HW 3
EGEE 518 Digital Signal Processing I
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1 my solution

①

Definitions

① auto correlation $R_{xx}(n,n+m)$: Measures the similarity of R.P. $X(t)$ at time n and $X(t)$ at later time $n+m$.

$$R_{xx}(n,n+m) = E\{X(n)\ X^*(n+m)\}$$

② stationary process.

This is a random process whose statistics do not change with shift in time origin.

③ Wide Sense Stationary process:

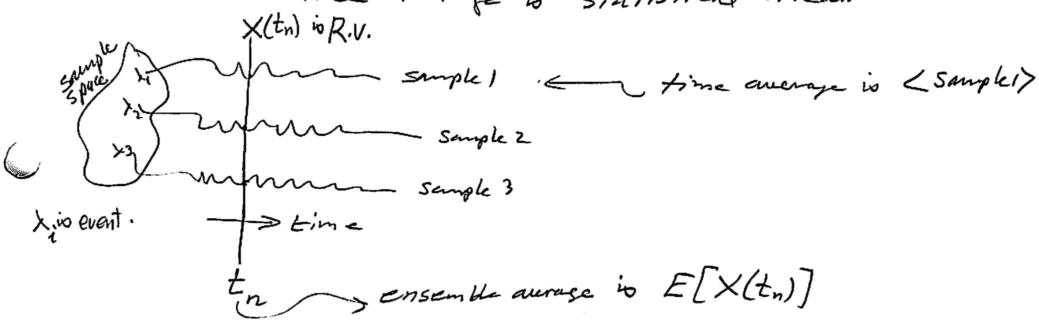
This is a random process $X(t)$ which statisfies the following Conditions:

1. its mean is Constant. i.e $E[X] =$ Constant.
2. autocorrelation depends only on time interval 'm. i.e $R_{xx}(n,n+m) = R_{xx}(m)$.

notice that stationary process is WSS, but WSS is not neccessarly stationary. i.e

④ Time averages, Ensemble averages

Time averages is the average of the sample sequence, while Ensemble average is statistical mean.

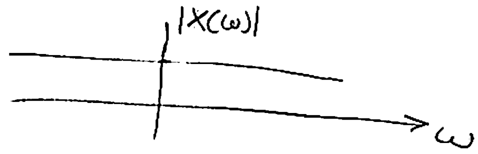


(2)

(5) White Noise;

this is a R.P. whose power spectral density is constant. i.e. power contained in a frequency bandwidth B is the same regardless of where this bandwidth is centered.

"flat" spectrum implies $X(t)$ is white Noise process.



the above is a description in the frequency domain. in the time domain, $\Phi_{XX}(m) = \delta(m)$ i.e. the autocorrelation is nonzero only if time interval is zero. i.e $X(t)$ only correlates with itself at zero time delay. so all R.V. that belong to a white noise process are uncorrelated with each others if time interval is nonzero.

(6) Erogdic Process;

this is a R.P. whose statistics taken from the time samples are the same as statistics taken from Ensembles.

for example. we say a process is Ergodic in the mean, then $E\{X(t)\} = \langle X(t) \rangle$

statiscal sample. expected value of R.V.

time average.. mean of a sample (or time series)

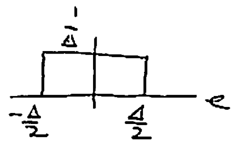
the above equality is in the <u>limit</u>, i.e as the time series length increases. and the statiscal mean is when the Number of time seriess increase as well.

#3 ③

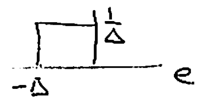
$y(n) = Q[x(n)] = x(n) + e(n)$ → quantization Error

$e(n)$ is white Noise.

Pdf for rounding is uniform



Pdf for truncation is



a) Find mean and Variance due to rounding
b) " " " " " truncation.

Answer

a) $m_e = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e \cdot f(e)\, de. = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{e}{\Delta} de = \frac{1}{\Delta}\left(\frac{e^2}{2}\right)_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}$

$= \frac{1}{2\Delta}\left[\left(\frac{\Delta}{2}\right)^2 - \left(-\frac{\Delta}{2}\right)^2\right] = \frac{1}{2\Delta}(0) = \boxed{0}$

$E[e^2] = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 f(e)\, de = \frac{1}{\Delta}\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 de = \frac{1}{\Delta}\left(\frac{e^3}{3}\right)_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}$

$= \frac{1}{3\Delta}\left[\left(\frac{\Delta}{2}\right)^3 - \left(-\frac{\Delta}{2}\right)^3\right] = \frac{1}{3\Delta}\left[\frac{\Delta^3}{8} + \frac{\Delta^3}{8}\right] = \frac{1}{3\Delta}\left[\frac{\Delta^3}{4}\right]$

$= \boxed{\frac{\Delta^2}{12}}$

so $\sigma^2 = E[e^2] - (E[e])^2 = \frac{\Delta^2}{12} - 0^2 = \boxed{\frac{\Delta^2}{12}}$

b) $m_e = \int_{-\Delta}^{0} e\, f(e)\, de = \int_{-\Delta}^{0} e \frac{1}{\Delta} de = \frac{1}{\Delta}\left(\frac{e^2}{2}\right)_{-\Delta}^{0} = \frac{1}{2\Delta}\left(0^2 - (-\Delta)^2\right)$

$= \frac{1}{2\Delta}(0 - \Delta^2) = \boxed{-\frac{\Delta}{2}}$

$E[e^2] = \int_{-\Delta}^{0} e^2 f(e) de = \int_{-\Delta}^{0} e^2 \frac{1}{\Delta} de = \frac{1}{\Delta}\left[\frac{e^3}{3}\right]_{-\Delta}^{0}$

$= \frac{1}{3\Delta}\left[0^3 - (-\Delta)^3\right] = \boxed{\frac{\Delta^2}{3}}$

so $\sigma^2 = E[e^2] - (E[e])^2 = \frac{\Delta^2}{3} - \left(-\frac{\Delta}{2}\right)^2 = \frac{\Delta^2}{3} - \frac{\Delta^2}{4} = \frac{4\Delta^2 - 3\Delta^2}{12} = \boxed{\frac{\Delta^2}{12}}$

#4 let $e(n)$ white Noise sequence. Let $s(n)$ uncorrelated sequence to $e(n)$. Show that $y(n) = s(n)\,e(n)$ is white. i.e $E[\,y(n)\;y(n+m)\,] = A\,\delta(m)$. (4)

Answer

$$E[\,y(n)\;y(n+m)\,] = E[\,s(n)e(n)\quad s(n+m)e(n+m)\,]$$

$$= E[\,s(n)s(n+m)\quad e(n)e(n+m)\,]$$

since $e(n)$ and $s(n)$ are uncorrelated, hence independent, then we can write the above as

$$= E[\,s(n)\,s(n+m)\,]\;E[\,e(n)\,e(n+m)\,]$$

but $e(n)$ is white. hence $\Phi_{ee}(n,m) = E[\,e(n)\;e(n+m)\,] = \boxed{\delta(m)}$ by definition of white signal.

hence $\Phi_{yy}(n,m) = E[\,s(n)\;s(n+m)\,]\;\delta(m)$.

Now, when $m=0$, $\Phi_{yy}(n,m) = E[\,s(n)\;s(n)\,]\cdot 1.$

Since $S(n)$ is uncorrelated with white Noise, then $m_s = 0$

Since $S(n)$ is also white.

hence $E[\,s^2(n)\,] =$ Total average power in $S(n)$

$= A$ some constant.

hence when $m=0$, $\Phi_{yy}(n,m) = A$

when $m \neq 0$ $\Phi_{yy}(n,m) = E[\,s(n)\;s(n+m)\,]\cdot 0$

$= 0$

Therefor $\boxed{\Phi_{yy}(n,m) = A\,\delta(m)}$

Since $\Phi_{yy}(n,m)$ is function of only m, it is white signal.

#6 Consider 2 real stationary random processes $\{X_n\}$ and $\{y_n\}$, with mean m_x, m_y, and variance σ_x^2, σ_y^2. (5)

(a) $\gamma_{xx}(m)$ this is autocovariance.

$$\gamma_{xx}(m) = E\{(x(n)-m_x)(x^*(n+m) - m_x^*)\}$$
$$= E\{ x(n)x^*(n+m) - m_x x(n) - m_x x^*(n+m) + m_x^2 \}$$
$$= E\{x(n)\, x^*(n+m)\} - m_x E\{x(n)\} - m_x E\{x^*(n+m)\} + m_x^2.$$
$$= \Phi_{xx}(n,n+m) - m_x^2 - m_x E\{x^*(n+m)\} + \cancel{m_x^2}$$
$$= \Phi_{xx}(n,n+m) - m_x E\{x^*(n+m)\}.$$

but $\{x_n\}$ is stationary, so its statistics do not change with shift of time origin. hence $E\{x^*(n+m)\} = E\{x^*(n)\} = m_x$.
so above becomes

$$\gamma_{xx}(m) = \Phi_{xx}(n,n+m) - m_x^2.$$

but $\Phi_{xx}(n,n+m) = \Phi_{xx}(m)$ since stationary hence

$$\boxed{\gamma_{xx}(m) = \Phi_{xx}(m) - m_x^2}$$

$$\gamma_{xy}(m) = E[(x(n)-m_x)(y^*(n+m) - m_y^*)]$$
$$= E[x(n)\, y^*(n+m) - m_y^* x(n) - m_x y^*(n+m) + m_y^* m_x]$$
$$= E\{x(n)\, y^*(n+m)\} - m_y^* E\{x(n)\} - m_x E\{y^*(n+m)\} + m_y^* m_x$$

but due to stationarity, $E\{y^*(n+m)\} = m_y$. so above becomes

$$= E\{X(n)\, y^*(n+m)\} - m_y m_x - m_x m_y + m_y m_x$$
$$= E\{X(n)\, y^*(n+m)\} - m_y m_x.$$

but $E\{x(n)\, y^*(n+m)\} = \Phi_{xy}(m)$ since stationary

so $\boxed{\gamma_{xy}(m) = \Phi_{xy}(m) - m_y m_x}$ $\longrightarrow$

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(b) $\Phi_{xx}(0) = E\{x(n)\,x^*(n+m)\}$

but $m=0$. hence

$\Phi_{xx}(0) = E\{x(n)\,x^*(n)\} = E\{x^2(n)\}$.

= mean square.

$\gamma_{xx}(0) = E\{(x(n)-m_x)(x^*(n+m)-m_x^*)\}$

but $m=0$ so

$$\gamma_{xx}(0) = E\{(x(n)-m_x)(x^*(n)-m_x^*)\}$$

$$= E\{x^2(n) - x(n)m_x - m_x x^*(n) + m_x^2\}$$

$$= E\{x^2(n)\} - m_x E\{x(n)\} - m_x E\{x^*(n)\} + m_x^2$$

$$= E\{x^2(n)\} - m_x^2 - \cancel{m_x^2} + \cancel{m_x^2}$$

$$= E\{x^2(n)\} - m_x^2$$

but this is the definition of σ_x^2. hence

$$\boxed{\gamma_{xx}(0) = \sigma_x^2}$$

$\longrightarrow$

(c) $\phi_{xx}(m) = E\{x(n)\, x^*(n+m)\} = E\{x^*_{n+m}\, x_n\} = \left(E\{x_{n+m} x_n^*\}\right)^*$ 7

$= \Phi^*_{xx}(-m)$

if process is real, then $\Phi^*_{xx}(-m) = \Phi_{xx}(-m)$.

i.e. $\Phi_{xx}(m) = \Phi_{xx}(-m)$

$\gamma_{xx}(m) = E\{(x(n)-m_x)\,(x^*(n+m)-m^*_x)\}$

$= \Phi_{xx}(m) - m_x m_x^*$ (from part (a)). 1

$= \Phi^*_{xx}(-m) - m_x m^*_x$. (using result above).

$= \left(E\{x_{n+m} X_n^*\}\right)^* - m_x m_x^*$

$= E\{x^*_{n+m}\, x_n\} - m_x m_x^*$

$= \left(E\{x_{n+m}\, x_n^*\} - m_x^* m_x\right)^*$

$= \gamma^*_{xx}(-m)$

if Real process, then $\gamma^*_{xx}(-m) = \gamma_{xx}(-m)$. $\Rightarrow$ $\boxed{\gamma_{xx}(m) = \gamma_{xx}(-m)}$

$\cancel{\Phi_{xy}(m) = E\{(x(n)-m_x)(y(n+m)-m_y)\}}$

$\cancel{= E\{x(n)\, y(n+m) - m_y x(n) - m_x y(n+m) + m_x m_y\}}$

$= E\{\cancel{x(n)\, y(n+m)}\} \cancel{- m_y m_x - m_x m_y} + m_x m_y$

$\cancel{= E\{x(n)\, y(n+m)\} - m_y m_x}$

But $\Phi_{yx}(-m) = E\{(y(n)-m_y)\,(x(n-m)-m_x)\}$

$= \cancel{E\{y(n)\, x(n-m)\} - m_x m_y - m_y m_x} + m_y m_x$

$= E\{y(n)\, x(n-m)\} - m_x m_y$

Since sequences $x(n)$ and $y(n)$ are real, ~~then~~ $\cancel{\Phi^*_{yx}(-m) =} \Phi_{yx}(-m)$.

so $\Phi^*_{yx}(-m) = E\{x(n-m)\, y(n)\} - m_x m_y$ $\longrightarrow$

part (c) Cont.

show that $\Phi_{xy}(m) = \Phi^*_{yx}(-m)$.

$$\Phi_{xy}(m) = E\{X_n y^*_{n+m}\} = E\{y^*_{n+m} x_n\} = \left(E\{y_{n+m} x^*_n\}\right)^*$$

$$= \Phi^*_{yx}(-m)$$

Show that $\gamma_{xy}(m) = \gamma^*_{yx}(-m)$

$$\gamma_{xy}(m) = E\{(X_n - m_x)(y^*_{n+m} - m^*_y)\}$$

$$= E\{(y^*_{n+m} - m^*_y)(X_n - m_x)\}$$

$$= \left(E\{(y_{n+m} - m_y)(x^*_n - m^*_x)\}\right)^*$$

$$= \gamma^*_{yx}(-m)$$

part (d) 9

show that $|\Phi_{xy}(m)| \leq \sqrt{\Phi_{xx}(0)\,\Phi_{yy}(0)}$

$$\Phi_{xy}(m) = E\{x_n\, y^*_{n+m}\}$$
$$\Phi_{xx}(0) = E\{x_n^2\}$$
$$\Phi_{yy}(0) = E\{y_n^2\}.$$

we did this in class as follows:

$$0 \leq E\{(x_n + a\,y_{n+m})^2\} = E\{x_n^2 + a^2 y_{n+m}^2 + 2a\,x_n\, y_{n+m}\}$$
$$= E(x_n^2) + a^2 E(y_{n+m}^2) + 2a\,E(x_n\, y_{n+m})$$
$$= \Phi_{xx}(0) + a^2\,\Phi_{yy}(0) + 2a\,\Phi_{xy}(m) \quad \equiv Ax^2+Bx+C$$

this is a quadratic equation that is ≥ 0 always.
hence can't have 2 real roots. i.e
discriminant ≤ 0. i.e

where $A = \Phi_{yy}(0)$, $B = 2\Phi_{xy}(m)$, $C = \Phi_{xx}(0)$.

but discriminant is $B^2 - 4AC$

so
$$4\Phi_{xy}^2(m) - 4\,\Phi_{yy}(0)\,\Phi_{xx}(0) \leq 0.$$

i.e
$$\Phi_{xy}^2(m) \leq \Phi_{yy}(0)\,\Phi_{xx}(0)$$

i.e
$$\boxed{|\Phi_{xy}(m)| \leq \sqrt{\Phi_{yy}(0)\,\Phi_{xx}(0)}}$$

2 key solution

H.W. #3 Sol.

1. a) Autocorrelation sequence : $\phi_{xx}(n,m)$ is defined by

$$\phi_{xx}(n,m) = E\{x_n x_m^*\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_n x_m^* \, P_{x_n x_m}(X_n, n, X_m, m)\, dX_n dX_m$$

b) A random process $\{x_n\}$ is a stationary process if its statistics are not affected by a shift in the time origin. i.e., x_n and x_m have the same statistics for all n and m

c) A stationary random process in the wide sense mean

(i) The mean is constant

(ii) the autocorrelation (2^{nd} order statistics) depend only on the time difference between the random variables

d) Time average of a random process $\{x_n\}$ is defined as

$$\langle x_n \rangle = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x_n$$

Ensemble average of a random process $\{x_n\}$ is defined as

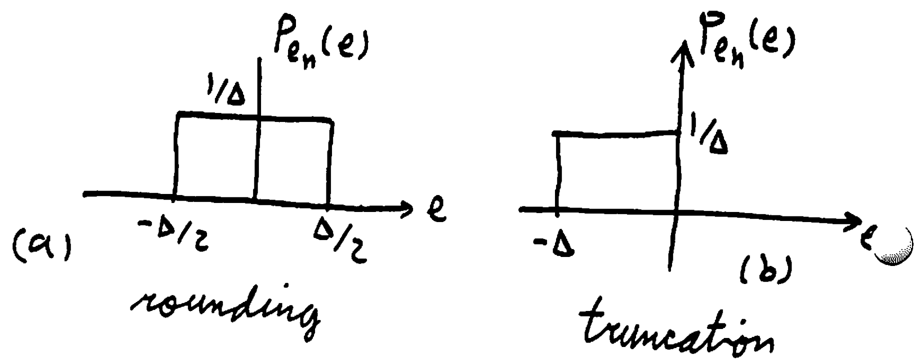
$$m_{x_n} = E\{x_n\} = \int_{-\infty}^{\infty} x \, P_{x_n}(x, n)\, dx$$

e) White noise is a random process in which all the random variables are independent. with zero mean

$$\Phi_{xx}(m) = \sigma_x^2 \, \delta(m)$$

f) A random process for which the time averages equal the ensemble averages is called an ergodic process.

② <u>8.3</u>



Prob. distribution

a) Mean & variance, rounding

$$m_e = \int_{-\infty}^{\infty} e \, P_{e_n}(e)\, de = \int_{-\Delta/2}^{\Delta/2} e \, \frac{1}{\Delta} de = \frac{1}{\Delta} \, \frac{e^2}{2} \Big|_{-\Delta/2}^{\Delta/2} = 0$$

$$\sigma_e^2 = E\{e_n^2\} = \int_{-\infty}^{\infty} e^2 P_{e_n}(e)\, de = \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} e^2 de$$

$$= \frac{e^3}{3\Delta} \Big|_{-\Delta/2}^{\Delta/2} = \frac{1}{3\Delta} \, 2 \, \frac{\Delta^3}{8} = \Delta^2/12$$

b) For truncation

$$m_e = \frac{1}{\Delta}\int_{-\Delta}^{0} e\,de = \frac{1}{\Delta}\left.\frac{e^2}{2}\right|_{-\Delta}^{0} = -\Delta/2$$

$$\sigma_e^2 = E\left\{\left(e_n+\frac{\Delta}{2}\right)^2\right\} = E\{e_n^2\}+\frac{\Delta^2}{4}+2\frac{\Delta}{2}E\{e_n\}$$

$$= E\{e_n^2\}+\frac{\Delta^2}{4}-\frac{\Delta^2}{2} = E\{e_n^2\}-\frac{\Delta^2}{4}$$

$$\sigma_e^2 = \frac{1}{\Delta}\int_{-\Delta}^{0} e^2\,de - \frac{\Delta^2}{4} = \frac{1}{\Delta}\left.\frac{e^3}{3}\right|_{-\Delta}^{0} - \frac{\Delta^2}{4} = \frac{\Delta}{1}$$

(3) 8.4 $e(n)$: white noise seq.
$s(n)$: uncorrelated with e_n

show $y(n) = s(n)\,e(n)$ is white ; i.e.

$$E\{y(n)\,y(n+m)\} = \underbrace{A}_{\text{const.}}\,\delta(m)$$

Sol.

$e(n)$ white $\Rightarrow E\{e(n)\,e(n+m)\} = \sigma_e^2\,\delta(m)$

uncorrelated $E\{e(n)\,y(m)\} = E\{e(n)\}E\{y(m)\}$

$$E\{y(n)\,y(n+m)\} = E\{s(n)\,e(n)\,s(n+m)\,e(n+m)\}$$

assume $s(n)$ is WSS or white noise

$$= E\{s(n)\,s(n+m)\,e(n)\,e(n+m)\}$$

$$= E\{s(n)\,s(n+m)\}\,E\{e(n)\,e(n+m)\}$$

$$= E\{s(n)\,s(n+m)\}\,\sigma_e^2\,\delta(m)$$

$$= \sigma_s^2\,\sigma_e^2\,\delta(m)$$

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8.6 Consider the two real stationary random processes $\{x_n\}$ and $\{y_n\}$. with means m_x and m_y and variances σ_x^2 and σ_y^2.

Show the following

(a) $\gamma_{xx}(m) = \phi_{xx}(m) - m_x^2$ & $\gamma_{xy}(m) = \phi_{xy}(m) - m_x m_y$

$$\gamma_{xx}(m) = E[(x_n - m_x)(x_{n+m} - m_x)]$$
$$= E[x_n x_{n+m}] - m_x E[x_{n+m}] - m_x E[x_n] + m_x m_x$$
$$= \phi_{xx}(m) - m_x m_x - m_x m_x + m_x m_x$$
$$= \phi_{xx}(m) - m_x^2$$

$$\gamma_{xy}(m) = E[(x_n - m_x)(y_{n+m} - m_y)]$$
$$= E[x_n y_{n+m}] - m_x m_y - m_y m_x + m_x m_y$$
$$= \phi_{xy}(m) - m_x m_y$$

(b) $\phi_{xx}(0) =$ mean square & $\gamma_{xx}(0) = \sigma_x^2$

$$\phi_{xx}(m) = E[x_n x_{n+m}] =$$
$$\phi_{xx}(0) = E[x_n x_n] = \text{mean square}$$
$$\gamma_{xx}(m) = E[(x_n - m_x)(x_{n+m} - m_x)]$$
$$\gamma_{xx}(0) = E[(x_n - m_x)^2] = \sigma_x^2$$

(c) $\phi_{xx}(m) = \phi_{xx}(-m)$

$$\phi_{xx}(-m) = (E[x_n x_{n-m}])$$

let $n' = n - m$

$$\phi_{xx}(-m) = (E[x_{n'+m} x_{n'}]) = E[x_{n'} x_{n'+m}]$$
$$= \phi_{xx}(m)$$

$\gamma_{xx}(m) = \gamma_{xx}(-m)$

$$\gamma_{xx}(-m) = (E[(x_n - m_x)(x_{n-m} - m_x)])$$
$$= (E[(x_{n'+m} - m_x)(x_{n'} - m_x)])$$
$$= E[(x_{n'} - m_x)(x_{n'+m} - m_x)]$$
$$= \gamma_{xx}(m)$$

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$$\phi_{xy}(m) = \phi_{yx}(-m)$$

$$\begin{aligned}\phi_{yx}(-m) &= (E[(y_n - m_y)(x_{n-m} - m_x)])\\ &= (E[(y_{n'+m} - m_y)(x_{n'} - m_x)])\\ &= E[(x_{n'} - m_x)(y_{n'+m} - m_y)]\\ &= \phi_{xy}(m)\end{aligned}$$

$$\gamma_{xy}(m) = \gamma_{yx}(-m)$$

$$\begin{aligned}\gamma_{yx}(-m) &= (E[(y_n - m_y)(x_{y-m} - m_x)])\\ &= (E[(y_{n'+m} - m_y)(x_{n'} - m_x)])\\ &= E[(x_{n'} - m_x)(y_{n'+m} - m_y)]\\ &= \gamma_{xy}(m).\end{aligned}$$

(d) Consider the inequality

$$E\left\{\left(\frac{x_n}{(E[x_n^2])^{1/2}} - \frac{y_{n+m}}{(E[y_{n+m}^2])^{1/2}}\right)^2\right\} \geq 0$$

This is true since the quantity inside the brackets is >0 for all m and n.

Now

$$E\left[\frac{x_n^2}{E[x_n^2]}\right] + E\left[\frac{y_{n+m}^2}{E[y_{n+m}^2]}\right] - 2\frac{E[x_n y_{n+m}]}{(E[x_n^2])^{1/2}(E[y_{n+m}^2])^{1/2}} \geq 0$$

This can be written as

$$\frac{\phi_{xx}(0)}{\phi_{xx}(0)} + \frac{\phi_{yy}(0)}{\phi_{yy}(0)} - \frac{2\,\phi_{xy}(m)}{\phi_{xx}^{1/2}(0)\;\phi_{yy}^{1/2}(0)} \geq 0$$

$$\frac{\phi_{xy}(m)}{\phi_{xx}^{1/2}(0)\,\phi_{yy}^{1/2}(0)} \leq 1 \quad \Rightarrow \quad \boxed{[\phi_{xx}(0)\,\phi_{yy}(0)]^{1/2} \geq \phi_{xy}(m)}$$

Now if we replace x_n by $(x_n - m_x)$ and y_{n+m} by $(y_{n+m} - m_y)$ in the inequality we can manipulate it in the same way to get .

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$$\boxed{[\gamma_{xx}(0)\,\gamma_{yy}(0)]^{1/2} \geq \gamma_{xy}(m)}$$

Letting $y_n = x_n$ we can specialize these inequalities to

$$\boxed{\begin{array}{c} \phi_{xx}(0) \geq \phi_{xx}(m) \\ \hline \gamma_{xx}(0) \geq \gamma_{xx}(m) \end{array}}$$

(e) Let $y_n = x_{n-n_0}$

$$\underline{\phi_{yy}(m)} = E[y_n\, y_{n+m}]$$
$$= E[x_{n-n_0}\, x_{n+m-n_0}]$$
$$= \underline{\phi_{xx}(m)}$$

Obviously $\underline{\gamma_{yy}(m) = \gamma_{xx}(m)}$ for the same reasons.

(f) Let $\gamma_{xx}(m) \longleftrightarrow \Gamma_{xx}(z)$

$$\gamma_{xy}(m) \longleftrightarrow \Gamma_{xy}(z)$$

$\Gamma_{xx}(z) \triangleq \sum_m \gamma_{xx}(m)\, z^{-m} \Rightarrow$ **(1)** $\quad \gamma_{xx}(m) = \frac{1}{2\pi j} \oint_c \Gamma_{xx}(z)\, z^{m-1}\, dz$

$$\underline{\gamma_{xx}(0)} = \boxed{\sigma_x^2 = \frac{1}{2\pi j} \oint_c \Gamma_{xx}(z)\, z^{-1}\, dz}$$

(2) We have shown that $\gamma_{xx}(m) = \gamma_{xx}(-m)$

Therefore $\quad \Gamma_{xx}(z) = \sum_{m=-\infty}^{\infty} \gamma_{xx}(m)\, z^{-m}$

$$\underline{\Gamma_{xx}(z^{-1})} = \sum_{m=-\infty}^{\infty} \gamma_{xx}(m)\, z^{m} = \sum_{p=-\infty}^{\infty} \gamma_{xx}(-p)\, z^{-p}$$

$$= \sum_{m=-\infty}^{\infty} \gamma_{xx}(m)\, z^{-m} = \underline{\Gamma_{xx}(z)}.$$

$p \to m \Rightarrow$

use $\gamma_{xy}(m) = \gamma^*_{yx}(-m)$

$$\Gamma_{xy}(z) = \sum_{m=-\infty}^{\infty} \gamma_{xy}(m) z^{-m} = \sum_{m=-\infty}^{\infty} \gamma^*_{yx}(-m) z^{-m}$$

$$= \left(\sum_{\ell=-\infty}^{\infty} \gamma_{yx}(\ell) z^{*\ell} \right)^*$$

$$= \left(\sum_{\ell=-\infty}^{\infty} \gamma_{yx}(\ell) \left(z^{*^{-1}}\right)^{-\ell} \right)^* = \Gamma^*_{yx}(1/z^*$$