26/30

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HW # 5
Math 501

Section 4.2

1, 5, 13, 27, 30, 33, 39, 47

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section 4.2 #1

Prove:

① if U is upper triangular and invertible, then U^{-1} is upper triangular

<u>Answer</u> proof by induction.

Let $U_{1\times 1}$ be upper triangle for $\underline{n=1}$ $\Rightarrow$ true since U^{-1} is obviously upper triangle.

<u>Assume true for $n\times n$.</u> i.e $U_{n\times n}$ is upper triangle with an inverse if exists which is $U^{-1}_{n\times n}$ is also upper triangle.

Now need to show that this will remain true for $(n+1)\times(n+1)$ case.

let $A_{(n+1)}$ be inverse of $U_{(n+1)}$. hence A must satisfy $UA=I$, and $AU=I$.

Consider the right inverse first.

$$U_{(n+1)\times(n+1)}\ A_{(n+1)(n+1)} = I_{(n+1)\times(n+1)}$$

now divide both matrices as follows:

$$\left[\begin{array}{c|c} U_{n\times n} & \begin{matrix} b_1\\ b_2\\ \vdots\\ b_n\end{matrix} \\ \hline 0\,0\cdots 0 & a \end{array}\right] \left[\begin{array}{c|c} A_{n\times n} & \begin{matrix} c_1\\ c_2\\ \vdots\\ c_n\end{matrix} \\ \hline d_1\, d_2 \cdots d_n & y \end{array}\right] = \left[\begin{array}{c|c} I_{n\times n} & 0_{n\times 1} \\ \hline 0_{1\times n} & 1 \end{array}\right]$$

Call the vector $[b_1\, b_2 \cdots b_n]^T = \underline{b}$, call $[c_1\, c_2 \cdots c_n]^T = \underline{c}$, call $[d_1\, d_2 \cdots d_n]^T = \underline{d}$.
where 'a' above is number, and 'y' above is number. so we can redraw this as

$$\left[\begin{array}{c|c} U & \bar{b} \\ \hline \bar{o} & a \end{array}\right] \left[\begin{array}{c|c} A & \bar{c} \\ \hline \bar{d} & y \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 1 \end{array}\right] \Rightarrow \boxed{\text{our goal is to show that } \bar{d}=0,\ y\neq 0}$$

hence we have by multiplication of matrices $U_{n+1}\, A_{n+1}$

$$UA + \bar{b}\bar{d} = I \quad \text{——} ①$$
$$U\bar{c} + \bar{b}y = \bar{o} \quad \text{——} ②$$
$$\bar{o}A + a\bar{d} = \bar{o} \quad \text{——} ③$$
$$\bar{o}\,\bar{c} + ay = 1 \quad \text{——} ④$$

these 4 equations result if we work out the multiplication of $(U_{n+1})(A_{n+1})$ as row x column standard method.

From ③ $\Rightarrow a\bar{d}=0$
From ④ $\Rightarrow ay=1$
one possibility to satisfy $a\bar{d}=0$ is that $a=0$. but if $a=0$, then Last row of $U_{n+1}=0$, hence NO inverse exist $\Rightarrow \boxed{a\neq 0}$ sbut U^{-1} exist. hence

Since $a\neq 0$ then $\boxed{\bar{d} \text{ must be zero}}$ and since $ay=1$ then $\boxed{y=\frac{1}{a}\neq 0}$

therefor we showed that $\bar{d} = 0$ and $y \neq 0$. hence A_{n+1} is

but since we assumed that $A_{n\times n}$ is the inverse of $U_{n\times n}$ and is upper trianglar, then we have shown that $A_{(n+1)\times(n+1)}$ is also upper triangular. and since $U_{n\times 1} A_{n+1} = I_{n+1}$, its is the inverse of U_{n+1}.

we need to also show this for left inverse. $A_{n\times 1} U_{n\times 1} = I_{n\times 1}$ but the argument will follow the same lines.

QED.

Section 4.2 #1

2) Show that inverse of unit lower triangular matrix is unit lower triangular.

Answer a unit lower triangular matrix is a lower triangular matrix which has '1' on the diagonal:  zero here. like this.

need to show that given U, then its inverse is also like this.

do proof by an induction similar to last problem.

this is clearly true for $n=1$ case.

assume true for $U_{n\times n}$ i.e. $U_{n\times n}$ has an inverse $A_{n\times n}$ which is also a unit lower triangular.

Now we need to show this is true for $U_{(n+1)\times(n+1)}$.

Consider right inverse first: $U_{(n+1)(n+1)}\, A_{(n+1)(n+1)} = I_{(n+1)\times(n+1)}$

divide Matrices as before:

$$\begin{bmatrix} U_{n\times n} & \bar{o} \\ \bar{b} & a \end{bmatrix} \begin{bmatrix} A_{n\times n} & \bar{c} \\ \bar{d} & y \end{bmatrix} = \begin{bmatrix} I_{n\times n} & \bar{o} \\ \bar{o} & 1 \end{bmatrix}$$

carry the multiplication we obtain

$$\left.\begin{aligned} UA + \bar{o}\bar{d} &= I \quad &(1)\\ U\bar{c} + \bar{o}y &= \bar{o} \quad &(2)\\ \bar{b}A + a\bar{d} &= \bar{o} \quad &(3)\\ \bar{b}\bar{c} + ay &= 1 \quad &(4)\end{aligned}\right\}$$

Need to show that $\boxed{\bar{c}=0,\ y=1}$

$$\left.\begin{aligned}&\text{From } (1) \Rightarrow UA = I\\ &\text{From } (2) \Rightarrow U\bar{c} = \bar{o}\end{aligned}\right\}$$ since A exist, then $A\neq 0$ i.e. $U\neq 0$. hence it must be that $\boxed{\bar{c}=0}$

now need to show that $y=1$ to complete proof.

from (4) since $\bar{c}=0$ we get $\boxed{ay=1}$ but $a=1$ since U_{n+1} is unit lower triangular. $\Rightarrow \boxed{y=1}$

Therefor $\boxed{A_{n+1} \text{ is unit lower triangular}}$ QED

(need to show $AV=I$ also, but it follows similar argument.)

section 4.2 # 1

(c) show that product of 2 upper/lower triangular matrices is upper/lower triangular.

use definition of matrix multiplication:

$$\underset{n\times K}{C} = \underset{n\times m}{A}\ \underset{m\times K}{B}$$

where $C_{ij} = \sum_{q=1}^{m} A_{iq} B_{qj}$

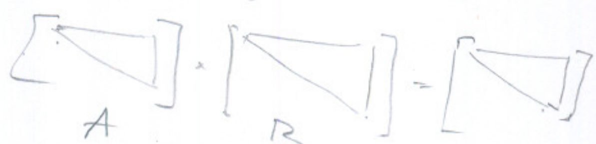
row of A * column of B

for example $C_{11} = \sum_{q=1}^{m} A_{1q} B_{q1} = A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31}$

and $C_{12} = \sum_{q=1}^{m} A_{1q} B_{q2} = A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32}$

ok verified.

now that matrix multiplication is defined, consider the first case: product of 2 upper triangular matrices:



since upper triangular, then

$A_{iq} = 0$ if $i > q$. i.e row number is > column number.

and $B_{qj} = 0$ if $q > j$

but $C_{ij} = \sum_{q=1}^{m} A_{iq} B_{qj}$.

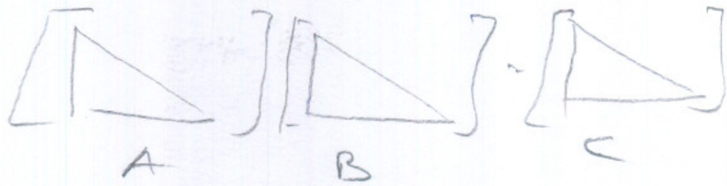
Then when $i > q$ or $q > j$ we have $A_{iq} B_{qj} = 0$.

Therefor when $i > q > j$ we have $A_{iq} B_{qj} = 0$

Therfor when $\boxed{i > j}$ we have $A_{iq} B_{qj} = 0$

$\Longrightarrow$ C is upper triangular as well. QED →

similar argument for the 2 lower triangular matrices.



hence $A_{iq} = 0$ if $i < q$.

and $B_{qj} = 0$ if $q < j$

hence $C_{ij} = 0$ if $i < q < j$

or $C_{ij} = 0$ if $i < j$

but this means C is lower triangular.

section 4.2 # 5

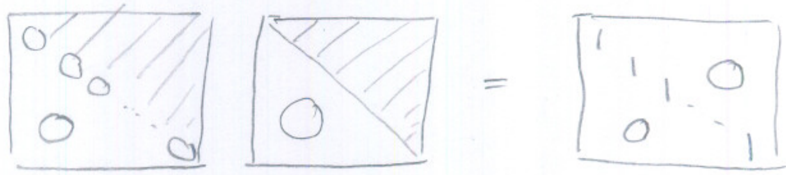
Proof that an upper or lower triangular Matrix is nonsingular iff its diagonal elements are all $\neq 0$.

proof.

$\Rightarrow$ direction: show that if U is invertible then diagonal elements are all nonzero.

by contradiction: assume that U is upper triangle, and it has all its diagonal elements $= 0$ and it has an Inverse, say A. then we have $UA = I$

i.e



U A

Note I have used the fact that A is upper triangular. because we proved in problem 4.2 # 1 that an upper triangle will have an inverse which is also an upper triangle. (if inverse exist).

OK now we continue. Carry out the matrix multiplication, we obtain for first row of U:

$$u_{11} a_{11} + u_{12} a_{21} + \cdots + u_{1n} a_{n1} = \overset{I(1,1)}{1} \quad \text{--- } (1)$$

but $a_{21} = a_{31} = \cdots = a_{n1} = 0$ since A is upper triangle.

hence (1) becomes $u_{11} a_{11} = 1$.

but we assumed that $u_{11} = 0 \Rightarrow$ Not possible since $u_{11}a_{11} = 1$.

hence $\boxed{u_{11} \text{ Can Not be Zero.}}$

Now do the same thing to get an equation for $I(2,2)$. this equation will show that u_{22} Can Not be Zero.

we continue this way to show that u_{ii} Can NOT be Zero $\longrightarrow$

will show for $I(2,2)$:

to find $I(2,2)=1$ we multiply 2^{nd} row of U by 2^{nd} column of A :

$$\cancel{u_{21}}^{=0} a_{12} + u_{22}a_{22} + \cancel{u_{23}a_{32}} + \cancel{\cdots} + \cancel{u_{2n}a_{n2}} = \overset{I(2,2)}{1}$$

but $u_{21}=0$ since U is upper triangular.

and $a_{32}=a_{42}=\cdots=a_{n2}=0$ since A is also upper triangl.

hence we have

$$\boxed{u_{22}a_{22} = 1}$$

but we assumed that $u_{22}=0$ which is not possible since $u_{22}a_{22}=1 \Rightarrow u_{22}\neq 0$.

etc... for $I(3,3)=1$, $I(4,4)=1$, ...

therefor our assumption is wrong.

$$\boxed{\text{hence if } U \text{ is upper triangular and is Invertible then } \underline{\text{NONE}} \text{ of its diagonal elements can be zero.}}$$

(Ps. proof for Lower triangular follows the same approach)

$\boxed{\text{To proof the} \Leftarrow \text{direction}}$: i.e need to show that if U has None of its diagonal elements $=0$ then it must be invertible.

$\longrightarrow$

but in problem 4.2 #1 we proofed that

if U is upper triangle then it has an Inverse

(which happened to be upper triangle as well).

the difference is that in 4.2 #1 we assumed that

U was invertible. but this assumption was only

need to be able to say that $U(n,n) \neq 0$ (this

is the element 'a' in diagram in 4.2 #1 solution).

but in this proof, we assumed that $U(n,n) \neq 0$ already.

therefor, we can follow the same exact

steps as in 4.2 #1, and we can just

use the fact that $U(n,n) \neq 0$, then we

showe that U has an inverse.

hence we just showed that if all diagonal

elements of U are none zero, it must have

an inverse.

So we proved "$\Longrightarrow$" and "$\Longleftarrow$" directions.

QED

section 4.2 # 13

Show that every matrix of Form $A = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ has LU Factorization.

Does it have LU Factorization in which L is unit lower triangular?

Answer

this matrix $A = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ have pivot $A_{11} = 0$, so exchange rows. it becoms

$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ which is in an upper triangular form U.

the L matrix is simple I_2 here.

hence we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

Note since we did a permutation on A before, we should include a Permutation Matrix P to indicate this. i.e the LU decomposition should be written as

$$\boxed{PLU = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}}_{U}} \; ?$$

$$= \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \leftarrow \text{the original } A.$$

a

Since $L = I_2$ in this case, and I_2 can be considered a unit lower triangular (it has 1 on diagonal, and zeros above diagonal) then answer is $\boxed{\text{YES}}$

section 4.2 # 27

if A is positive definite, does it follow that A^{-1} is also positive definite?

Answer

conjugate transpose.

a positive definite matrix is A s.t. $\underline{x}^H A \underline{x} > 0$ for all $\underline{x}$

a positive definite matrix have all its eigenvalues > 0.

one can ask: what happens to the eigenvalues of a matrix when it is inverted?

The inverse eigenvalues theorm says that if λ is an eigenvalue of A then λ^{-1} is an eigenvalue of A^{-1}.

Since all $\lambda > 0$, then $\frac{1}{\lambda} > 0$. hence A^{-1} is

$\boxed{\text{positive definite}}$

PS. I used the theorem "Inverse Eigenvalues" to prove this problem. but I did not proof this theorem. I hope this is ok. I just read about this theorem in another reference which I need to study its proof more.

Section 4.2 #30

Find LU Factorization of $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix}$

Pivot

Pivot

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix} \xrightarrow{l_{31} = \frac{1}{3}} \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 3 & -\frac{1}{3} \end{bmatrix} \xrightarrow{l_{32} = -3} \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & \frac{26}{3} \end{bmatrix} \leftarrow U$$

so $$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & -3 & 1 \end{bmatrix} \leftarrow L$$

So $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & -3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & \frac{26}{3} \end{bmatrix}$$

section 4.2 # 33

Suppose that the nonsingular matrix A has cholesky Factorization. what can be said about $|A|$?

Answer

Since A has cholesky decomposition, then A is symmetric positive definite.

Therefor A has all its eigenvalues $\lambda_i > 0$.

since $|A| =$ product of λ_i

then $\boxed{|A| > 0}$

$\prod l_{ii}^2$

$A = L^T L \quad ; \quad L = \begin{bmatrix} l_{11} & & 0 \\ & \ddots & \\ & & l_{nn} \end{bmatrix}$

$\det(A) = \prod_{i=1}^{n} l_{ii}^2$

section 4.2 #39

matrix A is positive/symmetric find $\sqrt{A}$ if $A=\begin{bmatrix} 13 & 10 \\ 10 & 17 \end{bmatrix}$

Answer

need to find $[X]$ such that $[X][X]=A$.

let $[X]=\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$.

so we get $\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}=\begin{bmatrix} 13 & 10 \\ 10 & 17 \end{bmatrix}$

$$\left.\begin{aligned} x_{11}^2 + x_{12}x_{21} &= 13 \quad \text{—— (1)} \\ x_{11}x_{12} + x_{12}x_{22} &= 10 \quad \text{—— (2)} \\ x_{21}x_{11} + x_{22}x_{21} &= 10 \quad \text{—— (3)} \\ x_{21}x_{12} + x_{22}^2 &= 17 \quad \text{—— (4)} \end{aligned}\right\}$$

4 equations
4 unknowns.

Solving on computer, ~~there~~ are the 4 roots;

$$X_1=\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \\ -\frac{5}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \end{bmatrix}$$

$$X_2=\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix}$$

$$X_3=\begin{bmatrix} -\frac{12}{\sqrt{13}} & -\frac{5}{\sqrt{13}} \\ -\frac{5}{\sqrt{13}} & -\frac{14}{\sqrt{13}} \end{bmatrix}$$

$$X_3=\begin{bmatrix} \frac{12}{\sqrt{13}} & \frac{5}{\sqrt{13}} \\ \frac{5}{\sqrt{13}} & \frac{14}{\sqrt{13}} \end{bmatrix}$$

i.e $X_1^2=A$

and $X_2^2=A$

$X_3^2=A$

$X_4^2=A$.

section 4.2 # 47

if A has Doolittle Factorization, what is simple formula for the determinant of A ?

Answer

$|A| =$ products of all pivots in the diagonal elements of the U matrix.

So $\boxed{|A| = \prod_{i=1}^{N} U(i,i)}$ where here N is the size of the matrix A.

note: the eigenvalues of A are along the diagonal of U after doolittle factorization.

note: the above also shows why when one eigenvalue is zero, then the determinant must be zero also.