Study notes. Math 504 (Simulation), CSUF. spring 2008

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Contents

1 some code I wrote for testing things...

1. `expected_test.m`
2. `solving_3_dot_6,nb`
3. A small note on using the left eigenvector of the one step transition matrix for a regular chain to determine the limiting distribution `trying_8_1_problem.nb` `trying_8_1_problem.pdf`

2 Definitions

Source of this block unknown and lost from the net:

Continuous Time Markov Chains Most of our models will be formulated as continuous time Markov chains. On this

2.1 Regular finite M.C.

definition 1: There exist some n such that $P^{(n)}$ has all positive entries
definition 2: A regular finite chain is one which is irreducible and aperiodic
Notice that this means regular chain has NO transient states.

2.2 irreducible M.C.

A M.C. which contains one and only one closed set of states. Note that for finite MC, this means all the states are recurrent. In otherwords, its state space contains no proper subset that is closed.

2.3 Stationary distribution

This is the state vector π which contains the probability of each state that the MC could be in the long term. For an irreducible MC, this is independent of the starting $\pi^{(0)}$, however, for a reducible MC, the Stationary distribution will be different for different initial $\pi^{(0)}$

2.4 recurrent state

1. $f_{ii} = 1$. In otherwords, the probability of reaching state i eventually, starting from state i is always certain.
2. $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$, in otherwords, since sum diverges, this means the probability to return back to i starting from i will always exist, not matter how large n is (i.e. sum terms never reach all zeros after some limiting value n)

2.5 transient state

1. $f_{ii} < 1$. In otherwords, the probability of reaching state i eventually, starting from state i is not certain. i.e. there will be a chance that starting from i, chain will never again get back to state i.

2. $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$, in otherwords, since sum converges, this means the probability to return back to i starting from i will NOT always exist (i.e. sum terms reach all zeros after some limiting value n)

2.6 Positive recurrent state

A recurrent state where the expected number of steps to return back to the state is finite.

2.7 Null recurrent state

A recurrent state where the expected number of steps to return back to the state is infinite.

2.8 Period of state

GCD of the integers n such that $p_{ii}^{(n)} > 0$. In otherwords, find all the steps MC will take to return back to the same state, then find the GCD of these values. If the GCD is 1, then the period is 1 and the state is called Aperiodic (does not have a period).

2.9 Ergodic state

A state which is Aperiodic and positive recurrent. i.e. a recurrent state (with finite number of steps to return) but it has no period.

2.10 First entrance time T_{ij}

The number of steps needed to reach state j (first time) starting from transient state i

2.11 $f_{ij}^{(n)}$

This is the probability that it will take n steps to first reach state j starting from transient state i. i..e $f_{ij}^{(n)} = P\left(T_{ij} = n\right)$.

2.12 f_{ij}

This is the probability of reaching state j (for first time) when starting from transient state i. Hence $f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} = P\left(T_{ij} < \infty\right)$

2.13 Closed set

A set of states, where if MC enters one of them, it can't reach a state outside this set. i.e. $P_{ij} = 0$ whenever $i \in S$ and $j \notin S$, then set S is called closed set.

2.14 Absorbing M.C.

All none-transient states are absorbing states. Hence the P matrix looks like $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ R & R & q_{11} & q_{12} \\ R & R & q_{21} & q_{22} \end{bmatrix}$ i.e. $\begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$

2.15 Q Matrix

Properties of a Q matrix are: There is at least one row which sums to less than 1. And there is a way to reach such row(s) from other others. and $Q^n \to 0$ as $n \to \infty$

2.16 Balance equations

3 HOW TO finite Markov chain

3.1 How to find $f_{ij}^{(n)}$

This is the probability it will take n steps to first reach state j from state i. In Below $C(J)$ means the closed set which contains the state j and T means the transient set

$i \in C(J), j \in C(J)$	use formula (1) below
$i \in C(J), j \notin C(J)$	$f_{ij}^{(n)} = 0$
$i \in T$, j Absorbing state	Calculate $A^m = Q^m R$ where $m = n - 1$ then the i, j entry of A^m gives $f_{ij}^{(n)}$
$i \in T, j \in T$	Normally we are interested in finding expected number of visits to j before absorbing. i.e $E\left(V_{ij}\right)$. see below. Otherwise use (1)

We can use $p_{ij}^{(n)}$. Notice the subtle difference between $f_{ij}^{(n)}$ and $p_{ij}^{(n)}$.
$f_{ij}^{(n)}$ gives the probability of needing n steps to first reach j from i, while $p_{ij}^{(n)}$ gives the probability of being in state j after n steps leaving i. So with $p_{ij}^{(n)}$ could have reached state j before n steps, but left state j and moved around, then came back, as long as after n steps exactly MC will be in state j. With $f_{ij}^{(n)}$ this is not allowed. The chain must reach state j the very first time in n steps from leaving i. So in a sense, $f_{ij}^{(n)}$ is a more strict probability. Using the recursive formula

$$p_{ij}^{(n)} = f_{ij}^{(n)} + f_{ij}^{(n-1)} p_{jj}^{(1)} + f_{ij}^{(n-2)} p_{jj}^{(2)} + \cdots + f_{ij}^{(1)} p_{jj}^{(n-1)} \tag{1}$$

We can calculate $f_{ij}^{(n)}$. We see that $f_{ij}^{(1)} = p_{ij}^{(1)}$ and so $f_{ij}^{(2)} = p_{ij}^{(2)} - f_{ij}^{(1)} p_{jj}^{(1)}$ and also

$$\begin{aligned}
f_{ij}^{(3)} &= p_{ij}^{(3)} - f_{ij}^{(2)} p_{jj}^{(1)} - f_{ij}^{(1)} p_{jj}^{(2)} \\
&= p_{ij}^{(3)} - \left(p_{ij}^{(2)} - f_{ij}^{(1)} p_{jj}^{(1)}\right) p_{jj}^{(1)} - f_{ij}^{(1)} p_{jj}^{(2)} \\
&= p_{ij}^{(3)} - \left(p_{ij}^{(2)} p_{jj}^{(1)} - p_{ij}^{(1)} \left[p_{jj}^{(1)}\right]^2\right) - p_{ij}^{(1)} p_{jj}^{(2)} \\
&= p_{ij}^{(3)} - p_{ij}^{(2)} p_{jj}^{(1)} + p_{ij}^{(1)} \left[p_{jj}^{(1)}\right]^2 - p_{ij}^{(1)} p_{jj}^{(2)}
\end{aligned}$$

and

$$\begin{aligned}
f_{ij}^{(4)} &= p_{ij}^{(4)} - f_{ij}^{(3)} p_{jj}^{(1)} - f_{ij}^{(2)} p_{jj}^{(2)} - f_{ij}^{(1)} p_{jj}^{(3)} \\
&= p_{ij}^{(4)} - \left(p_{ij}^{(3)} - p_{ij}^{(2)} p_{jj}^{(1)} + p_{ij}^{(1)} \left[p_{jj}^{(1)}\right]^2 - p_{ij}^{(1)} p_{jj}^{(2)}\right) p_{jj}^{(1)} - \left(p_{ij}^{(2)} - f_{ij}^{(1)} p_{jj}^{(1)}\right) p_{jj}^{(2)} - f_{ij}^{(1)} p_{jj}^{(3)} \\
&= p_{ij}^{(4)} - \left(p_{ij}^{(3)} p_{jj}^{(1)} - p_{ij}^{(2)} \left[p_{jj}^{(1)}\right]^2 + p_{ij}^{(1)} \left[p_{jj}^{(1)}\right]^3 - p_{ij}^{(1)} p_{jj}^{(1)} p_{jj}^{(2)}\right) - \left(p_{ij}^{(2)} p_{jj}^{(2)} - p_{ij}^{(1)} p_{jj}^{(1)} p_{jj}^{(2)}\right) - p_{ij}^{(1)} p_{jj}^{(3)} \\
&= p_{ij}^{(4)} - p_{ij}^{(3)} p_{jj}^{(1)} + p_{ij}^{(2)} \left[p_{jj}^{(1)}\right]^2 - p_{ij}^{(1)} \left[p_{jj}^{(1)}\right]^3 + p_{ij}^{(1)} p_{jj}^{(1)} p_{jj}^{(2)} - p_{ij}^{(2)} p_{jj}^{(2)} + p_{ij}^{(1)} p_{jj}^{(1)} p_{jj}^{(2)} - p_{ij}^{(1)} p_{jj}^{(3)} \\
&= p_{ij}^{(4)} - p_{ij}^{(3)} p_{jj}^{(1)} - p_{ij}^{(2)} p_{jj}^{(2)} + p_{ij}^{(2)} \left[p_{jj}^{(1)}\right]^2 - p_{ij}^{(1)} \left[p_{jj}^{(1)}\right]^3 + 2 p_{ij}^{(1)} p_{jj}^{(1)} p_{jj}^{(2)} - p_{ij}^{(1)} p_{jj}^{(3)}
\end{aligned}$$

etc...
Hence knowing just the P matrix, we can always obtain values of the f_{ij} for any powers
However, using the following formula, from lecture notes 6.2 is easier

$$A^{(n)} = Q^n R$$

the i, j entry of $A^{(n)}$ gives the probability of taking $n+1$ steps to first reaching j when starting from transient state i . So use this formula. Just note this formula works only when i is transient.

question : If i is NOT transient, and we asked to find what is the prob. it will take n steps to first reach state j from state i. Then use (1). right?

3.2 How to find f_{ij}

This is the probability that chain will eventually reach state j given it starts in state i

$i \in C(J)$ $j \in C(J)$	$f_{ij} = 1$
$i \in C(J)$ $j \notin C(J)$	$f_{ij} = 0$
$i \in T$ j recurrent but not absorbent, hence in a closed set with other states	Use formula in page 5.5 lecture notes $f_{ij} = \sum_{k \in T} p_{ik} f_{kj} + \sum_{k \in C(J)} p_{ik}$ $F = (I-Q)^{-1} Z$, hence just needs to find Z z_{ij} =probability that transient state i will reach class that contains j
$i \in T$ j is an absorbent	$f_{ij} = \left[(I-Q)^{-1} R\right]_{i,j}$
$i \in T$ $j \in T$	We know eventually $p_{ij} = 0$ for $i, j \in Q$, but can we talk about f_{ij} here?

3.3 How to find $E\left(V_{ij}\right)$ the expected number of visits to j before absorbing?

Here, $i \in T$ and $j \in T$ Then

$$E\left(V_{ij}\right) = (I-Q)^{-1}_{i,j}$$

The above gives the average number of visits to state j (also transient) before chain is absorbed for first time.
question: Note that if chain is regular, then all states communicates with each others and then $i \in R, j \in R$ and so $E\left(V_{ij}\right)$ can be found from the stationary distribution π^{∞} , right?

3.4 How to find average number of steps $E\left(T_{ij}\right)$ between state i and state j?

$regular : i \in R, j \in R$	$E\left(T_{ij}\right) = 1 + \sum_{k \neq j} p_{ik} E\left(T_{kj}\right)$ if $i = j$ then $E\left(T_{ij}\right) = \frac{1}{w_{jj}}$ where w is the stationary probability vector
$i \in T, j \in T$	does not make sense to ask this here?

3.5 How to find number of visits to a transient state?

Number of visits to transient state is a geometric distribution.

$$\Pr(n) = f_{ii}^{n-1}(1 - f_{ii})$$

The expected number of visits to transient state i is

$$E(X) = \frac{1}{1 - f_{ii}}$$

where f_{ii} is the probability of visiting state i if chain starts in state i

4 Some useful formulas

$\lim_{n\to\infty}\left(1 - \frac{z}{n}\right)^n = e^{-z}$
$\lim_{h\to 0}(1 - \lambda h + o(h))^{\frac{t}{h}} = e^{-\lambda t}$

4.1 Law of total probability

$$\Pr(A) = \sum \Pr(A|B_i)\Pr(B_i)$$

4.2 Conditional (Bayes) formula

$$\Pr(A|B) = \frac{\Pr(A,B)}{\Pr(B)}$$

4.3 Inverse of a 2 by 2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$$

4.4 $\omega_j = \frac{1}{\mu_j}$

The above says that for a regular finite MC, where a stationary probability exist (and is unique), then it is inverse of the mean number of steps between visits μ_j to state j in steady state.

4.5 $\omega = \omega P$

The above says that for a regular M.C. there exist a stationary probability distribution ω

4.6 Poisson random variable

N is number of events that occur over some period of time.
N is a Poisson random variable if

1. $N(0) = 0$

2. Independent increments

3. $P(N = n) = \frac{\lambda^n e^{-\lambda}}{n!}$

Where λ is the average number of events that occur over the same period that we are asking for the probability of this number of events to occur. Hence remember to adjust λ accordingly if we are given λ as rate (i.e. per unit time).

4.7 Poisson random Process

$N(t)$ is a Poisson random variable if

1. $N(0) = 0$
2. Independent increments
3. $P(N(t) = n) = \frac{(\lambda t)^n e^{-(\lambda t)}}{n!}$

Where λ is the average number of events that occur in one unit time. So $N(t)$ is random variable which is the number of events that occur during interval of length t

The probability that ONE event occure in the next h interval, when the interval is very small, is $\lambda h + o(h)$

This can be seen by setting $n = 1$ in the definition and using series expansion for $e^{-(\lambda h)}$ and then letting $h \rightarrow 0$
Expected value of Poisson random variable: $E(N) = \lambda$. For a process, $E(N(t)) = \lambda t$ where λ is the rate.

4.8 Exponential random variable

T is random variable which is the time between events where the number of events occur as Poisson distribution,
pdf: $f(t) = \lambda e^{-\lambda t}$

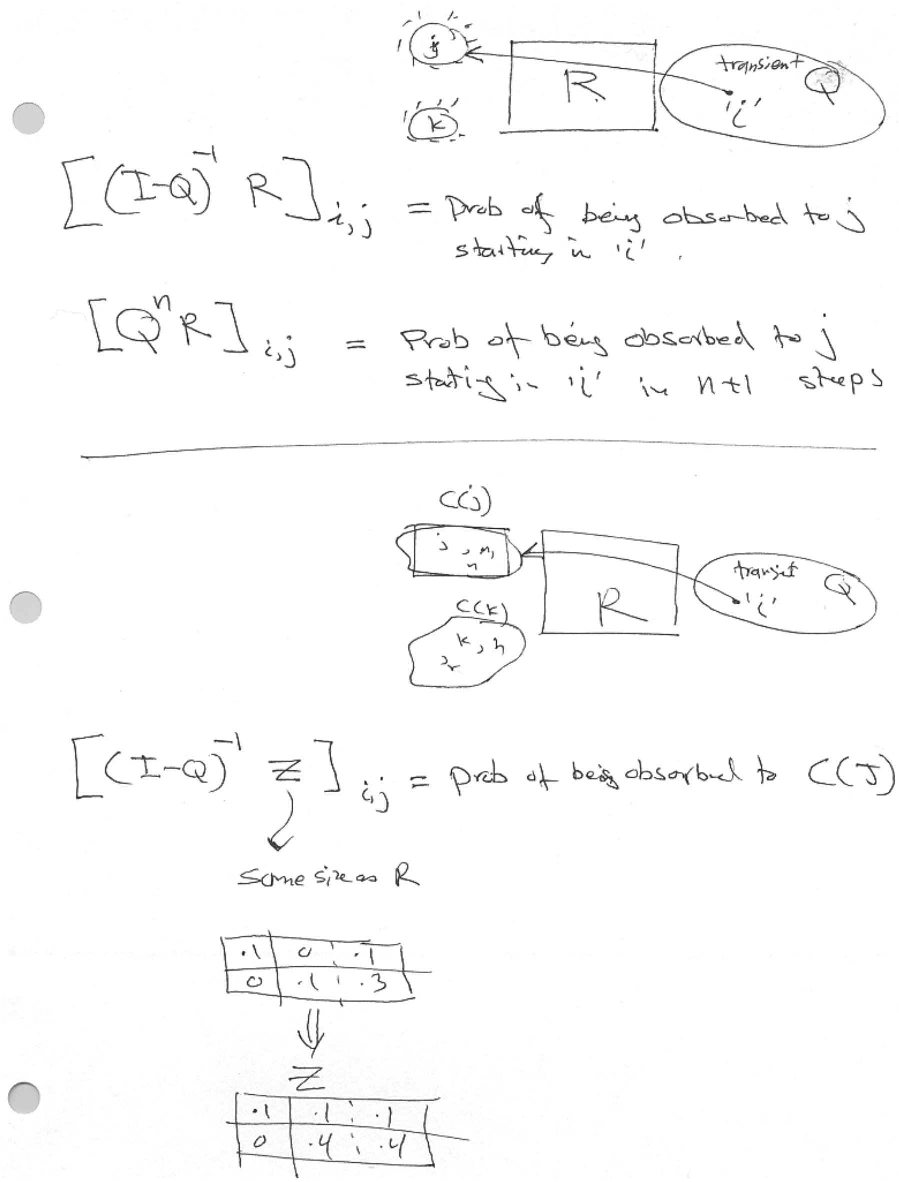
$$P(T > t) = \int_t^\infty \lambda e^{-\lambda s} ds = e^{-\lambda t}$$

$$P(T < t) = \int_0^t \lambda e^{-\lambda s} ds = -\left[e^{-\lambda s}\right]_0^t = -\left[e^{-\lambda t} - 1\right] = 1 - e^{-\lambda t}$$

pdf=derivative of CDF

Probability that the waiting time for n events to occur $\leq t$ is a GAMMA distribution. $g_n(t) = \frac{\lambda}{(n-1)!}(\lambda t)^{n-1} e^{-\lambda t}$

5 Diagram to help understanding



5.1 Continouse time Markov chain

$$p_{ii}(h) = 1 - v_i h + o(h)$$
$$p_{ij}(h) = q_{ij} h + o(h) \qquad i \neq j$$

v_i is the parameter (rate) for the exponential distributed random variable which represents the time in that state. Hence The probability that system remains in state i for time larger than t is given by

$$\Pr(T_i > t) = e^{-v_i t}$$

∘) Jump probability $Q_{ij} = \frac{q_{ij}}{v_i}$ for $i \neq j$. This is the probability of going from state i to state j (once the process leaves state i)
∘) FOrward Komogolv equation
$P'(t) = P(t) Q$, let $z(t) = z(0) P(t)$, hence $z'(t) = z(0) P'(t)$, hence $z'(t) = z(0) P(t) Q$ therefore

$$z'(t) = z(t)Q$$

∘) Balance equations

$$\pi_j v_j = \sum_{k \neq j} q_{kj} \pi_k$$

This is 'flow out' = 'flow in'.
This equation can also be obtaind more easily I think from $\pi Q = \mathbf{0}$ Where Q is the matrix made up from the $q's$ and the $v's$ on the diagonal. Just write then down, and at the end add $\pi_0 + \pi_1 + \cdots = 1$ to find π_0