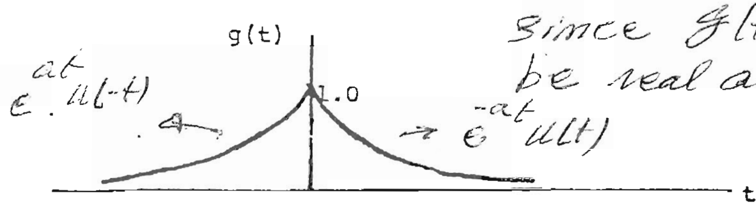
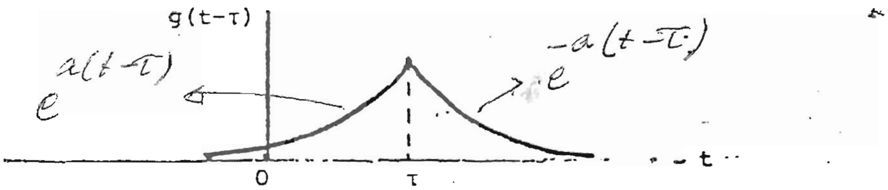
2.30 a) see handout page (28)

2.30 b) $g = \exp(-a|t|) = e^{-at}u(t) + e^{at}u(-t)$ with $a > 0$

since $g(t)$ is real then $R_g(\tau)$ will be real and even $\Rightarrow R_g(-\tau) = R_g(\tau)$





Therefore, for τ>0, $\Rightarrow R_g(\tau) = \int_{-\infty}^{+\infty} g(t)\,g(t-\tau)\,dt$

$$R_g(\tau) = \int_{-\infty}^{0} \exp(at)\exp[a(t-\tau)]dt$$

$$+ \int_{0}^{\tau} \exp(-at)\exp[a(t-\tau)]dt$$

$$+ \int_{\tau}^{\infty} \exp(-at)\exp[-a(t-\tau)]dt$$

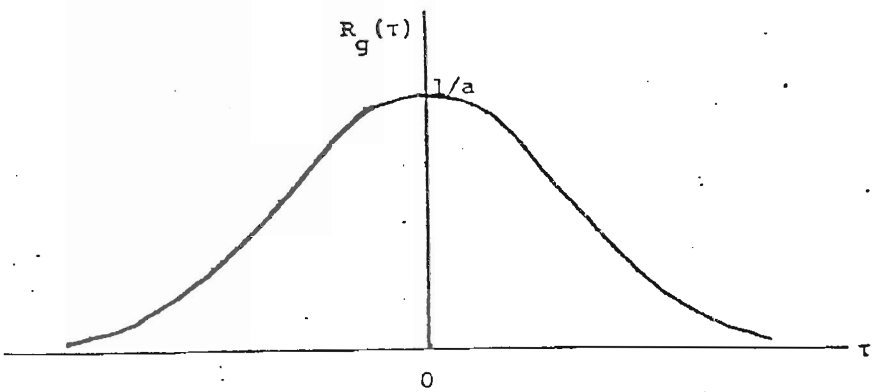
$$= \frac{1}{2a}\exp(-a\tau) + \tau\exp(-a\tau) + \frac{1}{2a}\exp(-a\tau)$$

$$= \left(\frac{1}{a} + \tau\right)\exp(-a\tau)$$

Since $R_g(-\tau) = R_g(\tau)$, we may express $R_g(\tau)$ for all τ as follows:

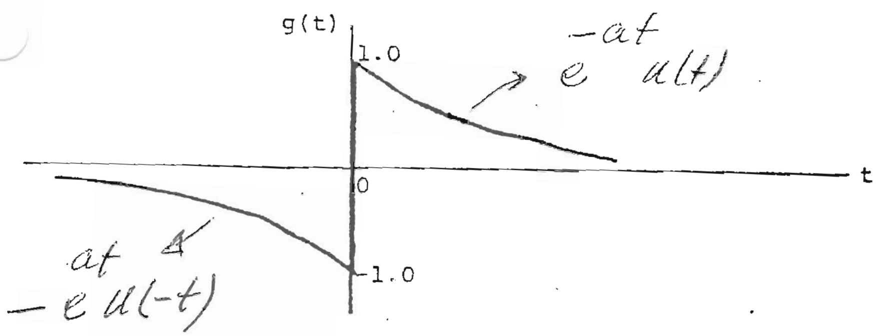
$$R_g(\tau) = \left(\frac{1}{a} + |\tau|\right)\exp(-a|\tau|)$$

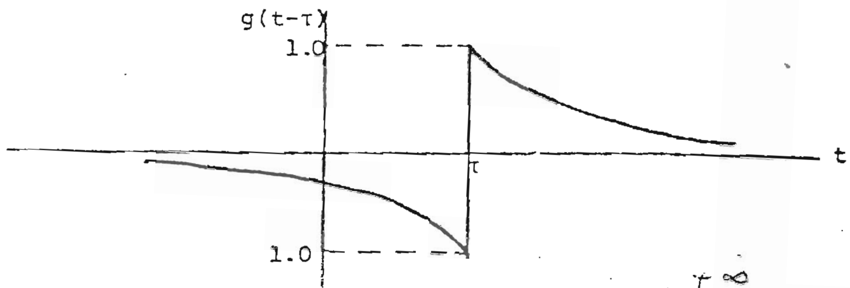
which is illustrated below:



(c) $g(t) = \exp(-at)u(t) - \exp(at)u(-t)$, $a > 0$, $g(t)$ is real.

For τ>0, we have





Therefore, for $\tau>0$, $R_g(\tau) = \int_{-\infty}^{+\infty} g(t)\,g(t-\tau)\,dt$

$$R_g(\tau) = \int_{-\infty}^{0} \exp(at)\exp[-a(t-\tau)]dt$$

$$-\int_{0}^{\tau} \exp(-at)\exp[a(t-\tau)]dt$$

$$+\int_{\tau}^{\infty} \exp(-at)\exp[-a(t-\tau)]dt$$

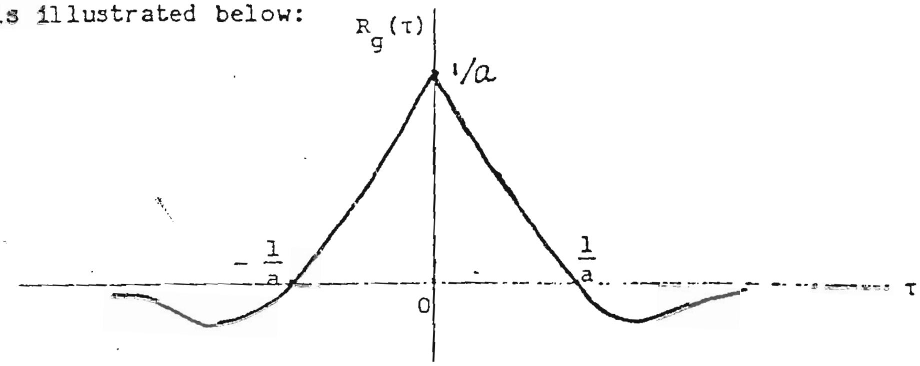
$$= \frac{1}{2a}\exp(-a\tau)-\tau\exp(-a\tau)+\frac{1}{2a}\exp(-a\tau)$$

$$= \left(\frac{1}{a}-\tau\right)\exp(-a\tau)$$

Since $R_g(-\tau) = R_g(\tau)$, we may express $R_g(\tau)$ for all τ as follows:

$$R_g(\tau) = \left(\frac{1}{a}-|\tau|\right)\exp(-a|\tau|)$$

which is illustrated below:



2.32) A $\mathrm{sinc}(2Wt) \rightleftharpoons \frac{A}{2W}\,\mathrm{rect}(\frac{f}{2W}) = G(f)$

Since,

$$R_g(\tau) \rightleftharpoons |G(f)|^2,$$

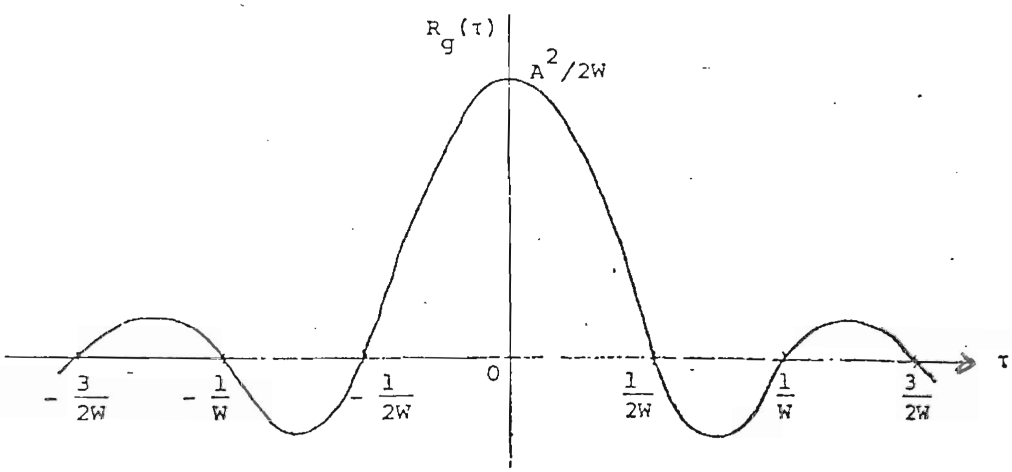
it follows that for the given sinc pulse

$$R_g(\tau) \rightleftharpoons \frac{A^2}{4W^2}\,\mathrm{rect}\left(\frac{f}{2W}\right)$$

Therefore,

$$R_g(\tau) = \frac{A^2}{2W}\,\mathrm{sinc}(2W\tau)$$

which is shown illustrated below:



Problem 2.33 → See also page (4)

$$G(f) = |\mathrm{sinc}(f)|$$

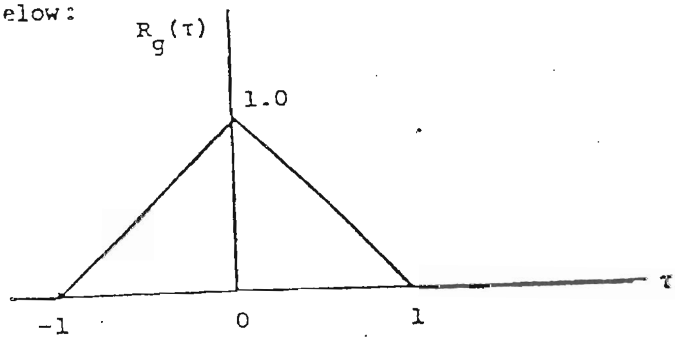
Therefore,

$$|G(f)|^2 = \mathrm{sinc}^2(f) \xleftrightarrow{F.T} R_g(\tau)$$

The function $\mathrm{sinc}^2(f)$ represents the Fourier transform of a triangular pulse of unit amplitude and width 2 seconds, centered at the origin. Therefore,

$$R_g(\tau) = \begin{cases} 1-|\tau|, & |\tau|<1 \\ 0, & |\tau|>1 \end{cases}$$

which is illustrated below:



2.23 : (second method) $\qquad G(f) = |\text{sinc}(f)|$

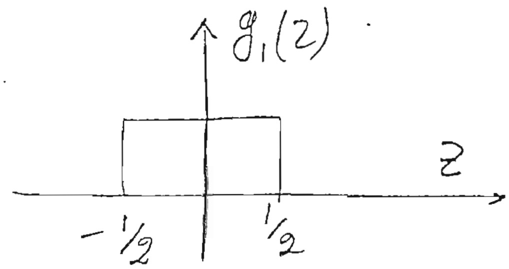
$$R_g(\tau) \xleftrightarrow{F.T.} |G(f)|^2 = \text{sinc}^2(f)$$

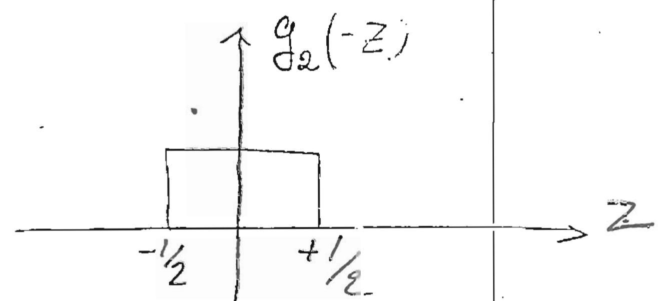
$$= \underbrace{\text{sinc}(f)}_{G_1(f)} \cdot \underbrace{\text{sinc}(f)}_{G_2(f)}$$

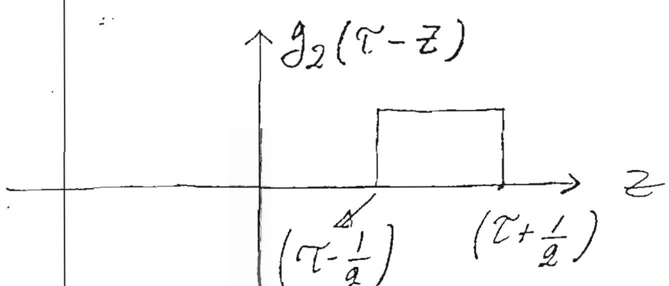
Therefore $\quad R_g(\tau) = g_1(\tau) \oplus g_2(\tau)$

where $\quad g_1(\tau) = g_2(\tau) = F^{-1}[\text{sinc}(f)] = \text{rect}(\tau)$

$$R_g(\tau) = \int_{-\infty}^{+\infty} g_1(z)\, g_2(\tau - z)\, dz$$

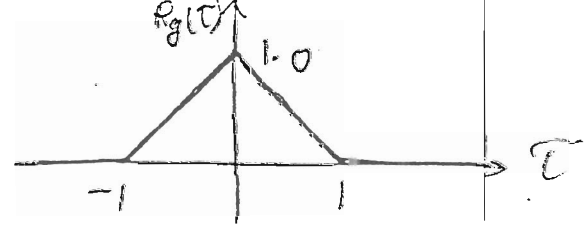






After computing the convolution we have

$$R_g(\tau) = \begin{cases} 0 & \tau < -1 \\ 1+\tau & -1 \le \tau < 0 \\ 1-\tau & 0 \le \tau < 1 \\ & \tau > 1 \end{cases} \Rightarrow$$



2.35

(a) $g(t) = A_0 + A_1 \cos(2\pi f_1 t+\theta)+A_2 \cos(2\pi f_2 t+\theta)$

Therefore:

$$G(f) = A_0\,\delta(f) + \frac{A_1}{2}[\delta(f-f_1)\exp(j\theta)+\delta(f+f_1)\exp(-j\theta)]$$

$$+\frac{A_2}{2}[\delta(f-f_2)\exp(j\theta)+\delta(f+f_2)\exp(-j\theta)]$$

and

$$|G(f)|^2 = A_0^2\,\delta(f) + \frac{A_1^2}{4}[\delta(f-f_1)+\delta(f+f_1)] + \frac{A_2^2}{4}[\delta(f-f_2)+\delta(f+f_2)]$$

Since $R_g(\tau) \rightleftharpoons |G(f)|^2$

it follows that

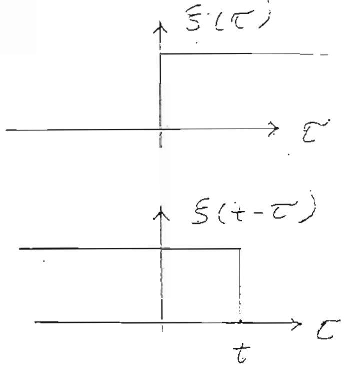
$$R_g(\tau) = A_0^2 + \frac{A_1^2}{2}\cos(2\pi f_1\tau) + \frac{A_2^2}{2}\cos(2\pi f_2\tau)$$

(b) $R_g(0) = A_0^2 + \frac{A_1^2}{2} + \frac{A_2^2}{2}$

(c) We see that $R_g(\tau)$ depends only on the dc component A_0, the amplitudes A_1 and A_2 of the two sinusoidal components and their frequencies f_1 and f_2. The phase information contained in the phase angles of the two sinusoidal components is completely lost when evaluating $R_g(\tau)$.

Extra prob # 2) Evaluate the following convolutions:

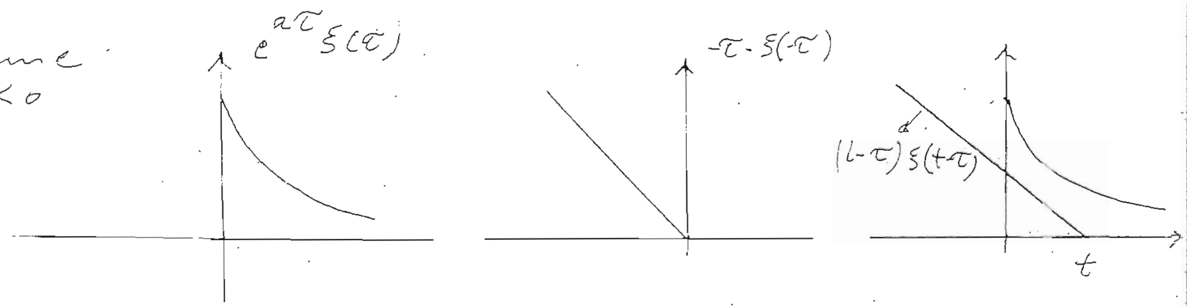
a) $\xi(t) * \xi(t) = \int_{-\infty}^{+\infty} \xi(\tau)\,\xi(t-\tau)\,d\tau$



$= \int_0^t 1 \cdot d\tau = \begin{cases} t, & \text{for } t \geq 0 \\ 0 & \text{"} \quad t < 0 \end{cases}$

b) $y(t) = t\,\xi(t) * e^{at}\,\xi(t) = \int_{-\infty}^{+\infty} e^{a\tau}\,\xi(\tau) \cdot (t-\tau)\,\xi(t-\tau)\,d\tau$

Assume $a < 0$



$y(t) = \int_0^t e^{a\tau}(t-\tau)\,d\tau = \frac{1}{a} e^{a\tau}(t-\tau)\Big|_0^t + \frac{1}{a}\int_0^t e^{a\tau}\,d\tau$

$= \frac{1}{a^2}\left(e^{at} - 1\right) - \frac{t}{a}$

c) $e^{at}\,\xi(t) * e^{at}\,\xi(t) =$

$= \int_0^t e^{a\tau} \cdot e^{a(t-\tau)}\,d\tau = \int_0^t e^{at}$

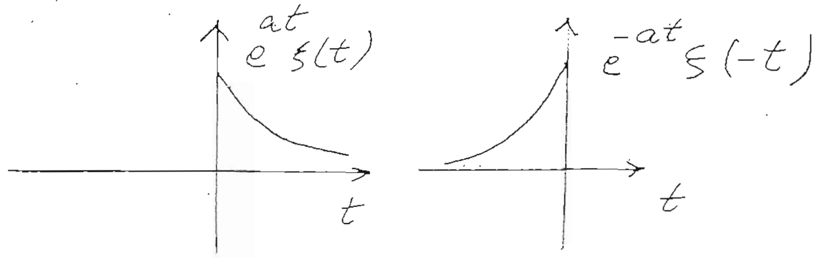
$= \begin{cases} t\,e^{at} & t \geq 0 \\ 0 & t < 0 \end{cases}$

d) Find the following convolution

(Extra problem:) $e^{at}\xi(t) * e^{-at}\xi(-t)$

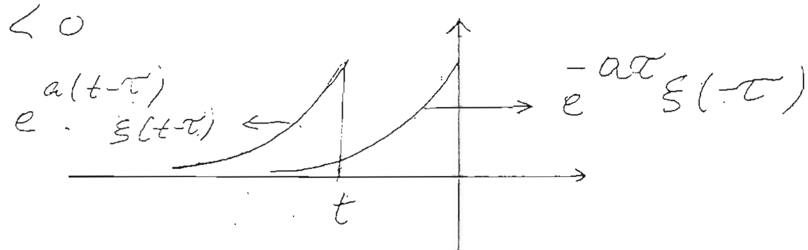
Note: The parameter a must be negative otherwise the convolution integral will not converge.

$\underline{a<0}$



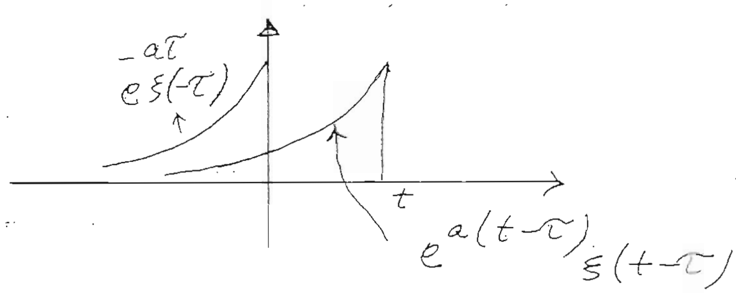
$$y(t) = e^{at}\xi(t) * e^{-at}\xi(-t) = \int_{-\infty}^{+\infty} e^{a(t-\tau)}\xi(t-\tau)\cdot e^{-a\tau}\xi(-\tau)d\tau$$

1) for $t<0$



$$y(t) = \int_{-\infty}^{t} e^{a(t-\tau)} e^{-a\tau} d\tau = \frac{-e^{-at}}{2a}$$

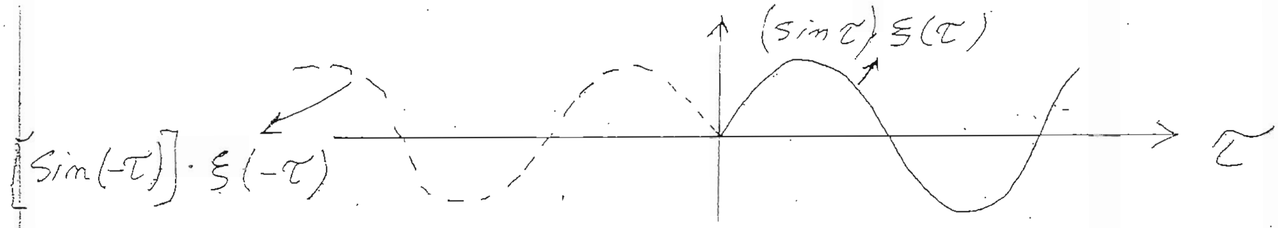
2) for $t \geq 0$



$$y(t) = \int_{-\infty}^{0} e^{a(t-\tau)}\cdot e^{-a\tau} d\tau = -\frac{e^{at}}{2a}$$

Thus: $y(t) = \begin{cases} -\frac{e^{-at}}{2a} & t<0 \\ -e^{at} & t\geq 0 \end{cases} \Rightarrow y(t) = -\frac{e^{a|t|}}{2a}$

e) $y(t) = (\sin t)\cdot \xi(t) * \sin t\, \xi(t)$



$y(t) = 0 \quad \text{for } t \leq 0$

$$y(t) = \int_0^t \sin\tau \cdot \sin(t-\tau)\, d\tau \quad \text{for } t > 0$$

$$= \frac{1}{2}\sin t - \frac{1}{2} t \cos t$$

$$y(t) = \int_0^t \frac{1}{2}\left[\cos(t-\tau-\tau) - \cos(t-\tau+\tau)\right] d\tau$$

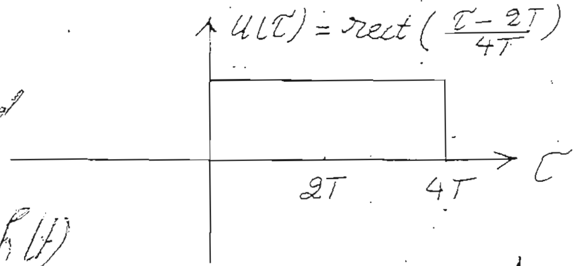
$$= \frac{1}{2}\int_0^t \left[\cos(t-2\tau) - \cos t\right] d\tau = \frac{1}{2}\left[-\frac{1}{2}\sin(t-2\tau) - \tau\cos t\right]_0^t$$

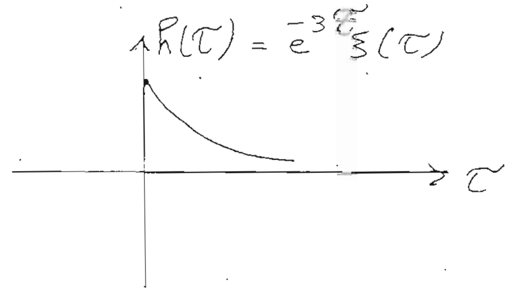
$$= \frac{1}{2}\left[-\frac{1}{2}\sin(-t) + \frac{1}{2}\sin(t) - t\cos t\right] = \frac{1}{2}\left[\sin t - t\cos t\right]$$

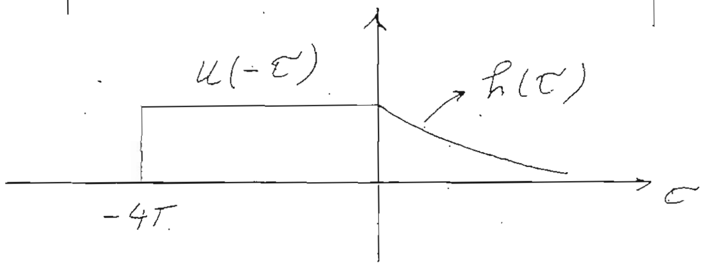
f)

Given $u(t)$ and h find

$y(t) = u(t) \circledast h(t)$





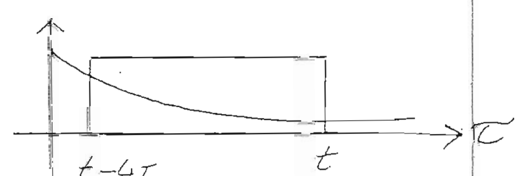


a) For $t < 0 \Rightarrow y(t) = 0$

b) For $0 \leq t < 4T$

$$y(t) = \int_0^t 1 \cdot e^{-3\tau}\, d\tau = -\frac{e^{-3\tau}}{3}\Bigg|_0^t = \frac{1 - e^{-3t}}{3}$$

c) For $t \geq 4T$



$$y(t) = \int_{t-4T}^{t} e^{-3\tau}\, d\tau = \frac{1}{3}\left(e^{-3t+12T} - e^{-3t}\right)$$

$$= \frac{e^{-3t}}{3}\left(e^{12T} - 1\right)$$

From a, b, c we have