HW11, Math 307. CSUF. Spring 2007.

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Contents

1 Section 5.6, problem 1

problem: If B is similar to A and C is similar to B, show that C is similar to A. What matrices are similar to I?
answer:
Since B is similar to A and C is similar to B, then we have the following

$$S_1^{-1}CS_1 = B \tag{1}$$
$$S_2^{-1}BS_2 = A \tag{2}$$

From (1) and (2)

$$\begin{aligned} S_2^{-1} B S_2 &= A \\ S_2^{-1} \left(S_1^{-1} C S_1 \right) S_2 &= A \\ \left(S_2^{-1} S_1^{-1} \right) C \left(S_1 S_2 \right) &= A \\ \left(S_1 S_2 \right)^{-1} C \left(S_1 S_2 \right) &= A \end{aligned}$$

Let $S_1 S_2 = S_3$, hence the above becomes

$$S_3^{-1} C S_3 = A$$

Hence C is similar to A. Now for the second part. We write

$$\begin{aligned} S^{-1} A S &= I \\ S^{-1} A &= S \end{aligned}$$

So A must be I, hence only I is similar to I.

2 Section 5.6 problem 2

problem: Describe in words all the matrices that are similar to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and find 2 of them

answer:
Let A be the above matrix. The above matrix represents a reflection across the x-axis. Hence Reflection across the y axis will be similar to it. Any multiple of this reflection matrix will also be similar to A.

Since reflection across the y-axis is $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ then this B matrix is similar to A

Then any multiple of B is also similar to A, such as $\begin{pmatrix} -10 & 0 \\ 0 & 10 \end{pmatrix}$ and $\begin{pmatrix} -20 & 0 \\ 0 & 20 \end{pmatrix}$

3 Section 5.6, problem 5

Problem: show (if B is invertible) then BA is similar to AB
answer: we want to show that $M^{-1}(BA)M = AB$
Let $M^{-1}(BA)M = H$, i.e. let $BA\tilde{}H$, and try to show that $H = AB$

$$\begin{aligned}
M^{-1}(BA)M &= H \\
(BA)M &= MH \\
BA &= MHM^{-1} \\
A &= B^{-1}MHM^{-1} \\
AB &= B^{-1}MHM^{-1}B \\
AB &= \left(B^{-1}M\right) H \left(M^{-1}B\right) \\
AB &= \left(M^{-1}B\right)^{-1} H \left(M^{-1}B\right)
\end{aligned}$$

Let $M^{-1}B = Z$, hence the above becomes

$$AB = Z^{-1}HZ$$

Then $H\tilde{}AB$
But we started by stating that $H\tilde{}BA$, and since if $r_1\tilde{}r_2$ and $r_2\tilde{}r_3$ then $r_1\tilde{}r_3$ then we showed $BA\tilde{}AB$.

4 Section 5.6 problem 18

problem: find normal matrix ($NN^H = N^H N$) that is not Hermitian, skew symmetric, unitary, or diagonal. Show that all permutation matrices are normal
answer:

5 Section 6.1, problem 1

problem: quadratic $f = x^2 + 4xy + 2y^2$ has saddle point at origin, despite that its coefficients are positive. Write f as difference of 2 squares
answer: Let $f = (ax + by)^2 - (cx + dy)^2$, hence

$$\begin{aligned} f &= (ax + by)^2 - (cx + dy)^2 \\ &= a^2x^2 + b^2y^2 + 2abxy - \left(c^2x^2 + d^2y^2 + 2cdxy\right) \\ &= a^2x^2 + b^2y^2 + 2abxy - c^2x^2 - d^2y^2 - 2cdxy \\ &= x^2\left(a^2 - c^2\right) + y^2\left(b^2 - d^2\right) + xy\left(2ab - 2cd\right) \end{aligned}$$

Hence, compare coefficients, we have $a^2 - c^2 = 1, b^2 - d^2 = 2, 2ab - 2cd = 4$
so $ab - cd = 2$.
Let $c = 1$, then we have
$a^2 = 2$,$b^2 - d^2 = 2, 2ab - 2d = 4$
3 equations in 3 unknown. Solve with computer for speed (running out of time!) I get one of the solutions as
$d = 0, a = -\sqrt{2}, b = -\sqrt{2}$
So $f = (ax + by)^2 - (cx + dy)^2 = \boxed{\left(-\sqrt{2}x + -\sqrt{2}y\right)^2 - (x)^2}$

6 Section 6.1, problem 8

problem: decide for or against PD for these matrices, write out corresponding $f = x^T Ax$

Answer: I use $a > 0$, and $ac > b^2$ test where $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$\begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} \rightarrow 1 > 0, 5 > 9$ no, $\boxed{\text{Not PD}} \rightarrow f = ax^2 = 2bxy + cy^2 \rightarrow \boxed{f = x^2 + 6xy + 3y}$

$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \rightarrow a > 0, 1 > 1$,no, $\boxed{\text{Not PD}} \rightarrow f = ax^2 = 2bxy + cy^2 \rightarrow \boxed{f = x^2 - 2xy + y}$

$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \rightarrow a > 0, 10 > 9$,yes, $\boxed{\text{PD}} \rightarrow f = ax^2 = 2bxy + cy^2 \rightarrow \boxed{f = 2x^2 + 6xy + 5y}$

$\begin{pmatrix} -1 & 2 \\ 2 & -8 \end{pmatrix} \rightarrow -1 > 0, no \boxed{\text{Not PD}} \rightarrow f = ax^2 + 2bxy + cy^2 \rightarrow \boxed{f = -x^2 + 4xy - 8y}$

For (b) we have $f = x^2 - 2xy + y$, if $y = \frac{x^2}{2x-1}$ then $f = x^2 - 2x\frac{x^2}{2x-1} + \frac{x^2}{2x-1} = 0$, hence I plot this:

And along the lines shown is $f = 0$

7 Section 6.1, problem 3

problem: if A is 2x2 symmetric matrix, passes test that a>0 , $ac > b^2$ solve equation $\det(A - \lambda I) = 0$ and show that eigenvalues are >0
answer:
Matrix is PD, then

$$\begin{aligned} det\left(\begin{pmatrix} a & b \\ b & c \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= 0 \\ \left|\begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix}\right| &= 0 \\ (a-\lambda)(c-\lambda) - b^2 &= 0 \\ ac - a\lambda - c\lambda + \lambda^2 &= 0 \\ \lambda^2 + \lambda(-a-c) + ac &= 0 \end{aligned}$$

Hence $\lambda_1 = a, \lambda_2 = c$
But $a > 0$, so $\lambda_1 > 0$, and given $ac >$ positive quantity b^2, then $\lambda_2 = c \longrightarrow \lambda_2 > 0$

8 Section 6.1 problem 5

(a) For which numbers b is $\begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix}$ PD?

$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is PD is $a > 0$ and $ac > b^2$

for PD need $ac > b^2$, hence need $9 > b^2$ ie. $b < 3$ and $b > -3$, so $\boxed{-3 < b < 3}$

(b)Factor $A = LDL^T$ when b is in the range above

$\begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix} \rightarrow l_{21} = b \rightarrow U = \begin{pmatrix} 1 & b \\ 0 & 9 - b^2 \end{pmatrix}$

So $L = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 9 - b^2 \end{pmatrix}, L^T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

(c) What is the minimum of $f(x, y) = \frac{1}{2}\left(x^2 + 2bxy + 9y^2\right) - y$ when in this range

when $f(x, y) = \frac{1}{2}\left(x^2 + 2bxy + 9y^2\right) - y = \frac{1}{2}x^2 + bxy + \frac{9}{2}y^2 - y$

$\frac{\partial f}{\partial x} = x + by = 0, \frac{\partial f}{\partial y} = bx + 9y - 1 = 0$

Hence $\begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & b \\ 0 & 9 - b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Hence $\boxed{y = \frac{1}{9-b^2}}$,and $x + by = 0 \rightarrow \boxed{x = -\frac{b}{9-b^2}}$

So $f(x, y) = \frac{1}{2}\left(x^2 + 2bxy + 9y^2\right) - y$

Hence $f(x, y) \rightarrow \frac{1}{2}\left(\left(-\frac{b}{9-b^2}\right)^2 + 2b\left(-\frac{b}{9-b^2}\right)\left(\frac{1}{9-b^2}\right) + 9\left(\frac{1}{9-b^2}\right)^2\right) - \left(\frac{1}{9-b^2}\right) = \frac{1}{2(b^2-9)}$

So minimum is $\boxed{\frac{1}{2(b^2-9)}}$

(d)When $b = 3$, we see that we get $\frac{1}{0} = \infty$ so no minimum

9 Section 6.1 problem 17

Problem: If A has independent columns then $A^T A$ is square and symmetric and invertible. Rewrite $\vec{x}^T A^T A\vec{x}$ to show why it is positive except when $\vec{x} = 0$, then $A^T A$ is PD
answer: $\vec{x}^T \left(A^T A\right) \vec{x} = (A\vec{x})^T A\vec{x}$
Let $A\vec{x} = \vec{b}$, then the above is $\vec{b}^T \vec{b} = \left\|\vec{b}\right\|^2$, which is positive quantity except when $\vec{b} = \vec{0}$, which occurs when $A\vec{x} = \vec{b} = \vec{0}$ which happens only when $\vec{x} = \vec{0}$, since A is invertible.
Hence $A^T A$ is positive definite except when $\vec{x} = 0$

10 Section 6.2, problem 7

problem: If $A = Q\Lambda Q^T$ is P.D. then $R = Q\sqrt{\Lambda}Q^T$ is its S.P.D. square root. Why does R have positive eigenvalues? Compute R and verify $R^2 = A$ for $A = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$, $A = \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix}$

answer:

For $A = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$

Given R is P.D. (problem said so), Hence $\vec{x}^T R\vec{x} > 0$ for all $\vec{x} \neq 0$

Now (assuming in all that follows that $x \neq 0$)

$$\begin{aligned} R\vec{x} &= \lambda\vec{x} \\ \vec{x}^T R\vec{x} &= x^T \lambda\vec{x} \\ \vec{x}^T R\vec{x} &= \lambda \|\vec{x}\|^2 \end{aligned}$$

Since $\vec{x}^T R\vec{x} > 0$ then $\lambda \|\vec{x}\|^2 > 0$, and since $\|\vec{x}\|^2 > 0$ hence $\boxed{\lambda > 0}$

To compute R we first need to find Q.

$A = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} \longrightarrow l_{21} = \frac{6}{10} \longrightarrow \begin{pmatrix} 10 & 6 \\ 6 - \frac{6}{10} \times 10 & 10 - \frac{6}{10} \times 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 10 & 6 \\ 0 & \frac{32}{5} \end{pmatrix}$

Hence $L = \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix}$, $U = \begin{pmatrix} 10 & 6 \\ 0 & \frac{32}{5} \end{pmatrix}$

Then

$$\begin{aligned} LDU &= \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix} \begin{pmatrix} 1 & \frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix}^T \end{aligned}$$

Hence we see that $Q = L = \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix}$, $\Lambda = D = \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix}$, $Q^T = L^T$

Since A is SPD, then $A = R^T R$ and $A = Q\Lambda Q^T$, hence we can take $R = \sqrt{\Lambda}Q^T$

$$\begin{aligned} R = \sqrt{\Lambda}Q^T &= \sqrt{\begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix}} \begin{pmatrix} 1 & \frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{\frac{32}{5}} \end{pmatrix} \begin{pmatrix} 1 & \frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{10} & \frac{3}{5}\sqrt{2}\sqrt{5} \\ 0 & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \end{aligned}$$

Verify that $R^T R = A$

$$\begin{aligned} R^T R &= \begin{pmatrix} \sqrt{10} & 0 \\ \frac{3}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \begin{pmatrix} \sqrt{10} & \frac{3}{5}\sqrt{2}\sqrt{5} \\ 0 & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \\ &= \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} \end{aligned}$$

verified oK.

Now do the same for $A = \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix}$

$A = \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix} \longrightarrow l_{21} = \frac{-6}{10} \longrightarrow U = \begin{pmatrix} 10 & -6 \\ -6 - \frac{-6}{10} \times 10 & 10 - \frac{-6}{10} \times -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 10 & -6 \\ 0 & \frac{32}{5} \end{pmatrix}$:

Hence $L = \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix}$, $U = \begin{pmatrix} 10 & -6 \\ 0 & \frac{32}{5} \end{pmatrix}$

Then

$$\begin{aligned} LDU &= \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix} \begin{pmatrix} 1 & -\frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix}^T \end{aligned}$$

Hence we see that $Q = L = \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix}$, $\Lambda = D = \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix}$, $Q^T = L^T$

Then now we find R

Since A is SPD, then $A = R^T R$ and $A = Q \Lambda Q^T$, hence we can take $R = \sqrt{\Lambda} Q^T$

$$\begin{aligned} R = \sqrt{\Lambda} Q^T &= \sqrt{\begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix}} \begin{pmatrix} 1 & -\frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{\frac{32}{5}} \end{pmatrix} \begin{pmatrix} 1 & -\frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{10} & -\frac{3}{5}\sqrt{2}\sqrt{5} \\ 0 & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \end{aligned}$$

Verify that $R^T R = A$

$$\begin{aligned} R^T R &= \begin{pmatrix} \sqrt{10} & 0 \\ -\frac{3}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \begin{pmatrix} \sqrt{10} & -\frac{3}{5}\sqrt{2}\sqrt{5} \\ 0 & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \\ &= \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix} \end{aligned}$$

verified oK.

11 Section 6.2, problem 4

Show from the eigenvalues that if A is P.D. so is A^2 and so is A^{-1}
answer:
Given A is PD. Hence Eigenvalues of A are positive.
Let eigenvalue of A be λ_A
Let $B = A^2$
Let eigenvalue of B be λ_B
We need to show that $\lambda_B > 0$
Now

$$\begin{aligned} Bx &= \lambda_B x \\ A^2 x &= \lambda_B x \\ AAx &= \lambda_B x \\ A\lambda_A x &= \lambda_B x \\ \lambda_A A x &= \lambda_B x \\ \lambda_A \lambda_A x &= \lambda_B x \end{aligned}$$

From the last statement above we can now say

$$\lambda_A^2 = \lambda_B$$

Hence $\lambda_B > 0$, hence by theorem 6B which says that if all eigenvalues are positive then the matrix is PD, then in this case the matrix B which is A^2 is PD. QED
Now for A^{-1}

$$Ax = \lambda_A x$$

pre multiply both sides by A^{-1}

$$\begin{aligned} \overbrace{A^{-1}A}^{I} x &= A^{-1}\lambda_A x \\ x &= A^{-1}\lambda_A x \\ \frac{1}{\lambda_A} x &= A^{-1} x \end{aligned}$$

i.e.

$$A^{-1}x = \frac{1}{\lambda_A} x$$

Hence eigenvalue of A^{-1} is $\frac{1}{\lambda_A}$. And since $\lambda_A > 0$, then so is $\frac{1}{\lambda_A}$, and by theorem 6B again, since all eigenvalues are positive then A^{-1} is P.D.

12 Section 6.2, problem 6

From the pivots, eigenvalues, eigenvectors of $A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$, write A as $R^T R$ in 3 ways

1. $\left(L\sqrt{D}\right)\left(\sqrt{D}L^T\right)$
2. $\left(Q\sqrt{\Lambda}\right)\left(\sqrt{\Lambda}Q^T\right)$
3. $\left(Q\sqrt{\Lambda}Q^T\right)\left(Q\sqrt{\Lambda}Q^T\right)$

Answer:
First find if A is PD or not. Since this is a 2 by 2 matrix, a simple test is to look at the quantity $a^2 - bc$ and if it is positive, and if a is also positive, then the matrix is PD

$$\begin{aligned} A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ a &= 5 > 0 \\ a^2 - bc &= 25 - 16 \\ &= 9 > 0 \end{aligned}$$

hence A is P.D.
Then it can be written as $R^T R$ where R is full rank square matrix.
1) Since A is symmetric $P.D.$, then it has choleskly decomposition CC^T where $C = L\sqrt{D}$, and $C^T = \sqrt{D}L^T$ (the pivots are positive in the D matrix diagonal, so we can take their square root)
Then we write $A = R^T R = \left(L\sqrt{D}\right)\left(\sqrt{D}L^T\right)$ where $R = \left(\sqrt{D}L^T\right)$
$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \rightarrow l_{21} = \frac{4}{5} \rightarrow U = \begin{pmatrix} 5 & 4 \\ 0 & 5 - \frac{4}{5} \times 4 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 0 & \frac{9}{5} \end{pmatrix}$
Hence $L = \begin{pmatrix} 1 & 0 \\ \frac{4}{5} & 1 \end{pmatrix}, U = \begin{pmatrix} 5 & 4 \\ 0 & \frac{9}{5} \end{pmatrix} \rightarrow LDU = \begin{pmatrix} 1 & 0 \\ \frac{4}{5} & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & \frac{9}{5} \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{pmatrix}$
Hence $R = \sqrt{\begin{pmatrix} 5 & 0 \\ 0 & \frac{9}{5} \end{pmatrix}} \begin{pmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{\frac{9}{5}} \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{5} & \frac{4}{5}\sqrt{5} \\ 0 & \frac{3}{5}\sqrt{5} \end{pmatrix}$
Hence

$$A = \overbrace{\begin{pmatrix} \sqrt{5} & 0 \\ \frac{4}{5}\sqrt{5} & \frac{3}{5}\sqrt{5} \end{pmatrix}}^{L\sqrt{D}} \overbrace{\begin{pmatrix} \sqrt{5} & \frac{4}{5}\sqrt{5} \\ 0 & \frac{3}{5}\sqrt{5} \end{pmatrix}}^{\sqrt{D}L^T}$$

2)From $A = Q\Lambda Q^T$ where Q is the matrix which contains as its columns the normalized eigenvectors of A and Λ contains in its diagonal the eigenvalues of A. First start by finding eigenvalues and eigenvectors of A
$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \rightarrow$ eigenvectors:$\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\} \leftrightarrow 1, \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \leftrightarrow 9$
Hence $Q = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ $\rightarrow$normalize columns$\rightarrow$ $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$

So, verify first that the above is correct:

$A = \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 9\end{pmatrix}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{2}\times\begin{pmatrix}10 & 8\\ 8 & 10\end{pmatrix} = \begin{pmatrix}5 & 4\\ 4 & 5\end{pmatrix}$

Correct. So we write $R = \left(\sqrt{\Lambda}Q^T\right) = \sqrt{\begin{pmatrix}1 & 0\\ 0 & 9\end{pmatrix}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\frac{1}{\sqrt{2}} = \begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\frac{1}{\sqrt{2}} = \begin{pmatrix}-1 & 1\\ 3 & 3\end{pmatrix}\frac{1}{\sqrt{2}}$

Hence

$$\begin{aligned} A &= R^T R \\ &= \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 3\\ 1 & 3\end{pmatrix}}^{R^T=Q\sqrt{\Lambda}}\underbrace{\begin{pmatrix}-1 & 1\\ 3 & 3\end{pmatrix}\frac{1}{\sqrt{2}}}^{R=\sqrt{\Lambda}Q^T} \end{aligned}$$

3) now find $R = \left(Q\sqrt{\Lambda}Q^T\right)$

$R = \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\sqrt{\begin{pmatrix}1 & 0\\ 0 & 9\end{pmatrix}}\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$

Hence

$$\begin{aligned} A &= R^T R \\ &= \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}^T\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} \\ &= \overbrace{\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}}^{R^T=Q\sqrt{\Lambda}Q^T}\overbrace{\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}}^{R=Q\sqrt{\Lambda}Q^T} \end{aligned}$$

13 Section 6.2 problem 8

problem: if A is SPD and C is nonsignular, prove that $B = C^T AC$ is also SPD
solution: Since A is SPD, then it has positive eigenvalues.
Since B is similar to A (given), then B has the same eigenvalues as A, Hence B also has all its eigenvalues positive.
Hence by theorem 6B, B is symmetric positive definite.