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HW # 9

Math 501

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Problem # 9, Section 4.7

Computer Assignment 4/4/2007

Iterative Eigen values

Power

Inverse Power

Shifted Power

Shifted Inverse Power

problem # 9, section 4.7

let A be $n\times n$, and A-orthonormal system exist, show that A is SPD.

Solution outline: express $\bar{x}$ in terms of basis $u^{(i)}$, start with definition of $\bar{x}^T A x$, show this must be >0. utilize property of A-orthonormal to cancel terms:

solution: given A $n\times n$

given A-orthonormal system $[u^1|u^2\cdots|u^n]$

i.e $\langle u^i, Au^j\rangle = \delta_{ij}$

$$(1)\quad \text{i.e}\left\{\begin{array}{lll} (u^1)^T A u^1 = 1 & (u^2)^T A u^1 = 0 & \cdots (u^n)^T A u^1 = 0 \\ (u^1)^T A u^2 = 0, & (u^2)^T A u^2 = 1 & (u^n)^T A u^2 = 0 \\ (u^1)^T A u^3 = 0 & \vdots & \vdots \\ \vdots & & \\ (u^1)^T A u^n = 0 & (u^2)^T A u^n = 0 \cdots & (u^n)^T A u^n = 1 \end{array}\right.$$

now A is SPD if $\bar{x}^T A x > 0$ $(\bar{x}\neq 0)$

since u^i are Basis in $\mathbb{R}^n$, we can write vector $\bar{x}$ in terms of these basis. so

$$\bar{x} = \langle\bar{x},\bar{u}^1\rangle\bar{u}^1 + \langle\bar{x},\bar{u}^2\rangle\bar{u}^2 + \cdots + \langle\bar{x},\bar{u}^n\rangle\bar{u}^n$$

hence using this, we now write $\bar{x}^T A\bar{x}$ and compare to (1)

$$\bar{x}^T A\bar{x} = \left(\langle\bar{x},\bar{u}^1\rangle\bar{u}^1 + \langle\bar{x},\bar{u}^2\rangle\bar{u}^2 + \cdots + \langle\bar{x},\bar{u}^n\rangle\bar{u}^n\right)^T A\left(\langle\bar{x},\bar{u}^1\rangle\bar{u}^1 + \cdots + \langle\bar{x},\bar{u}^n\rangle\bar{u}^n\right)$$

let $\langle\bar{x},\bar{u}^1\rangle = a_1$, $\langle\bar{x},u^2\rangle = a_2$ etc... these are the coordinates of $\bar{x}$ in this subspace. so we write

$$\bar{x}^T A\bar{x} = (a_1\bar{u}^1 + a_2\bar{u}^2 + \cdots + a_n\bar{u}^n)^T A(a_1\bar{u}^1 + a_2\bar{u}^2 + \cdots + a_n\bar{u}^n)$$

$$= (a_1\bar{u}^1 + a_2\bar{u}^2 + \cdots + a_n\bar{u}^n)^T(A a_1\bar{u}^1 + A a_2\bar{u}^2 + \cdots + A a_n\bar{u}^n)$$

$$= a_1^2(\bar{u}^1)^T A u^1 + a_1a_2\bar{u}^{1T} A\bar{u}^2 + a_1a_3\bar{u}^{1T} A\bar{u}^3 + \cdots + a_1a_n(\bar{u}^1)^T A\bar{u}^n$$
$$+ a_2a_1(\bar{u}^2)^T A u^1 + a_2^2(\bar{u}^2)^T A\bar{u}^2 + a_2a_3(u^2)^T A\bar{u}^3 + \cdots + a_2a_n(\bar{u}^2)^T A\bar{u}^n$$
$$\cdots + \cdots + a_na_1(\bar{u}^n)^T A u^1 + a_na_2(\bar{u}^n)^T A\bar{u}^2 + \cdots + \cdots + a_n^2(\bar{u}^n)^T A\bar{u}^n \rightarrow$$

So we see the pattern for $\bar{x}^T A \bar{x}$ as

$$= a_1^2 (\bar{u}^1)^T A \bar{u}^1 + \cdots + \cdots \quad (2)$$
$$+ a_2^2 (\bar{u}^2)^T A \bar{u}^2 + \cdots$$
$$\cdots + a_3^2 (\bar{u}^3)^T A \bar{u}^3 + \cdots$$

But since from (1) we see that $(u^1)^T A \bar{u}^1 = 1$, $(u^2)^T A \bar{u}^2 = 1, \ldots, (u^n)^T A u^n = 1$ and everything else is zero, then (2) can be written as

$$\bar{x}^T A x = a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2$$

but the a_i are the coordinates of $\bar{x}$ in R^n. and for $\bar{x} \neq \bar{0}$, then at least one coordinate must not vanish. hence $\bar{x}^T A x \neq 0$. in addition, since the coordinates are all squared hence it is positive sum.

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$$\Longrightarrow \boxed{\bar{x}^T A x > 0} \quad \text{i.e. } A \text{ is SPD}$$

QED

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Math 501 – Numerical Analysis & Computation – Dr. Lee - Spring 2007

Computer Assignment 04/04/2007

1) Implement in MATLAB the following iterative Eigen methods:
   a) Power
   b) Inverse Power
   c) Shifted Power
   d) Shifted Inverse Power

2) Suppose $A = \begin{bmatrix} 4 & 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 1 & 0 \\ 2 & 1 & 4 & 1 & 2 \\ 0 & 1 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 & 4 \end{bmatrix}$ and start with $\vec{w}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, find with 10 iterations:

   a) The dominant eigenpair of A
   b) The least dominant eigenpair of A
   c) The eigenpair of A farthest to 2
   d) The eigenpair of A closest to 6.5

```
%
% MATH 501, Computer Assignment 04/04/2007
% by Nasser Abbasi
%
% test script to run problem 2 on the data given

A=[4 2 1 0 0;
   1 4 1 1 0;
   2 1 4 1 2;
   0 1 1 4 1;
   0 0 1 2 4];
initialEigenVectorGuess=[1 1 1 1 1];

%use Matlab to see the values to verify aginst

[v,l]=eig(A)

%Set parameters
maxIter=10;
delErr=0.0001;  %not specified, try these
delEps=0.0001;  %not specified, try these

[lambda]=nma_power(A,initialEigenVectorGuess,maxIter,delErr,delEps);
fprintf('---------- power result. Eigenvalue=%f\n',lambda);

[lambda]=nma_inverse_power(A,initialEigenVectorGuess,maxIter,delErr,delEps);
fprintf('---------- power inverse result. Eigenvalue=%f\n',lambda);

[lambda,k]=nma_shifted_power(A,initialEigenVectorGuess,maxIter,delErr,delEps,2)
;
lambda=2+lambda;
fprintf('---------- power shifted result. Eigenvalue=%f\n',lambda);

[lambda,k]=nma_inverse_shifted_power(A,initialEigenVectorGuess,maxIter,delErr,d
elEps,6.5);
lambda=6.5+lambda;
fprintf('---------- power inverse shifted result. Eigenvalue=%f\n',lambda);
```

TEST RESULT

```
v =

   -0.3861   -0.6325    0.0000    0.2405   -0.5000
   -0.3861   -0.3162    0.4082    0.2405    0.5000
   -0.6354    0.0000   -0.8165   -0.8767   -0.0000
   -0.3861    0.3162    0.4082    0.2405   -0.5000
   -0.3861    0.6325    0.0000    0.2405    0.5000

l =

    7.6458         0         0         0         0
         0    5.0000         0         0         0
         0         0    3.0000         0         0
         0         0         0    2.3542         0
         0         0         0         0    2.0000

---------- power result. Eigenvalue=7.645769
---------- power inverse result. Eigenvalue=2.354139
---------- power shifted result. Eigenvalue=7.645755
---------- power inverse shifted result. Eigenvalue=7.645724
```

```
function
[lambda_new,k]=nma_inverse_shifted_power(A,initialEigenVectorGuess,maxIter,delE
rr,delEps,mu)
%
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% by Nasser Abbasi
%
% IMPLEMENT iterative Inverse  shifted power method

w_old=initialEigenVectorGuess(:);
lambda_old=rand;
In=eye(size(A,1));
A=inv(A-mu*In);

for k=1:maxIter
    w_old=w_old/norm(w_old);
    w_new=A*w_old;
    lambda_new=dot(w_old,w_new);

    if norm(w_new-w_old)<delErr || abs(lambda_old-lambda_new)<delEps
        break;
    else
        w_old=w_new;
        lambda_old=lambda_new;
    end
end

lambda_new=1/lambda_new;
```

==

```
function [lambda_new,k]=nma_shifted_power(A,initialEigenVectorGuess,...
    maxIter,delErr,delEps,mu)
%
% MATH 501, Computer Assignment 04/04/2007
% by Nasser Abbasi
%
% IMPLEMENT iterative  shifted power method

w_old=initialEigenVectorGuess(:);
lambda_old=mu;
In=eye(size(A,1));
A=A-mu*In;

for k=1:maxIter
    w_old=w_old/norm(w_old);
    w_new=A*w_old;
    lambda_new=dot(w_old,w_new);

    if norm(w_new-w_old)<delErr || abs(lambda_old-lambda_new)<delEps
        break;
    else
        w_old=w_new;
        lambda_old=lambda_new;
    end
end
```