HW # 2

Math 501

CSUF Spring 2007

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Section 3.2 # 17

(C) $\left|\frac{1}{\sqrt{n+1}}\right| \overset{?}{\leq} C \left|\frac{1}{\sqrt{n}}\right|^2$

$$\frac{1}{\sqrt{n+1}} \overset{?}{\leq} C \frac{1}{n}$$

Since as $n \to \infty$, $n > \sqrt{n+1}$

then Not possible to find a C. for any C we pick, eventually n will get large enough such that $\frac{1}{n} < \frac{1}{\sqrt{n+1}}$

$\Rightarrow$ $\boxed{\text{NOT quadratic Convergence}}$

Section 3.2 # 17

(d) $\frac{1}{e^n}$

$$\frac{1}{e^{n+1}} \overset{?}{\leq} c\left(\frac{1}{e^n}\right)^2$$

$$\frac{1}{e^{n+1}} \overset{?}{\leq} c\,\frac{1}{e^{2n}}$$

as $n \to \infty$, $e^{2n} \gg e^{n+1}$

hence no matter how small C we pick, eventually $C\left(\frac{1}{e^{2n}}\right)$ will get larger than $\frac{1}{e^{n+1}}$

$\Rightarrow$ $\boxed{\text{NOT quadratic}}$

Section 3.2 # 17

1e) $\frac{1}{n^n}$

$$\frac{1}{(n+1)^{(n+1)}} \overset{?}{\leq} C \left(\frac{1}{n^n}\right)^2$$

$$\frac{1}{(n+1)^n (n+1)} \overset{?}{\leq} \frac{1}{n^{2n}}$$

as $n \to \infty$ the above is approximated to

$$\frac{1}{(n^n)n} \overset{?}{\leq} C \frac{1}{n^{2n}}$$

$$\frac{1}{n^{n+1}} \overset{?}{\leq} C \frac{1}{n^{2n}}$$

hence we see that for large n, $\frac{1}{n^{2n}} > \frac{1}{n^{n+1}}$

so no matter how small C we pick

eventually $\frac{1}{n^{n+1}}$ will grow larger than $C \frac{1}{n^{2n}}$

$\Rightarrow$ $\boxed{\text{NOT quadratic}}$

Section 3.2

\# 19

Prove that if r is zero of order K of function f, then quadratic convergence in Newton iteration will be restored by making this modification

$$x_{n+1} = x_n - K \frac{f(x_n)}{f'(x_n)}$$

another attempt.

Answer

$$e_n = x_n - r$$

$$e_{n+1} = x_{n+1} - r$$

$$e_{n+1} = \left[x_n - K\frac{f(x_n)}{f'(x_n)}\right] - r$$

$$e_{n+1} = [x_n - r] - K\frac{f(x_n)}{f'(x_n)}$$

$$\boxed{e_{n+1} = e_n - K\frac{f(x_n)}{f'(x_n)}} \rightarrow e_{n+1} = \frac{e_n f'(x_n) - K f(x_n)}{f'(x_n)}$$

From taylor series

$$f(x_n + e_n) = f(r) = 0 = f(x_n) - e_n f'(x_n) + \frac{e_n^2}{2!} f''(x_n)$$

$$- \frac{e_n^3 f'''(x_n)}{3!} + \cdots + \frac{e_n^K}{K!} f^{(K)}(\xi_K)$$

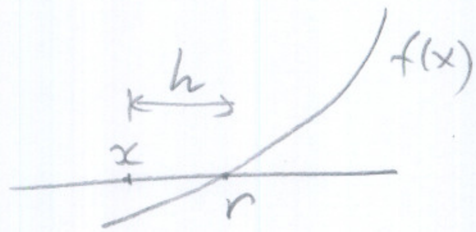
so $\quad 0 = f(x_n) - e_n f'(x_n) + \frac{e_n^2}{2!} f''(x_n) - \cdots + \frac{e_n^K}{K!} f^{(K)}(\xi_K)$

if a function $f(x)$ has zero at $x=r$ of order k,

then, I think, this means

$$f(r)=0$$
$$f'(r)=0$$
$$f^{(2)}(r)=0$$
$$\vdots$$
$$f^{(k)}(r)=0$$
$$f^{(k+1)}(r)\neq 0$$

$$f(r)=f(x+h)=0=f(x)+hf'(x)$$
$$\Rightarrow h=-\frac{f(x)}{f'(x)}$$
$$x_{n+1}=x_n-\frac{f(x)}{f'(x)}$$



so expanding in taylor

$$f(x+h)=f(r)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\cdots$$
$$0=f(x)+hf'(x)+\cdots$$

also $$f'(r)=0=f'(x)+hf''(x)+\frac{h^2}{2!}f'''(x)+\cdots$$

also $$f''(r)=0=f''(x)+hf'''(x)+\frac{h^2}{2!}f''''(x)+\cdots$$

$$\vdots$$

$$f^{k}(r)=0=f^{k}(x)+hf^{(k+1)}(x)+\frac{h^2}{2!}f^{(k+2)}(x)+\cdots$$

---

so $h=-\frac{f(x)}{f'(x)}$ from first eq. ignoring $O(h^2)$

$h=-\frac{f'(x)}{f''(x)}$ from 2^{nd} eq. ignoring $O(h^3)$

$\vdots$

$h=-\frac{f^{(k)}(x)}{f^{(k+1)}(x)}$ from k eq ignoring $O(h^k)$ terms need to show that $f^{(k)}(x) \sim k \frac{f(x)}{f'(x)} \frac{f^{(k+1)}(x)}{f^{(k)}(x)}$.

if can do that, then it will

simplify to

$$h = -\frac{k f(x)}{f'(x)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{k f(x)}{f'(x)}.$$

but not sure how to do

this. ~~need~~ more time :

Section 3.2 # 23.

(a) perform 2 Newton iteration on

$$\begin{cases} 4x_1^2 - x_2^2 = 0 \\ 4x_1x_2^2 - x_1 = 1 \end{cases} \qquad \text{starting at } \begin{matrix} X_1 = 0 \\ X_2 = 1 \end{matrix}$$

$$[X]_{k+1} = [X]_k - [J^{-1}]_k [F]_k .$$

$$[F] = \begin{bmatrix} 4x_1^2 - x_2^2 \\ 4x_1x_2^2 - x_1 - 1 \end{bmatrix}, \quad [J] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 8X_1 & -2X_2 \\ 4x_2^2 - 1 & 8x_1x_2 \end{bmatrix}$$

Step 1. $k=0$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}}{6} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \underset{2\times 2}{\begin{bmatrix} 0 & \frac{1}{3} \\ -\frac{1}{2} & 0 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ +\frac{1}{2} \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{2} \end{bmatrix} \begin{matrix} \to x_1 \\ \to x_2 \end{matrix}}$$

Step 2 $k=1$

$$\cancel{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} -\frac{8}{3} & -3 \\ 4\left(\frac{3}{2}\right)^2 - 1 & 8\left(-\frac{1}{3}\right)\left(\frac{3}{2}\right) \end{bmatrix}^{-1} \begin{bmatrix} 4\left(-\frac{1}{3}\right)^2 - \left(\frac{3}{2}\right)^2 \\ 4\left(-\frac{1}{3}\right)\left(\frac{3}{2}\right)^2 - \left(-\frac{1}{3}\right) - 1 \end{bmatrix}}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_1 - [J]_1^{-1} [F]_1$$

$$= \begin{bmatrix} -\frac{1}{3} \\ 0.5 \end{bmatrix} - \begin{bmatrix} 8(-\frac{1}{3}) & -2(0.5) \\ 4(0.5)^2-1 & 8(-\frac{1}{3})(0.5) \end{bmatrix}^{-1} \begin{bmatrix} 4(-\frac{1}{3})^2 - (0.5)^2 \\ 4(-\frac{1}{3})(0.5)^2 - (-\frac{1}{3}) - 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 \\ 0.5 \end{bmatrix} - \begin{bmatrix} -2.66667 & -1 \\ 0 & -1.33332 \end{bmatrix}^{-1} \begin{bmatrix} 0.194443 \\ -0.33332 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 \\ 0.5 \end{bmatrix} - \frac{\begin{bmatrix} -1.33332 & 1 \\ 0 & -2.66667 \end{bmatrix}}{3.55555} \begin{bmatrix} ? \\ \ \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 \\ 0.5 \end{bmatrix} - \begin{bmatrix} 0.375 & 1.8 \\ 0 & -0.74999 \end{bmatrix} \begin{bmatrix} 0.194443 \\ -0.33332 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 \\ 0.5 \end{bmatrix} - \begin{bmatrix} -0.527076 \\ 0.24997 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 0.19375 \\ 0.25003 \end{bmatrix}}$$

Section 3.2 # 22

Starting with $(0,0,1)$ carry out Newton method on

$$\begin{cases} xy - z^2 = 1 \\ xyz - x^2 + y^2 = 2 \\ e^x - e^y + z = 3 \end{cases}$$

$$f_1(x,y,z) = xy - z^2 - 1$$
$$f_2(x,y,z) = xyz - x^2 + y^2 - 2 \quad \Rightarrow F_k = \begin{bmatrix} x^k y^k - (z^k)^2 - 1 \\ x^k y^k z^k - (x^k)^2 + (y^k)^2 - 2 \\ e^{x^k} - e^{y^k} + z^k - 3 \end{bmatrix}$$
$$f_3(x,y,z) = e^x - e^y + z - 3$$

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} x^k \\ y^k \\ z^k \end{bmatrix} - J_{(k)}^{-1} F_{(k)}$$

and $$J_k = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}_{=(x^k, y^k, z^k)}$$

$$\frac{\partial f_1}{\partial x} = y, \quad \frac{\partial f_1}{\partial y} = x, \quad \frac{\partial f_1}{\partial z} = -2z$$

$$\frac{\partial f_2}{\partial x} = yz - 2x, \quad \frac{\partial f_2}{\partial y} = xz + 2y, \quad \frac{\partial f_2}{\partial z} = xy$$

$$\frac{\partial f_3}{\partial x} = e^x, \quad \frac{\partial f_3}{\partial y} = -e^y, \quad \frac{\partial f_3}{\partial z} = 1$$

$$\Rightarrow \quad J_k = \begin{bmatrix} y & x & -2z \\ yz - 2x & xz + 2y & xy \\ e^x & -e^y & 1 \end{bmatrix}_{\substack{x = x^k \\ y = y^k \\ z = z^k}} \longrightarrow$$

$x^0=0,\ y^0=0\ ,\ z^0=1.$

so $F^0 = \begin{bmatrix} xy - z^2 - 1 \\ xyz - x^2 + y^2 - 2 \\ e^x - e^y + z - 3 \end{bmatrix}_{\substack{x=0\\y=0\\z=1}} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$

$[X]_{\substack{x=0\\y=0\\z=1}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$[J]_{\substack{x=0\\y=0\\z=1}} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$ ← problem.

so $[X]_{\substack{x=x^1\\y=y^1\\z=z^1}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - [J]^{-1}_{\substack{x=0\\y=0\\z=1}} \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$

Since $[J]$ has a row with all zero $\Rightarrow$

can <u>NOT</u> inverse.

this means the point $(0,0,1)$ we selected has "zero slope". Newton method Fails here. we need to have $f'(\cdot) \neq 0$ to process Newton method.

Section 3.2 # 23

(b) Starting with (1,1)

$$\begin{cases} xy^2 + x^2y + x^4 = 3 \\ x^3y^5 - 2x^5y - x^2 = -2. \end{cases}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_1 = \begin{bmatrix} x \\ y \end{bmatrix}_0 - \begin{bmatrix} J^{-1} \end{bmatrix}_0 \begin{bmatrix} F \end{bmatrix}_0$$

$$\{F\} = \begin{bmatrix} xy^2 + x^2y + x^4 - 3 \\ x^3y^5 - 2x^5y - x^2 + 2 \end{bmatrix}$$

$$[J] = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} y^2 + 2xy + 4x^3 & 2xy + x^2 \\ 3x^2y^5 - 10x^4y - 2x & 5x^3y^4 - 2x^5 \end{bmatrix}$$

so $\begin{bmatrix} x \\ y \end{bmatrix}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1+2+4 & 2+1 \\ 3-10-2 & 5-2 \end{bmatrix}^{-1} \begin{bmatrix} 1+1+1-3 \\ 1-2-1+2 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 7 & 3 \\ -9 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Zero!

so $\boxed{\begin{bmatrix} x \\ y \end{bmatrix}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ so this means $\begin{bmatrix} x \\ y \end{bmatrix}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as well.

Point starting from is already Zero! of system.

```
function y=nma_HW2_section_2_2_number_2_FINAL(x)
%
% MATH 501, HW2.  CSUF
%
% problem section 2.2, computer problem 2.
%
% EXAMPLE OUTPUT FROM RUN:
%
% >> nma_HW2_section_2_2_number_2_FINAL(0.00000001)
% near bad region
%
% ans =
%
%      0
%
% >>
% >>
% >> nma_HW2_section_2_2_number_2_FINAL(pi)
%
% ans =
%
%    0.20264236728468
%
% >> nma_HW2_section_2_2_number_2_FINAL(2*pi)
% near bad region
%
% ans =
%
%      0
%
% >>
%
%


epsilon=0.00001;  % or use something like 10^6*eps  HOW CLOSE to bad point
multiple = 2*pi;

    if  rem(abs(x),multiple)<=epsilon
        fprintf('near bad region\n');
        y=f_series(x);
    else
        y=(1-cos(x))/x^2;
    end

end

%%%%%%%%%%%%%%%%%%%%%%
%
% This function evaluates (1-cos(x))/x^2 by
% expansion of taylor series of cos() around
% the x-point for 10 terms. This is done to
% avoid L.O.S.
%
%%%%%%%%%%%%%%%%%%%%%%%%
function f=f_series(x)
  syms z;

  f= (1- taylor(cos(z),10,x))/x^2;
```

```
  f=double(subs(f,z,x));
end

```

```
> # SECTION 2.2, computer Problem 8
  # Nasser Abbasi, 2/6/07
  # MAPLE 10 on windows XP
  #
> restart;                    # clear all variables and start new maple session
  UseHardwareFloats := true; #make sure we use IEEE HW floating points

  analyze:= proc(a,b,wrong_answer)
      local correct_answer,relative_error,absolute_error;

      correct_answer:=evalf(a/b);
      relative_error:=abs(  (correct_answer - wrong_answer) /correct_answer ):
      absolute_error:=abs(  correct_answer - wrong_answer ) :

      printf("correct_answer is %16.12f\n",correct_answer);
      printf("wrong_answer is %16.12f\n",wrong_answer);

      printf("absolute error is %16.12f\n",absolute_error);
      printf("relative error is %16.12f%%\n",relative_error);
  end proc;

> analyze(5505001,294911,18.66600092909);

correct_answer is  18.666651970000
wrong_answer is  18.666000929090
absolute error is   0.000651040000
relative error is   0.000034877170%
> analyze(4.999999,14.999999,0.333329);

correct_answer is   0.333333288900
wrong_answer is   0.333329000000
absolute error is   0.000004288900
relative error is   0.000012866702%
> analyze(4195835,3145727,1.33382);

correct_answer is   1.333820449000
wrong_answer is   1.333820000000
absolute error is   0.000000449000
relative error is   0.000000336627%
> #
  #We see that one should use the relative error as the correct measure of
  #accuracy of calculations. In some cases above (case 1 and 3) the absolute
  #error was greater than the relative error, while in others (case (2))
  #it was less. The relative erorr is the correct measure to use.
> #
```

Section 2.3

#2 $\begin{cases} x_0 = 1 & x_1 = 0.9 \\ x_{n+1} = -0.2\,x_n + 0.99\,x_{n-1} \end{cases}$

10/10

$P(E) = 0 \Rightarrow E^2 + 0.2E - 0.99 = 0$

roots are $\boxed{\lambda_1 = -1.1 \ , \ \lambda_2 = 0.9}$

hence general solution is $\boxed{x_n = A\lambda_1^n + B\lambda_2^n}$

Now Find A, B from I.C.

$n = 0 \ , \ x_0 = 1 \Rightarrow 1 = A + B$

$n = 1 \ , \ x_1 = 0.9 \Rightarrow 0.9 = A(-1.1) + B(0.9)$

Solve for A, B $\Rightarrow A = 0 \ , \ B = 1$

hence analytical solution is $\boxed{x_n = 0.9^n}$

notice that due to initial Condition, $A = 0$ has <u>removed</u> the <u>bad</u> root λ_1 which was <u>unstable</u>.

the Computation will be stable.

section 2.3

\#4 the condition number of $f(x) = x^{\alpha}$ is independent of x. what is the condition number?

$$CN = x \frac{f'(x)}{f(x)} = x \frac{\alpha x^{\alpha-1}}{x^{\alpha}}$$

$$= x \alpha \frac{x^{\alpha} x^{-1}}{x^{\alpha}} = \boxed{\alpha}$$

section 2.3

#5 what are the condition numbers for the following functions? when are they large?

(a) $(x-1)^\alpha$

$$CN = x\frac{f'(x)}{f(x)} = \frac{x\,\alpha(x-1)^{\alpha-1}}{(x-1)^\alpha} = x\alpha\frac{(x-1)^\alpha(x-1)^{-1}}{(x-1)^\alpha}$$

$$= \frac{x\alpha}{(x-1)}$$

when $\boxed{x \approx 1, \quad CN \to \infty}$

(b) $\ln x$

$$CN = x\frac{f'(x)}{f(x)} = \frac{x\,\frac{1}{x}}{\ln x} = \frac{1}{\ln x}$$

when $|\ln x| \to 0$ $\quad CN \to \infty$.

i.e. when $\boxed{x \approx 1 \quad CN \to \infty}$

(c) $\sin x$

$$CN = \frac{x f'(x)}{f(x)} = \frac{x\cos x}{\sin x} = \frac{x}{\tan x}$$

when $\boxed{x = \pm n\pi}$ for $n = 1, 2, \ldots$ we have $\tan x = 0$ and $x \neq 0$ hence $\boxed{CN \to \infty}$

Section 2.3

(d) e^x

$$CN = \frac{x f'(x)}{f(x)} = \frac{x e^x}{e^x} = x$$

so C.N. depends on x. for large x, $CN \to \infty$.

(e) $x^{-1} e^x$

$$CN = \frac{x f'(x)}{f(x)} = e^{-x}\left(-\frac{e^x}{x^2} + \frac{e^x}{x}\right)x^2$$

$$= x - 1$$

so CN depends on x.

$\boxed{CN \to \infty \quad \text{when } x \to \infty}$

(f) $\frac{1}{\cos x}$

$$CN = x\frac{f'(x)}{f(x)} = x \tan x$$

so at $\boxed{x = \pm n\frac{\pi}{2}}$ for $n = 1, 2, 3, \dots$

$\boxed{CN \to \infty}$

Section 2.3

#7

Show that the recurrence relation

$$x_n = 2x_{n-1} + x_{n-2}$$

has a general solution of form

$$x_n = A\lambda^n + B\mu^n$$

Is the recurrence relation a good way to compute x_n from arbitrary initial values x_0, x_1 ?

Answer

$$E^2 x_{n-2} = 2E\, x_{n-2} + E^0 x_{n-2}$$

$$\Rightarrow \quad (E^2 - 2E - 1) = 0$$

$$\boxed{\lambda_1 = 1+\sqrt{2}, \quad \lambda_2 = 1-\sqrt{2}}$$

since simple roots then solution is

$$x_n = A\lambda_1^n + B\lambda_2^n$$

let $\lambda_1 \equiv \lambda$, $\lambda_2 \equiv \mu$ we rewrite as

$$\boxed{x_n = A\lambda^n + B\mu^n}$$

where $\lambda = 1+\sqrt{2}$, $\mu = 1-\sqrt{2}$

notice λ unstable root.

The recurrence relation is <u>NOT</u> a good way to compute x_n from arbitrary I.C. due to possible L.O.S.

Section 2.3

#8.

Fibonacci Seq $\begin{cases} r_0 = 1 \quad r_1 = 1 \\ r_{n+1} = r_n + r_{n-1} \end{cases}$

$P(E) = (E^2 - E - 1) = 0$

Solution is $\boxed{\lambda_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}, \quad \lambda_2 = \frac{1}{2} - \frac{\sqrt{5}}{2}}$

simple roots $\Rightarrow$ general Solution $\boxed{r_n = A\lambda_1^n + B\lambda_2^n}$

when $n=0 \Rightarrow 1 = A + B$

when $n=1 \Rightarrow 1 = A\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) + B\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)$

Solve for $A, B \Longrightarrow$ $\boxed{\begin{array}{l} A = \frac{1}{2} + \frac{\sqrt{5}}{10} \\ B = \frac{1}{2} - \frac{\sqrt{5}}{10} \end{array}}$

hence general solution is $\boxed{r_n = \left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right)\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^n + \left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right)\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^n}$

now need to show that $\frac{2r_n}{r_{n-1}}$ converges to $1 + \sqrt{5}$

$\longrightarrow$

$$\frac{2 r_n}{r_{n-1}} = 2\,\frac{A\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)^n + B\left(\frac{1}{2}-\frac{\sqrt5}{2}\right)^n}{A\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)^{n-1} + B\left(\frac{1}{2}-\frac{\sqrt5}{2}\right)^{n-1}} = 2\,\frac{A\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)^n + B\left(\frac{1}{2}-\frac{\sqrt5}{2}\right)^n}{A\frac{\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)^n}{\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)} + B\frac{\left(\frac{1}{2}-\frac{\sqrt5}{2}\right)^n}{\left(\frac{1}{2}-\frac{\sqrt5}{2}\right)}}$$

note $\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)\left(\frac{1}{2}-\frac{\sqrt5}{2}\right) = -1$

$$= -2\,\frac{\left[A\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)^n + B\left(\frac{1}{2}-\frac{\sqrt5}{2}\right)^n\right]}{A\left(\frac{1}{2}-\frac{\sqrt5}{2}\right)\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)^n + B\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)\left(\frac{1}{2}-\frac{\sqrt5}{2}\right)^n}$$

But $A = \frac{1}{2}+\frac{\sqrt5}{10}$

$B = \frac{1}{2}-\frac{\sqrt5}{10}$

$$= -2\,\frac{\left[\left(\frac{1}{2}+\frac{\sqrt5}{10}\right)\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)^n + \left(\frac{1}{2}-\frac{\sqrt5}{10}\right)\left(\frac{1}{2}-\frac{\sqrt5}{2}\right)^n\right]}{\left(\frac{1}{2}+\frac{\sqrt5}{10}\right)\left(\frac{1}{2}-\frac{\sqrt5}{2}\right)\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)^n + \left(\frac{1}{2}-\frac{\sqrt5}{10}\right)\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)\left(\frac{1}{2}-\frac{\sqrt5}{2}\right)^n} \quad \text{—— } (1)$$

<u>note</u> $\left(\frac{1}{2}+\frac{\sqrt5}{10}\right)\left(\frac{1}{2}-\frac{\sqrt5}{2}\right) = \frac{1}{4}-\frac{\sqrt5}{4}+\frac{\sqrt5}{20}-\frac{5}{20} = \frac{-5\sqrt5+\sqrt5}{20} = -\frac{4\sqrt5}{20} = \boxed{-\frac{\sqrt5}{5}}$

and $\left(\frac{1}{2}-\frac{\sqrt5}{10}\right)\left(\frac{1}{2}+\frac{\sqrt5}{2}\right) = \frac{1}{4}+\frac{\sqrt5}{4}-\frac{\sqrt5}{20}-\frac{5}{20} = \frac{5\sqrt5-\sqrt5}{20} = \frac{4\sqrt5}{20} = \boxed{\frac{\sqrt5}{5}}$

so (1) becomes.

$$= -2\,\frac{\left[\left(\frac{1}{2}+\frac{\sqrt5}{10}\right)\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)^n + \left(\frac{1}{2}-\frac{\sqrt5}{10}\right)\left(\frac{1}{2}-\frac{\sqrt5}{2}\right)^n\right]}{\frac{\sqrt5}{5}\left(\left(\frac{1}{2}-\frac{\sqrt5}{2}\right)^n - \left(\frac{1}{2}+\frac{\sqrt5}{2}\right)^n\right)}$$

$$= +2\left(\frac{5}{\sqrt5}\right)\frac{\left[\left(\frac{5+\sqrt5}{10}\right)\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)^n + \left(\frac{5-\sqrt5}{10}\right)\left(\frac{1}{2}-\frac{\sqrt5}{2}\right)^n\right]}{\left(\frac{1}{2}+\frac{\sqrt5}{2}\right)^n + \left(\frac{1}{2}-\frac{\sqrt5}{2}\right)^n}$$

getting close →

take $\frac{5+\sqrt{5}}{10}$ as common factor from Numerator

$$\cancel{2}\left(\frac{5}{\sqrt{5}}\right)\left(\frac{5+\sqrt{5}}{\cancel{10}\,5}\right)\frac{\left[\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^n+\frac{(5-\sqrt{5})}{(5+\sqrt{5})}\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n\right]}{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^n+\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n}$$

$$=\frac{5+\sqrt{5}}{\sqrt{5}}\left[\frac{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^n}{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^n+\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n}+\left(\frac{5-\sqrt{5}}{5+\sqrt{5}}\right)\frac{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n}{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^n+\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n}\right]$$

So

$$=\frac{5+\sqrt{5}}{\sqrt{5}}\left[\frac{1}{1+\frac{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n}{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^n}}+\left(\frac{5-\sqrt{5}}{5+\sqrt{5}}\right)\frac{1}{1+\frac{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^n}{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n}}\right]$$

now take limit as $n\to\infty$. we see that

$$=\frac{5+\sqrt{5}}{\sqrt{5}}\left[\frac{1}{1+\underbrace{\lim_{n\to\infty}\left(\frac{\frac{1}{2}-\frac{\sqrt{5}}{2}}{\frac{1}{2}+\frac{\sqrt{5}}{2}}\right)^n}_{\to 0 \text{ since term}<1}}+\left(\frac{5-\sqrt{5}}{5+\sqrt{5}}\right)\frac{1}{1+\underbrace{\lim_{n\to\infty}\left(\frac{\frac{1}{2}+\frac{\sqrt{5}}{2}}{\frac{1}{2}-\frac{\sqrt{5}}{2}}\right)^n}_{\to\infty \text{ since term}>1}}\right]$$

so we set $\frac{5+\sqrt{5}}{\sqrt{5}}\left[\frac{1}{1+0}+\frac{\sqrt{5}-\sqrt{5}}{5+\sqrt{5}}\cancel{\frac{1}{1+\infty}}=0\right]$

we set $\frac{5+\sqrt{5}}{\sqrt{5}}=\frac{5\sqrt{5}+5}{5}=\boxed{1+\sqrt{5}}$

I hope we do not get such problem in exam

Section 3.1

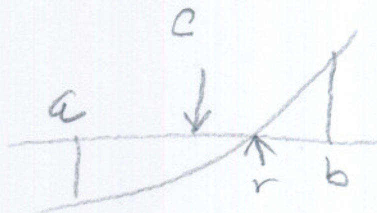
#2 Consider bisection method, interval $[1.5, 3.5]$.

(a) what is width of interval after n steps?

(b) what is maximum distance possible between root r and the midpoint of this interval?

Answer

(a)



after one iteration, width $= \frac{1}{2}[a,b]$

after 2 iterations, width $= \frac{1}{2}\left(\frac{1}{2}[a,b]\right)$.

so after n iterations, width $= \left(\frac{1}{2}\right)^n [a,b]$.

i.e. $\left(\frac{1}{2}\right)^n (3.5-1.5) = \boxed{2\left(\frac{1}{2}\right)^n}$

(b) max possible distance is when root is on edge of interval.

So max distance $= \frac{|a-b|}{2} = \boxed{1}$

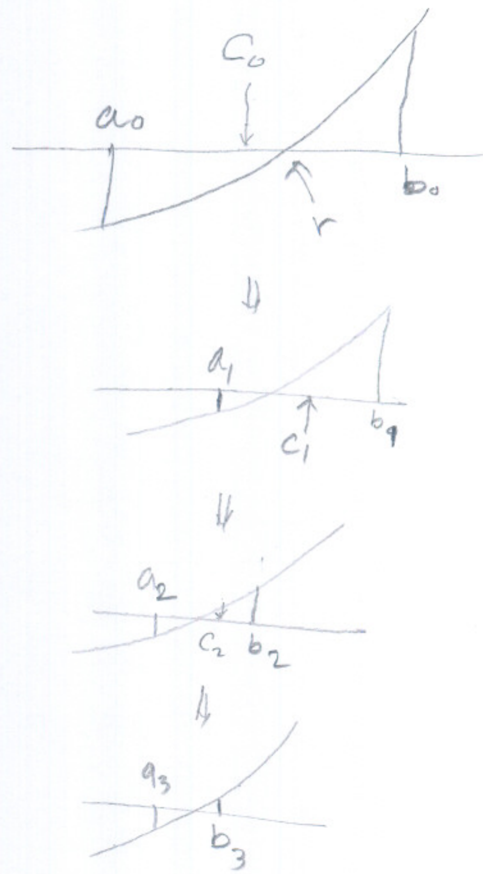
section 3.1

14

let bisection method be applied to Continuous function resulting in intervals $[a_0, b_0]$, $[a_1, b_1]$, ...
let $r = \lim_{n \to \infty} a_n$. which of these statements Can be false?

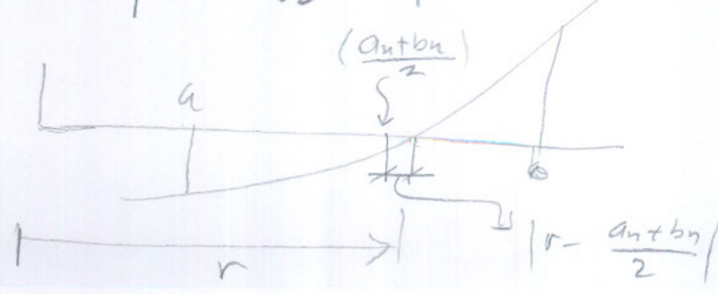
(a) $a_0 \le a_1 \le a_2 \le \dots$

this stament $\boxed{\text{Can NOT}}$ be false. the "left" edge 'a' will always be shifted to the right, and never can be shifted back to the "left".



(b) $\left| r - \frac{(a_n + b_n)}{2} \right| \le \frac{b_0 - a_0}{2^n}$ $n \ge 0$.

this $\boxed{\text{Can NOT}}$ be false. $\left| r - \frac{(a_n + b_n)}{2} \right|$



Section 3.1

(c) $\left| r - \frac{a_{n+1}+b_{n+1}}{2} \right| \leq \left| r - \frac{a_n+b_n}{2} \right| \quad n \geq 0$

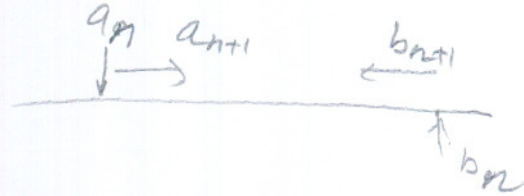
$\boxed{\text{Can NOT}}$ be false since $\frac{a_{n+1}+b_{n+1}}{2} \leq \frac{a_n+b_n}{2}$

since a_{n+1}, b_{n+1} are "closer" to each other than a_n, b_n to each other.

(d) $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \quad n \geq 0$

since $a_{n+1} \geq a_n$

and $b_{n+1} \leq b_n$



Therefore this means all points in interval $[a_{n+1}, b_{n+1}]$ are inside all points in interval $[a_n, b_n]$.

So this statment $\boxed{\text{Can NOT}}$ be false

(e) $|r - a_n| = O\left(\frac{1}{2}\right)^n$ as $n \to \infty$

root r is fixed. so "edge" a will move closer to it if r is somewhere inside interval. "a" can not move to r faster than $\frac{1}{2}$ the length of the interval. so taking the length of the initial interval as constant, then for root inside interval this statment is correct.

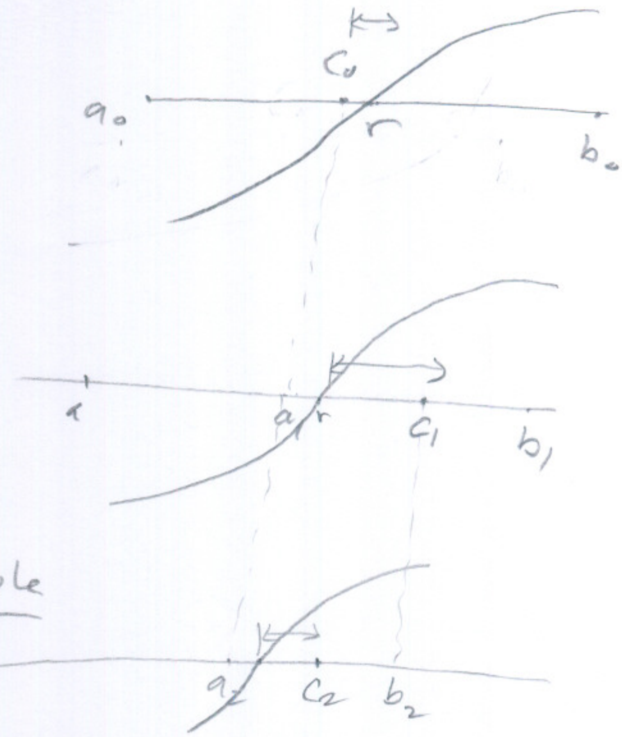
so statment $\boxed{\text{Can NOT}}$ be false

ps. do I need to worry about pathological cases when $r = a_0$?

Section 3.1

(4) $|r - c_n| < |r - c_{n-1}|$ $\quad n \geq 1$

c_n is the middle of the interval at step n.



We see that it is NOT possible that $|r - c_n| > |r - c_{n-1}|$

So this statement $\boxed{\text{Can NOT}}$ be false

P.S. for $n=0$, it is possible that

$|r - c_0| < |r - c_1|$ $\quad$ but for $n \geq 1$ Not possible.

Section 3.1 # 15

prove that point c computed in bisection is the point where line through

(a, sign(f(a))) and (b, sign(f(b)))

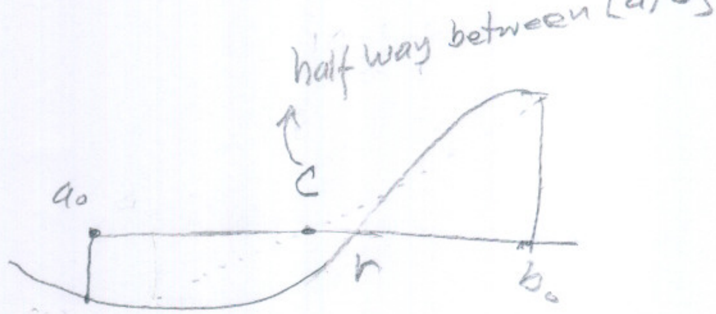
intersects x-axis.

Note

$\text{Sign}(f(a)) = 1$ if $f(a) > 0$

$\text{Sign}(f(a)) = -1$ if $f(a) < 0$.

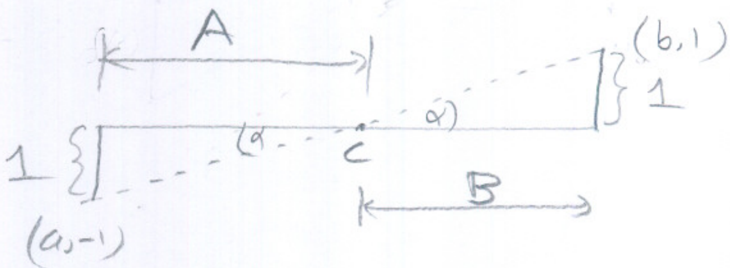
$\text{sign}(f(a)) = 0$ if $f(a) = 0$



Assume $f(a) < 0$ and $f(b) > 0$.
then the line is $(a, -1)$ and $(b, 1)$ as follows

need to show that

$A = B$.



since $\tan\alpha = \frac{1}{B}$

and also $\tan\alpha = \frac{1}{A}$

hence $\frac{1}{B} = \frac{1}{A} \Rightarrow B = A$ hence line intersect

x-axis at mid point, which is point C.

section 3.1

#16 suppose $|a_n - b_n| \le \lambda_n |a_{n-1} - b_{n-1}|$ for all n with $\lambda_n < 1$, final upper bound on $|a_n - b_n|$ in terms of $|a_0 - b_0|$ and $\lambda = \max_{1 \le i \le n} \{\lambda_i\}$

answer

$$|a_1 - b_1| \le \lambda_1 |a_0 - b_0|$$

and $|a_2 - b_2| \le \lambda_2 |a_1 - b_1| \le \lambda_2 \lambda_1 |a_0 - b_0|$

10/10

and $|a_3 - b_3| \le \lambda_3 |a_2 - b_2| \le \lambda_3 \lambda_2 \lambda_1 |a_0 - b_0|$.

hence

$$|a_n - b_n| \le \lambda_n |a_{n-1} - b_{n-1}|$$

$$|a_n - b_n| \le \overbrace{\lambda_n \lambda_{n-1} \lambda_{n-2} \cdots \lambda_1}^{\text{n-t these}} |a_0 - b_0|$$

So upper bound is $\boxed{\lambda^n}$ where λ is the $\boxed{\text{largest}}$ of all λ's.

i.e $\lambda = \max_{1 \le i \le n} \lambda_n$

```
> # SECTION 3.1, computer Problem 1
  # Nasser Abbasi, 2/6/07
  # MAPLE 10 on windows XP
  #
  #PROBLEM: Write and test the bisection method on the following
  # (a) x^-1 - tan(x) on [0,Pi/2]
  # (b) x^-1 -2^x on [0,1]
  # (c) 2^-x + exp(x) +2*cos(x) - 6 on [1,3]
  # (d) (x^3+4 x^2+3 x+5)(2x^3-9x^2+18x-2) on [0,4]

> restart;                        # clear all variables and start new maple session
  UseHardwareFloats := true; #make sure we use IEEE HW floating points

  bisection:= proc(leftPt,rightPt,M,yTol,xTol)
       local u,v,e,k,w,a,b,c;

       a:=leftPt;
       b:=rightPt;

       u:=f(a):
       v:=f(b):
       e:=b-a:

       printf("a=%f,b=%f,f(a)=%f,f(b)=%f\n",a,b,u,v);

       if sign(u)=sign(v) then
          return;
       end if;

       for k from 1 to M do
           e:=e/2;
           c:=a+e;
           w:=f(c);
           printf("k=%d,c=%f,w=%f,e=%f\n",k,c,w,e);

           if abs(e)<xTol or abs(w)<yTol then
              if(abs(e)<xTol) then
                 printf("reached X-tolerance\n");
              else
                 printf("reached Y-tolerance\n");
              end if;
              return;
           end if;

           if (sign(w)<>sign(u)) then
              b:=c;
              v:=w;
           else
             a:=c;
             u:=w;
           end if;
     end do;

  end proc;
```

```
> #CASE(a)
  xTol:=0.0001;
  yTol:=0.0001;
  MAX_ITER:=1000;
  f:=x-> 1/x - tan(x);   #[0,Pi/2]
  #plot(f(x),x=0..Pi/2);
  bisection(0,pi/2,MAX_ITER,xTol,yTol);
```

$$xTol := 0.0001$$

$$yTol := 0.0001$$

$$MAX\_ITER := 1000$$

$$f := x \rightarrow \frac{1}{x} - \tan(x)$$

```
Error, (in f) numeric exception: division by zero
```

```
> #CASE(b)
  f:=x-> 1/x - 1/2^2;   #[0,1]
  #plot(f(x),x=0..1);
  bisection(0,1,MAX_ITER,xTol,yTol);
```

$$f := x \rightarrow \frac{1}{x} - \frac{1}{4}$$

```
Error, (in f) numeric exception: division by zero
```

```
> #CASE(c)
  f:=x-> 1/2^x - exp(x) + 2*cos(x) -6;   #[1,3]
  #plot(f(x),x=1..3);
  bisection(1,3,MAX_ITER,xTol,yTol);
```

$$f := x \rightarrow \frac{1}{2^x} - \mathbf{e}^x + 2\cos(x) - 6$$

```
a=1.000000,b=3.000000,f(a)=-7.137677,f(b)=-27.940522
Error, (in bisection) unable to evaluate sign
```

```
> #CASE(d)
  xTol:=0.00001;
  yTol:=0.00001;
  MAX_ITER:=10000;

  f:=x-> (x^3 +4*x^2+3*x+5)/(2*x^3-9*x^2+18*x-2);   #[0,4]
  plot(f(x),x=0..0.5);
  bisection(0,4,MAX_ITER,xTol,yTol);
```

$$xTol := 0.00001$$

$$yTol := 0.00001$$

$$MAX\_ITER := 10000$$

$$f := x \rightarrow \frac{x^3 + 4x^2 + 3x + 5}{2x^3 - 9x^2 + 18x - 2}$$

```
> # SECTION 3.1, computer Problem 3
  # Nasser Abbasi, 2/6/07
  # MAPLE 10 on windows XP
  #
  #PROBLEM:
  # Find a root of f(x)=x-tan(x) in interval [1,2]

> #Use the bisection method written for previouse problem

  restart;                      # clear all variables and start new maple session
  UseHardwareFloats := true; #make sure we use IEEE HW floating points

  bisection:= proc(leftPt,rightPt,M,yTol,xTol)
      local u,v,e,k,w,a,b,c;

      a:=leftPt;
      b:=rightPt;

      u:=f(a):
      v:=f(b):
      e:=b-a:

      printf("a=%f,b=%f,f(a)=%f,f(b)=%f\n",a,b,u,v);

      if sign(u)=sign(v) then
         return;
      end if;

      for k from 1 to M do
          e:=e/2;
          c:=a+e;
          w:=f(c);
          printf("k=%d,c=%f,w=%f,e=%f\n",k,c,w,e);

          if abs(e)<xTol or abs(w)<yTol then
             if(abs(e)<xTol) then
                printf("reached X-tolerance\n");
             else
                printf("reached Y-tolerance\n");
             end if;
             return;
          end if;

          if (sign(w)<>sign(u)) then
             b:=c;
             v:=w;
          else
            a:=c;
            u:=w;
          end if;
     end do;

  end proc;
```

```
> # SECTION 2.2, computer Problem 1
  # Nasser Abbasi, 2/6/07
  # MAPLE 10 on windows XP
  #
  # PROBLEM: Write a program to compute
  #  f(x)= sqrt( x^2+1 ) -1;
  #  g(x)= x^2/( sqrt(x^2+1) +1 );
  # for data x = 8^(-n). Comment on result and which is more reliable
  #

  restart;                       # clear all variables and start new maple session
  UseHardwareFloats := true; #make sure we use IEEE HW floating points

  # now define the 2 functions

  f:= x -> sqrt( x^2+1 ) -1;
  g:= x -> x^2/( sqrt(x^2+1) +1 );

  # set some max iterations and define data to store results in
  MAX_ITER:=10:
  data:=Matrix(MAX_ITER,5):

  for n from 1 to MAX_ITER do
       x := 8^(-n);
       data[n,1]:= n;
       data[n,2]:= x;
       data[n,3]:= evalf(f(x));
       data[n,4]:= evalf(g(x));
       data[n,5]:= abs(data[n,3]-data[n,4]);  # difference |f(x)-g(x)|
  end:

  # Now display the data. The first column is n, second is x
  # third is f(x), 4th is g(x), 5th is |f(x)-g(x)|
  data;
```

$$UseHardwareFloats := true$$

$$f := x \rightarrow \sqrt{x^2+1} - 1$$

$$g := x \rightarrow \frac{x^2}{\sqrt{x^2+1}+1}$$

→ back

$$
\begin{array}{c}
\begin{array}{ccccc} n & x & f(x) & g(x) & |f(x)-g(x)| \end{array} \\
\begin{bmatrix}
1 & \frac{1}{8} & 0.007782218 & 0.007782218539 & 0.539\ 10^{-9} \\
2 & \frac{1}{64} & 0.000122063 & 0.0001220628628 & 0.1372\ 10^{-9} \\
3 & \frac{1}{512} & 0.1907\ 10^{-5} & 0.1907346814\ 10^{-5} & 0.346814\ 10^{-9} \\
4 & \frac{1}{4096} & 0.30\ 10^{-7} & 0.2980232194\ 10^{-7} & 0.19767806\ 10^{-9} \\
5 & \frac{1}{32768} & 0. & 0.4656612873\ 10^{-9} & 0.4656612873\ 10^{-9} \\
6 & \frac{1}{262144} & 0. & 0.7275957615\ 10^{-11} & 0.7275957615\ 10^{-11} \\
7 & \frac{1}{2097152} & 0. & 0.1136868377\ 10^{-12} & 0.1136868377\ 10^{-12} \\
8 & \frac{1}{16777216} & 0. & 0.1776356840\ 10^{-14} & 0.1776356840\ 10^{-14} \\
9 & \frac{1}{134217728} & 0. & 0.2775557562\ 10^{-16} & 0.2775557562\ 10^{-16} \\
10 & \frac{1}{1073741824} & 0. & 0.4336808690\ 10^{-18} & 0.4336808690\ 10^{-18}
\end{bmatrix}
\end{array}
$$

```
> # We see from the above that g(x) is more reliable. For example at iteration
  n=5,
  # f(x) gave zero as a result.
  # g(x) is more reliable since it was rewritten to avoid L.O.S. problem with x
  # is close to zero
>
```

section 2.2 problem 9

18/20

(a) $\sqrt{x^2+1} - x$

$$(\sqrt{x^2+1} - x)\frac{(\sqrt{x^2+1}+x)}{(\sqrt{x^2+1}+x)} = \frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x} = \boxed{\frac{1}{\sqrt{x^2+1}+x}}$$

(b) $\log_{10} x - \log_{10} y = \boxed{\log \frac{x}{y}}$ when $x \sim y$

$\Rightarrow \log 1 = 0$

(c) $x^{-3}(\sin x - x)$

Per solution on page 58, we see that

$$\sin x - x \Rightarrow \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - x \Rightarrow \left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

so $x^{-3}(\sin x - x) = -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots$

per page 58, we say

For $|x| \geq 1.9$ use $\boxed{x^{-3}(\sin x - x)}$

for $|x| < 1.9$ use $\boxed{-\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \frac{x^6}{9!}}$

d) $\sqrt{x+2} - \sqrt{x} = (\sqrt{x+2} - \sqrt{x})\frac{(\sqrt{x+2}+\sqrt{x})}{(\sqrt{x+2}+\sqrt{x})} = \frac{(x+2)-x}{\sqrt{x+2}+\sqrt{x}} = \boxed{\frac{2}{\sqrt{x+2}+\sqrt{x}}}$

e) $e^x - e$. when $x \approx 1$ we get numbers close to each others.

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots\right) - e = (1-e) + \left(x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots\right)$$

f) $\log x - 1$ $\boxed{\text{Note } \log_{10} x = \frac{\ln x}{\ln 10}}$

when $x \approx 10$ we get numbers close to each other.

$$\log_{10} x = \frac{1}{\ln 10}\left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots\right].$$

$$\text{so } \log_{10} x - 1 = \frac{(x-1)}{\ln 10} - 1 + \frac{1}{\ln 10}\left[-\frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots\right]$$

$$= \frac{x-1-\ln 10}{\ln 10} + \frac{1}{\ln 10}\left[-\frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots\right]$$

(g) $\frac{\cos x - e^{-x}}{\sin x}$ when $x \approx 0$ loss of significance.

$$= \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - \left(1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots\right)}{\sin x}$$

$$= \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)}{\sin x}$$

$$= \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots}{\sin x}$$

$$= \frac{x - x^2 + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{2}{6!}x^6 \cdots}{\sin x} = \frac{x\left(1 - x + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots\right)}{\sin x}$$

Section 2.2

#9

(h) $\sin x - \tan x$

when $x \approx n\pi$ where $n = \pm$ integer. then L.O.S.

express in taylor series.

$$\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) - \left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots\right)$$

$$= x^3\left(\frac{1}{3} - \frac{1}{6}\right) + x^5\left(\frac{1}{5!} - \frac{2}{15}\right) + x^7\left(\frac{17}{315} - \frac{1}{7!}\right) + \cdots$$

$$= -\frac{1}{2}x^3 - \frac{1}{8}x^5 - \frac{13}{240}x^7 - \cdots\cdots$$

OK

$$= \boxed{x^3\left(-\frac{1}{2} - \frac{1}{8}x^2 - \frac{13}{240}x^4 - \cdots\cdots\right)}$$

No L.O.S. when $x \approx 0$.

So use above for $x \approx 0$. For say N terms where N is TBD depending on desired accuracy.

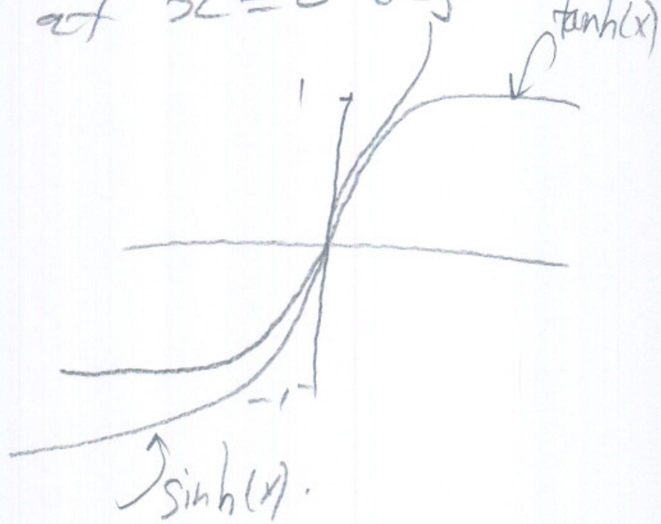
for x away from $x \approx n\pi$, then can use $\sin x - \tan x$ ok.

Section 2.2

\# 9

(i) $\sinh(x) - \tanh(x)$.

a plot of $\sinh(x)$ and $\tanh(x)$ shows they are close to each others at $x=0$ orig.



Taylor: $\sinh(x) =$

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots$$

hence $\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right) - \left(x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots\right)$

$$= x^3\left(\frac{1}{3!} + \frac{1}{3}\right) + x^5\left(\frac{1}{5!} - \frac{2}{15}\right) + x^7\left(\frac{1}{7!} + \frac{17}{315}\right) + \cdots$$

$$= x^3\left(\frac{1}{2}\right) + x^5\left(-\frac{1}{8}\right) + x^7\left(\frac{13}{240}\right) + x^9\left(-\frac{529}{24192}\right) + \cdots$$

$$= x^3\left[\frac{1}{2} - \frac{1}{8}x^2 + \frac{13}{240}x^4 - \cdots\right]$$

when $x \approx 0$ use ↗ else use $\sinh(x) - \tanh(x)$.

Section 2.2

#9

(j) $\ln\left(x+\sqrt{x^2+1}\right)$.

No subtraction. NO L.O.S.

Section 2.2

#12

(a) $\dfrac{(1-x)}{(1+x)} - \dfrac{1}{(3x+1)}$

One case to consider is $(1-x)$. we have L.O.S. when $x \approx 1$

another case to consider is when $\dfrac{(1-x)}{(1+x)} \approx \dfrac{1}{(3x+1)}$, then we have L.O.S. when $x \approx 0$ and when $x \approx \frac{1}{3}$.

So 3 regions: $x \approx 1$, $x \approx 0$, $x \approx \frac{1}{3}$.

$$\frac{(1-x)}{(1+x)} - \frac{1}{3x+1} = \frac{(1-x)(3x+1)-(1+x)}{(1+x)(3x+1)} = \frac{3x+1-3x^2-x-1-x}{(1+x)(3x+1)}$$

$$= \frac{x-3x^2}{(1+x)(3x+1)} = \boxed{\frac{x(1-3x)}{(1+x)(3x+1)}}$$

Now test the 3 regions on this new form.

when $x \approx 1$ ok. No L.O.S.

when $x \approx 0$ ok. No L.O.S.

when $x \approx \frac{1}{3}$ we have L.O.S. using this form. So for Case $x \approx \frac{1}{3}$ use original form.

Conclusion

$x \approx 1$ or $x \approx 0$ use $\dfrac{x(1-3x)}{(1+x)(3x+1)}$

when $x \approx \frac{1}{3}$ use $\dfrac{(1-x)}{(1+x)} - \dfrac{1}{(3x+1)}$

section 2.2

12 (b)

$$\sqrt{x+\frac{1}{x}} - \sqrt{x-\left(\frac{1}{x}\right)} \Rightarrow \text{L.O.S. when } x \approx 0$$

$$= \left(\sqrt{x+\frac{1}{x}} - \sqrt{x-\frac{1}{x}}\right) \frac{\left(\sqrt{x+\frac{1}{x}} + \sqrt{x-\frac{1}{x}}\right)}{\left(\sqrt{x+\frac{1}{x}} + \sqrt{x-\frac{1}{x}}\right)}$$

$$= \frac{\left(x+\frac{1}{x}\right) - \left(x-\frac{1}{x}\right)}{\sqrt{x+\frac{1}{x}} + \sqrt{x-\frac{1}{x}}} = \frac{\frac{2}{x}}{\sqrt{x+\frac{1}{x}} + \sqrt{x-\frac{1}{x}}}$$

so when $x \approx 0$ use $\dfrac{2}{x\left(\sqrt{x+\frac{1}{x}} + \sqrt{x-\frac{1}{x}}\right)}$

else use $\sqrt{x+\frac{1}{x}} - \sqrt{x-\frac{1}{x}}$

section 2.2

#12 (c) $e^x - \cos x - \sin x$

when $x \approx 0$ we get L.O.S. since $\approx 1-1$

write taylor Series for each term.

$$\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\cdots\right)-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots\right)-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)$$

$$= \boxed{2\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\frac{x^{10}}{10!}+\frac{x^{11}}{11!}+\cdots\right)}$$

so when $x \approx 0$ use expression. Select up to $\frac{x^{11}}{11!}$ term.

else use original expression $e^x - \cos x - \sin x$.

section 2.2

#16 $f(x) = -e^{-2x} + e^{x}$

express in power series

$$= -\left(1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \cdots\right) + \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= \left(-1 + 2x - \frac{4x^2}{2!} + \frac{8x^3}{3!} \cdots\right) + \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= 3x - \frac{3}{2}x^2 + \frac{1}{2}x^3 - \cdots$$

$$= 3x\left(1 - \frac{x}{2}\right) + \frac{1}{2}x^3 - \cdots$$

∴ for small x $\left(3x - \frac{3}{2}x^2\right)$ is more accurate than $3x$

Since we are using 2 terms in expansion.

hence answer is $\boxed{3x\left(1 - \frac{x}{2}\right)}$

section 2.2

\# 21

Find a way to accurately compute $f(x) = x + e^{x} + e^{-x}$ for small x.

where is the loss?

$$f(x) = x + \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) + \left(1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \cdots\right)$$

$$= x + \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right)$$

$$= x + \left(2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + 2\frac{x^6}{6!} + \cdots\right)$$

$$= x + 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)$$

$$= \boxed{x + 2\cos x}$$