HW2, Math 307. CSUF. Spring 2007.

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1 Section 1.5, Problem 2

Problem: What multiple l_{32} of row 2 of A will elimination subtract from row 3 of A? Use the factored form $A = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \boxed{4} & 1 \end{bmatrix}}^{L} \overbrace{\begin{bmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix}}^{U}$ what will be the pivots? will a row exchange be required?

Solution:

$l_{32} = 4$, hence elimination will subtract 4 times row 2 from row 3.

Looking at the U matrix, we see the pivots along the diagonal of the matrix: $\begin{bmatrix} \boxed{5} & 7 & 8 \\ 0 & \boxed{2} & 3 \\ 0 & 0 & \boxed{6} \end{bmatrix}$

To find out if a row exchange will be needed or not, first determine A

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 8 \\ 10 & 16 & 19 \\ 5 & 15 & 26 \end{bmatrix}$$

Carry the first elimination, we get

$$A = \begin{bmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 8 & 18 \end{bmatrix}$$

Hence, $\boxed{\text{there would not be a need for a row exchange.}}$

2 Section 1.5, Problem 5

Problem: Factor A into LU and write down the upper triangular system $Ux = c$ which appears after elimination for

$$Ax = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

Answer:

$$A = \begin{bmatrix} \boxed{2} & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \overset{l_{21}=0}{\Longrightarrow} \begin{bmatrix} \boxed{2} & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \overset{l_{31}=3}{\Longrightarrow} \begin{bmatrix} 2 & 3 & 3 \\ 0 & \boxed{5} & 7 \\ 0 & 0 & -1 \end{bmatrix} \overset{l_{32}=0}{\Longrightarrow} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & \boxed{-1} \end{bmatrix}$$

Hence

$$\begin{aligned} L &= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}} \end{aligned}$$

and

$$U = \boxed{\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}}$$

Hence now we can write $Ax = b$ as $(LU)\,x = b$, or $L\,(Ux) = b$, Where $Ux = c$
We can solve for c by solving $Lc = b$, then we solve for x by solving $Ux = c$

3 Section 1.5, Problem 6

Problem: Find E^2 and E^8 and E^{-1} if $E = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$

Answer:

$E^2(A)$ means $E(E(A))$, which means we first subtract -6 times first row from second row of A, then apply E to this result again, subtracting -6 times first row from the second row of the resulting matrix.

Hence the net result is subtracting -12 times first row from the second row of the original matrix A.

Hence in general, we write

$$E^n = \begin{bmatrix} 1 & 0 \\ n \times 6 & 1 \end{bmatrix}$$

Therefore

$$E^2 = \begin{bmatrix} 1 & 0 \\ 2 \times 6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 12 & 1 \end{bmatrix}$$

$$E^8 = \begin{bmatrix} 1 & 0 \\ 8 \times 6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 48 & 1 \end{bmatrix}$$

$$E^{-1} = \begin{bmatrix} 1 & 0 \\ -1 \times 6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}$$

4 Section 1.5, Problem 9

Problem: (a) Under what conditions is the following product non singular?

$$A=\underbrace{\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}d_1&0&0\\0&d_2&0\\0&0&d_3\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}}_{V}$$

(b) Solve the system $Ax=b$ starting with $Lc=b$

$$\underbrace{\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}}_{L}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}=b$$

Solution:

(a) Since $U=[D]\,[V]=\begin{bmatrix}d_1&0&0\\0&d_2&0\\0&0&d_3\end{bmatrix}\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}=\begin{bmatrix}d_1&-d_1&0\\0&d_2&-d2\\0&0&d_3\end{bmatrix}$

Hence the elements along the diagonal of U are the pivots. Then if any one of d_1, d_2, d_3 is zero, then A will be non-singular

Hence for A to be non singular, then $\boxed{d_1 \text{ and } d_2 \text{ and } d_3 \text{ must all be nonzero.}}$

(b)

$$Ax=b$$

$$L\overbrace{Ux}^{c}=b$$

$Lc=b$ where $Ux=c$

hence starting with $Lc=b$ we write

$$\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Solve for c by back substitution process $\Rightarrow\boxed{c_1=0,\ c_2=0,\ c_3=1}$

Hence now we write $[U]\,x=c$ or

$$[D]\,[V]\,x=c$$

$$\overbrace{\begin{bmatrix}d_1&-d_1&0\\0&d_2&-d2\\0&0&d_3\end{bmatrix}}^{[D][V]}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\overbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}^{c}$$

Solve for x by back substitution process $\Rightarrow$

$$x_3 = \frac{1}{d_3}$$

and from second row, $d_2 x_2 - d_2 x_3 = 0 \Rightarrow d_2 x_2 - \frac{d_2}{d_3} = 0$, hence

$$\begin{aligned} x_2 &= \left(\frac{d_2}{d_3}\right)\frac{1}{d_2} \\ &= \frac{1}{d_3} \end{aligned}$$

and from the first row, we have $x_1 d_1 - x_2 d_1 = 0$, hence $x_1 d_1 = \frac{d_1}{d_3} \Rightarrow x_1 = \frac{1}{d_3}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{d_3} \\ \frac{1}{d_3} \\ \frac{1}{d_3} \end{bmatrix}$$

5 Section 1.5, problem 24

Problem: What three elimination matrices E_{21}, E_{31}, E_{32} put A into upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by $E_{32}^{-1}, E_{31}^{-1}, E_{21}^{-1}$ to factor A into LU where $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

Solution:

$$\overbrace{\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}}^{A} \overset{l_{21}=2}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix} \overset{l_{31}=3}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix} \overset{l_{32}=2}{\Longrightarrow} \overbrace{\begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{2} & 0 \\ 0 & 0 & \boxed{2} \end{bmatrix}}^{U}$$

Hence $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l_{31} & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -l_{32} & 1 \end{bmatrix}$

i.e. $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$$\begin{aligned} L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} &= \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{E_{21}^{-1}} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}}^{E_{31}^{-1}} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}}^{E_{32}^{-1}} \\ &= \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{E_{21}^{-1}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \\ &= \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}}^{L} \end{aligned}$$

Hence $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

6 Section 1.5, problem 42

Problem: If P_1 and P_2 are permutation matrices, so is P_1P_2. This still has the rows of I in some order. Give examples with $P_1P_2 \neq P_2P_1$, and examples of $P_3P_4 = P_4P_3$

Solution:

A permutation matrix exchanges one row with another. It is used when the pivot is zero. Assume P_1 exchanges row i with row j. Assume P_2 exchanges row k with row l. Hence P_1P_2 exchanges row k with row l and next exchanges row i with row j of the resulting matrix.

For specific examples, Let P_2 exchange second row with third row, and let P_1 exchange second row with third row. Given $A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$ hence $P_1P_2(A) = P_1P_2 \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = P_1 \begin{bmatrix} R_1 \\ R_3 \\ R_2 \end{bmatrix} = \begin{bmatrix} R_3 \\ R_1 \\ R_2 \end{bmatrix}$

While $P_2P_1(A) = P_2P_1 \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = P_2 \begin{bmatrix} R_2 \\ R_1 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_2 \\ R_3 \\ R_1 \end{bmatrix}$

We see that the result is not the same. Hence in this example $P_1P_2 \neq P_2P_1$

Now assume we have $A = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, and let P_3be an exchange of the first and second rows, while P_4 be an exchange of the third and 4th row. In this case we will see that $P_3P_4 = P_4P_3$

$\boxed{\textit{Hence the rule is as follows}}$: If P_1 exchanges row i with row j,and P_2 exchanges row k with row l. Then $P_1P_2 = P_2P_1$ only when i, j, k, l are all not equal. (not counting the trivial case when $i = j, k = l$)

Specific examples

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P_1P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \neq P_2P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

While

$$P_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow P_3P_4 = P_4P_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$