HW2, EGME 431 (Mechanical Vibration)

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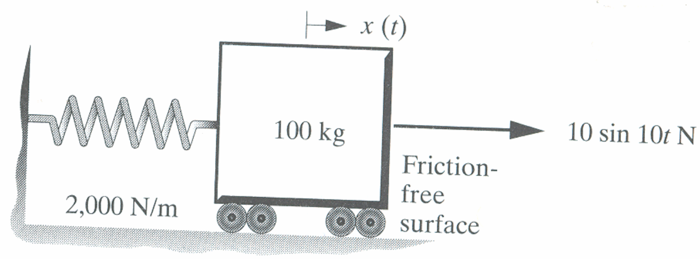
Contents

1 Problem 2.7

2.7. Consider the system in Figure P2.7, write the equation of motion, and calculate the response assuming (a) that the system is initially at rest, and (b) that the system has an initial displacement of 0.05 m.



Solution sketch: Obtain the Lagrangian, find EQM, solve in terms of general initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$, then solve parts (a) and (b) using this general solution.

This is one degree of freedom system. Using x as the generalized coordinates, we first obtain the Lagrangian L

$$L = T - U$$

Where

$$\begin{aligned}
T &= \frac{1}{2}m\dot{x}^2 \\
U &= \frac{1}{2}kx^2 \\
L &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \\
\frac{\partial L}{\partial \dot{x}} &= m\dot{x} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= m\ddot{x} \\
\frac{\partial L}{\partial x} &= -kx
\end{aligned}$$

Hence the EQM is (using the Lagrangian equation), and $F = 10$ and $\omega = 10$ rad/sec. (the forcing frequency)

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= F\sin\omega t \\
m\ddot{x} + kx &= F\sin\omega t \\
\ddot{x} + \frac{k}{m}x &= \frac{F}{m}\sin\omega t \\
\ddot{x} + \omega_n^2 x &= \frac{F}{m}\sin\omega t \qquad (1)
\end{aligned}$$

Where $\omega_n^2 = \frac{k}{m}$. The solution is

$$x(t) = x_h(t) + x_p(t) \qquad (2)$$

To obtain $x_h t(t)$

$$\ddot{x}_h(t) + \omega_n^2 x_h(t) = 0$$

Assume $x_h(t) = e^{\lambda t}$ and substitute in the above ODE we obtain the characteristic equation

$$\begin{aligned}
\lambda^2 + \omega_n^2 &= 0 \\
\lambda &= \pm j\omega_n
\end{aligned}$$

Hence

$$x_h(t) = A\cos\omega_n t + B\sin\omega_n t \qquad (3)$$

Guess

$$\begin{aligned}
x_p(t) &= c_1\cos\omega t + c_2\sin\omega t \\
\dot{x}_p(t) &= -\omega c_1\sin\omega t + \omega c_2\cos\omega t \\
\ddot{x}_p(t) &= -\omega^2 c_1\cos\omega t - \omega^2 c_2\sin\omega t
\end{aligned}$$

Notice, the above guess is valid only under the $\boxed{\text{condition that } \omega \neq \omega_n}$ which is the case in this problem. Now, substitute the above 3 equations into (1) we obtain

$$\begin{aligned}
\left(-\omega^2 c_1\cos\omega t - \omega^2 c_2\sin\omega t\right) + \omega_n^2\left(c_1\cos\omega t + c_2\sin\omega t\right) &= \frac{F}{m}\sin\omega t \\
\sin\omega t\left(-\omega^2 c_2 + \omega_n^2 c_2\right) + \cos\omega t\left(-\omega^2 c_1 + \omega_n^2 c_1\right) &= \frac{F}{m}\sin\omega t
\end{aligned}$$

By comparing coefficients, we obtain

$$c_2\left(\omega_n^2-\omega^2\right)=\frac{F}{m}$$
$$c_2=\frac{F/m}{\left(\omega_n^2-\omega^2\right)}$$

and $c_1=0$, hence

$$\boxed{x_p(t)=\frac{F/m}{\left(\omega_n^2-\omega^2\right)}\sin\omega t}$$

Then from (2) we obtain

$$\begin{aligned}x(t)&=x_h(t)+x_p(t)\\&=x_h(t)+\frac{F/m}{\left(\omega_n^2-\omega^2\right)}\sin\omega t\end{aligned}$$

Using (3) in the above

$$x(t)=A\cos\omega_n t+B\sin\omega_n t+\frac{F/m}{\left(\omega_n^2-\omega^2\right)}\sin\omega t \tag{4}$$

Now assume $x(0)=x_0$ and $\dot{x}(0)=v_0$ For the condition $x(0)=x_0$ we obtain

$$\boxed{x_0=A}$$

For the condition $\dot{x}(0)=v_0$ we obtain

$$\begin{aligned}\dot{x}(t)&=-A\omega_n\sin\omega_n t+B\omega_n\cos\omega_n t+\omega\frac{F/m}{\left(\omega_n^2-\omega^2\right)}\cos\omega t\\ \dot{x}(0)&=v_0=B\omega_n+\omega\frac{F/m}{\left(\omega_n^2-\omega^2\right)}\end{aligned}$$

Hence

$$\boxed{B=\frac{v_0}{\omega_n}-\frac{\omega}{\omega_n}\frac{F/m}{\left(\omega_n^2-\omega^2\right)}}$$

Hence (4) can be written as

$$x(t)=x_0\cos\omega_n t+\left(\frac{v_0}{\omega_n}-\frac{\omega}{\omega_n}\frac{F/m}{\left(\omega_n^2-\omega^2\right)}\right)\sin\omega_n t+\frac{F/m}{\left(\omega_n^2-\omega^2\right)}\sin\omega t$$

Let $\frac{\omega}{\omega_n}=r$, the above becomes

$$x(t)=x_0\cos\omega_n t+\left(\frac{v_0}{\omega_n}-\frac{\frac{F}{m}r}{\omega_n^2\left(1-r^2\right)}\right)\sin\omega_n t+\frac{F/m}{\omega_n^2\left(1-r^2\right)}\sin\omega t$$

But $\omega_n^2=\frac{k}{m}$ hence

$$x(t)=x_0\cos\omega_n t+\left(\frac{v_0}{\omega_n}-\frac{\frac{F}{m}r}{\frac{k}{m}\left(1-r^2\right)}\right)\sin\omega_n t+\frac{\frac{F}{m}}{\frac{k}{m}\left(1-r^2\right)}\sin\omega t$$

Therefore, the general solution is

$$x(t)=x_0\cos\omega_n t+\left(\frac{v_0}{\omega_n}-\frac{F}{k}\frac{r}{\left(1-r^2\right)}\right)\sin\omega_n t+\frac{F}{k}\frac{1}{\left(1-r^2\right)}\sin\omega t \tag{5}$$

1.1 Part(a)

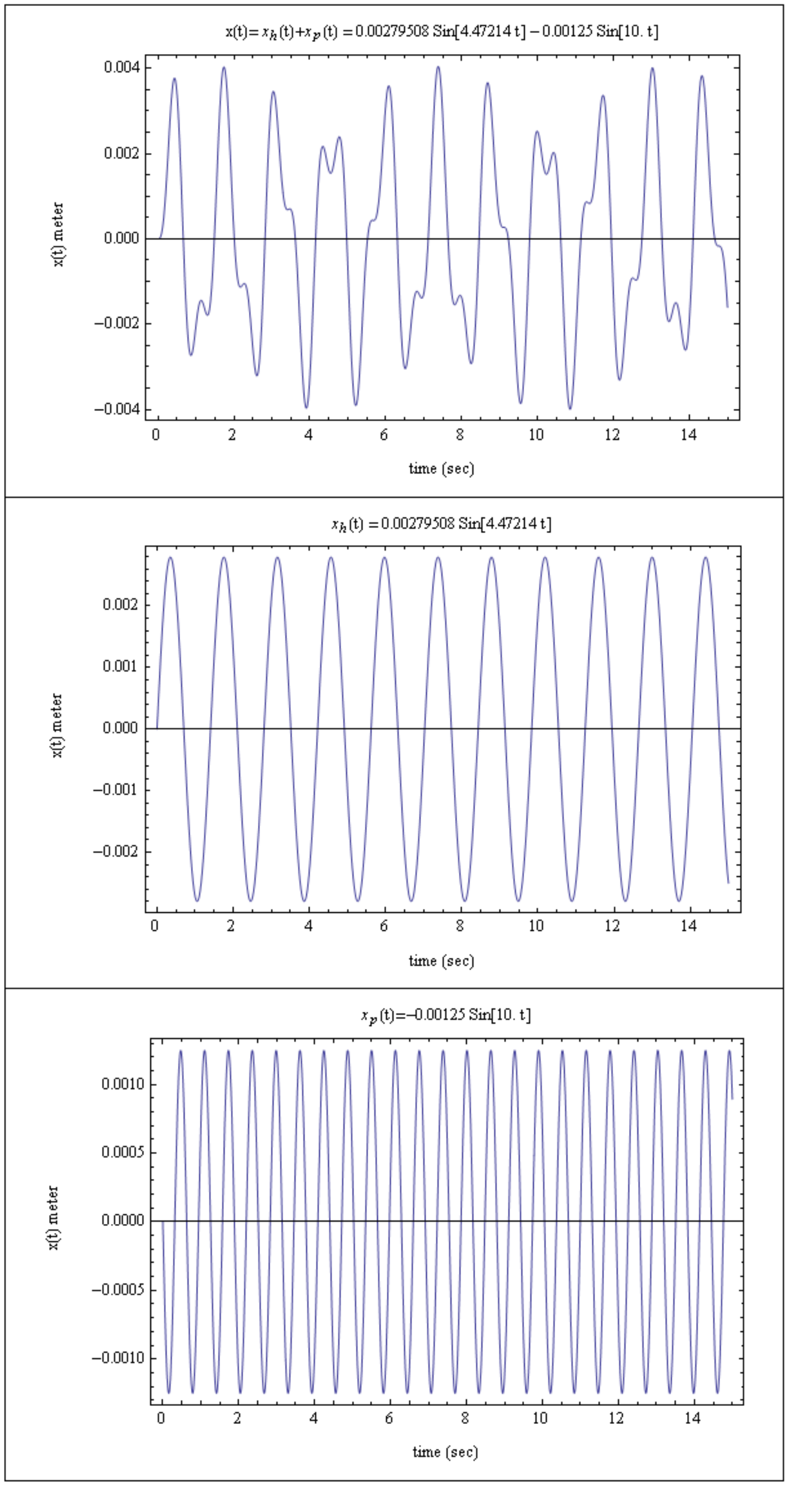
When $x_0 = 0, v_0 = 0$ we obtain from (5)

$$x(t) = \left(-\frac{F}{k}\frac{r}{(1-r^2)}\right)\sin\omega_n t + \frac{F}{k}\frac{1}{(1-r^2)}\sin\omega t \tag{6}$$

Substitute numerical values, and plot the solution. $F = 10, \omega = 10$ rad/sec.$, k = 2000, m = 100, \omega_n = \sqrt{\frac{2000}{100}} = 4.4721, r = \frac{\omega}{\omega_n} = \frac{10}{4.4721} = 2.2361$, then equation (6) becomes

$$\begin{aligned} x(t) &= \left(-\frac{10}{2000}\frac{2.2361}{(1-2.2361^2)}\right)\sin 4.4721t + \frac{10}{2000}\frac{1}{(1-2.2361^2)}\sin 10t \\ &= 0.002795\sin 4.4721t - 0.001250\sin 10t \end{aligned}$$

In the following plot, we show the homogeneous solution and the particular solution separately, then show the general solution.



1.2 Part(b)

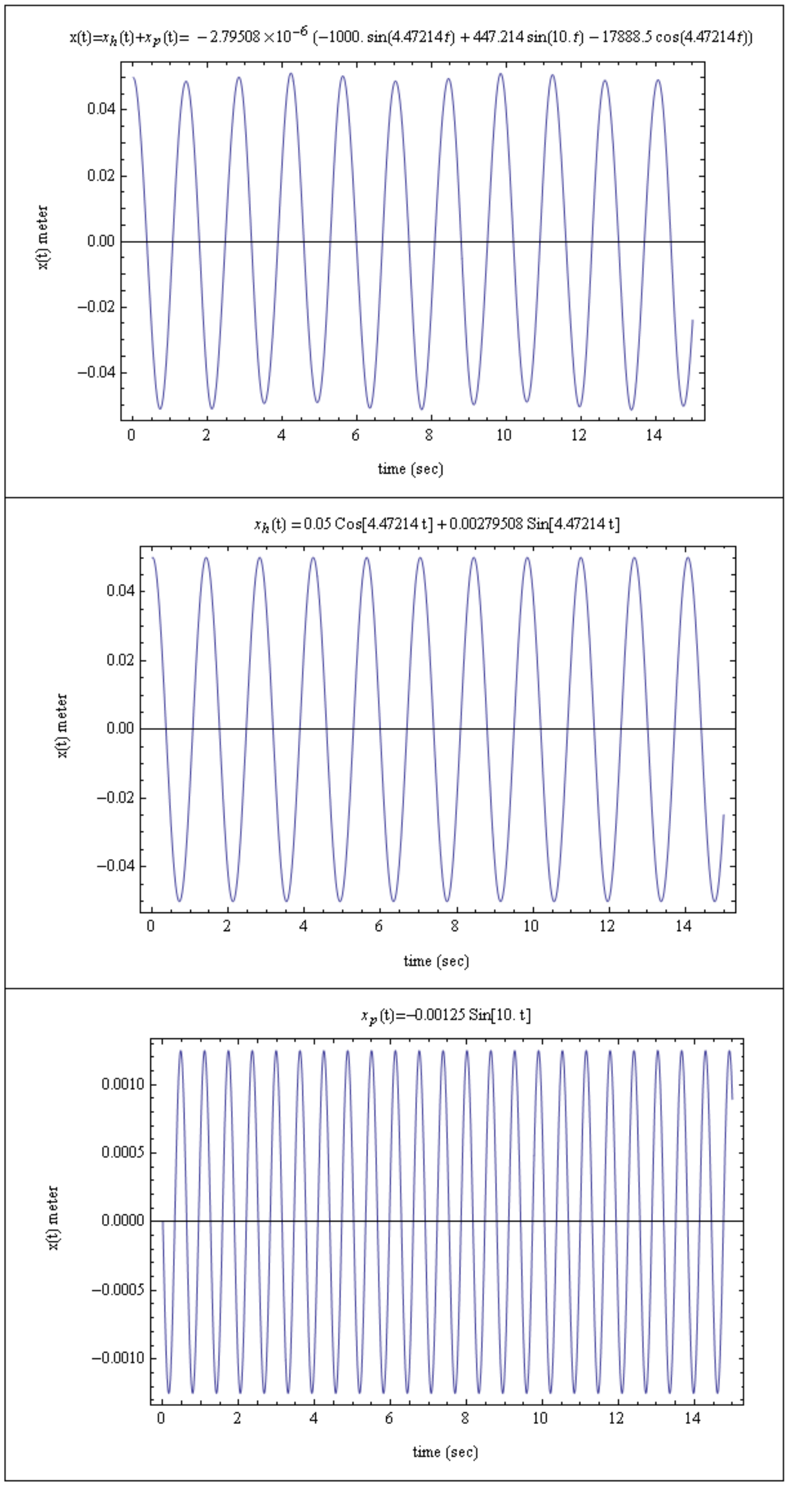
When $x_0 = 0.05$ and $v_0 = 0$ we obtain from (5)

$$x(t) = 0.05 \cos \omega_n t + \left(-\frac{F}{k} \frac{r}{(1-r^2)} \right) \sin \omega_n t + \frac{F}{k} \frac{1}{(1-r^2)} \sin \omega t$$

Substitute numerical values found in part(a), then the solution becomes

$$\begin{aligned} x(t) &= 0.05\cos 4.472\,1t + \left(-\frac{10}{2000}\frac{2.236\,1}{(1-2.236\,1^2)}\right)\sin 4.472\,1t + \frac{10}{2000}\frac{1}{(1-2.236\,1^2)}\sin 10t \\ &= 0.05\cos 4.472\,1t + 0.002795\sin 4.472\,1t - 0.001250\sin 10t \end{aligned}$$

In the following plot, we show the homogeneous solution and the particular solution separately, then show the general solution.



2 Problem 2.10

2.10. Compute the initial conditions such that the response of

$$m\ddot{x} + kx = F_0 \cos \omega t$$

oscillates at only one frequency (ω).

Following the approach taken in problem 2.7, the EQM is

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m} \cos \omega t$$

And $x(t) = x_h(t) + x_p(t)$ where $x_h(t) = A\cos \omega_n t + B \sin \omega_n t$. For $x_p(t)$, guess $x_p(t) = c_1 \cos \omega t + c_2 \sin \omega t$ and following the same steps in problem 2.7, we obtain

$$\sin \omega t \left(-\omega^2 c_2 + \omega_n^2 c_2\right) + \cos \omega t \left(-\omega^2 c_1 + \omega_n^2 c_1\right) = \frac{F_0}{m} \cos \omega t$$

Notice that the above guess is valid only under the $\boxed{\text{condition that } \omega \neq \omega_n}$. Compare coefficients, we find $c_2 = 0$ and

$$c_1 = \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2}$$

Hence

$$x_p(t) = \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2} \cos \omega t$$

Then, the general solution is

$$x(t) = A\cos \omega_n t + B \sin \omega_n t + \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2} \cos \omega t \tag{1}$$

Let, at $t = 0$, $x(0) = x_0$, and $\dot{x}(0) = v_0$,then from (1), we find

$$x_0 = A + \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2}$$
$$A = x_0 - \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2}$$

And since

$$\dot{x}(t) = -A\omega_n \sin \omega_n t + B\omega_n \cos \omega_n t - \omega \frac{F_0}{m} \frac{1}{\omega_n^2 - \omega^2} \sin \omega t$$

Then

$$v_0 = B\omega_n$$
$$B = \frac{v_0}{\omega_n}$$

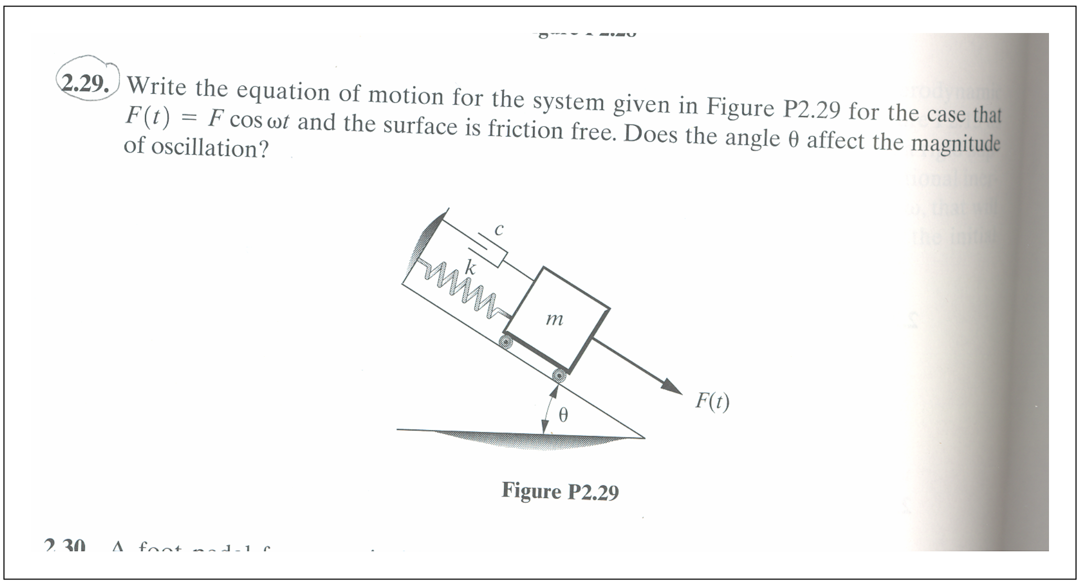
Therefore, the general solution is (from (1))

$$\boxed{x(t) = \left(x_0 - \frac{F_0}{m}\frac{1}{\omega_n^2 - \omega^2}\right)\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t + \frac{F_0}{m}\frac{1}{\omega_n^2 - \omega^2}\cos\omega t}$$

To make the response oscillate at frequency ω only, we can set $v_0 = 0$ to eliminate the $\sin\omega_n t$, and set $x_0 = \frac{F_0}{m}\frac{1}{\omega_n^2 - \omega^2}$ to eliminate the $\cos\omega_n t$ term. Hence, the initial conditions are

$$v_0 = 0$$
$$x_0 = \frac{F_0}{m}\frac{1}{\omega_n^2 - \omega^2}$$

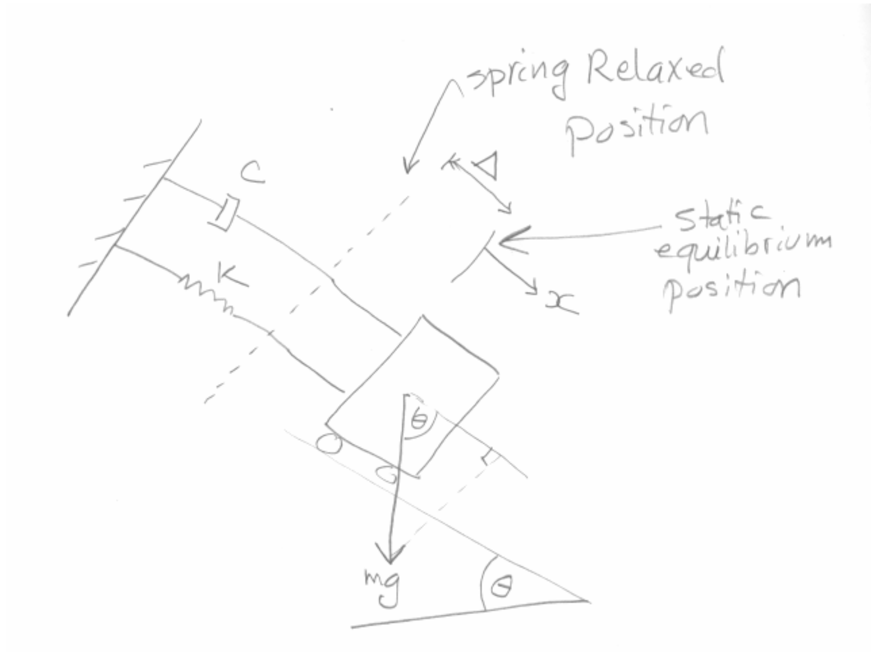
3 Problem 2.29



2.29. Write the equation of motion for the system given in Figure P2.29 for the case that $F(t) = F\cos\omega t$ and the surface is friction free. Does the angle θ affect the magnitude of oscillation?

Figure P2.29

This is one degree of freedom system. Using x along the inclined surface as the generalized coordinates, we first obtain the Lagrangian L



We first note that $k\Delta = mg\cos\theta$ and the mass will lose potential as it slides down the surface. We measure everything from the relaxed position (not the static equilibrium.) This is done to show more clearly that the angle do not affect the solution.

$$L = T - U$$

Where

$$\begin{aligned} T &= \frac{1}{2}m\left(\frac{d}{dt}(x+\Delta)\right)^2 \\ &= \frac{1}{2}m\dot{x}^2 \\ U &= \frac{1}{2}k(x+\Delta)^2 - mg(x+\Delta)\cos\theta \end{aligned}$$

Hence

$$\begin{aligned} L &= \frac{1}{2}m\dot{x}^2 - \left(\frac{1}{2}k(x+\Delta)^2 - mg(x+\Delta)\cos\theta\right) \\ &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x+\Delta)^2 + mgx\cos\theta + mg\Delta\cos\theta \\ \frac{\partial L}{\partial \dot{x}} &= m\dot{x} \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= m\ddot{x} \\ \frac{\partial L}{\partial x} &= -k(x+\Delta) + mg\cos\theta \\ &= -kx - k\Delta + mg\cos\theta \end{aligned}$$

But $k\Delta = mg\cos\theta$, hence the above reduces to

$$\frac{\partial L}{\partial x} = -kx$$

Hence the EQM is (using the Lagrangian equation)

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= F\cos\omega t \\
m\ddot{x} + kx &= F\cos\omega t \\
\ddot{x} + \omega_n^2 x &= \frac{F}{m}\cos\omega t \qquad (1)
\end{aligned}
$$

Where $\omega_n^2 = \frac{k}{m}$. We see that the angle θ is not in the EQM. Hence the solution does not involve θ and the $\boxed{\text{oscillation magnitude is not affected by the angle}}$. Intuitively, the reason for this is because the angle effect is already counted for to reach the static equilibrium. Once the system is in static equilibrium, the angle no longer matters as far as the solution is concerned.

4 Problem 2.46

2.46. Consider Example 2.4.1 for car 1, illustrated in Figure P2.46, if three passengers totaling 200 kg are riding in the car. Calculate the effect of the mass of the passengers on the deflection at 20, 80, 100, and 150 km/h. What is the effect of the added passenger mass on car 2?

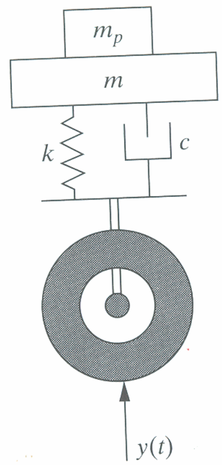


Figure P2.46 Model of a car suspension with the mass of the occupants, m_p, included.

From example 2.4.1, we note the following table

TABLE 2.1 COMPARISON OF CAR VELOCITY, FREQUENCY, AND DISPLACEMENT FOR TWO DIFFERENT CARS

Speed (km/h)	ω_b	r_1	r_2	x_1 (cm)	x_2 (cm)
20	5.817	0.923	1.158	3.19	2.32
80	23.271	3.692	4.632	0.12	0.07
100	29.088	4.615	5.79	0.09	0.05
150	43.633	6.923	8.686	0.05	0.03

Also, from example 2.4.1, the mass of car 1 is $1007kg$ and the mass of car 2 is $1585kg$. Hence we write

$$m_1 = 1007kg$$
$$m_2 = 1585kg$$

To find the deflection of the car, we use equation 2.70 in the book, which is

$$X = Y\sqrt{\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}$$

Where X is the magnitude of the steady state deflection and Y is the magnitude of the base deflection, which is given as 0.01 meters in the example.

Hence, for each speed, we calculate ω_p and then we find $\omega_n = \sqrt{\frac{k}{m_1+m_p}}$ and then find $r = \frac{\omega_p}{\omega_n}$ and then find $\xi = \frac{c}{2\sqrt{k(m_1+m_p)}}$ and then using equation(1), we calculate X. This is done for each different speed (all for car m_1). Next, we do the same for car m_2. These calculation are shown in the following table. Note also that $\boxed{c = 2000 \text{ N s/m}}$ as given in the example and $\boxed{k = 4 \times 10^4 \text{N/m}}$

car 1

$v\ (km/h)$	$\omega_p = 0.2909\ v$	$\omega_n = \sqrt{\frac{k}{m_1+m_p}}$	$r = \frac{\omega_p}{\omega_n}$	$\xi = \frac{c}{2\sqrt{k(m_1+m_p)}}$	$X = Y\sqrt{\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}\ (cm)$
20	5.818	5.7567	1.0106	0.14392	3.571
80	23.272	5.7567	4.0426	0.14392	0.0997
100	29.09	5.7567	5.0532	0.14392	0.0718
150	43.635	5.7567	7.5799	0.14392	0.0425

car 2

$v\ (km/h)$	$\omega_p = 0.2909\ v$	$\omega_n = \sqrt{\frac{k}{m_2+m_p}}$	$r = \frac{\omega_p}{\omega_n}$	$\xi = \frac{c}{2\sqrt{k(m_2+m_p)}}$	$X = Y\sqrt{\frac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}\ (cm)$
20	5.818	4.7338	1.229	0.11835	1.7726
80	23.272	4.7338	4.9161	0.11835	0.066141
100	29.09	4.7338	6.1452	0.11835	0.04797
150	43.635	4.7338	9.2178	0.11835	0.02857

Observations: The heavier car (car 2) has smaller defection (X values) for all speeds. Adding passengers, causes ω_n to change. This results in making the deflection smaller when passengers are in the car as compared without them. Heaver cars and heavier passenger results in smaller deflection values. For the lighter car however, adding the passenger did not result in smaller deflection for all speeds. For speed $v = 20$, adding the passenger caused a larger defection (3.19 vs. 3.571). As car 1 speed became larger, the deflection became smaller for both cars.

So, in conclusion: lighter cars have larger deflections at bumps, and the faster the car, the smaller the deflection.

5 Problem 2.57

2.57. Consider a typical unbalanced machine problem as given in Figure P2.57 with a machine mass of 120 kg, a mount stiffness of 800 kN/m, and a damping value of 500 kg/s. The amplitude of the out-of-balance force is measured to be 374 N at a running speed of 3000 rev/min. (a) Determine the amplitude of motion due to the out of balance. (b) If the out-of-balance mass is estimated to be 1% of the total mass, estimate the value of *e*.

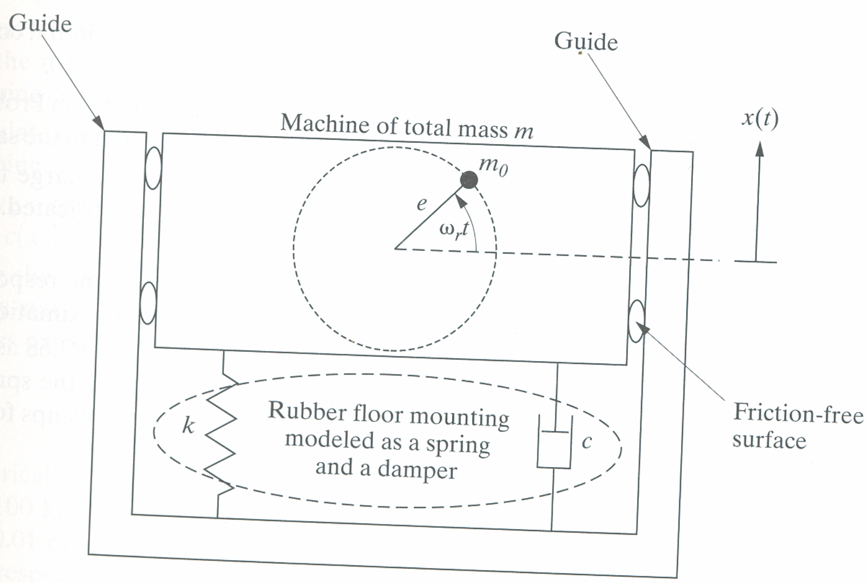


Figure P2.57 Typical unbalanced machine problem.

Given $m = 120\ kg, k = 800 \times 10^3$N/M, $c = 500$ kg/s, and mass m_0 has angular speed of $\omega_r = \frac{3000\times 2\pi}{60} = 100\pi$ radians per seconds

5.1 Part(a)

The rotating mass will cause a downward force as the result of the centripetal force $m_0 e\omega_r^2 \sin(\omega_r t)$. Hence the reaction to this force on the machine will be in the upward direction. Hence

$$F_r = m_0 e\omega_r^2 \sin(\omega_r t)$$

Hence the machine equation of motion is

$$\begin{aligned} m\ddot{x} + c\dot{x} + kx &= m_0 e\omega_r^2 \sin(\omega_r t) \\ \ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x &= \frac{m_0}{m} e\omega_r^2 \sin(\omega_r t) \qquad (1) \end{aligned}$$

By guessing $x_p = X\sin(\omega_r t - \theta)$ then we find that (The method of undetermined coefficients is used, derivation is show in text book at page 115)

$$X = \frac{m_o e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \tag{2}$$

This is the maximum magnitude of motion in steady state. In the above, $r = \frac{\omega_r}{\omega_n}$. Hence to find X we substitute the given values in the above expression. We first note that we are told that $m_0 e\omega_r^2 = 374N$, hence $m_o e = \frac{374}{\omega_r^2}$ but we found that $\omega_r = 100\pi$ rad/sec, hence

$$m_o e = \frac{374}{(100\,\pi)^2} = 0.0037894$$

And

$$r = \frac{\omega_r}{\omega_n} = \frac{100\pi}{\sqrt{\frac{k}{m}}} = \frac{100\,\pi}{\sqrt{\frac{800\times 10^3}{120}}} = 3.8476$$

And

$$\xi = \frac{c}{2\sqrt{km}} = \frac{500}{2\sqrt{800\times 10^3 \times 120}} = 0.025516$$

Substitute into (2) we obtain

$$\begin{aligned} X &= \left(\frac{0.0037894}{120}\right) \frac{3.8476^2}{\sqrt{(1-3.8476^2)^2 + (2\times 0.025516\times 3.8476)^2}} \\ &= 3.3863\times 10^{-5}\ meter \end{aligned}$$

5.2 Part(b)

We are told that $m_o = 0.01\times m$, hence $m_o = 0.01\times 120 = 1.2kg$. And since we are told that $m_0 e\omega_r^2 = 374N$, then

$$e = \frac{374}{m_0\omega_r^2} = \frac{374}{1.2\times (100\pi)^2} = 3.1578\times 10^{-3} \text{ meter}$$