Random Signals

All useful message signals appear random to the receiver, since the receiver does not know, a priori, which of the possible message signals will be transmitted. Also, the noise superimposed to the desired signal is random. Therefore, we need an efficient description of random signals.

properties of a random variable, X:

1) The distribution function $F_X(x)$ of the random variable X is given by:

$$F_X(x) \triangleq P_r[X \le x] \quad (33)$$

This is the probability that the value taken by the R.V. X is less than or equal to a real number x.

$F_X(x)$ has the following properties:

a) $0 \le F_X(x) \le 1$

b) $F_X(x_1) \le F_X(x_2) \quad \text{if } x_1 \le x_2$

c) $F_X(-\infty) = 0$

d) $F_X(+\infty) = 1$

2) probability density function (pdf) of the random variable X; $f_X(x)$:

The pdf and the distribution function of the R.V X are related to each other by:

$$f_X(x) = \frac{dF_X(x)}{dx} \qquad (34) \qquad \text{or}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x_1)\,dx_1 \qquad (35)$$

Thus:

$$P[x_1 \leq X \leq x_2] = P[X \leq x_2] - P[X \leq x_1] = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)\,dx \qquad (36)$$

Properties of $f_X(x)$

a) $f_X(x) \geq 0$

b) $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$

3) Ensemble Averages:

a) The mean value m_X, of a continuous random variable X, is defined by:

$$m_X \triangleq E\{X\} = \int_{-\infty}^{+\infty} x f_X(x)\,dx \qquad (37)$$

b) The nth moment:

$$E\{X^n\} \triangleq \int_{-\infty}^{+\infty} x^n f_X(x)\,dx \qquad (38)$$

c) The mean square value or the power

$$P = E\{x^2\} \triangleq \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx \qquad (39)$$

d) Variance of the R.V. X ;

$$\sigma_X^2 = Var[X] \triangleq E\{(X - m_X)^2\} = \int_{-\infty}^{+\infty} (x - m_X)^2 f_X(x)\, dx \qquad (40)$$

The variance σ_X^2 is a measure of the randomness of the random variable X.

$E\{(x-\mu)^2\} = E\{x^2 + \mu^2 - 2x\mu\}$
$= E\{x^2\} + E\{\mu^2\} - 2\mu E\{x\}$
$= E\{x^2\} + \mu^2 - 2\mu^2$

e) One can verify that:

$$\sigma_X^2 = E\{x^2\} - m_X^2 \qquad (41) \quad \text{where:}$$

$E\{x^2\}$ represents the total power (average),
m_X^2 " the DC power and
σ_X^2 " " AC power (average)

4) Random Processes (R.P.)

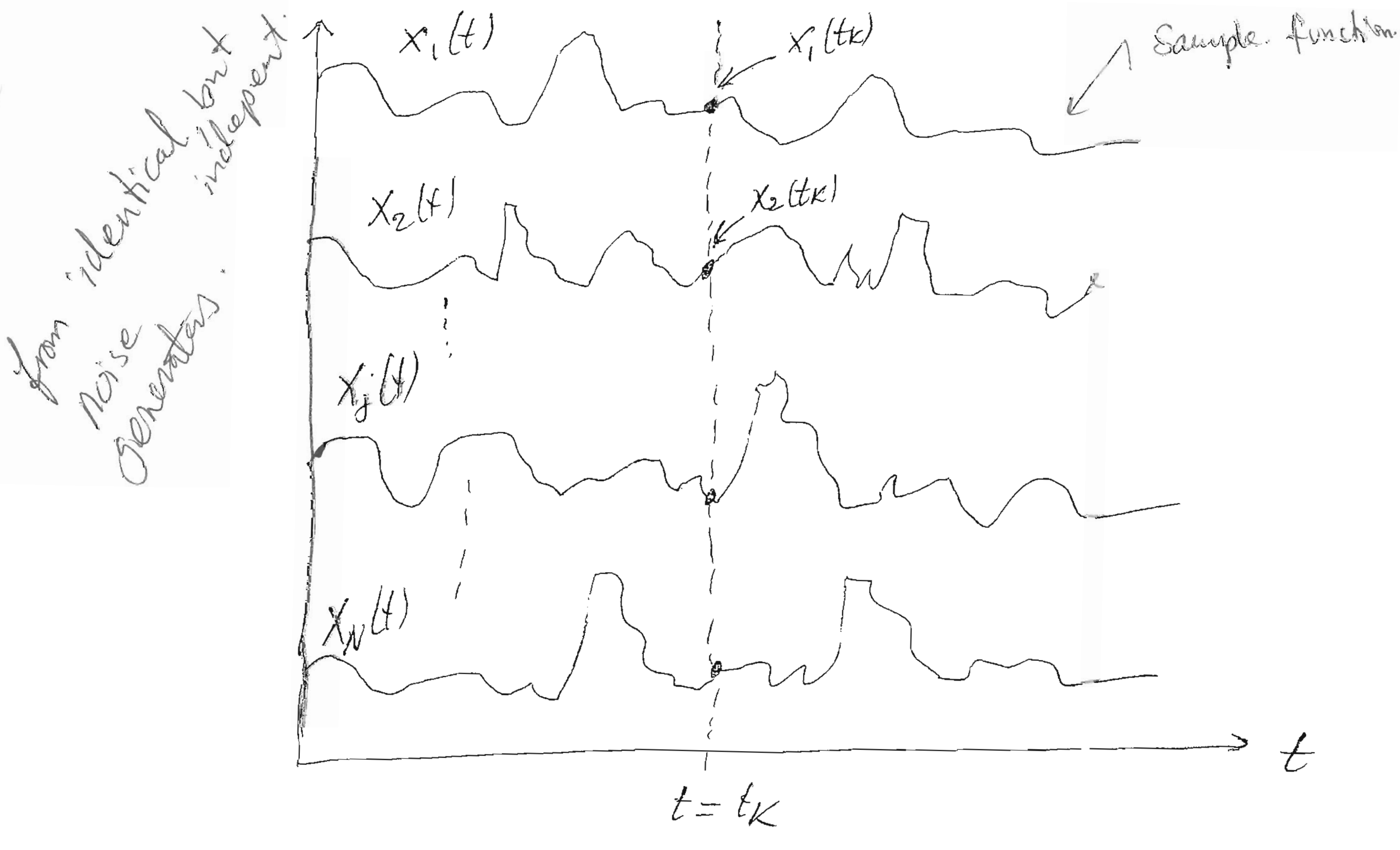
A random process $X(A,t)$ is a function of two variables; a random event A and time. The following figure shows a R.P., which consist of N sample functions of time, $\{X_j(t)\}$. Each of the sample function can be viewed as the output of a different noise generator (That is distinct but identical noise generators).

- For a specific event A_j, that is when the event A_j is known, we have a single function of time (sample function) $X(A_j, t) \triangleq X_j(t)$. The totality of all sample function is called an ensemble or R.P.

- For a specific time t_K, $X(A, t_K)$ is a random variab $X(t_K)$ whose value depends on the event.

- For a specific event $A = A_j$ and a specific time $t = t_K$, $X(A_j, t_K)$ is a number.

From now on we will use $X(t)$ to describe the R.P. $X(A, t)$.



Random Noise Process

next time

statistical Averages of a Random Process:

Because the value of a RP at any future time is unknown, a RP whose distribution function are continuous can be described statistically with a prob. density function (pdf). In general the form of the pdf of a RP will be different at diffent times. In most cases it is not possible to determine empirically the probability distribution functio of a R.P. However, a partial description consisting of its mean and its autocorrelation function are sufficient for the needs of communication systems.

; we define the mean of the R.P. $X(t)$ as:

$$m_X(t_K) = E\{X(t_K)\} \triangleq \int_{-\infty}^{+\infty} x f_{X_K}(x)\,dx \qquad (42)$$

where $X(t_K)$ is the Random Variable obtained by observing the R.P. at time $t = t_K$ and $f_{X_K}(x)$ is the pdf over the ensemble of events at $t = t_K$.

the pdf over the ensemble of events at $t=t_K$.

b) Autocorrelation function of the R.P. $X(t)$:

$$R_X(t_1, t_2) \triangleq E\{X(t_1)X(t_2)\} \qquad (43)$$

where $X(t_1)$ and $X(t_2)$ are R. Varables obtained by observing the R.P. $X(t)$ at time t_1 and t_2.

$R_X(t_1, t_2)$ is a measure of similarity of the two samples $X(t_1)$ and $X(t_2)$ of the same R.P.

stationarity:

(time Invariant?)

a) A R.P. $x(t)$ is said to be stationary in strict sence (SSS) if none of its statistic are affected by a shift in the time origin.

b) A R.P. $x(t)$ is said to be stationary in wide sense (WSS) if its mean and its autocorrelation function do not change with a shift in the time origin. That is:

$$E\{x(t)\} = m_x = \text{constant}$$

$$R_x(t_1, t_2) = R_x(t_2 - t_1) \triangleq R_x(\tau)$$

Note that: $(SSS) \underset{\not\leftarrow}{\longrightarrow} WSS$

The autocorrelation of WSS is defined as:

$$R_x(\tau) \triangleq E\{x(t)x^*(t+\tau)\} \quad \text{or}$$

$$R_x(\tau) \triangleq E\{x(t)x^*(t-\tau)\} \qquad (44)$$

Properties of $R_x(\tau)$ for WSS:

1) $R_x(\tau) = R_x(-\tau)$: If the R.P. $x(t)$ is real, then $R_x(\tau)$ is real and even.

2) $R_x(\tau) \leq R_x(0)$

3) $R_x(\tau) \xleftrightarrow{F.T} S_x(f)$ P.S.D

4) $R_x(0) = E\{x^2(t)\} \triangleq P_{av}$

Time Averaging and Ergodicity:

To find m_X and $R_X(\tau)$ by ensemble averaging, we have to average over all sample function of the R.P. This would require the knowledge of I° and II° order joint probability density functions, which are not generally available.

we will consider a particular class of R.P. known as ergodic process, where its time averages equal its ensemble averages, and its statistical properties can be obtained by time averaging a single sample function of the process. Note for a R.P. to be ergodic it must be SSS.

a) we say a R.P. $X(t)$ is ergodic in mean if and only if:

$$m_X = E\{X(t)\} = \langle X(t) \rangle \quad (45) \quad \text{where}$$

$$E\{X(t)\} \triangleq \int_{-\infty}^{+\infty} x f_X(x)dx \quad (46) \quad \text{ensemble average}$$

$$\langle X(t) \rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)dt \quad (47) \quad \text{time average}$$

b) we say a R.P. $X(t)$ is ergodic in autocorrelation iff:

$$E\{X(t)X(t+\tau)\} = \langle X(t)X(t+\tau)\rangle \quad (48)$$

where:

$$R_X(\tau) = E\{X(t)X(t+\tau)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{X(t)X(t+\tau)}(x,y)dxdy \quad (49)$$

$$R_X(\tau) = \langle X(t)X(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} X(t)X(t+\tau)dt$$

For an ergodic process, fundamental electrical engineering parameters, such as dc value, rms value, and average power can be related to the moments of an ergodic process:

a) $m_X = E\{X(t)\}$ is the dc level of the signal

b) m_X^2 is the normalized power in the dc component

c) $E\{X^2(t)\}$ is the total average normalized power

d) $\sqrt{E\{X^2(t)\}}$ is the root mean square (rms) value

e) σ_X^2 is the variance or the average normalized power in the ac component. where $\sigma_X^2 = E\{X^2(t)\} - m_X^2$.

Properties of the P.S.D of a R.P. $X(t)$

If the process $X(t)$ is real then:

a) $S_X(f) \geq 0$ and is always real valued.

b) $S_X(f) = S_X(-f)$

c) $S_X(f) \xleftrightarrow{FT} R_X(\tau)$

d) $P_X = \int_{-\infty}^{+\infty} S(f)\,df = R_X(0)$

Example: consider the sample function $X(t) = A\cos(\omega_0 t + \theta)$ where A and f_0 are constant and θ is a random variable uniformly distributed over $(0, 2\pi)$, that is:

$$f_\theta(\alpha) = \begin{cases} \frac{1}{2\pi} & 0 \leq \alpha \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

a) verify that $x(t)$ is ergodic in mean.
b) " " " " " " autocorrelation.

a) $$m_x = E\{x(t)\} = \int_{-\infty}^{+\infty} A\cos(\omega_0 t+\alpha) f_\theta(\alpha)d\alpha$$

$$= \frac{A}{2\pi}\int_{-\infty}^{+\infty} \cos(\omega_0 t+\alpha)d\alpha = 0$$

$$\langle x(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A\cos(\omega_0 t+\theta)dt$$

$$= \lim_{T\to\infty} \frac{A}{T}\cdot\frac{1}{2\pi f_0}\left[\sin(2\pi f_0 t+\theta)\right]_{-T/2}^{T/2}$$

$$= \lim_{T\to\infty} \frac{A}{T}\cdot\frac{1}{2\pi f_0}\left\{\underbrace{\sin(\pi f_0 T+\theta) - \sin(-\pi f_0 T+\theta)}_{\leq 2}\right] = 0$$

Thus $E\{x(t)\} = \langle x(t)\rangle \Rightarrow$ Ergodic in mean.

b)

$$R_x(\tau) = E\{x(t)\,x(t+\tau)\}$$

$$= E\{A^2\cos(2\pi f_0 t+\theta)\cos(2\pi f_0 t + 2\pi f_0\tau + \theta)\}$$