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HW # 9

section 5.3

math 501

5.3 #2

Prove that if A is $m \times n$ matrix of rank n, then $A^H A$ is nonsingular

$\underset{(n \times m)}{A^H} \underset{(m \times n)}{A}$ = is matrix that square of order $n \times n$.

need to show $A^H A$ Ras

but since Rank $= n$, then

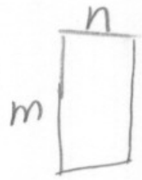
we have n L.I. columns, and n L.I rows. so Matrix is full column Rank and full Row Rank.

$\Longrightarrow$ Non Singular.

5.3 #3

Prove that if A is $m \times n$ matrix of rank n, then $A^H A$ is Hermitian and positive definite.

A Hermitian Matrix is square matrix in which $A = A^H$

$A =$  since rank = # of columns, then # of column $\leq$ number of rows m

Positive definite means $x^T A x > 0$ $\quad x \neq 0$.

$$\underset{(n \times m)}{A^H}\ \underset{(m \times n)}{A} \text{ is } n \times n \text{ Matrix.}$$

need to show that $\underline{(A^H A)} = (A^H A)^H$

$$= A^H (A^H)^H$$

$$= \underline{A^H A}$$

so it is Hermitian. Now to show it is positive definite.

need to show that $x^T (A^H A) x > 0$ ①

Since $A^H A$ is Hermitian, Then $(A^H A) = (A^H A)^H = A^H A$

so ① is $x^T A^H A x = (x^T A^H) A x = (Ax)^H \underline{Ax}$

but $Ax \xrightarrow{\text{mapped}}$ vector in Range of A. Say $\bar{b} \neq 0$ if fully Column Rank

so $(Ax)^H Ax = (\bar{b})^H b = \|\bar{b}\| > 0$ QED.

5.3 # 14

If U, V are unitary, does it follow that $\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$ is unitary?

U is unitary if $UU^H = U^H U = I_n$

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}^H = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} U^2 & 0 \\ 0 & V^2 \end{pmatrix}$$

for this to be I_n, we must have $U^2 = I_n$ and $V^2 = I_n$. i.e we need $UU = I_n$ and $VV = I_n$.

but this is not necessarily true. We are told only that $UU^H = I_n$, $VV^H = I_n$.

So it does **NOT** follow

5.3 # 16 use Householder algorithm to do QR

$$\begin{pmatrix} 0 & -4 \\ 0 & 0 \\ -5 & -2 \end{pmatrix} \leftarrow \text{Notice rank} < m.$$

$$\beta = -\|A_1\|_2 = \left\| \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} \right\| = -5$$

$$\alpha = \frac{\sqrt{2}}{\|A_1 - \beta \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\|_2} = \frac{\sqrt{2}}{\left\| \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\|_2} = 0.1394$$

$$v = \alpha (A_1 - \beta e^{(1)}) = 0.1394 \left(\begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1.2306 \\ 0 \\ -.6968 \end{pmatrix}$$

so $U_1 = I - vv^H$

$$= \begin{pmatrix} -5.145 & 0 & 0.8575 \\ 0 & 1 & 0 \\ 0.8575 & 0 & 0.5145 \end{pmatrix}$$

$$U_1 A = \begin{pmatrix} -5.83 & 0.343 \\ 0 & 0 \\ 0 & -4.459 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 \\ -4.459 \end{pmatrix}$$

$$\beta = -\|A_1\|_2 = -\left\| \begin{pmatrix} 0 \\ -4.459 \end{pmatrix} \right\|_2 = -4.459$$

$$\alpha = \frac{\sqrt{2}}{\|A_1 - \beta \begin{pmatrix} 1 \\ 0 \end{pmatrix}\|_2} = \frac{\sqrt{2}}{\left\| \begin{pmatrix} 0 \\ -4.459 \end{pmatrix} + 4.459 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_2} = 0.2243$$

$$v = \alpha (A_1 - \beta e^{(1)}) = 0.2243 \left(\begin{pmatrix} 0 \\ -4.459 \end{pmatrix} + 4.459 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow$$

So $U_2 = I - vv^H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

R

so $U_2 U_1 A = \boxed{\begin{pmatrix} -5.831 & 0.3430 \\ 0 & -4.459 \\ 0 & 0 \end{pmatrix}}$

$$Q = U_1^H U_2^H = \begin{pmatrix} -5.145 & 0 & 0.8575 \\ 0 & 1 & 0 \\ 0.8575 & 0 & 0.5145 \end{pmatrix}^H \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^H$$

$$\boxed{Q = \begin{pmatrix} -0.5145 & 0.875 & 0 \\ 0 & 0 & 1 \\ 0.8575 & 0.5145 & 0 \end{pmatrix}}$$

5.3 # 20 let D be diagonal matrix and U unitary matrix, under what hypotheses on D can we infer that DU is unitary?

Unitary Matrix is $n \times n$, satisfying $U^H U = U U^H = I_n$

so matrix is unitary iff its inverse = its conjugate transpose.

if DU is unitary, then $(DU)^H (DU) = I_n$

i.e $U^H D^H D U = I_n$

since $U^H U = I_n$, then we need $D^H D = I_n$ to infer that DU is unitary.

also we want $(DU)(DU)^H = I_n$

i.e $\underbrace{D U U^H}_{I_n} D^H = I_n$

$DD^H = I_n$

I_n since we know U is unitary.

So we require that $D^H D = I_n$

and $DD^H = I_n$.

So we require D to be unitary also.

5.3 #29

give example of vectors x, y for which

$$\|x+y\|_2^2 = \|x\|_2^2 + \|y\|_2^2 \quad \text{and} \quad \langle x, y \rangle \neq 0.$$

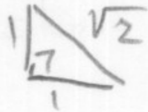
Laws of Cosine $\|x+y\|_2^2 = \|x\|_2^2 + \|y\|_2^2 - 2\|x\|_2\|y\|_2 \cos\theta$

angle between $\bar{x}, \bar{y}$.

So we are told $2\|x\|_2\|y\|_2 \cos\theta = 0$.

but $\cos\theta$ can't be zero since we are told that

$\langle x, y \rangle \neq 0$ when $\langle x, y \rangle = \|x\| \, \|y\| \cos\theta$.

Since can't have a 

, since $\cos 90^\circ = 0$.

So the other choice is to have $\bar{x}$ or $\bar{y} = 0$ but

this makes $\langle x, y \rangle = 0$ also.

$\therefore$ Impossible to solve this problem.

?

5.3 # 30

Find L.S. $(x\ y)\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix} = (3\ 0\ 1)$

$1\times 2 \quad 2\times 3 \quad 1\times 3$

$(3x+2y \quad 2x+3y \quad x+2y) = (3 \quad 0 \quad 1)$

i.e $\begin{array}{l} 3x+2y=3 \\ 2x+3y=0 \\ x+2y=1 \end{array} \rightarrow \underset{A}{\begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 1 & 2 \end{pmatrix}} \underset{x}{\begin{pmatrix} x \\ y \end{pmatrix}} = \underset{b}{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}}$.

so Least squares Solution is

$\hat{x} = (A^TA)^{-1}A^Tb = \begin{pmatrix} 1.3810 \\ -0.6667 \end{pmatrix}$

i.e $\boxed{\begin{array}{l} x = 1.3810 \\ y = -0.6667 \end{array}}$

I used formula for least squares I learned in 307. which is

$$Ax = b$$

$$A^TA\hat{x} = A^Tb$$

$$\boxed{\hat{x} = (A^TA)^{-1}A^Tb}$$

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QR for $\begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

find β. $= -\|A_1\|_2 = -\left\| \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\|_2 = \boxed{-5}$

$$\alpha = \frac{\sqrt{2}}{\|A_1 - \beta e^{(1)}\|_2} = \frac{\sqrt{2}}{\left\| \begin{pmatrix} 3 \\ 4 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_2} = \frac{\sqrt{2}}{\left\| \begin{pmatrix} 8 \\ 4 \end{pmatrix} \right\|_2}$$

$$= \frac{\sqrt{2}}{8.9443} = \boxed{0.1581}$$

so $v = \alpha (A_1 - \beta e^{(1)}) = 0.1581 \left(\begin{pmatrix} 3 \\ 4 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$

$$= 0.1581 \begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 1.2648 \\ 0.6324 \end{pmatrix}$$

so first Unitary factor $U_1 = I - vv^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1.2648 \\ 0.6324 \end{pmatrix} v^H$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1.2648 \\ 0.6324 \end{pmatrix} \begin{pmatrix} 1.2648 & 0.6324 \end{pmatrix}$$

$$U_1 = \begin{pmatrix} -0.6 & -0.8 \\ -0.8 & 0.6 \end{pmatrix}$$

$$U_1 A = \begin{pmatrix} -5 & -5.2 & -6.6 \\ 0 & 1.4 & 1.2 \end{pmatrix}$$

so $\boxed{Q = U_1^H = \begin{pmatrix} -0.6 & -0.8 \\ -0.8 & 0.6 \end{pmatrix}}$

$$R = U_1 A = \begin{pmatrix} -5 & -5.2 & -6.6 \\ 0 & 1.4 & 1.2 \end{pmatrix}$$