omega_n^2-\omega^2} \\ B &= \frac{\omega}{\omega_n}\frac{\frac{F_0}{m}}{\omega^2-\omega_n^2} \end{aligned}$$

Hence the solution becomes

$$
\begin{aligned}
x_1(t) &= \frac{\omega}{\omega_n}\frac{\frac{F_0}{m}}{\omega^2-\omega_n^2}\sin\omega_n t+\frac{\frac{F_0}{m}}{\omega_n^2-\omega^2}\sin\omega t\\
&= \frac{\frac{F_0}{m}}{\omega_n^2-\omega^2}\left(\sin\omega t-\frac{\omega}{\omega_n}\sin\omega_n t\right)\\
&= \frac{\frac{F_0}{m}}{\omega_n^2\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega t-\frac{\omega}{\omega_n}\sin\omega_n t\right)\\
&= \frac{\frac{F_0}{m}}{\frac{k}{m}\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega t-\frac{\omega}{\omega_n}\sin\omega_n t\right)
\end{aligned}
$$

Hence

$$
x_1(t)=\frac{\frac{F_0}{k}}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega t-\frac{\omega}{\omega_n}\sin\omega_n t\right)\qquad 0<t\leq t_1 \tag{1}
$$

Now we need to find where the maximum is. Take derivative, and set it to zero, we obtain

$$
x_1'(t)=\frac{F_0}{k}\frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}(\omega\cos\omega t-\omega\cos\omega_n t)=0
$$

For $\omega\neq\omega_n$, we need to solve

$$
\cos\omega t-\cos\omega_n t=0
$$

Using $\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$, then the above becomes

$$
\begin{aligned}
-2\sin\left(\frac{(\omega+\omega_n)t}{2}\right)\sin\left(\frac{(\omega-\omega_n)t}{2}\right)&=0\\
\sin\left(\frac{(\omega+\omega_n)t}{2}\right)\sin\left(\frac{(\omega-\omega_n)t}{2}\right)&=0
\end{aligned}
$$

Hence, either $\frac{(\omega+\omega_n)t_p}{2}=n\pi$ or $\frac{(\omega-\omega_n)t_p}{2}=n\pi$for $n=\pm1,\pm2,\cdots$ or the time t_p which makes the maximum $x(t)$ is one of the following

$$
t_p=\begin{cases}\frac{2n\pi}{\omega+\omega_n} & \\ & n=\pm1,\pm2,\cdots\\ \frac{2n\pi}{\omega-\omega_n} & \end{cases}
$$

We now need to find which one of the above 2 solution gives a larger maximum. Using the first solution

$t_p = \frac{2n\pi}{\omega+\omega_n}$, then (1) becomes

$$x_{\max}\left(t_{p_1}\right) = \frac{F_0}{k\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega\left(\frac{2n\pi}{\omega+\omega_n}\right) - \frac{\omega}{\omega_n}\sin\omega_n\left(\frac{2n\pi}{\omega+\omega_n}\right)\right)$$
$$= \frac{F_0}{k\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\left(\frac{2n\pi\frac{\omega}{\omega_n}}{1+\frac{\omega}{\omega_n}}\right) - \frac{\omega}{\omega_n}\sin\left(\frac{2n\pi}{1+\frac{\omega}{\omega_n}}\right)\right)$$

And at $t_p = \frac{2n\pi}{\omega-\omega_n}$, then (1) becomes

$$x_{\max}\left(t_{p_2}\right) = \frac{F_0}{k\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega\left(\frac{2n\pi}{\omega-\omega_n}\right) - \frac{\omega}{\omega_n}\sin\omega_n\left(\frac{2n\pi}{\omega-\omega_n}\right)\right)$$
$$= \frac{F_0}{k\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\left(\frac{2n\pi\frac{\omega}{\omega_n}}{\frac{\omega}{\omega_n}-1}\right) - \frac{\omega}{\omega_n}\sin\left(\frac{2n\pi}{\frac{\omega}{\omega_n}-1}\right)\right)$$

Need now to find which of the above is larger. Let us take the difference and see if the result is positive or negative (is there an easier way?)

$$x\left(t_{p_1}\right) - x\left(t_{p_2}\right) = \frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\left(\frac{2n\pi\frac{\omega}{\omega_n}}{1+\frac{\omega}{\omega_n}}\right) - \sin\left(\frac{2n\pi\frac{\omega}{\omega_n}}{\frac{\omega}{\omega_n}-1}\right) - \frac{\omega}{\omega_n}\sin\left(\frac{2n\pi}{1+\frac{\omega}{\omega_n}}\right) + \frac{\omega}{\omega_n}\sin\left(\frac{2n\pi}{\frac{\omega}{\omega_n}-1}\right)\right)$$
$$= \frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\left(\frac{2n\pi\frac{\omega}{\omega_n}}{1+\frac{\omega}{\omega_n}}\right) - \sin\left(\frac{2n\pi\frac{\omega}{\omega_n}}{\frac{\omega}{\omega_n}-1}\right) - \frac{\omega}{\omega_n}\sin\left(\frac{2n\pi}{1+\frac{\omega}{\omega_n}}\right) + \frac{\omega}{\omega_n}\sin\left(\frac{2n\pi}{\frac{\omega}{\omega_n}-1}\right)\right)$$

Not sure how to continue. Now let us look at $t > t_1$. The solution here is

$$x_2(t) = A\cos\omega_n t + B\sin\omega_n t$$

But with IC given by $x_1(t_1)$ and $x_1'(t_1)$, hence from (1)

$$x_1(t_1) = \frac{\frac{F_0}{k}}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega t_1 - \frac{\omega}{\omega_n}\sin\omega_n t_1\right)$$

and

$$x_1'(t_1) = \frac{F_0}{k}\frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\omega\cos\omega t_1 - \omega\cos\omega_n t_1\right)$$

Hence

$$x_2(t_1) = A\cos\omega_n t_1 + B\sin\omega_n t_1 = \frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega t_1 - \frac{\omega}{\omega_n}\sin\omega_n t_1\right) \tag{3}$$

And

$$x_2'(t_1) = -\omega_n A \sin\omega_n t + B\omega_n \cos\omega_n t = \frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\omega\cos\omega t_1 - \omega\cos\omega_n t_1\right) \tag{4}$$

We need to solve (3) and (4) for A and B. Combining (3) and (4) we obtain

$$\begin{bmatrix} \cos\omega_n t_1 & \sin\omega_n t_1 \\ -\omega_n\sin\omega_n t & \omega_n\cos\omega_n t \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\sin\omega t_1 - \frac{\omega}{\omega_n}\sin\omega_n t_1\right) \\ \frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left(\omega\cos\omega t_1 - \omega\cos\omega_n t_1\right) \end{bmatrix}$$

This is in the form $Ax = b$, solve for x, we obtain

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\omega_n}\begin{bmatrix} \omega_n\cos\omega_n t_1 & -\sin\omega_n t_1 \\ \omega_n\sin\omega_n t & \cos\omega_n t \end{bmatrix}\begin{bmatrix} \left(\sin\omega t_1 - \frac{\omega}{\omega_n}\sin\omega_n t_1\right) \\ \left(\omega\cos\omega t_1 - \omega\cos\omega_n t_1\right) \end{bmatrix}\left(\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\right)$$

Hence

$$\begin{aligned} A &= \frac{1}{\omega_n}\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left[\omega_n\cos\omega_n t_1\left(\sin\omega t_1 - \frac{\omega}{\omega_n}\sin\omega_n t_1\right) - \sin\omega_n t_1\left(\omega\cos\omega t_1 - \omega\cos\omega_n t_1\right)\right] \\ &= \frac{1}{\omega_n}\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left[\omega_n\cos\omega_n t_1\sin\omega t_1 - \omega\cos\omega_n t_1\sin\omega_n t_1 - \omega\sin\omega_n t_1\cos\omega t_1 + \omega\sin\omega_n t_1\cos\omega_n t_1\right] \\ &= \frac{1}{\omega_n}\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left[\omega_n\cos\omega_n t_1\sin\omega t_1 - \omega\sin\omega_n t_1\cos\omega t_1\right] \end{aligned}$$

And

$$\begin{aligned} B &= \frac{1}{\omega_n}\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left[\omega_n\sin\omega_n t_1\left(\sin\omega t_1 - \frac{\omega}{\omega_n}\sin\omega_n t_1\right) + \cos\omega_n t_1\left(\omega\cos\omega t_1 - \omega\cos\omega_n t_1\right)\right] \\ &= \frac{1}{\omega_n}\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left[\omega_n\sin\omega_n t_1\sin\omega t_1 - \omega\sin\omega_n t_1\sin\omega_n t_1 + \omega\cos\omega_n t_1\cos\omega t_1 - \omega\cos\omega_n t_1\cos\omega_n t_1\right] \\ &= \frac{1}{\omega_n}\frac{F_0/k}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}\left[\omega_n\sin\omega_n t_1\sin\omega t_1 - \omega\sin\omega_n t_1\sin\omega_n t_1 + \omega\cos\omega_n t_1\cos\omega t_1 - \omega\cos\omega_n t_1\cos\omega_n t_1\right] \end{aligned}$$

Ask about the above, why can't I get the answer shown in notes?

9 Solving 3.44 using convolution

To find the response $x(t)$ use convolution. Since this is an undamped system, then the impulse response is

$$h(t) = \frac{1}{m\omega_n}\sin\omega_n t$$

Hence, for $0 \leq t \leq t_1$

$$\begin{aligned}
x(t) &= \int_0^t F(\tau)\,h(t-\tau)\,d\tau \\
&= \int_0^t \left(F_0 \sin\frac{\pi\tau}{t_1}\right)\frac{1}{m\omega_n}\sin\omega_n(t-\tau)\,d\tau \\
&= \frac{F_0}{m\omega_n}\int_0^t \sin\frac{\pi\tau}{t_1}\sin\omega_n(t-\tau)\,d\tau
\end{aligned}$$

Using $\sin A\sin B = \frac{1}{2}(\cos(A-B)-\cos(A+B))$, then

$$\sin\overbrace{\frac{\pi\tau}{t_1}}^{A}\sin\overbrace{\omega_n(t-\tau)}^{B} = \frac{1}{2}\cos\left(\frac{\pi\tau}{t_1}-\omega_n(t-\tau)\right)-\frac{1}{2}\cos\left(\frac{\pi\tau}{t_1}+\omega_n(t-\tau)\right)$$

Hence the convolution integral becomes

$$\begin{aligned}
x(t) &= \frac{F_0}{m\omega_n}\int_0^t \frac{1}{2}\cos\left(\frac{\pi\tau}{t_1}-\omega_n(t-\tau)\right)-\frac{1}{2}\cos\left(\frac{\pi\tau}{t_1}+\omega_n(t-\tau)\right)d\tau \\
&= \frac{F_0}{2m\omega_n}\left[\int_0^t \cos\left(\frac{\pi\tau}{t_1}-\omega_n(t-\tau)\right)d\tau-\int_0^t \cos\left(\frac{\pi\tau}{t_1}+\omega_n(t-\tau)\right)\right] \\
&= \frac{F_0}{2m\omega_n}\left\{\left[\frac{\sin\left(\frac{\pi\tau}{t_1}-\omega_n(t-\tau)\right)}{\frac{\pi}{t_1}+\omega_n}\right]_0^t-\left[\frac{\sin\left(\frac{\pi\tau}{t_1}+\omega_n(t-\tau)\right)}{\frac{\pi}{t_1}-\omega_n}\right]_0^t\right\} \\
&= \frac{F_0}{2m\omega_n}\left\{\left(\frac{\sin\left(\frac{\pi t}{t_1}\right)}{\frac{\pi}{t_1}+\omega_n}-\frac{\sin(-\omega_n t)}{\frac{\pi}{t_1}+\omega_n}\right)-\left(\frac{\sin\left(\frac{\pi t}{t_1}\right)}{\frac{\pi}{t_1}-\omega_n}-\frac{\sin(\omega_n t)}{\frac{\pi}{t_1}-\omega_n}\right)\right\} \\
&= \frac{F_0}{2m\omega_n}\left\{\frac{\sin\frac{\pi t}{t_1}}{\frac{\pi}{t_1}+\omega_n}+\frac{\sin\omega_n t}{\frac{\pi}{t_1}+\omega_n}-\frac{\sin\frac{\pi t}{t_1}}{\frac{\pi}{t_1}-\omega_n}+\frac{\sin\omega_n t}{\frac{\pi}{t_1}-\omega_n}\right\}
\end{aligned}$$

And for $t > t_1$

$$\begin{aligned}
x(t) &= \int_0^{t_1} F(\tau)\,h(t-\tau)\,d\tau \;+\int_{t_1}^t 0\times h(t-\tau)\,d\tau \\
&= \int_0^{t_1}\left(F_0\sin\frac{\pi\tau}{t_1}\right)\frac{1}{m\omega_n}\sin\omega_n(t-\tau)\,d\tau \\
&= \frac{F_0}{m\omega_n}\int_0^{t_1}\sin\frac{\pi\tau}{t_1}\sin\omega_n(t-\tau)\,d\tau
\end{aligned}$$

As was done earlier, perform integration by parts, we obtain

$$
\begin{aligned}
x(t) &= \frac{F_0}{2m\omega_n}\left\{\left[\frac{\sin\left(\frac{\pi\tau}{t_1}-\omega_n(t-\tau)\right)}{\frac{\pi}{t_1}+\omega_n}\right]_0^{t_1}-\left[\frac{\sin\left(\frac{\pi\tau}{t_1}+\omega_n(t-\tau)\right)}{\frac{\pi}{t_1}-\omega_n}\right]_0^{t_1}\right\}\\
&= \frac{F_0}{2m\omega_n}\left\{\frac{\sin(\pi-\omega_n(t-t_1))}{\frac{\pi}{t_1}+\omega_n}-\frac{\sin(-\omega_n t)}{\frac{\pi}{t_1}+\omega_n}-\frac{\sin(\pi+\omega_n(t-t_1))}{\frac{\pi}{t_1}-\omega_n}+\frac{\sin(\omega_n t)}{\frac{\pi}{t_1}-\omega_n}\right\}\\
&= \frac{F_0}{2m\omega_n}\left\{\frac{\sin(\pi-\omega_n(t-t_1))}{\frac{\pi}{t_1}+\omega_n}+\frac{\sin(\omega_n t)}{\frac{\pi}{t_1}+\omega_n}-\frac{\sin(\pi+\omega_n(t-t_1))}{\frac{\pi}{t_1}-\omega_n}+\frac{\sin(\omega_n t)}{\frac{\pi}{t_1}-\omega_n}\right\}
\end{aligned}
$$

But $\sin(\pi-\alpha)=\sin\alpha$ and $\sin(\pi+\alpha)=-\sin\alpha$, hence the above becomes

$$
x(t)=\frac{F_0}{2m\omega_n}\left\{\frac{\sin\omega_n(t-t_1)}{\frac{\pi}{t_1}+\omega_n}+\frac{\sin(\omega_n t)}{\frac{\pi}{t_1}+\omega_n}+\frac{\sin\omega_n(t-t_1)}{\frac{\pi}{t_1}-\omega_n}+\frac{\sin(\omega_n t)}{\frac{\pi}{t_1}-\omega_n}\right\}
$$

Therefore, the final solution is

$$
x(t)=\begin{cases}\frac{F_0}{2m\omega_n}\left\{\frac{\sin\frac{\pi t}{t_1}}{\frac{\pi}{t_1}+\omega_n}+\frac{\sin\omega_n t}{\frac{\pi}{t_1}+\omega_n}-\frac{\sin\frac{\pi t}{t_1}}{\frac{\pi}{t_1}-\omega_n}+\frac{\sin\omega_n t}{\frac{\pi}{t_1}-\omega_n}\right\} & 0\le t\le t_1\\ \frac{F_0}{2m\omega_n}\left\{\frac{\sin\omega_n(t-t_1)}{\frac{\pi}{t_1}+\omega_n}+\frac{\sin(\omega_n t)}{\frac{\pi}{t_1}+\omega_n}+\frac{\sin\omega_n(t-t_1)}{\frac{\pi}{t_1}-\omega_n}+\frac{\sin(\omega_n t)}{\frac{\pi}{t_1}-\omega_n}\right\} & t>t_1\end{cases} \tag{1}
$$

We can simplify the above more as follows

$$
x(t)=\begin{cases}\frac{F_0}{2m\omega_n}\left\{\sin\frac{\pi t}{t_1}\left(\frac{1}{\frac{\pi}{t_1}+\omega_n}-\frac{1}{\frac{\pi}{t_1}-\omega_n}\right)+\sin\omega_n t\left(\frac{1}{\frac{\pi}{t_1}+\omega_n}+\frac{1}{\frac{\pi}{t_1}-\omega_n}\right)\right\} & 0\le t\le t_1\\ \frac{F_0}{2m\omega_n}\left\{\sin\omega_n(t-t_1)\left(\frac{1}{\frac{\pi}{t_1}+\omega_n}+\frac{1}{\frac{\pi}{t_1}-\omega_n}\right)+\sin(\omega_n t)\left(\frac{1}{\frac{\pi}{t_1}+\omega_n}+\frac{1}{\frac{\pi}{t_1}-\omega_n}\right)\right\} & t>t_1\end{cases} \tag{1}
$$

Hence

$$
x(t)=\begin{cases}\frac{F_0}{2m\omega_n}\left\{\sin\frac{\pi t}{t_1}\left(\frac{\left(\frac{\pi}{t_1}-\omega_n\right)-\left(\frac{\pi}{t_1}+\omega_n\right)}{\left(\frac{\pi}{t_1}+\omega_n\right)\left(\frac{\pi}{t_1}-\omega_n\right)}\right)+\sin\omega_n t\left(\frac{\left(\frac{\pi}{t_1}-\omega_n\right)+\left(\frac{\pi}{t_1}+\omega_n\right)}{\left(\frac{\pi}{t_1}+\omega_n\right)\left(\frac{\pi}{t_1}-\omega_n\right)}\right)\right\} & 0\le t\le t_1\\ \frac{F_0}{2m\omega_n}\left\{\sin\omega_n(t-t_1)\left(\frac{\left(\frac{\pi}{t_1}-\omega_n\right)+\left(\frac{\pi}{t_1}+\omega_n\right)}{\left(\frac{\pi}{t_1}+\omega_n\right)\left(\frac{\pi}{t_1}-\omega_n\right)}\right)+\sin(\omega_n t)\left(\frac{\left(\frac{\pi}{t_1}-\omega_n\right)+\left(\frac{\pi}{t_1}+\omega_n\right)}{\left(\frac{\pi}{t_1}+\omega_n\right)\left(\frac{\pi}{t_1}-\omega_n\right)}\right)\right\} & t>t_1\end{cases} \tag{1}
$$

Hence

$$
x(t)=\begin{cases}\frac{F_0}{2m\omega_n}\left\{\sin\frac{\pi t}{t_1}\left(\frac{-2\omega_n}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)+\sin\omega_n t\left(\frac{2\frac{\pi}{t_1}}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\right\} & 0\le t\le t_1\\ \frac{F_0}{2m\omega_n}\left\{\sin\omega_n(t-t_1)\left(\frac{2\frac{\pi}{t_1}}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)+\sin(\omega_n t)\left(\frac{2\frac{\pi}{t_1}}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\right\} & t>t_1\end{cases} \tag{1}
$$

or

$$x(t)=\begin{cases}\left(\frac{1}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{2m\omega_n}\left\{-2\omega_n\sin\frac{\pi t}{t_1}+2\frac{\pi}{t_1}\sin\omega_n t\right\} & 0\le t\le t_1\\ \left(\frac{\omega}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\sin\omega_n(t-t_1)-\sin(\omega_n t)\right\} & t>t_1\end{cases}\tag{1}$$

Hence

$$x(t)=\begin{cases}\left(\frac{1}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{-\omega_n\sin\frac{\pi t}{t_1}+\frac{\pi}{t_1}\sin\omega_n t\right\} & 0\le t\le t_1\\ \left(\frac{\omega}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\sin\omega_n(t-t_1)-\sin\omega_n t\right\} & t>t_1\end{cases}\tag{1}$$

To find where $x_{\max}$ is, we need to find $x_{\max}$. Take the derivative, we obtain

$$\dot{x}(t)=\begin{cases}\left(\frac{1}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{-\omega_n\frac{\pi}{t_1}\cos\frac{\pi t}{t_1}+\frac{\pi}{t_1}\omega_n\cos\omega_n t\right\} & 0\le t\le t_1\\ \left(\frac{\omega}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\omega_n\cos\omega_n(t-t_1)-\omega_n\cos\omega_n t\right\} & t>t_1\end{cases}$$

Now let $\dot{x}(t)=0$ for $t>t_1$ to find t_{peak}.

$$\left(\frac{\omega}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\omega_n\cos\omega_n(t_p-t_1)-\omega_n\cos\omega_n t_p\right\}=0$$

$$\cos\omega_n(t_p-t_1)-\cos\omega_n t_p=0\tag{2}$$

But

$$\cos\omega_n(t_p-t_1)=\cos\omega_n t_p\cos\omega_n t_1+\sin\omega_n t_p\sin\omega_n t_1$$

Substitute the above into (2) we obtain

$$(\cos\omega_n t_p\cos\omega_n t_1+\sin\omega_n t_p\sin\omega_n t_1)-\cos\omega_n t_p=0$$

Divide by $\cos\omega_n t_p$

$$\cos\omega_n t_1+\tan\omega_n t_p\sin\omega_n t_1-1=0$$

$$\tan\omega_n t_p=\frac{(1-\cos\omega_n t_1)}{\sin\omega_n t_1}$$

$$\omega_n t_p=\tan^{-1}\left(\frac{1-\cos\omega_n t_1}{\sin\omega_n t_1}\right)$$

Hence, the hypotenuse is $\sqrt{(1-\cos\omega_n t_1)^2+\sin^2\omega_n t_1}=\sqrt{2(1-\cos\omega_n t_1)}$ and so $\sin\omega_n t_p=-\sqrt{\frac{1}{2}(1-\cos\omega_n t_1)}$ and $\cos\omega_n t_p=\frac{-\sin\omega_n t_1}{\sqrt{2(1-\cos\omega_n t_1)}}$ and using these into (1) we find $x_{\max}$ when $t>t_1$ as

$$x_{\max}=\left(\frac{\omega}{\left(\frac{\pi}{t_1}\right)^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\sin\omega_n(t_p-t_1)-\sin\omega_n t_p\right\}$$

But $\sin\omega_n(t_p - t_1) = \sin\omega_n t_p \cos\omega_n t_1 - \cos\omega_n t_p \sin\omega_n t_1$, hence

$$\begin{aligned}
x_{\max} &= \left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\sin\omega_n t_p\cos\omega_n t_1 - \cos\omega_n t_p\sin\omega_n t_1 - \sin\omega_n t_p\right\}\\
&= \left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{-\sqrt{\frac{1}{2}(1-\cos\omega_n t_1)}\cos\omega_n t_1 + \frac{\sin\omega_n t_1}{\sqrt{2(1-\cos\omega_n t_1)}}\sin\omega_n t_1 + \sqrt{\frac{1}{2}(1-\cos\omega_n t_1)}\right\}\\
&= \left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\frac{-(1-\cos\omega_n t_1)\cos\omega_n t_1 + \sin^2\omega_n t_1 + (1-\cos\omega_n t_1)}{\sqrt{2(1-\cos\omega_n t_1)}}\right\}\\
&= \left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\frac{-\cos\omega_n t_1 + \cos^2\omega_n t_1 + \sin^2\omega_n t_1 + 1 - \cos\omega_n t_1}{\sqrt{2(1-\cos\omega_n t_1)}}\right\}\\
&= \left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{F_0}{m\omega_n}\left\{\frac{-\cos\omega_n t_1 + 1 + 1 - \cos\omega_n t_1}{\sqrt{2(1-\cos\omega_n t_1)}}\right\}\\
&= \left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{2F_0}{m\omega_n}\left\{\frac{1-\cos\omega_n t_1}{\sqrt{2(1-\cos\omega_n t_1)}}\right\}\\
&= \left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{F_0}{2m\omega_n}\left\{\sqrt{1-\cos\omega_n t_1}\right\}
\end{aligned}$$

Hence

$$\begin{aligned}
x_{\max} &= \left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{F_0}{2m\omega_n}\sqrt{1-\cos\omega_n t_1}\\
&= \left(\frac{\omega}{\omega^2-\omega_n^2}\right)\frac{\omega_n F_0}{2m\omega_n^2}\sqrt{1-\cos\omega_n t_1}\\
&= \left(\frac{\frac{\omega}{\omega_n}}{\left(\left(\frac{\omega}{\omega_n}\right)^2 - 1\right)}\right)\frac{F_0}{2k}\sqrt{1-\cos\omega_n t_1}
\end{aligned}$$

Hence

$$\boxed{x_{\max}\frac{k}{F_0} = \left(\frac{r}{r^2-1}\right)\frac{1}{2}\sqrt{1-\cos\omega_n t_1}}$$

Where $r = \frac{\omega}{\omega_n}$

A plot of $x_{\max}\frac{k}{F_0}$ vs. r gives the response spectrum

10 Problem 3.49

Problem Calculate the compliance transfer function for a system described by $ax'''' + bx''' + cx'' + dx' + ex = f(t)$ where $f(t)$ is the input and $x(t)$ is the displacement.

Answer

Take Laplace transform (assuming zero IC) we obtain

$$as^4X(s) + bs^3X(s) + cs^2X(s) + dsX(s) + eX(s) = F(s)$$

Hence

$$X(s)\left[as^4 + bs^3 + cs^2 + ds + e\right] = F(s)$$

Hence

$$\boxed{H(s) = \frac{X(s)}{F(s)} = \frac{1}{as^4+bs^3+cs^2+ds+e}}$$

11 Problem 3.50

Problem

Calculate the frequency response function for the system of problem 3.49 for $a = 1, b = 4, c = 11, d = 16, e = 8$

Answer

$$\begin{aligned} H(s) &= \frac{1}{as^4 + bs^3 + cs^2 + ds + e} \\ &= \frac{1}{s^4 + 4s^3 + 11s^2 + 16s + 11} \end{aligned}$$

Let $s = j\omega$

$$\begin{aligned} H(j\omega) &= \frac{1}{(j\omega)^4 + 4(j\omega)^3 + 11(j\omega)^2 + 16(j\omega) + 11} \\ &= \frac{1}{\omega^4 - 4j\omega^3 - 11\omega^2 + 16j\omega + 11} \\ &= \frac{1}{(\omega^4 - 11\omega^2 + 11) + j(16\omega - 4\omega^3)} \end{aligned}$$

Hence

$$\begin{aligned} |H(j\omega)| &= \frac{1}{\sqrt{(\omega^4 - 11\omega^2 + 11)^2 + (16\omega - 4\omega^3)^2}} \\ &= \frac{1}{\sqrt{\omega^8 - 6\omega^6 + 15\omega^4 + 14\omega^2 + 121}} \end{aligned}$$

and

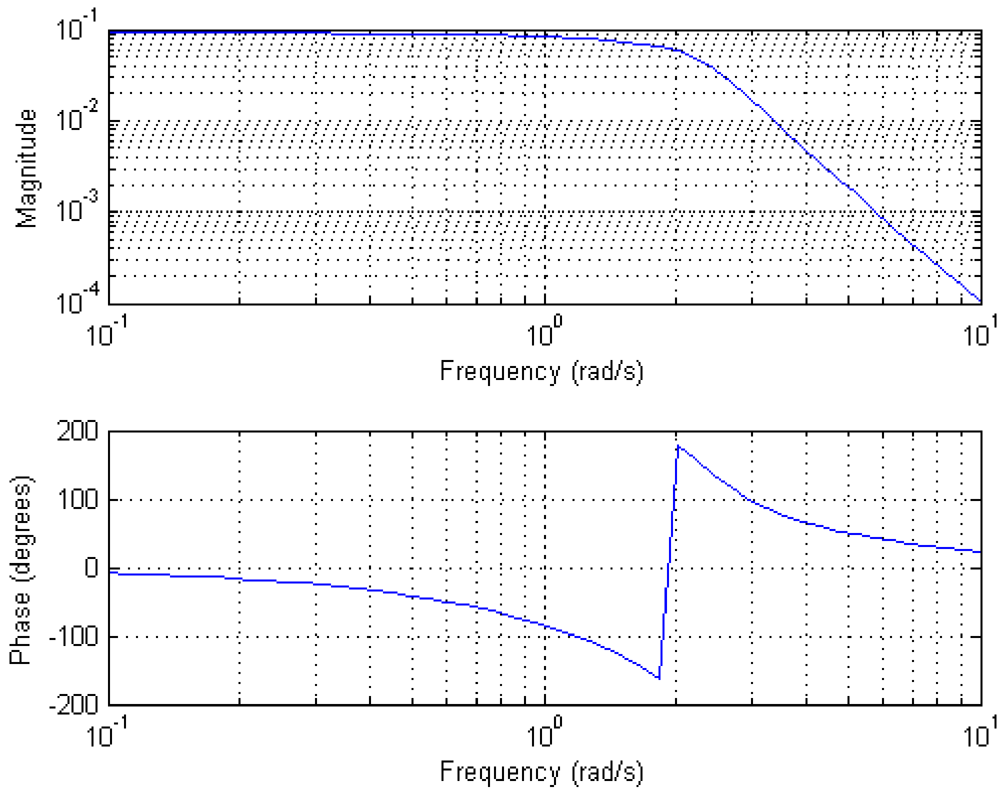
$$Phase(H(j\omega)) = -\tan^{-1}\left(\frac{16\omega - 4\omega^3}{\omega^4 - 11\omega^2 + 11}\right)$$

This is a plot of the magnitude and phase

```
EDU>> num=1;
EDU>> den=[1 4 11 16 11]
EDU>> sys=tf(num,den)

Transfer function:
                1
--------------------------------
s^4 + 4 s^3 + 11 s^2 + 16 s + 11

EDU>> w = logspace(-1,1);
EDU>> freqs(num,den,w)
```



Magnitude
Frequency (rad/s)
Phase (degrees)
Frequency (rad/s)