9-6

$$y(n) = \alpha y(n-1) + x(n)$$

variables & coefficients : sign-&-magnitude

results of mult.'s : truncated

$$\Rightarrow \quad w(n) = Q[\alpha w(n-1)] + x(n)$$

$Q[\cdot]$: sign-&-mag. truncation.

possibility of a zero-input limit cycle

$$|w(n)| = |w(n-1)| \quad \forall\ n$$

show that if the ideal sys. is stable, then no zero-input limit cycle can exist. Is the same true for 2's complement truncation ?

Sol.

To have zero-input limit cycle

$$|w(n)| = |w(n-1)|$$

or

$$|Q[\alpha w(n-1)]| = |w(n-1)| \qquad (1)$$

stable sys. $\Rightarrow |\alpha| < 1$

$$\Rightarrow \quad |\alpha w(n-1)| < |w(n-1)| \qquad (2)$$

a) For sign-&-mag. truncation.

$$-2^{-b} < Q(x) - x \leq 0 \qquad x \geq 0$$

$$0 \leq Q(x) - x < 2^{-b} \qquad x < 0$$

add to notes

$\Rightarrow |Q(x)| \leqslant |x| \quad$ for $x \geqslant 0$ or $x < 0$

Let $x = \alpha W(n-1)$

$\Rightarrow |Q[\alpha W(n-1)]| \leqslant |\alpha W(n-1)| \quad (3)$

(3) & (2) $\Rightarrow |Q[\alpha W(n-1)]| \leqslant |\alpha W(n-1)| < |W(n-1)|$

Since (1) is not satisfied no zero input limit cycle is possible.

b) For $Q[\cdot]$ = two's complement

$$-2^{-b} < Q(x) - x \leqslant 0 \qquad \forall x$$

If $\underline{x > 0}$ $\quad x \geqslant Q[x]$ or $|x| \geqslant |Q[x]|$ (4)

If $\underline{x < 0}$ $\quad |Q[x]| \geqslant |x| \quad$ (5)

For $\alpha W(n-1) > 0$

$$|Q[\alpha W(n-1)]| \underset{4}{\leqslant} |\alpha W(n-1)| \underset{2}{<} |W(n-1)|$$

$\Rightarrow$ no limit cycle : (1) is not satisfied

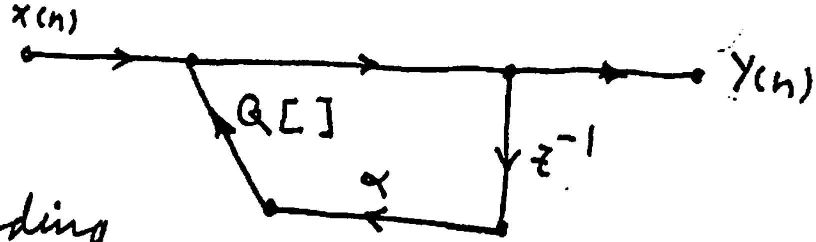
For $\alpha W(n-1) < 0$

$|\alpha W(n-1)| \leqslant |Q[\alpha W(n-1)]| \quad$ by (5)

and $|\alpha W(n-1)| < |W(n-1)| \quad$ by (2)

Possible that $|Q[\alpha W(n-1)]| = |W(n-1)|$ for $\alpha W(n-1) < 0 \Rightarrow$ limit cycle

9-7



Q[ ] : rounding

Fixed-pt. fractions, b bits

zero input $-$ $y(-1) = A$ initial cond.

Dead band : $A \ni |Q[\alpha A]| = A$

a) dead band in terms of α and β

b) For $b = 6$, $A = 1/16$ sketch $y(n)$ for $\alpha = \begin{cases} 15/16 \\ -15/16 \end{cases}$

c) For $b = 6$, $A = 1/2$ sketch $y(n)$ for $\alpha = -15/16$

Sol.

$$y(n) = Q[\alpha y(n-1)] + x(n) \qquad (x(n) = 0)$$

Rounding : $$-\frac{2^{-b}}{2} < Q[\alpha w(n-1)] - \alpha w(n-1) \leq \frac{2^{-b}}{2}$$

If filter is in the dead band

$$-\frac{2^{-b}}{2} < Q[\alpha A] - \alpha A \leq \frac{2^{-b}}{2}$$

or $$|Q[\alpha A] - \alpha A| \leq \frac{2^{-b}}{2}$$

In a limit cycle $|Q[\alpha A]| = A$

$$\Rightarrow |Q[\alpha A]| - |\alpha A| \leq |Q[\alpha A] - \alpha A| \leq \frac{1}{2} 2^{-b}$$

$$\Rightarrow |A| - |\alpha||A| \leq \frac{1}{2} 2^{-b}$$

$$\Rightarrow |A| \leqslant \frac{\frac{1}{2}\,2^{-b}}{1-|\alpha|}$$

b) $b=6 \quad 2^{-b}=1/64 \quad |\alpha| = 15/16 \quad 1-|\alpha| = 1/16$

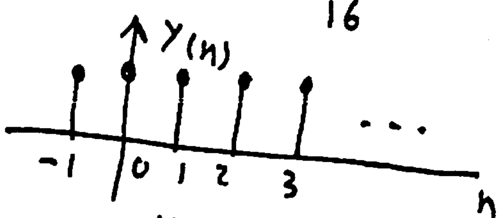
$$|A| \leqslant \frac{\frac{1}{2}\cdot\frac{1}{64}}{\frac{1}{16}} = 1/8 \quad \underline{\text{dead band}}$$

Thus for $A = 1/16$ the system starts immediately in the limit cycle.

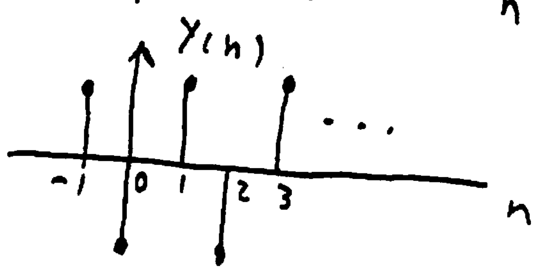
$$\alpha = \frac{15}{16} \quad Y(n) = Q\left[\alpha\, Y(n-1)\right] = Q\left[\frac{15}{16}\cdot\frac{1}{16}\right] = Q\left[\frac{15}{256}\right] = \frac{1}{16}$$

$$\alpha = -\frac{15}{16} \quad Y(n) = Q\left[-\frac{15}{16}\cdot\frac{1}{16}\right] = \begin{cases} -\frac{1}{16} & n \text{ even} \\ \frac{1}{16} & n \text{ odd} \end{cases} \quad \underline{\text{rounding!}}$$

$$Y(-1) = \frac{1}{16}$$



$\alpha = 15/16$

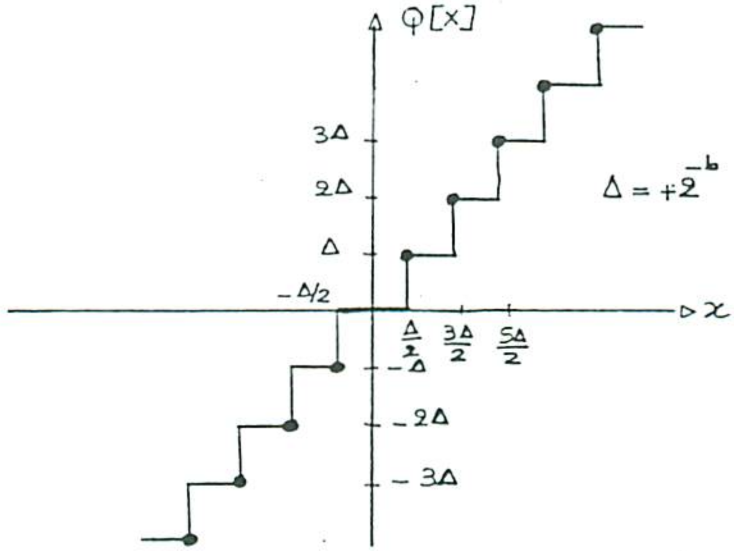


$\alpha = -15/16$

c) $b=6 \quad A = 1/2 \quad \alpha = -\frac{15}{16} \quad \Rightarrow$ same dead band

$$Y(n) = Q\left[-\frac{15}{16}\cdot Y(n-1)\right]$$

5/6



Thus we have: $W(0) = Q\left[-\frac{1}{2}\,\frac{15}{16}\right] = Q\left[-59\,\frac{\Delta}{2} - \frac{\Delta}{2}\right] = -30\Delta$

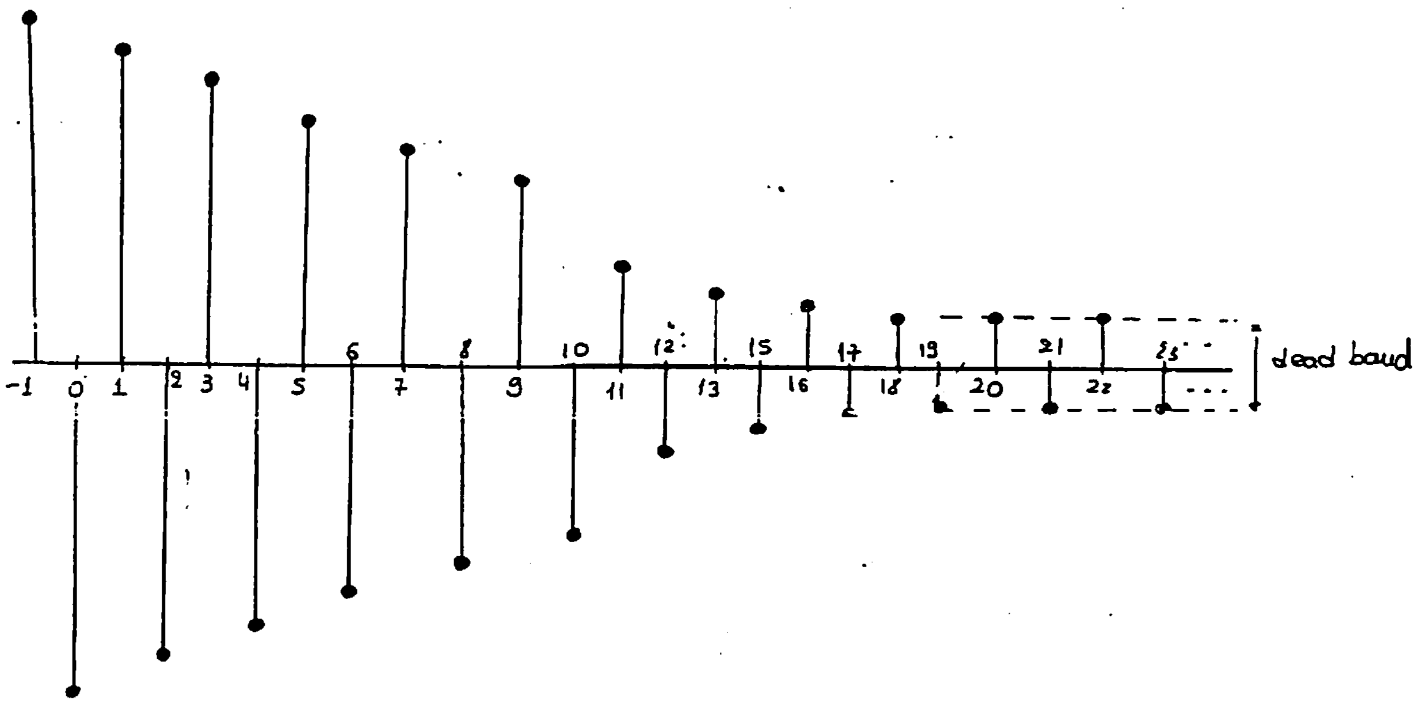
$$W(1) = Q\left[\frac{15}{16}\cdot 30\Delta\right] = Q\left[56\,\frac{\Delta}{2} + \frac{1}{4}\,\frac{\Delta}{2}\right] = 28\Delta$$

Hence we repeat the above procedure and we get:

$W(-1) = 32/64$
$W(0) = -30/64$
$W(1) = 28/64$
$W(2) = -26/64$
$W(3) = 24/64$ rounding up $\cancel{Q\left[\frac{-52.5}{128}\right] =}$ → Round down!
$Q\left[\frac{24.37}{64}\right]$!
$W(4) = -23/64$
$W(5) = 22/64$
$W(6) = -21/64$
$W(7) = 20/64$
$W(8) = -19/64$
$W(9) = 18/64$
$W(10) = -17/64$
$W(11) = 16/64$
$W(12) = -15/64$
$W(13) = 14/64$
$W(14) = -13/64$
$W(15) = 12/64$

$W(16) = -11/64$
$W(17) = 10/64$
$W(18) = -9/64$
$W(19) = 8/64$ ← rounding up
$W(20) = -8/64$
$W(21) = 8/64$
$W(22) = -8/64$

The output will be:



11.1

$$C_{xx}(m) = \frac{1}{N} \sum_{n=0}^{N-|m|-1} x(n)\,x(n+m) \qquad |m| \leq N-1$$

show that

$$I_N(w) = \frac{1}{N} \left| X(e^{jw}) \right|^2$$

$$I_N(w) = \sum_{m=-(N-1)}^{N-1} C_{xx}(m)\, e^{-jwm}$$

sol.

$$C_{xx}(m) = \frac{1}{N}\; x(n) * x(-n)$$

$$x(-n) \xrightarrow{F.T.} X(e^{-jw}) = X^*(e^{jw}) \qquad \text{For } x(n) \text{ real}$$

$$\Rightarrow I_N(e^{jw}) = \frac{1}{N} X(e^{jw})\, X^*(e^{jw}) = \frac{1}{N} \left| X(e^{jw}) \right|^2$$

or

$$I_N(w) \triangleq \sum_{m=-(N-1)}^{N-1} C_{xx}(m)\, e^{-jwm}$$

$$= \sum_{m=-(N-1)}^{N-1} \left[\frac{1}{N} \sum_{n=0}^{N-|m|-1} x(n)\, x(n+m) \right] e^{-jwm}$$

$$= \frac{1}{N} \sum_{n=0}^{N-|m|-1} x(n) \sum_{m=-(N-1)}^{N-1} x(n+m)\, e^{-jwm}$$

$$= \frac{1}{N} \sum_{n=0}^{N-|m|-1} x(n) \sum_{\ell=n-(N-1)}^{n+(N-1)} x(\ell)\, e^{-jw\ell}\, e^{jwn}$$

$$= \frac{1}{N} \sum_{n=0}^{N-|m|-1} x(n)\, e^{jwn} \sum_{\ell=n-(N-1)}^{n+(N-1)} x(\ell)\, e^{-jw\ell}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{jwn} \sum_{\ell=0}^{N-1} x(\ell)\, e^{-jw\ell} \qquad \text{since } x(n)=0 \text{ for } n<0 \;\&\; n \geq N$$

$$I_N(w) = \frac{1}{N}\left[\sum_{n=0}^{N-1} x(n)\, e^{-jwn}\right]^* \sum_{\ell=0}^{N-1} x(\ell)\, e^{-jw\ell}$$

$$= \frac{1}{N}\left| X(e^{jw}) \right|^2$$

11.2 $S_{xx}(w) = \sum_{m=-(M-1)}^{M-1} C_{xx}(m)\, W(m)\, e^{-jwm}$

$W(m)$ of length $2M-1$

show that $E\{S_{xx}(w)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E\{I_N(\theta)\}\, W(e^{j(w-\theta)})\, d\theta$

$$\begin{cases} W(m) = 0 & |m| \geq 2M \\ C_{xx}(m) = 0 & \text{for } |m| \geq M \end{cases}$$

Knowing these we can say

$$S_{xx}(w) = \sum_{m=-\infty}^{\infty} C_{xx}(m)\, W(m)\, e^{-jwm}$$

$$= \mathcal{F}\{C_{xx}(m)\, W(m)\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{F}\{C_{xx}(m)\}\, W(e^{j(w-\theta)})\, d\theta \quad \text{conv}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} I_N(\theta)\, W(e^{j(w-\theta)})\, d\theta$$

$$E\{S_{xx}(w)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E\{I_N(\theta)\}\, W(e^{j(w-\theta)})\, d\theta$$