HW # 4
Math 501

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Section 3.5 # 1, 2, 3, 5, 6, 10

Section 4.1 # 15, 16, 17, 18

Section 4.2 # 1, 5, 13, 27, 30, 33, 39, 47

Computer Assignment
02/14/07.
LU Factorization.

Please Note I solved section 4.2
but email said we can
hand section 4.2 next
week.

section 3.5 # 1

use horner's algorithm to find $P(4)$ where

$$P(z) = 3z^5 - 7z^4 - 5z^3 + z^2 - 8z + 2$$

Answer

use arrangment

$$
\begin{array}{c|cccccc}
 & a_n & a_{n-1} & a_{n-2} & \cdots & & a_0 \\
z_0 & & z_0 b_{n-1} & z_0 b_{n-2} & & & z_0 b_0 \\
\hline
 & b_{n-1} & b_{n-2} & \cdots & & b_0 & \boxed{P(z_0)}
\end{array}
$$

Therfor, $n = 5$, we have

$$
\begin{array}{c|cccccc}
 & 3 & -7 & -5 & 1 & -8 & 2 \\
4 & & 12 & 20 & 60 & 244 & 944 \\
\hline
 & 3 & 5 & 15 & 61 & 236 & \boxed{946}
\end{array}
$$

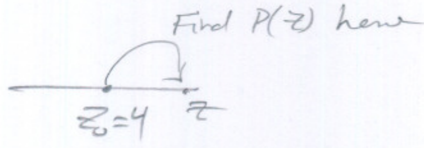
so $\boxed{P(4) = 946}$

Section 3.5 # 2

For $p(z) = 3z^5 - 7z^4 - 5z^3 + z^2 - 8z + 2$

Find it expression in taylor series about $\boxed{z_0 = 4}$

Answer



$$P(z) = P(z_0) + (z - z_0)P'(z_0) + \frac{(z-z_0)^2 P''(z_0)}{2!}$$

$$+ \frac{(z-z_0)^3 P'''(z_0)}{3!} + \cdots + R_n(\xi_n)$$

$p'(z) = 15z^4 - 28z^3 - 15z^2 + 2z - 8$ @ $z_0 = 4 \Rightarrow \boxed{1808}$

$p''(z) = 60z^3 - 84z^2 - 30z + 2$ @ $z_0 = 4 \Rightarrow \boxed{2378}$

$p'''(z) = 180z^2 - 168z - 30$ @ $z_0 = 4 \Rightarrow \boxed{2178}$

$p''''(z) = 360z - 168$ @ $z_0 = 4 \Rightarrow \boxed{1272}$

$p^{(5)}z = 360$

so $$P(z) = 946 + (z-4)1808 + \frac{(z-4)^2 \, 2378}{2!} + \frac{(z-4)^3 \, 2178}{3!}$$

$$+ \frac{(z-4)^4 \, 1272}{4!} + \frac{(z-4)^5 \, 360}{5!}$$

$$\boxed{P(z) = 946 + 1808(z-4) + 1189(z-4)^2 + 363(z-4)^3 + 53(z-4)^4 + 3(z-4)^5}$$

Section 3.5 #3

For $P(z) = 3z^5 - 7z^4 - 5z^3 + z^2 - 8z + 2$

start Newton method at $z_0 = 4$. what is z_1?

Answer.

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}.$$

$$z_1 = z_0 - \frac{f(z_0)}{f'(z_0)}.$$

From problem ① we found $P(1) = 946$

$P'(z_0)$ can be found again using Horner method.

	3	5	15	61	236	→ this line from problem ①
4		12	68	332	1572	
	3	17	83	393	$\boxed{1808}$	→ $P'(z_0)$

So $z_1 = 4 - \dfrac{946}{1808}$

$= \boxed{3.47677}$

Section 3.5 # 5

For $P(z) = 3z^5 - 7z^4 - 5z^3 + z^2 - 8z + 2$ find a disk centered at the origin that contains all roots.

Answer

using theorm that given $P(z) = a_0 + a_1 z + \cdots + a_n z^n$

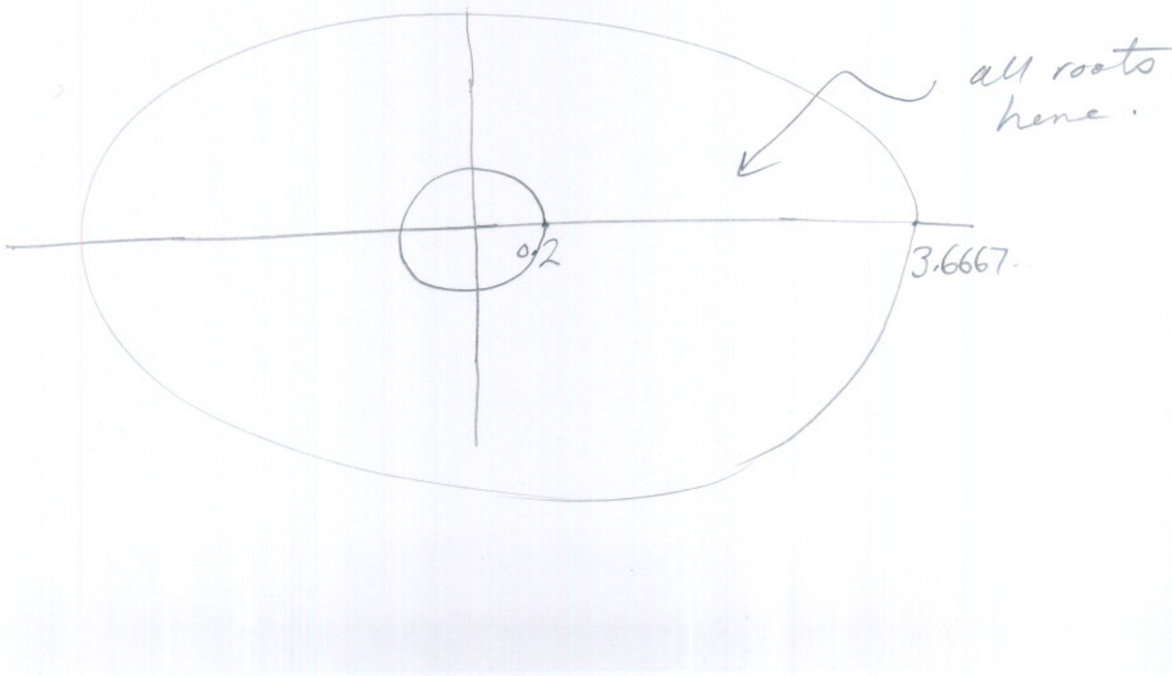
then $\rho_{max} = 1 + \frac{1}{|a_n|} \max_{0 \le j \le n} |a_j|$

where ρ_{max} is max distance from origin for any root to be at.

and $\rho_{min} = \dfrac{1}{1 + \frac{1}{|a_0|} \max_{1 \le j \le n} |a_j|}$

so $\rho_{max} = 1 + \frac{1}{3} \max\{3, |-7|, |-5|, 1, |-8|, 2\} = 1 + \frac{1}{3} \times 8 = 1 + \frac{8}{3} = \frac{3+8}{3} = \boxed{\frac{11}{3}} \approx 3.67$

$\rho_{min} = \dfrac{1}{1 + \frac{1}{2} 8} = \dfrac{1}{1+4} = \boxed{\frac{1}{5}}$



Section 3.5 # 6

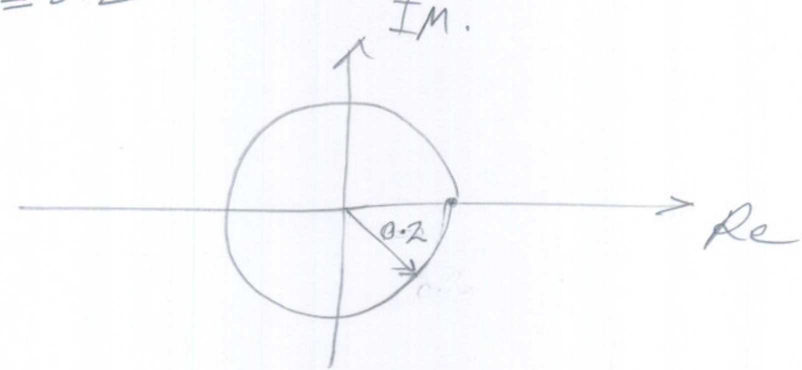
$p(z) = 3z^5 - 7z^4 - 5z^3 + z^2 - 8z + 2$

find disk centered at the origin that contains none of the roots.

answer

From problem #5 this is the disk of radius

$\rho_{min} = 0.2$



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Section 3.5 # 10

For $p(z) = 9z^4 - 7z^3 + z^2 - 2z + 5$

find $p(6)$, $p'(6)$, and the next point in Newton iteration starting at $z = 6$.

Answer using Horner method

$$
\begin{array}{c|ccccc}
 & a_4 & a_3 & a_2 & a_1 & a_0 \\
 & 9 & -7 & 1 & -2 & 5 \\
6 & & 54 & 282 & 1698 & 10176 \\
\hline
 & 9 & 47 & 283 & 1696 & \boxed{10181} \to P(6)
\end{array}
$$

$$
\begin{array}{c|cccc}
 & 9 & 47 & 283 & 1696 \\
6 & & 54 & 606 & 5334 \\
\hline
 & 9 & 101 & 889 & \boxed{7030} \to p'(6)
\end{array}
$$

$$z_1 = z_0 - \frac{f(z_0)}{f'(z_0)}$$

$$= 6 - \frac{10181}{7030} = \boxed{4.55178}$$

Section 4.1 #6

For what values of 'a' is this positive definite?

$$A = \begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix}$$

Answer

if all the eigenvalues of A are positive, the A is positive definite. hence

$$\begin{vmatrix} 1-\lambda & a & a \\ a & 1-\lambda & a \\ a & a & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - \lambda(-3+3a^2) + (3a^2 - 2a^3 - 1) = 0$$

is the charateristic equation.

roots of the above equation are

$$\boxed{\begin{aligned} \lambda_1 &= 2a+1 \\ \lambda_2 &= 1-a \\ \lambda_3 &= 1-a \end{aligned}}$$

hence from λ_2 we get that $1-a > 0$ i.e

$\boxed{a < 1}$

From $\lambda_1 = 2a+1 > 0$. hence $\boxed{a > -\frac{1}{2}}$

hence Combine both we get

$$\boxed{-\frac{1}{2} < a < 1}$$

Section 4.1 # 15

Are these positive definite?

(a) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

note Matrix A is +ve definite if $x^H A x > 0$ For any $\underline{x} \neq 0$.

$$[x_1 \quad x_2]\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1 + x_2 \qquad -x_1 + x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1(x_1 + x_2) + x_2(-x_1 + x_2)$$

$$= x_1^2 + x_1 x_2 - x_1 x_2 + x_2$$

$$= x_1^2 + x_2^2 \quad > 0$$ since not both x_1, x_2 are zero.

$\Longrightarrow$ $\boxed{\text{positive definite}}$

another way to show this is to find Eigenvalues and show that they are all > 0.

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = 0$$

so $1 + \lambda^2 - 2\lambda + 1 = 0 \qquad \lambda^2 - 2\lambda + 2 = 0$

$\Rightarrow \lambda = \dfrac{+2 \pm \sqrt{4 - 2\times 2}}{2} = 1 \qquad$ so $\lambda_{1,2} = 1 > 0$

$\Rightarrow$ positive definite.

$\longrightarrow$

(b) $A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix}$

since A is symmetric matrix, I can use sylvester criterion. which says that the symmetric matrix is +ve definite if all the upper left matrices have positive determinants. i.e Leading principle minors are positive.

i.e $\begin{bmatrix} \boxed{4} & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, $\begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, $\begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix}$

$\Downarrow$ $4 > 0$ ok

$\Downarrow$ $20 - 4 = 16 > 0$ ok

$\Downarrow$

$= 4(20-4) - 2(8-2) + 1(4-5)$

$= 4(16) - 2(\cancel{4}6) + (-1)$

$= 64 + 12 - 1 > 0$

ok.

ok

since all Leading minors $> 0 \implies \boxed{\text{Positive definite}}$

section 4.1 #17

A square Matrix is said to be skew-symmetric if $A^T = -A$. prove that if A is skew-symmetric, then $x^T A x = 0$ For all x.

Answer

$$= [x_1 \ x_2 \cdots x_n] \underset{n\times n}{\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & & \\ a_{2n} & \cdots & a_{nn} \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}$$

$1\times n$

note here that

$$\boxed{a_{ij} = -a_{ji} \quad \text{and} \quad a_{ii} = 0}$$

$$= \underset{1\times n}{\left[\sum_{i=1}^{n} x_i a_{i1}, \ \sum_{i=1}^{n} x_i a_{i2}, \ldots, \sum_{i=1}^{n} x_i a_{in} \right]} \underset{n\times n}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}$$

$$= \left(x_1 \sum_{i=1}^{n} x_i a_{i1}\right) + \left(x_2 \sum_{i=1}^{n} x_i a_{i2}\right) + \cdots + \left(x_n \sum_{i=1}^{n} x_i a_{in}\right)$$

$$= \sum_{i=1}^{n} x_1 x_i a_{i1} + \sum x_2 x_i a_{i2} + \cdots + \sum x_n x_i a_{in}$$

$$= \sum_{i=1}^{n} x_1 x_i a_{i1} + x_2 x_i a_{i2} + \cdots + x_n x_i a_{in}$$

$=$

hence this sum has this pattern:

$$x_1 x_1 a_{11} + x_2 x_1 a_{12} + x_3 x_1 a_{13} + \cdots + x_n x_1 a_{1n}$$

$i=2$

$$x_1 x_2 a_{21} + \underbrace{\cancel{x_2 x_2 a_{22}}}_{=0} + x_3 x_2 a_{23} + \cdots + x_n x_2 a_{2n}$$

$i=3$

$$x_1 x_3 a_{31} + x_2 x_3 a_{32} + \underbrace{\cancel{x_3 x_3 a_{33}}}_{=0} + \cdots + x_n x_3 a_{3n}$$

$i=n$

$$x_1 x_n a_{n1} + x_2 x_n a_{n2} + x_3 x_n a_{n3} + \cdots + \underbrace{\cancel{x_n x_n a_{nn}}}_{=0}$$

$\begin{bmatrix} & a_{12} & a_{13} \\ a_{21} & & \\ a_{31} & & \end{bmatrix}$

since $a_{11} = a_{22} = a_{33} = \cdots = a_{nn} = 0$

and since $a_{ij} = -a_{ji}$ we rewrite above as by

Factoring out a_{ij} terms

$$a_{12}\underbrace{\cancel{(x_2 x_1 - x_1 x_2)}}_{=0} + a_{13}\underbrace{\cancel{(x_3 x_1 - x_1 x_3)}}_{=0} + a_{14}\underbrace{\cancel{(x_4 x_1 - x_1 x_4)}}_{=0} + \cdots + a_{1n}\underbrace{\cancel{(x_n x_1 - x_1 x_n)}}_{=0}$$

$$+ \quad a_{23}\underbrace{\cancel{(x_3 x_2 - x_2 x_3)}}_{=0} + a_{24}\underbrace{\cancel{(x_4 x_2 - x_2 x_4)}}_{=0} + a_{25}\underbrace{\cancel{(x_5 x_2 - x_2 x_5)}}_{=0} + \cdots + a_{2n}\underbrace{\cancel{(x_n x_2 - x_2 x_n)}}_{=0}$$

$$+ \quad a_{34}\underbrace{\cancel{(x_4 x_3 - x_3 x_4)}}_{=0} + a_{35}\underbrace{\cancel{(x_5 x_3 - x_3 x_5)}}_{=0} + a_{36}\underbrace{\cancel{(x_6 x_3 - x_3 x_6)}}_{=0} + \cdots + a_{3n}\underbrace{\cancel{(x_n x_3 - x_3 x_n)}}_{=0}$$

$\vdots$

$$a_{n-1,n}\underbrace{\cancel{(x_n x_{n-1} - x_{n-1} x_n)}}_{=0} = 0$$

To see this easier I used this diagram $\rightarrow$

$$\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & \\ a_{21} & 0 & a_{23} & a_{24} & \\ a_{31} & a_{32} & 0 & & \\ a_{41} & a_{42} & & 0 & \\ & & & & 0 \end{bmatrix}$$

hence $\boxed{\begin{array}{c} x^H A x = 0 \\ \text{For all } x \end{array}} \Rightarrow$ semi positive definite

Section 4.1 # 18

Prove that diagonal elements of skew symmetric matrix are zero. also prove that determinant is zero when the matrix is of odd order

Answer

since skew symmetric

$$\text{Let } A = \begin{bmatrix} a_{11} & -a_{21} & -a_{31} & \cdots & -a_{m1} \\ a_{21} & a_{22} & -a_{32} & \cdots & -a_{m2} \\ a_{31} & a_{32} & a_{33} & \cdots & \\ \vdots & & & & \\ a_{m1} & a_{m2} & & & a_{mm} \end{bmatrix}$$

Since skew symmetric, then $-A^T = A$

ok ok

$$\text{i.e } \begin{bmatrix} -a_{11} & -a_{21} & -a_{31} & & -a_{m1} \\ +a_{21} & -a_{22} & \cdots & & -a_{m2} \\ +a_{31} & & & & \\ \vdots & & & & \\ +a_{m1} & & \cdots & & -a_{mm} \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{21} & -a_{31} & \cdots & -a_{m1} \\ a_{21} & a_{22} & -a_{32} & \cdots & \\ \vdots & a_{32} & & & \\ a_{m1} & & \cdots & & a_{mm} \end{bmatrix}$$

so this is verified ok for all off diagonal elements

but for diagonal elements we have $\boxed{a_{ii} = -a_{ii}}$

this is only possible if $\boxed{a_{ii} = 0}$

Now need to show that determinant is zero if matrix is of odd order.

let A is skew symmetric of odd order

$$\begin{bmatrix} 0 & -a_{21} & -a_{31} & \cdots & -a_{n1} \\ a_{21} & 0 & -a_{32} & & \\ a_{31} & a_{32} & 0 & & \\ & & & & a_{nn} \\ a_{n1} & & & & \end{bmatrix}$$

where n is odd.

For $n=2$, we have $\begin{bmatrix} 0 & -a_{21} \\ a_{21} & 0 \end{bmatrix} \longrightarrow \det = -(-a_{21} \times a_{21}) = a_{21}^2 \neq 0$ (unless $a_{21}=0$).

$n=3$ $\begin{bmatrix} 0 & -a_{21} & -a_{31} \\ a_{21} & 0 & -a_{32} \\ a_{31} & a_{32} & 0 \end{bmatrix} = 0\begin{vmatrix} 0 & -a_{32} \\ a_{32} & 0 \end{vmatrix} + a_{21}\begin{vmatrix} a_{21} & -a_{32} \\ a_{31} & 0 \end{vmatrix} - a_{31}\begin{vmatrix} a_{21} & 0 \\ a_{31} & a_{32} \end{vmatrix}$

$$= a_{21}(a_{32}a_{31}) - a_{31}(a_{21}a_{32}) = 0$$

$n=4$ $\begin{vmatrix} 0 & -a_{21} & -a_{31} & -a_{41} \\ a_{21} & 0 & -a_{32} & -a_{43} \\ a_{31} & a_{32} & 0 & -a_{43} \\ a_{41} & a_{42} & a_{43} & 0 \end{vmatrix} = 0 + a_{21}\begin{vmatrix} a_{21} & -a_{32} & -a_{43} \\ a_{31} & 0 & -a_{43} \\ a_{41} & a_{43} & 0 \end{vmatrix} - a_{31}\begin{vmatrix} a_{21} & 0 & -a_{43} \\ a_{31} & a_{32} & -a_{43} \\ a_{41} & a_{42} & 0 \end{vmatrix} + a_{41}\begin{vmatrix} a_{21} & 0 & -a_{32} \\ a_{31} & a_{32} & 0 \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$

$$= a_{21}\left[a_{21}(a_{43}^2) + a_{32}(a_{43}a_{41}) - a_{43}(a_{31}a_{43})\right] - a_{31}\left(a_{21}(a_{43}a_{42}) - a_{43}(a_{31}a_{42} - a_{32}a_{41})\right)$$

$$+ a_{41}\left(a_{21}(a_{32}a_{43}) - a_{32}(a_{31}a_{42} - a_{32}a_{41})\right) \neq 0.$$

need to find a general pattern? even $\Rightarrow |A| \neq 0$

but showed true for $n=2$, $n=3$ $\Rightarrow$ odd $|A|=0$??

how to generalize?

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