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HW # 6
Math 501

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Section 4.3 # 1 (b), (e)

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Section 4.3 #1 (b)

With No Pivoting

$$\begin{bmatrix} 1 & 6 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{L_{21}=2} \begin{bmatrix} 1 & 6 & 0 \\ 0 & -11 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{L_{32}=-\frac{2}{11}} \begin{bmatrix} 1 & 6 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{so } \underline{L}\underline{U} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -\frac{2}{11} & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

solve

$Ax = b \Rightarrow \underline{L}\underline{U}x = b$ let $\underline{U}x = v$ then $\underline{L}v = b$

solve for v:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -\frac{2}{11} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \Rightarrow v_1 = 3, \quad 2v_1 + v_2 = 1 \Rightarrow v_2 = 1 - 6 = -5$$

$$-\frac{2}{11}v_2 + v_3 = 1 \Rightarrow v_3 = 1 + \frac{2}{11}(-5) = 1 - \frac{10}{11} = \frac{1}{11}$$

so $v = \begin{bmatrix} 3 \\ -5 \\ \frac{1}{11} \end{bmatrix}$. Now From $\underline{U}x = v$ solve for x:

$$\begin{bmatrix} 1 & 6 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ \frac{1}{11} \end{bmatrix} \Rightarrow x_3 = \frac{1}{11}, \quad -11x_2 = -5 \Rightarrow x_2 = \frac{5}{11},$$

$$x_1 + 6x_2 = 3 \Rightarrow x_1 = 3 - 6\left(\frac{5}{11}\right) = 3 - \frac{30}{11} = \frac{33-30}{11} = \frac{3}{11}$$

so Solution $\underline{X} = \begin{bmatrix} 3/11 \\ 5/11 \\ \frac{1}{11} \end{bmatrix} = \boxed{\frac{1}{11}\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}}$

Now redo the problem using scaled partial row pivoting and solve again

$\longrightarrow$

section 4.3 #1 (b)

with row pivoting

$$\begin{bmatrix} 1 & 6 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \div \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1 \\ 0 \end{bmatrix}$$

so $A^{(1)} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 2 & 1 \end{bmatrix}$, new scale also reorder: $\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$, $P = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

now apply G.E.: $\xrightarrow{L_{21} = 1/2} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5.5 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 5.5 \\ 2 \end{bmatrix} \div \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.9.. \\ 1 \end{bmatrix}$

so reorder again $A^{(2)} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 5.5 & 0 \end{bmatrix}$; $L^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$ just the multipliers.

and $P^{(2)} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \Longrightarrow P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

now apply G.E.: $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 5.5 & 0 \end{bmatrix} \xrightarrow{L_{32} = \frac{5.5}{2}} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -\frac{5.5}{2} \end{bmatrix}$

So $LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{5.5}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -\frac{5.5}{2} \end{bmatrix}$

so $PAx = Pb \Rightarrow \boxed{LUx = Pb}$ (because $PA = LU$)

Let $Ux = v$ then $Lv = Pb$. solve for v

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{5.5}{2} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{5.5}{2} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \Rightarrow \begin{matrix} v_1 = 1 \\ v_2 = 1 \\ v_3 = -.75 \end{matrix} \rightarrow v = \begin{bmatrix} 1 \\ 1 \\ -.75 \end{bmatrix}$$

$\longrightarrow$

Now $Ux = v$ so $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & \frac{-5.5}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -.25 \end{bmatrix}$

so $x_3 = \dfrac{-.25 \times 2}{-5.5} = \dfrac{-.5}{-5.5} = \dfrac{1}{11}$

$2x_2 + x_3 = 1 \Rightarrow x_2 = \dfrac{1 - \left(\frac{1}{11}\right)}{2} = \dfrac{\frac{11-1}{11}}{2} = \dfrac{10}{22} = \dfrac{5}{11}$

$2x_1 + x_2 = 1 \Rightarrow x_1 = \dfrac{1 - \left(\frac{5}{11}\right)}{2} = \dfrac{\frac{11-5}{11}}{2} = \dfrac{6}{22} = \dfrac{3}{11}$

so Solution $\boxed{\underline{X} = \frac{1}{11}\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}}$

Which matches solution found with No pivoting.

section 4.3 # 1 (e)

(e)
$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 4 & -9 & 2 & 1 \\ 8 & 16 & 6 & 5 \\ 2 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 14 \\ -3 \\ 0 \end{bmatrix}$$

LU without scaled Pivoting

$L_{21}=4$ $\longrightarrow$ $L_{31}=8$ $L_{41}=2$

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 1 \\ 0 & -9 & -6 & -3 \\ 0 & 16 & -10 & -3 \\ 0 & 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -19 \\ -4 \end{bmatrix}$$

$L_{32}=\frac{-16}{9}$ $\longrightarrow$ $L_{42}=-\frac{1}{3}$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & \boxed{-9} & -6 & -3 \\ 0 & 0 & \frac{-62}{3} & \frac{-25}{3} \\ 0 & 0 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -\frac{25}{3} \\ -\frac{31}{3} \end{bmatrix}$$

$L_{43}=\frac{6}{31}$ $\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -9 & -6 & -3 \\ 0 & 0 & \boxed{-\frac{62}{3}} & \frac{-25}{3} \\ 0 & 0 & 0 & \frac{-12}{31} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -\frac{25}{3} \\ -\frac{1111}{93} \end{bmatrix}$$

so
$$LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 8 & \frac{-16}{9} & 1 & 0 \\ 2 & -1/3 & \frac{6}{31} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -9 & -6 & -3 \\ 0 & 0 & -62/3 & -25/3 \\ 0 & 0 & 0 & \frac{-12}{31} \end{bmatrix}$$

$\longrightarrow$ solve

Now we solve

since $Ax=b$, then $\underline{L}UX=b$.

let $UX=v$, then $\underline{L}v=b$

<u>solve for v:</u>

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 8 & -\frac{16}{9} & 1 & 0 \\ 2 & -1/3 & 6/31 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 14 \\ -3 \\ 0 \end{bmatrix} \Rightarrow \boxed{v_1 = 2}$$

$4v_1 + v_2 = 14 \Rightarrow v_2 = 14 - 8 = \boxed{6}$

$8v_1 - \frac{16}{9}v_2 + v_3 = -3 \Rightarrow v_3 = -3 - 8(2) + \frac{16}{9}(6) = -\frac{25}{3}$.

and $2v_1 - \frac{1}{3}v_2 + \frac{6}{31}v_3 + v_4 = 0 \Rightarrow v_4 = \frac{-12}{31}$

so $v = \begin{bmatrix} 2 \\ 6 \\ -25/3 \\ -12/31 \end{bmatrix}$ Now solve for X from $UX = v$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -9 & -6 & -3 \\ 0 & 0 & -\frac{62}{3} & -25/3 \\ 0 & 0 & 0 & \frac{-12}{31} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -25/3 \\ -12/31 \end{bmatrix}$$

so $x_4 = \dfrac{\frac{-12}{31}}{\frac{-12}{31}} = 1$

$-\frac{62}{3}x_3 - \frac{25}{3}x_4 = -\frac{25}{3} \Rightarrow x_3 = 0$

$-9x_2 - 6x_3 - 3x_4 = 6 \Rightarrow x_2 = -1$

$x_1 + 2x_3 + x_4 = 2 \Rightarrow X_1 = 1$

so Solution is $\boxed{X = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}}$ $\longrightarrow$

Section 4.3 #1 (e) continue.

Now do LU but with row scaling

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 4 & -9 & 2 & 1 \\ 8 & 16 & 6 & 5 \\ 2 & 3 & 2 & 1 \end{bmatrix} \xrightarrow[\text{Vector}]{\text{scale}} \begin{bmatrix} 2 \\ 9 \\ 16 \\ 3 \end{bmatrix} \Longrightarrow \text{For First Column} \rightarrow \begin{bmatrix} 1 \\ 4 \\ 8 \\ 2 \end{bmatrix} \div \begin{bmatrix} 2 \\ 9 \\ 16 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.44\ldots \\ 0.5 \\ 0.66\ldots \end{bmatrix}$$

so reorder as follows: $4^{th}, 1^{st}, 3^{rd}, 2^{nd} \Longrightarrow P = \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \end{bmatrix}$.

$$\text{so } A^{(1)} = \begin{bmatrix} 2 & 3 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 8 & 16 & 6 & 5 \\ 4 & -9 & 2 & 1 \end{bmatrix}.$$

$$\xrightarrow[\substack{L_{31}=4 \\ L_{41}=2}]{L_{21}=\frac{1}{2}} \begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -\frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 4 & -2 & 1 \\ 0 & -15 & -2 & -1 \end{bmatrix}$$

now apply scaling to the submatrix shown

$$\Rightarrow \begin{bmatrix} -3/2 \\ 4 \\ -15 \end{bmatrix} \div \begin{bmatrix} 9 \\ 16 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ 1/4 \\ 5 \end{bmatrix}$$

so need to move 4^{th} to 2^{nd}, move 2^{nd} to 3^{rd}, move 3^{rd} to 4^{th}.

$$\text{so } P = \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 \\ 2 \\ 3 \\ 1 \end{bmatrix} \quad \text{also reorder } \underline{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix}$$

$$A^{(2)} = \begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & \boxed{-15} & -2 & -1 \\ 0 & 4 & -2 & 1 \\ 0 & -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 4 \\ \frac{1}{2} \end{bmatrix} \quad \text{with } P = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 1 \end{bmatrix} \text{ and } \underline{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix}.$$

now continue G.E. $\xrightarrow[L_{42}=\frac{1}{10}]{L_{32}=-\frac{4}{15}}$

$$\begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -15 & -2 & -1 \\ 0 & 0 & -\frac{38}{15} & \frac{11}{15} \\ 0 & 0 & \frac{6}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 24/15 \\ \frac{1}{5} \end{bmatrix} \longrightarrow$$

section 4.3 # 1 (e)

Now do but with row Pivoting:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 4 & -9 & 2 & 1 \\ 8 & 16 & 6 & 5 \\ 2 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 2 \\ 9 \\ 16 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 4 \\ 8 \\ 2 \end{bmatrix} \div \begin{bmatrix} 2 \\ 9 \\ 16 \\ 3 \end{bmatrix} = \begin{bmatrix} .5 \\ 0.44 \\ .5 \\ 0.66 \end{bmatrix}$$

so $A^{(1)} = \begin{bmatrix} \boxed{2} & 3 & 2 & 1 \\ 4 & -9 & 2 & 1 \\ 8 & 16 & 6 & 5 \\ 1 & 0 & 2 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, new scale. $\begin{bmatrix} 3 \\ 9 \\ 16 \\ 2 \end{bmatrix}$

now apply GE: $\xrightarrow[L_{41}=\frac{1}{2}]{L_{21}=2,\ L_{31}=4} \begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -15 & -2 & -1 \\ 0 & 4 & -2 & 1 \\ 0 & -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} -15 \\ 4 \\ -\frac{3}{2} \end{bmatrix} \div \begin{bmatrix} 9 \\ 16 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.66.. \\ .25 \\ .75 \end{bmatrix}$ largest No reordering

apply GE: $\begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & \boxed{-15} & -2 & -1 \\ 0 & 4 & -2 & 1 \\ 0 & -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix} \xrightarrow[L_{42}=\frac{3}{30}]{L_{32}=-\frac{4}{15}} \begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -15 & -2 & -1 \\ 0 & 0 & -\frac{38}{15} & \frac{11}{15} \\ 0 & 0 & \frac{12}{10} & \frac{6}{10} \end{bmatrix} \qquad \begin{bmatrix} -\frac{38}{15} \\ \frac{12}{10} \end{bmatrix} \div \begin{bmatrix} 16 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.158 \\ 0.6 \end{bmatrix}$

so reorder.

$$A = \begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -15 & -2 & -1 \\ 0 & 0 & -\frac{38}{15} & \frac{11}{15} \\ 0 & 0 & \frac{12}{10} & \frac{6}{10} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -15 & -2 & -1 \\ 0 & 0 & \frac{12}{10} & \frac{6}{10} \\ 0 & 0 & -\frac{38}{15} & \frac{11}{15} \end{bmatrix}, \quad P = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

and reorder $\underline{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & -\frac{4}{15} & 1 & 0 \\ \frac{1}{2} & \frac{3}{30} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{3}{30} & 1 & 0 \\ 4 & -\frac{4}{15} & 0 & 1 \end{bmatrix}$

$\longrightarrow$

now apply G.E.:

$$\begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -15 & -2 & -1 \\ 0 & 0 & \frac{12}{10} & \frac{6}{10} \\ 0 & 0 & -\frac{38}{15} & \frac{11}{15} \end{bmatrix} \xrightarrow{L_{43} = \frac{-19}{9}} \begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -15 & -2 & -1 \\ 0 & 0 & \frac{12}{10} & \frac{6}{10} \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Finished.. hence

$$LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{10} & 1 & 0 \\ 4 & \frac{-4}{15} & \frac{-19}{9} & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -15 & -2 & -1 \\ 0 & 0 & \frac{12}{10} & \frac{6}{10} \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$PA = LU$.

$Ax = b \Rightarrow PAx = Pb \Rightarrow \boxed{LUx = Pb}$

Let $Ux = v$. then $Lv = Pb$. solve for v:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{10} & 1 & 0 \\ 4 & \frac{-4}{15} & \frac{-19}{9} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P} \begin{bmatrix} 2 \\ 14 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ 2 \\ -3 \end{bmatrix}$$

so $\boxed{v_1 = 0}$, $\boxed{v_2 = 14}$, $\frac{1}{10}v_2 + v_3 = 2 \Rightarrow v_3 = 2 - \frac{1}{10}(14) = 2 - \frac{14}{10}$

$v_3 = \frac{20-14}{10} = \frac{6}{10} = \boxed{\frac{3}{5}}$

$-\frac{4}{15}v_2 - \frac{19}{9}v_3 + v_4 = -3 \Rightarrow v_4 = -3 + \frac{4}{15}(14) + \frac{19}{9}\left(\frac{3}{5}\right) = -3 + \frac{56}{15} + \frac{19}{15}$

so $v_4 = \frac{-45+56+19}{15} = \frac{30}{15} = \boxed{2}$

$$\Rightarrow v = \begin{bmatrix} 0 \\ 14 \\ \frac{3}{5} \\ 2 \end{bmatrix} \longrightarrow$$

now $Ux = v$

hence $\begin{bmatrix} 2 & 3 & 2 & 1 \\ 0 & -15 & -2 & -1 \\ 0 & 0 & \frac{12}{10} & \frac{6}{10} \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ 3/5 \\ 2 \end{bmatrix}$

so $x_4 = \boxed{1}$

$$\frac{12}{10} x_3 + \frac{6}{10} x_4 = \frac{3}{5} \Rightarrow x_3 = \frac{\frac{3}{5} - \frac{6}{10}(1)}{\frac{12}{10}} = \boxed{0}$$

$$-15x_2 - 2x_3 - x_4 = 14$$

so $x_2 = \dfrac{14+1}{-15} = \boxed{-1}$

$$2x_1 + 3x_2 + 2x_3 + x_4 = 0$$

so $x_1 = \dfrac{-3(-1) - (1)}{2} = \dfrac{3-1}{2} = \dfrac{2}{2} = \boxed{1}$

so $\underline{X} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ which matches solution found earlier

4.3 # 30

$$x_2 + 2x_3 = 1$$
$$2x_1 - x_2 = 2$$
$$2x_2 + x_3 = 3$$

Determine $PA = LU$

system is $\begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

so $\begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ $\quad \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \div \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

so $P = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ and $s = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

Now apply GE

$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow[\text{already zero}]{\text{1st col}} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \div \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} .5 \\ 1 \end{bmatrix}$

so $A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $s = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

Now apply G.E: $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{L_{32} = \frac{1}{2}} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}$ done.

so $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}$.

$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Verify $PA = LU$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Verified OK.

Now $\det(A) =$ multiplication of Pivots (diagonal element of U).

hence $\det(A) = (2)(2)(\frac{3}{2}) = \boxed{6}$

section 4.3 # 31

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 6 & 6 & 2 \\ -1 & 1 & 3 \end{bmatrix} \rightarrow \text{scale} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix} \div \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1/3 \end{bmatrix}$$

No flip needed.

$$A^{(1)} \xrightarrow[\ell_{31}=-\frac{1}{3}]{\ell_{21}=2} \begin{bmatrix} \boxed{3} & 2 & -1 \\ 0 & 2 & 4 \\ 0 & 5/3 & 8/3 \end{bmatrix} \rightarrow \underline{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 5/3 \end{bmatrix} \div \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 5/9 \end{bmatrix}$$

so flip

so $P = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and flip $\underline{L}$ also $\underline{L} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

so $A^{(2)} = \begin{bmatrix} 3 & 2 & -1 \\ 0 & \boxed{5/3} & 8/3 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{\ell_{32}=6/5} \begin{bmatrix} 3 & 2 & -1 \\ 0 & 5/3 & 8/3 \\ 0 & 0 & \frac{4}{15} \end{bmatrix}$

so $\underline{L}U = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 2 & \frac{6}{5} & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 0 & 5/3 & 8/3 \\ 0 & 0 & 4/15 \end{bmatrix}$.

Now $D = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{3}{5} & 0 \\ 0 & 0 & \frac{15}{4} \end{bmatrix}$, $U \rightarrow \begin{bmatrix} 1 & 2/3 & -1/3 \\ 0 & 1 & 8/5 \\ 0 & 0 & 1 \end{bmatrix}$

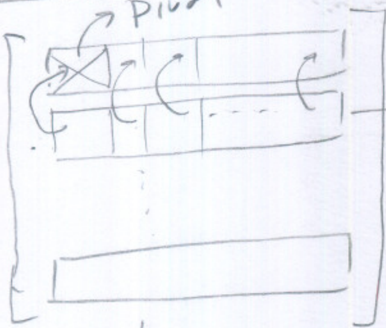
So

$PA = LDU$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 6 & 6 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 2 & \frac{6}{5} & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & \frac{3}{5} & 0 \\ 0 & 0 & \frac{15}{4} \end{bmatrix} \begin{bmatrix} 1 & 2/3 & -1/3 \\ 0 & 1 & 8/5 \\ 0 & 0 & 1 \end{bmatrix}$$

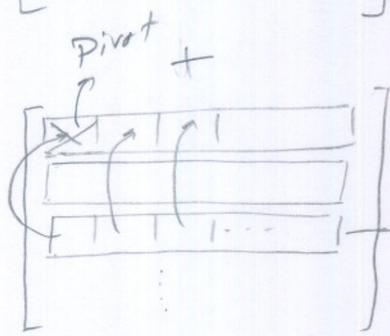
Section 4.3 # 39

given $A =$



$\to$ n multiplications by multiplier l_{21} where $l_{21} = \frac{a_{21}}{a_{11}}$ (modulus sign).

+



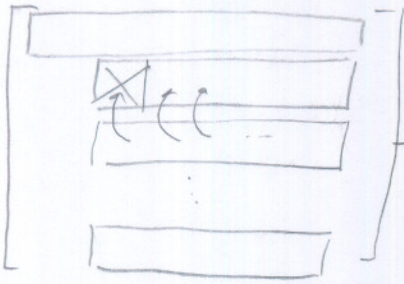
$\to$ n multiplications by multiplier l_{31} where $l_{31} = \frac{a_{31}}{a_{11}}$

⋮

do the above for $(n-1)$ rows. $\Rightarrow$ $n(n-1)$ multiplications $+(n-1)$ divisions $\left(\frac{a_{i1}}{a_{11}}\right)$ $i = 2 \ldots n$

Now we repeate the above for $(n-1)\times(n-1)$ Matrix

$(n-2)$ times



$\to (n-1)$ ops

+

$\to (n-1)$ ops.

⋮

$\Rightarrow$ $(n-1)(n-2)$ multiplications $+ (n-2)$ divisions

Continuous this way we obtain

multiplications: $n(n-1) + (n-1)(n-2) + \cdots + 3\cdot 2 + 2\cdot 1 < n^2 + (n-1)^2 + \cdots + 3^2 + 2^2$
$= \frac{n}{6}(n+1)(2n+1) - 1$
$\approx \boxed{\frac{n^3}{3}}$

divisions: $(n-1) + (n-2) + (n-3) + \cdots + 1$
$= \frac{n}{2}(n-1) \approx \boxed{\frac{1}{2}n^2}$ $\Rightarrow$ ops $\approx \boxed{\frac{1}{3}n^3 + \frac{1}{2}n^2}$

section 4.3 # 43

Prove that if P is permutation matrix, then $P^{-1} = P^T$

Solution

P has only one '1' in each row. and it has one '1' in each column..

$P(i,j) = 1$ means that row that was in row i was moved to row j.

Now it is possible that row j was moved back to i or it is possible that row j moved to new location row say k.

Case 1 row $i \to$ row j and row $j \to$ row i

in this case $P(i,j) = 1$ and $P(j,i) = 1$

in this case, to reverse the effect on this P acting on A, we need to have row $j \to$ row i and row $i \to$ row j

but this means $P^{-1} = P^T$ (becaus P^{-1} means to reverse the effect of P)

and due to symmetry, we see that P^T will move row j to row i and row i back to j.

Now consider the hard case 2 when row i moved to row j and row j moved to row k. but row k has to go somewhere, the cycle must terminate either in case 1 or eventually we reach row i again $\longrightarrow$

i.e Consider row $i \to$ row $j \to$ row k $\to$ back to row i

This general case illustrated as follows:

i.e $P(i,j)=1$, $P(j,k)=1$, $P(k,i)=1$ —— (1)

$$P = \begin{array}{c|ccc|} & & & \\ i & & 1 & \\ j & & & 1 \\ k & 1 & & \\ \hline & i & j & k \end{array}$$

how to inverse this? i.e we want P^{-1} such that when applied on $\tilde{A}$ to obtain original A

allready permuted.

So we want what in row i to go to row k and what in k to go back to row j and what in j to go back to row i. i.e we want

$$\begin{array}{c|ccc|} i & & & 1 \\ j & 1 & & \\ k & & 1 & \\ \hline & i & j & k \end{array} \Rightarrow P(i,k)=1, \quad P(k,j)=1, \quad P(j,i)=1 \text{ —— (2)}$$

compare (1) and (2) we see that they represent a symmetrical formation.

i.e P^{-1} is obtained by transposing P.

i.e $P^{-1} = P^T$

So I considered the 2 possible cases direct row exchanges and chain of row exchanges. These are only possible cases. note

In both $P^{-1}=P^T$

QED

another possible approach to proof:
$[P \mid I]$ and carry Gaussian-Jordan
elimination. This will result in $[I \mid P^{-1}]$.

and show that row operations need on P to
make the pivot 1 each time will case
RHS to have a form so that $P^{-1} = P^{T}$.
I attempted this approach but had
some difficulties. that is why
I provided previous solution.

HW # 6, section 4.3 # 45

if A is tridiagonal and P is permutation matrix, prove or disprove that PAP^{-1} is tridiagonal

Answer

First note that $P^{-1} = P^T$ From previous solution.

hence we need to analyse $\boxed{PAP^T}$

PA produces a matrix whose rows are exchanged according to permutation matrix. Call this matrix C.

hence $\boxed{C = PA}$

Now CP produced a matrix whose columns are exchanged.

Hence CP^T produces a matrix whose rows are exchanged.

Therefor For Final result of PAP^T to be restored back to A we must have that $CP^T = A$ i.e by post multiplying C by P^T we go back to A which is tridiagonal.

is $CP^T = A$?

or is $\underbrace{PA}_{C}P^T = A$? Now premultiply both sides by P^{-1}:

$\cancel{P^{-1}P}AP^T = P^{-1}A$

$AP^T = P^{-1}A$ or $\boxed{AP^T = P^T A}$ since $P^{-1} = P^T$

So, this is the condition for PAP^{-1} to be Tridiagonal.

but This is like asking is $AB = BA$? another proof

This is NOT true in general for Matrices. Hence PAP^{-1} NOT trid. QED section 4.3 #45

This is another proof if the previous one was not acceptable.

proof by showing one case to the contrary of it being tridiagonal.

let $A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 5 & 0 & 0 \\ 0 & 6 & 7 & 8 & 0 \\ 0 & 0 & 9 & 10 & 11 \\ 0 & 0 & 0 & 12 & 13 \end{bmatrix}$, let $P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

Then $PAP^T = \begin{bmatrix} 10 & 0 & 11 & 9 & 0 \\ 0 & 4 & 0 & 5 & 3 \\ 12 & 0 & 13 & 0 & 0 \\ 8 & 6 & 0 & 7 & 0 \\ 0 & 2 & 0 & 0 & 1 \end{bmatrix}$

which is <u>NOT</u> tridiagonal.

as it enough to show one case to contrary.

we concluded that in general

PAP^T is NOT tridigonal

even if A is.

QED