HW 2
Electronic Communication Systems
Fall 2008
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Contents

1 Questions

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Problem 2.30 Determine and sketch the autocorrelation functions of the following exponential pulses:

(a) $g(t) = \exp(-at)u(t)$
✓ (b) $g(t) = \exp(-a|t|)$
✓ (c) $g(t) = \exp(-at)u(t) - \exp(at)u(-t)$

Problem 2.32 Determine the autocorrelation function of the sinc pulse $A\,\mathrm{sinc}(2Wt)$, and sketch it.

Problem 2.33 The Fourier transform of a signal is defined by $|\mathrm{sinc}(f)|$. Show that the autocorrelation function of this signal is triangular in form.

(Hint : Find $|G(f)|^2$, then find $R_g(\tau)$)

Problem 2.35 Consider a signal $g(t)$ defined by

$$g(t) = A_0 + A_1 \cos(2\pi f_1 t + \theta) + A_2 \cos(2\pi f_2 t + \theta)$$

(a) Determine the autocorrelation function $R_g(\tau)$ of this signal.
(b) What is the value of $R_g(0)$?
(c) Has any information about $g(t)$ been lost in obtaining the autocorrelation function?

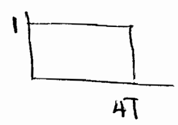
(Hint use Freq. domain approach.)

extra problem: do:

a) $\xi(t) \circledast \xi(t)$ where $\xi(t)$ is unit step function

b) $y(t) = t\,\xi(t) \circledast e^{at}\,\xi(t)$ $a<0$

c) $y(t) = u(t) \circledast h(t)$ where $u(t)$



and $h(t) = e^{-3t}u(t)$

2 Problem 2.30

Problem

Determine and sketch the autocorrelation function of the following

(b) $g(t) = e^{-a|t|}$

(c) $g(t) = e^{-at}u(t) - e^{at}u(-t)$

2.1 part(b)

$$g(t) = \begin{cases} e^{-at} & t > 0 \\ 1 & t = 0 \\ e^{at} & t < 0 \end{cases}$$

Assume $a > 0$ for the integral to be defined. From definition, autocorrelation of a function $g(t)$ is

$$R(\tau) = \int_{-\infty}^{\infty} g(t)\, g^*(t-\tau)\, dt$$

Since $g(t)$ in this case is real, then $g^*(t-\tau) = g(t-\tau)$, hence

$$R(\tau) = \int_{-\infty}^{\infty} g(t)\, g(t-\tau)\, dt$$

Consider the 3 cases, $\tau < 0$ and $\tau > 0$ and when $\tau = 0$

$\boxed{\text{case } \tau > 0}$

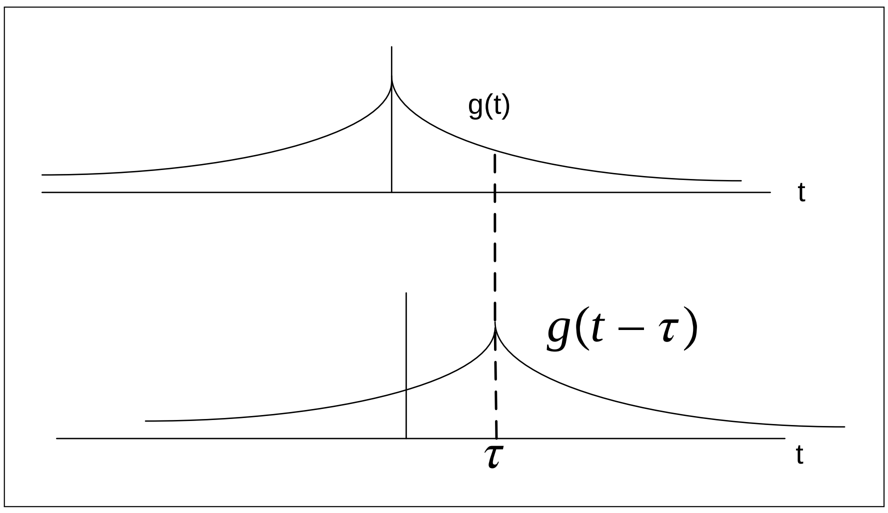


Figure 1: Case 1 Part b

Break the integral over the 3 regions, $\{-\infty, 0\}, \{0, \tau\}, \{\tau, \infty\}$

$$R(\tau) = \int_{-\infty}^{0} e^{at} e^{a(t-\tau)} dt + \int_{0}^{\tau} e^{-at} e^{a(t-\tau)} dt + \int_{\tau}^{\infty} e^{-at} e^{-a(t-\tau)} dt$$

But $\int_{-\infty}^{0} e^{at} e^{a(t-\tau)} dt = e^{-a\tau} \int_{-\infty}^{0} e^{2at} dt = e^{-a\tau} \frac{\left[e^{2at}\right]_{-\infty}^{0}}{2a} = e^{-a\tau} \frac{[1-0]}{2a} = \frac{e^{-a\tau}}{2a}$
and $\int_{0}^{\tau} e^{-at} e^{a(t-\tau)} dt = e^{-a\tau} \int_{0}^{\tau} 1 dt = \tau e^{-a\tau}$
and $\int_{\tau}^{\infty} e^{-at} e^{-a(t-\tau)} dt = e^{a\tau} \int_{\tau}^{\infty} e^{-2at} dt = e^{a\tau} \frac{\left[e^{-2at}\right]_{\tau}^{\infty}}{-2a} = e^{a\tau} \frac{\left[0-e^{-2a\tau}\right]}{-2a} = \frac{e^{-a\tau}}{2a}$
Hence for $\tau > 0$ we obtain

$$\begin{aligned} R(\tau) &= \frac{e^{-a\tau}}{2a} + \tau e^{-a\tau} + \frac{e^{-a\tau}}{2a} \\ &= \frac{e^{-a\tau}}{a} + \tau e^{-a\tau} \\ &= \boxed{e^{-a\tau}\left(\frac{1}{a} + \tau\right)} \end{aligned}$$

$\boxed{\text{case } \tau < 0}$

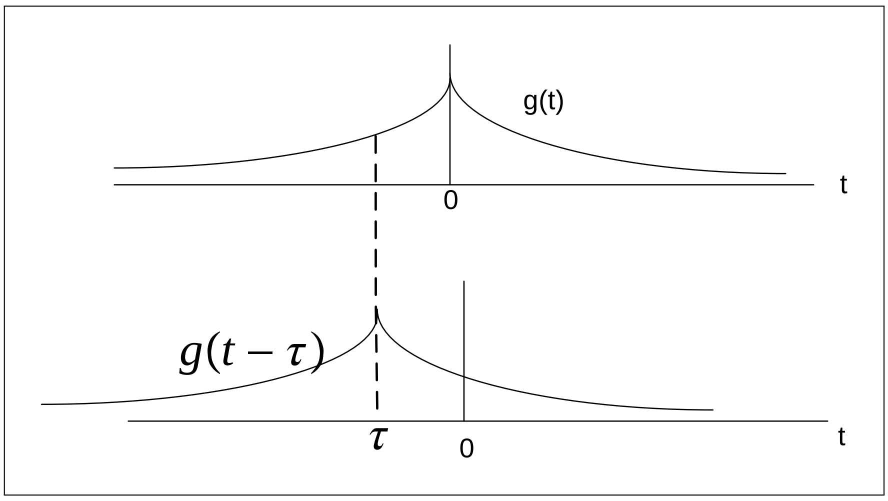


Figure 2: Case 2 Part b

Break the integral over the 3 regions, $\{-\infty,\tau\},\{\tau,0\},\{0,\infty\}$

$$R(\tau)=\int_{-\infty}^{\tau}e^{at}e^{a(t-\tau)}dt+\int_{\tau}^{0}e^{-at}e^{a(t-\tau)}dt+\int_{0}^{\infty}e^{-at}e^{-a(t-\tau)}dt$$

Now $\int_{-\infty}^{\tau}e^{at}e^{a(t-\tau)}dt=e^{-a\tau}\int_{-\infty}^{\tau}e^{2at}dt=e^{-a\tau}\frac{\left[e^{2at}\right]_{-\infty}^{\tau}}{2a}=e^{-a\tau}\frac{\left[e^{2a\tau}-0\right]}{2a}=\frac{e^{a\tau}}{2a}$
and $\int_{\tau}^{0}e^{-at}e^{a(t-\tau)}dt=e^{-a\tau}\int_{\tau}^{0}1dt=-\tau e^{-a\tau}$
and $\int_{0}^{\infty}e^{-at}e^{-a(t-\tau)}dt=e^{a\tau}\frac{\left[e^{-2at}\right]_{0}^{\infty}}{-2a}=\frac{e^{a\tau}}{-2a}(0-1)=\frac{e^{a\tau}}{2a}$
Hence

$$\begin{aligned}R(\tau)&=\frac{e^{a\tau}}{2a}-\tau e^{-a\tau}+\frac{e^{a\tau}}{2a}\\&=\boxed{e^{a\tau}\left(\frac{1}{a}-\tau\right)}\end{aligned}$$

$\boxed{\text{When } \tau=0}$

$R(0)$ gives the the maximum power in the signal $g(t)$. Now evaluate this

$$\begin{aligned}R(\tau)&=\int_{-\infty}^{0}e^{at}e^{at}dt+\int_{0}^{\infty}e^{-at}e^{-at}dt\\&=\frac{\left[e^{2at}\right]_{-\infty}^{0}}{2a}+\frac{\left[e^{-2at}\right]_{0}^{\infty}}{-2a}\\&=\frac{1}{a}\end{aligned}$$

Hence

$$R(\tau)=\begin{cases}e^{-a\tau}\left(\frac{1}{a}+\tau\right) & \tau>0\\ \frac{1}{a} & \tau=0\\ e^{a\tau}\left(\frac{1}{a}-\tau\right) & \tau<0\end{cases}$$

Or we could write

$$\boxed{R(\tau)=e^{-|\tau|a}\left(\frac{1}{a}-(-|\tau|\right)}$$

This is a plot of $R(\tau)$, first plot is for $a=1$ and the second for $a=4$

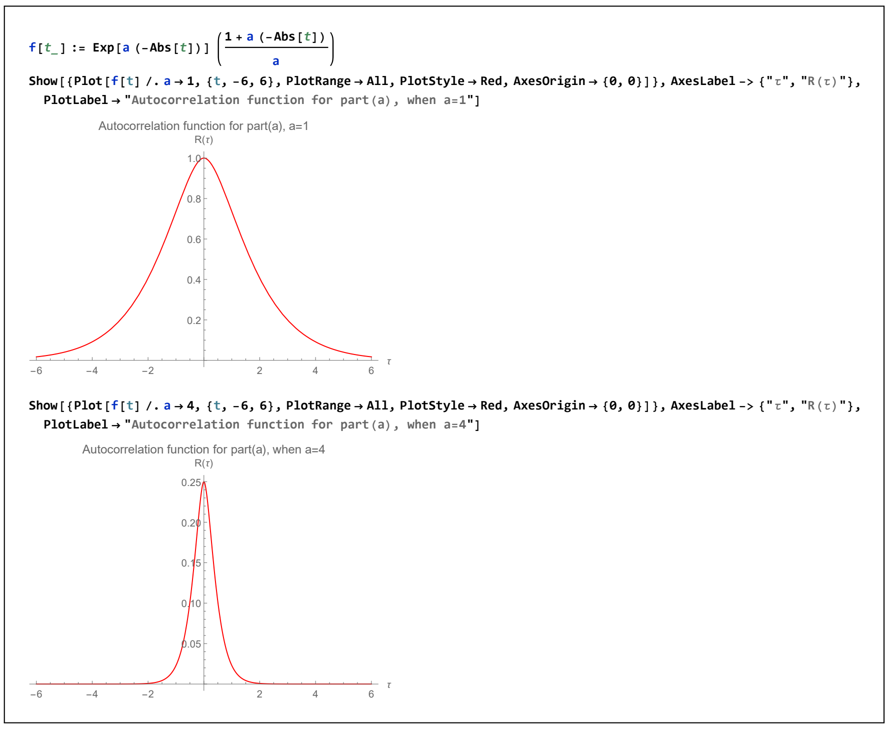


Figure 3: final part

2.2 part(c)

$$g(t) = e^{-at}u(t) - e^{at}u(-t)$$

Assume $a > 0$.

Consider the 3 cases, $\tau < 0$ and $\tau > 0$ and when $\tau = 0$

$\boxed{\text{case } \tau > 0}$

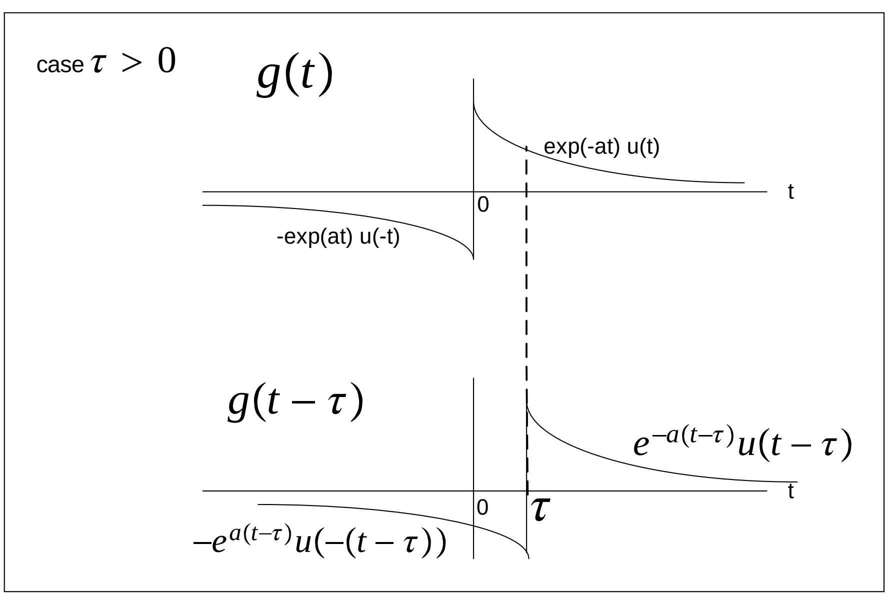


Figure 4: Case 1 Part c

Break the integral into 3 parts, $\{-\infty, 0\}, \{0, \tau\}, \{\tau, \infty\}$

$$
\begin{aligned}
R(\tau) &= \int_{-\infty}^{0} g(t)\,g(t-\tau)\,dt + \int_{0}^{\tau} g(t)\,g(t-\tau)\,dt + \int_{\tau}^{\infty} g(t)\,g(t-\tau)\,dt \\
&= \int_{-\infty}^{0} -e^{at}\left(-e^{a(t-\tau)}\right)dt + \int_{0}^{\tau} e^{-at}\left(-e^{a(t-\tau)}\right)dt + \int_{\tau}^{\infty} e^{-at}\left(e^{-a(t-\tau)}\right)dt \\
&= e^{-a\tau}\int_{-\infty}^{0} e^{2at}dt - e^{-a\tau}\int_{0}^{\tau} 1dt + e^{a\tau}\int_{\tau}^{\infty} e^{-2at}dt \\
&= e^{-a\tau}\frac{\left[e^{2at}\right]_{-\infty}^{0}}{2a} - \tau e^{-a\tau} + e^{a\tau}\frac{\left[e^{-2at}\right]_{\tau}^{\infty}}{-2a} \\
&= e^{-a\tau}\frac{[1-0]}{2a} - \tau e^{-a\tau} + e^{a\tau}\frac{\left[0-e^{-2a\tau}\right]}{-2a} \\
&= \frac{e^{-a\tau}}{2a} - \tau e^{-a\tau} + \frac{e^{-a\tau}}{2a} \\
&= e^{-a\tau}\left(\frac{1}{2a} - \tau + \frac{1}{2a}\right) \\
&= e^{-a\tau}\left(\frac{1}{a} - \tau\right)
\end{aligned}
$$

$\boxed{\text{case } \tau < 0}$

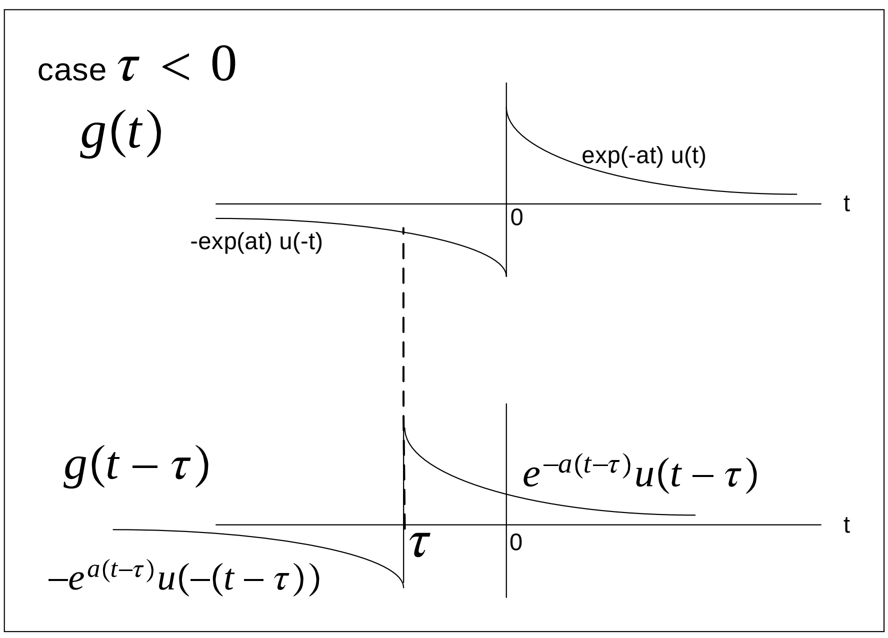


Figure 5: Case 2 Part c

Break the integral into 3 parts, $\{-\infty,\tau\},\{\tau,0\},\{0,\infty\}$

$$
\begin{aligned}
R(\tau) &= \int_{-\infty}^{\tau} g(t)\,g(t-\tau)\,dt + \int_{\tau}^{0} g(t)\,g(t-\tau)\,dt + \int_{0}^{\infty} g(t)\,g(t-\tau)\,dt \\
&= \int_{-\infty}^{\tau} -e^{at}\left(-e^{a(t-\tau)}\right)dt + \int_{\tau}^{0} -e^{at}e^{-a(t-\tau)}dt + \int_{0}^{\infty} e^{-at}e^{-a(t-\tau)}dt \\
&= e^{-a\tau}\int_{-\infty}^{\tau} e^{2at}dt - e^{a\tau}\int_{\tau}^{0} 1dt + e^{a\tau}\int_{0}^{\infty} e^{-2at}dt \\
&= e^{-a\tau}\frac{\left[e^{2at}\right]_{-\infty}^{\tau}}{2a} + \tau e^{a\tau} + e^{a\tau}\frac{\left[e^{-2at}\right]_{0}^{\infty}}{-2a} \\
&= e^{-a\tau}\frac{\left[e^{2a\tau}-0\right]}{2a} + \tau e^{a\tau} + e^{a\tau}\frac{[0-1]}{-2a} \\
&= \frac{e^{a\tau}}{2a} + \tau e^{a\tau} + \frac{e^{a\tau}}{2a} \\
&= e^{a\tau}\left(\frac{1}{a} + \tau\right)
\end{aligned}
$$

At $\tau = 0$, we see that $R(0) = \frac{1}{a}$, hence the final answer is

$$R(\tau) = \begin{cases} e^{-a\tau}\left(\frac{1}{a} - \tau\right) & \tau > 0 \\ \frac{1}{a} & \tau = 0 \\ e^{a\tau}\left(\frac{1}{a} + \tau\right) & \tau < 0 \end{cases}$$

Or we could write

$$\boxed{R(\tau) = e^{-|\tau| a}\left(\frac{1}{a} - |\tau|\right)}$$

This is a plot of $R(\tau)$, first plot is for $a = 1$ and the second for $a = 4$

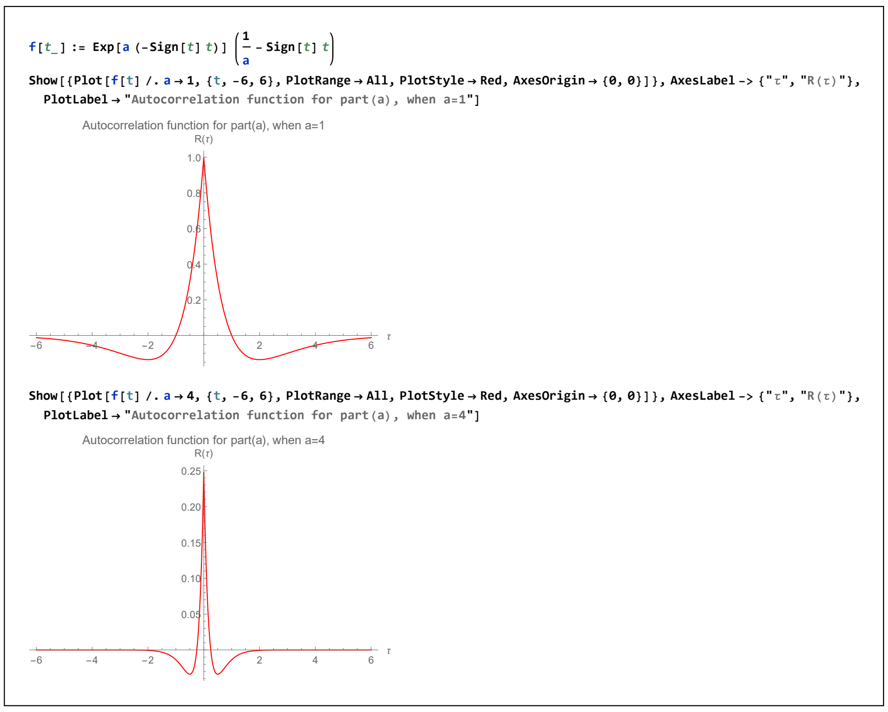


Figure 6: Part c

3 Problem 2.32

problem: Determine the autocorrelation function of $g(t) = A\,\text{sinc}(2Wt)$ and sketch it

solution:

$$R(\tau) = \int_{-\infty}^{\infty} g(t)\, g^*(t-\tau)\, dt$$

The above is difficult to do directly, hence we use the second method.

Since the function $g(t)$ is an energy function, hence $R(\tau)$ and the energy spectrum density $\Psi_g(f)$ of $g(t)$make a Fourier transform pairs.

$$R(\tau) \Leftrightarrow \Psi_g(f)$$

Therefore, to find $R(\tau)$, we first find $\Psi_g(f)$, then find the Inverse Fourier Transform of $\Psi_g(f)$, i.e.

$$R(\tau) = F^{-1}(\Psi_g(f)) \tag{1}$$

But

$$\Psi_g(f) = |G(f)|^2 \tag{2}$$

and we know that

$$A\,\text{sinc}\left(2Wt\right) \Leftrightarrow \frac{A}{2W} rect\left(\frac{f}{2W}\right)$$

Hence

$$G\left(f\right) = \frac{A}{2W} rect\left(\frac{f}{2W}\right)$$

The (2) becomes

$$\begin{aligned}\Psi_g\left(f\right) &= \left|\frac{A}{2W} rect\left(\frac{f}{2W}\right)\right|^2 \\ &= \left(\frac{A}{2W}\right)^2 \left|rect\left(\frac{f}{2W}\right)\right|^2\end{aligned}$$

But $\left|rect\left(\frac{f}{2W}\right)\right|^2 = rect\left(\frac{f}{2W}\right)$, since it has height of 1, so

$$\boxed{\Psi_g\left(f\right) = \left(\frac{A}{2W}\right)^2 rect\left(\frac{f}{2W}\right)}$$

Hence from (1)

$$\begin{aligned}R\left(\tau\right) &= F^{-1}\left(\left(\frac{A}{2W}\right)^2 rect\left(\frac{f}{2W}\right)\right) \\ &= \left(\frac{A}{2W}\right)^2 F^{-1}\left[rect\left(\frac{f}{2W}\right)\right]\end{aligned}$$

Hence

$$\boxed{R\left(\tau\right) = \left(\frac{A}{2W}\right)^2 \text{sinc}\left(2W\tau\right)}$$

This is a plot of the above function, for $W = 4$, and $A = 1$

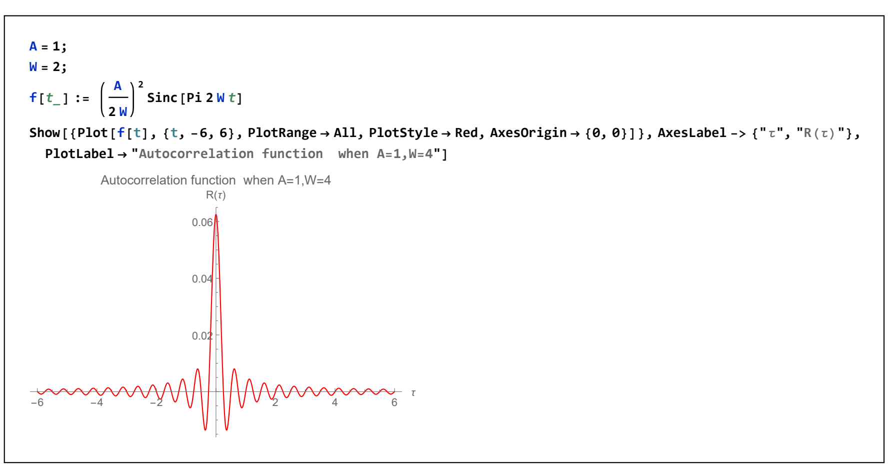


Figure 7: Plot for $W = 4$, and $A = 1$

4 Problem 2.33

The Fourier transform of a signal is defined by $\left|\text{sinc}\left(f\right)\right|$. Show that $R\left(\tau\right)$ of the signal is triangular in form.

Answer:

Since

$$R(\tau) \Leftrightarrow |G(f)|^2$$

Then

$$\begin{aligned} R(\tau) &\Leftrightarrow |\mathrm{sinc}(f)|^2 \\ &\Leftrightarrow \mathrm{sinc}^2(f) \end{aligned}$$

Hence to find $R(\tau)$ we need to find the inverse Fourier transform of $\mathrm{sinc}^2(f)$

But

$$\begin{aligned} F^{-1}\left(\mathrm{sinc}^2(f)\right) &= F^{-1}(\mathrm{sinc}(f) \times \mathrm{sinc}(f)) \\ &= F^{-1}\{\mathrm{sinc}(f)\} \otimes F^{-1}\{\mathrm{sinc}(f)\} \end{aligned}$$

But $F^{-1}\{\mathrm{sinc}(f)\} = rect(t)$, hence

$$\begin{aligned} F^{-1}\left(\mathrm{sinc}^2(f)\right) &= rect(t) \otimes rect(t) \\ &= \int_{-\infty}^{\infty} rect(\tau)\, rect(t-\tau)\, d\tau \end{aligned}$$

This integral has the value of $tri(t)$ (we also did this in class) Hence

$$tri(\tau) \Leftrightarrow \mathrm{sinc}^2(f)$$

Hence

$$R(\tau) = tri(\tau)$$

Where $tri(\tau)$ is the triangle function, defined as

$$tri(t) = \begin{cases} 1-|t| & |t| < 0 \\ 0 & otherwise \end{cases}$$

5 Problem 2.35

Consider the signal $g(t)$ defined by

$$g(t) = A_0 + A_1 \cos(2\pi f_1 t + \theta) + A_2 \cos(2\pi f_2 t + \theta)$$

(a) determine $R(\tau)$

(b) what is $R(0)$

(c) has any information been lose in obtaining $R(\tau)$?

Answer:

(a)

Take the Fourier transform of $g(t)$ we obtain

$$G(f) = A_0\delta(f) + \frac{A_1}{2}\left[e^{j\theta}\delta(f-f_1) + e^{-j\theta}\delta(f+f_1)\right] + \frac{A_2}{2}\left[e^{j\theta}\delta(f-f_2) + e^{-j\theta}\delta(f+f_2)\right]$$

Hence $|G(f)|^2 = G(f)\,G^*(f)$, so we need to find $G^*(f)$

$$G^*(f) = A_0\delta(f) + \frac{A_1}{2}\left[e^{-j\theta}\delta(f-f_1) + e^{j\theta}\delta(f+f_1)\right] + \frac{A_2}{2}\left[e^{-j\theta}\delta(f-f_2) + e^{j\theta}\delta(f+f_2)\right]$$

So

$$G(f)G^*(f) = A_0^2\delta(f) + \frac{A_1^2}{4}\left[\delta(f-f_1) + \delta(f+f_1)\right] + \frac{A_2^2}{4}\left[\delta(f-f_2) + \delta(f+f_2)\right]$$

So

$$S_g(f) = A_0^2\delta(f) + \frac{A_1^2}{4}\left[\delta(f-f_1) + \delta(f+f_1)\right] + \frac{A_2^2}{4}\left[\delta(f-f_2) + \delta(f+f_2)\right]$$

So

$$\begin{aligned} R(\tau) &= F^{-1}(S_g(f)) \\ &= F^{-1}\left(A_0^2\delta(f)\right) + \frac{A_1^2}{4}F^{-1}\left[\delta(f-f_1) + \delta(f+f_1)\right] + \frac{A_2^2}{4}F^{-1}\left[\delta(f-f_2) + \delta(f+f_2)\right] \end{aligned}$$

Hence

$$\boxed{R(\tau) = A_0^2 + \tfrac{A_1^2}{2}\cos 2\pi f_1\tau + \tfrac{A_2^2}{2}\cos 2\pi f_2\tau} \quad (1)$$

Part (b)

$$\begin{aligned} R(0) &= A_0^2 + \frac{A_1^2}{2} + \frac{A_2^2}{2} \\ &= \frac{1}{2}\left(2A_0^2 + A_1^2 + A_2^2\right) \end{aligned}$$

part(c)

In obtaining $R(\tau)$ we have lost the phase information in the original signal as can be seen from (1) above

6 extra Problem

(a) find $\xi(t) \otimes \xi(t)$ where $\xi(t)$ is unit step function

(b)Find $t\xi(t) \otimes e^{at}\xi(t)$ where $a > 0$

(c)find $u(t) \otimes h(t)$ where $h(t) = e^{-3t}u(t)$ and $u(t)$ is as shown

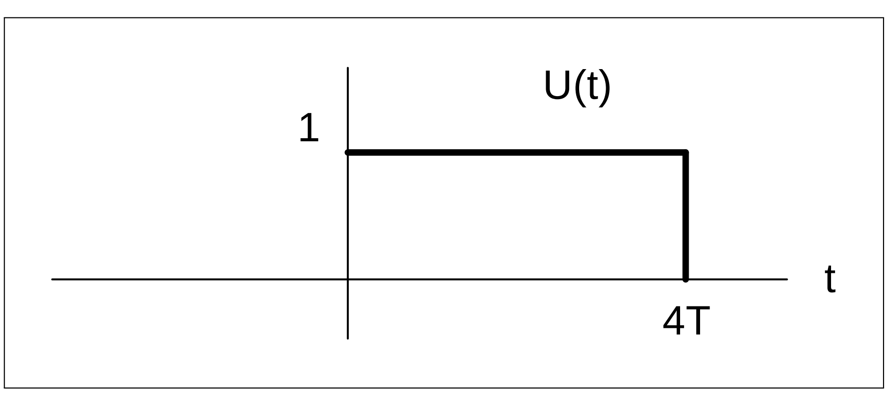


Figure 8: Extra problem

To DO

7 Key solution

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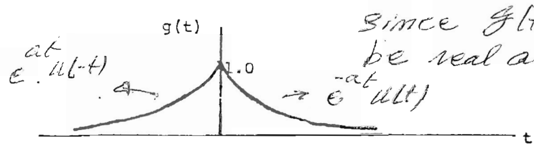
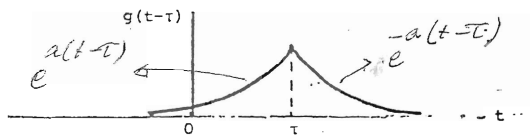
3.30 a) see handout page (28)

3.30 b) $g = \exp(-a|t|) = e^{-at}u(t) + e^{at}u(-t)$ with $a>0$

Since $g(t)$ is real then $R_g(\tau)$ will be real and even $\Rightarrow R_g(-\tau) = R_g(\tau)$





Therefore, for $\tau>0$, $\Rightarrow R_g(\tau) = \int_{-\infty}^{+\infty} g(t)\,g(t-\tau)\,dt$

$$R_g(\tau) = \int_{-\infty}^{0} \exp(at)\exp[a(t-\tau)]dt$$

$$+ \int_{0}^{\tau} \exp(-at)\exp[a(t-\tau)]dt$$

$$+ \int_{\tau}^{\infty} \exp(-at)\exp[-a(t-\tau)]dt$$

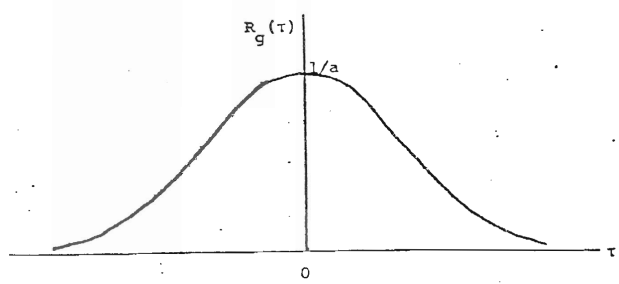
$$= \frac{1}{2a}\exp(-a\tau) + \tau\exp(-a\tau) + \frac{1}{2a}\exp(-a\tau)$$

$$= \left(\frac{1}{a} + \tau\right)\exp(-a\tau)$$

Since $R_g(-\tau) = R_g(\tau)$, we may express $R_g(\tau)$ for all τ as follows:

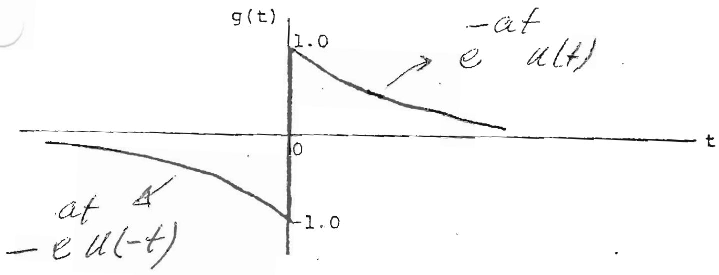
$$R_g(\tau) = \left(\frac{1}{a} + |\tau|\right)\exp(-a|\tau|)$$

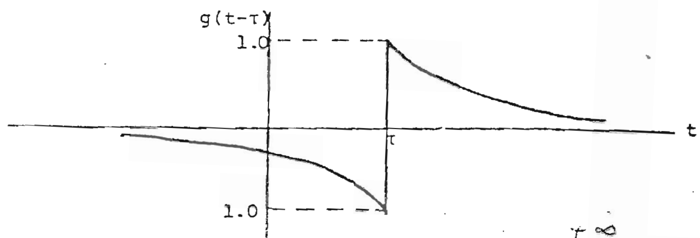
which is illustrated below:



(c) $g(t) = \exp(-at)u(t) - \exp(at)u(-t)$, $a>0$, $g(t)$ is real.

For $\tau>0$, we have





Therefore, for τ>0, $R_g(\tau) = \int_{-\infty}^{+\infty} g(t)\,g(t-\tau)\,dt$

$$R_g(\tau) = \int_{-\infty}^{0} \exp(at)\exp[a(t-\tau)]dt$$

$$- \int_{0}^{\tau} \exp(-at)\exp[a(t-\tau)]dt$$

$$+ \int_{\tau}^{\infty} \exp(-at)\exp[-a(t-\tau)]dt$$

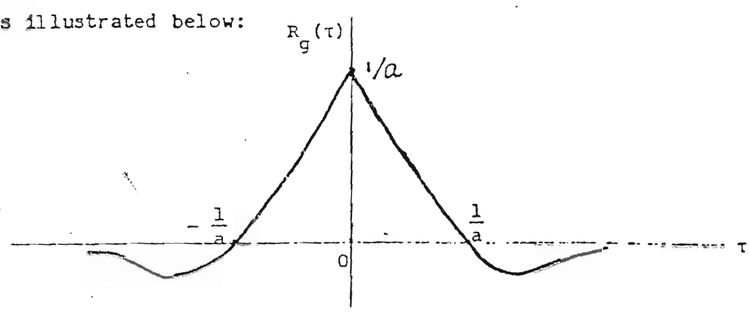
$$= \frac{1}{2a}\exp(-a\tau) - \tau\exp(-a\tau) + \frac{1}{2a}\exp(-a\tau)$$

$$= \left(\frac{1}{a} - \tau\right)\exp(-a\tau)$$

Since $R_g(-\tau) = R_g(\tau)$, we may express $R_g(\tau)$ for all τ as follows:

$$R_g(\tau) = \left(\frac{1}{a} - |\tau|\right)\exp(-a|\tau|)$$

which is illustrated below:



2.32) $A\,\mathrm{sinc}(2Wt) \rightleftharpoons \frac{A}{2W}\,\mathrm{rect}(\frac{f}{2W}) = G(f)$

page 3

Since,

$$R_g(\tau) \rightleftharpoons |G(f)|^2,$$

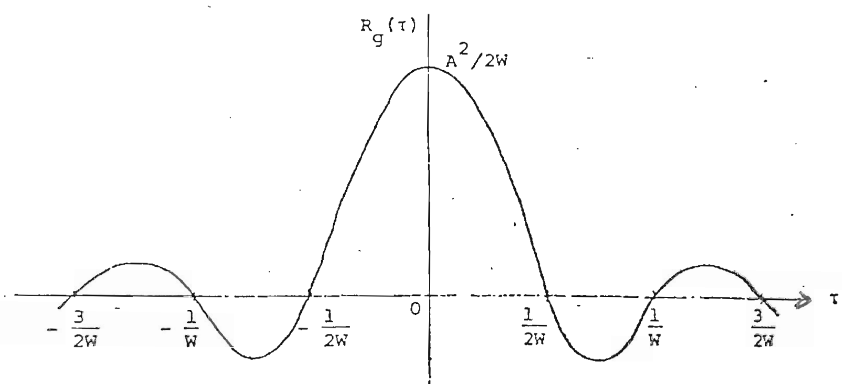
it follows that for the given sinc pulse

$$R_g(\tau) \rightleftharpoons \frac{A^2}{4W^2}\,\mathrm{rect}\left(\frac{f}{2W}\right)$$

Therefore,

$$R_g(\tau) = \frac{A^2}{2W}\,\mathrm{sinc}(2W\tau)$$

which is shown illustrated below:



Problem 2.33 → See also page (4)

$$G(f) = |\mathrm{sinc}(f)|$$

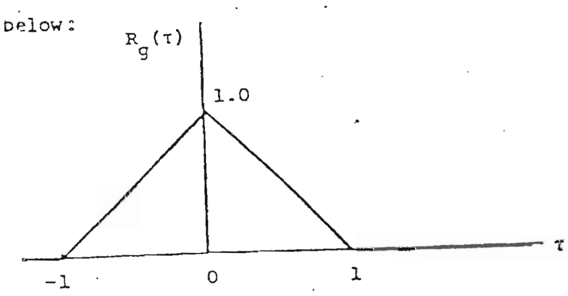
Therefore,

$$|G(f)|^2 = \mathrm{sinc}^2(f) \xleftrightarrow{F.T} R_g(\tau)$$

The function $\mathrm{sinc}^2(f)$ represents the Fourier transform of a triangular pulse of unit amplitude and width 2 seconds, centered at the origin. Therefore,

$$R_g(\tau) = \begin{cases} 1-|\tau|, & |\tau|<1 \\ 0, & |\tau|>1 \end{cases}$$

which is illustrated below:



2.23 : (second method)

$$G(f) = |\text{Sinc}(f)|$$

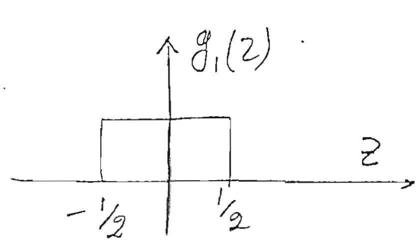
$$R_g(\tau) \xleftrightarrow{F.T.} |G(f)|^2 = \text{Sinc}^2(f)$$

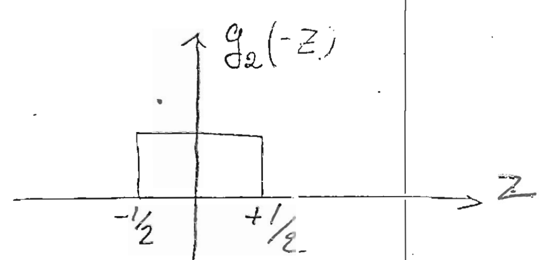
$$= \underbrace{\text{Sinc}(f)}_{G_1(f)} \cdot \underbrace{\text{Sinc}(f)}_{G_2(f)}$$

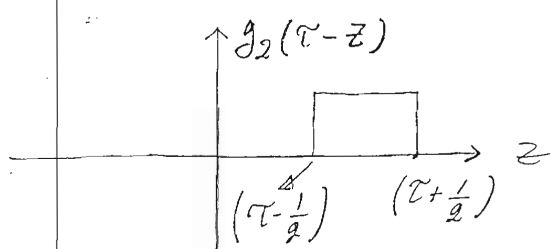
Therefore $R_g(\tau) = g_1(\tau) \circledast g_2(\tau)$

where $g_1(\tau) = g_2(\tau) = F^{-1}\left[\text{Sinc}(f)\right] = \text{rect}(\tau)$

$$R_g(\tau) = \int_{-\infty}^{+\infty} g_1(z)\, g_2(\tau - z)\, dz$$

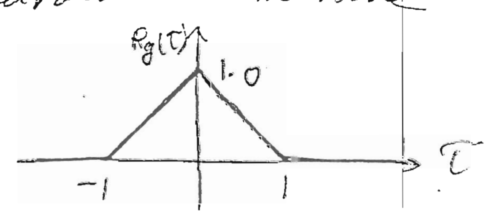






After computing the convolution we have

$$R_g(\tau) = \begin{cases} 0 & \tau < -1 \\ 1+\tau & -1 \le \tau < 0 \\ 1-\tau & 0 \le \tau < 1 \\ & \tau > 1 \end{cases} \Rightarrow$$



2.35

(a) $g(t) = A_0 + A_1 \cos(2\pi f_1 t+\theta)+A_2 \cos(2\pi f_2 t+\theta)$

Therefore:

$$G(f) = A_0\,\delta(f) + \frac{A_1}{2}\,[\delta(f-f_1)\exp(j\theta)+\delta(f+f_1)\exp(-j\theta)]$$

$$+ \frac{A_2}{2}\,[\delta(f-f_2)\exp(j\theta)+\delta(f+f_2)\exp(-j\theta)]$$

and

$$|G(f)|^2 = A_0^2\,\delta(f) + \frac{A_1^2}{4}\,[\delta(f-f_1)+\delta(f+f_1)] + \frac{A_2^2}{4}\,[\delta(f-f_2)+\delta(f+f_2)]$$

Since $R_g(\tau) \rightleftharpoons |G(f)|^2$

it follows that

$$R_g(\tau) = A_0^2 + \frac{A_1^2}{2}\cos(2\pi f_1\tau) + \frac{A_2^2}{2}\cos(2\pi f_2\tau)$$

(b) $R_g(0) = A_0^2 + \frac{A_1^2}{2} + \frac{A_2^2}{2}$

(c) We see that $R_g(\tau)$ depends only on the dc component A_0, the amplitudes A_1 and A_2 of the two sinusoidal components and their frequencies f_1 and f_2. The phase information contained in the phase angles of the two sinusoidal components is completely lost when evaluating $R_g(\tau)$.

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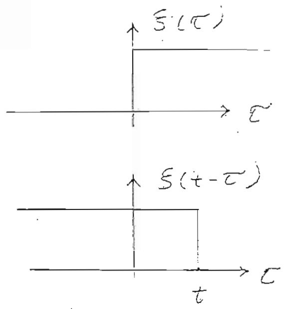
Extra prob #2) Evaluate the following convolutions:

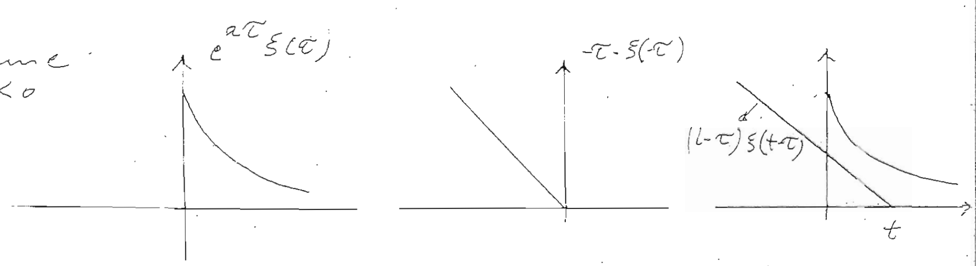
a) $\xi(t) * \xi(t) = \int_{-\infty}^{+\infty} \xi(\tau)\, \xi(t-\tau)\, d\tau$



$$= \int_0^t 1 \cdot d\tau = \begin{cases} t, & \text{for } t \geq 0 \\ 0, & t < 0 \end{cases}$$

b) $y(t) = t\,\xi(t) * e^{at}\xi(t) = \int_{-\infty}^{+\infty} e^{a\tau}\xi(\tau) \cdot (t-\tau)\,\xi(t-\tau)\, d\tau$

Assume $a < 0$



$$y(t) = \int_0^t e^{a\tau}(t-\tau)\, d\tau = \frac{1}{a} e^{a\tau}(t-\tau)\Big|_0^t + \frac{1}{a}\int_0^t e^{a\tau}\, d\tau$$

$$= \frac{1}{a^2}\left(e^{at} - 1\right) - \frac{t}{a}$$

c) $e^{at}\xi(t) * e^{at}\xi(t) =$

$$= \int_0^t e^{a\tau} \cdot e^{a(t-\tau)}\, d\tau = \int_0^t e^{at}$$

$$= \begin{cases} t\, e^{at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

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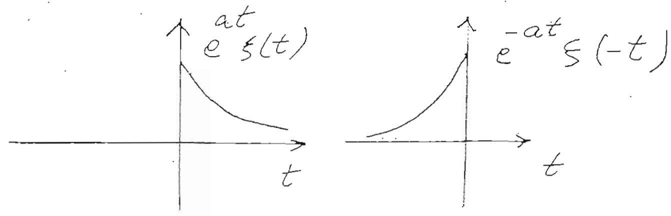
d) Find the following convolution

(Extra problem:) $e^{at}\xi(t) * e^{-at}\xi(-t)$

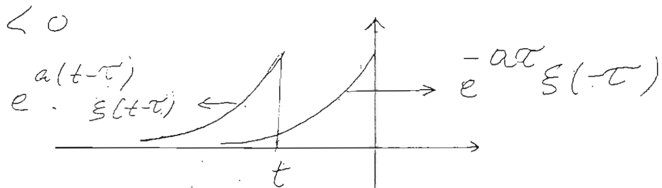
Note: The parameter a must be negative otherwise the convolution integral will not converge.

$a<0$



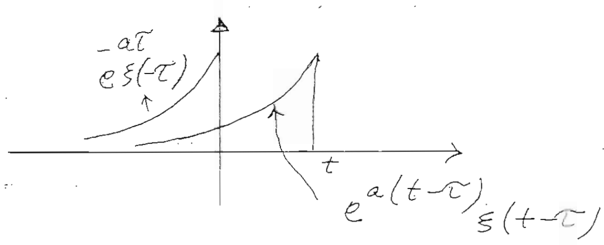
$$y(t) = e^{at}\xi(t) * e^{-at}\xi(-t) = \int_{-\infty}^{+\infty} e^{a(t-\tau)}\xi(t-\tau)\cdot e^{-a\tau}\xi(-\tau)\,d\tau$$

1) for $t<0$



$$y(t) = \int_{-\infty}^{t} e^{a(t-\tau)} e^{-a\tau}\,d\tau = \frac{-e^{-at}}{2a}$$

2) for $t \geq 0$

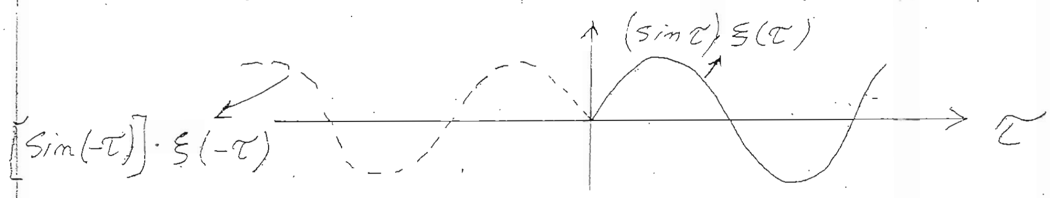


$$y(t) = \int_{-\infty}^{0} e^{a(t-\tau)}\cdot e^{-a\tau}\,d\tau = -\frac{e^{at}}{2a}$$

Thus: $y(t) = \begin{cases} -\frac{e^{-at}}{2a} & t<0 \\ -e^{at} & t\geq 0 \end{cases} \Rightarrow y(t) = -\frac{e^{a|t|}}{2a}$

EE 443	chapter	HW # 2	page

e) $y(t) = (\sin t)\cdot \xi(t) * \sin t\ \xi(t)$



$y(t) = 0 \quad \text{for } t \leq 0$

$$y(t) = \int_0^t \sin\tau \cdot \sin(t-\tau)\, d\tau \qquad \text{for } t > 0$$

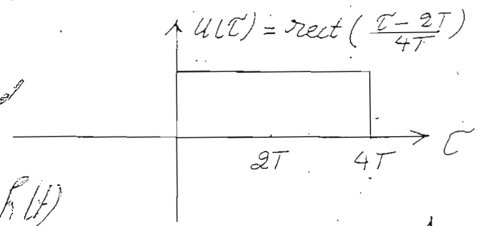
$$= \frac{1}{2}\sin t - \frac{1}{2} t \cos t$$

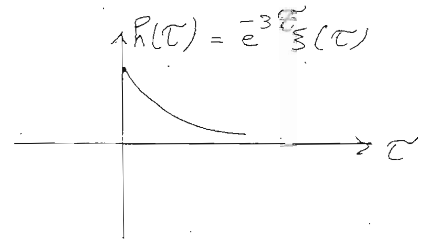
$$y(t) = \int_0^t \frac{1}{2}\left[\cos(t-\tau-\tau) - \cos(t-\tau+\tau)\right] d\tau$$

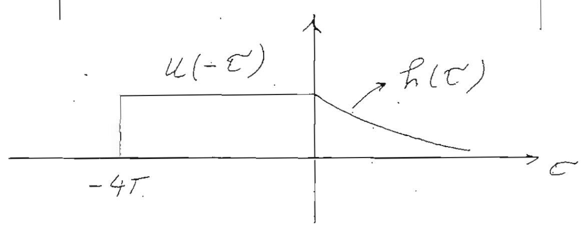
$$= \frac{1}{2}\int_0^t \left[\cos(t-2\tau) - \cos t\right] d\tau = \frac{1}{2}\left[-\frac{1}{2}\sin(t-2\tau) - \tau\cos t\right]_0^t$$

$$= \frac{1}{2}\left[-\frac{1}{2}\sin(-t) + \frac{1}{2}\sin(t) - t\cos t\right] = \frac{1}{2}\left[\sin t - t\cos t\right]$$

f) Given $u(t)$ and $h(t)$ find $y(t) = u(t) \otimes h(t)$





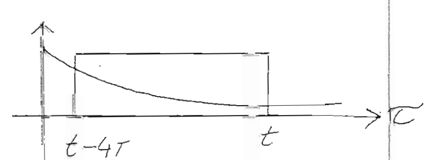


a) For $t < 0 \Rightarrow y(t) = 0$

b) For $0 \le t < 4T$

$$y(t) = \int_0^t 1 \cdot e^{-3\tau}\, d\tau = -\frac{e^{-3\tau}}{3}\Big|_0^t = \frac{1 - e^{-3t}}{3}$$

c) For $t \geq 4T$



$$y(t) = \int_{t-4T}^{t} e^{-3\tau}\, d\tau = \frac{1}{3}\left(e^{-3t+12T} - e^{-3t}\right)$$

$$= \frac{e^{-3t}}{3}\left(e^{12T} - 1\right)$$

From a, b, c we have

8 my graded HW

HW2, EGEE 443. CSUF, Fall 2008

Nasser Abbasi

September 18, 2008

19/20

1 Problem 2.30

Problem

Determine and sketch the autocorrelation function of the following

(b) $g(t) = e^{-a|t|}$

(c) $g(t) = e^{-at}u(t) - e^{at}u(-t)$

1.1 part(b)

$$g(t) = \begin{cases} e^{-at} & t > 0 \\ 1 & t = 0 \\ e^{at} & t < 0 \end{cases}$$

Assume $a > 0$ for the integral to be defined. From definition, autocorrelation of a function $g(t)$ is

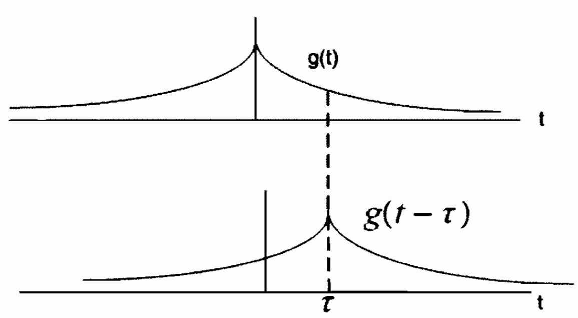
$$R(\tau) = \int_{-\infty}^{\infty} g(t)\, g^{*}(t-\tau)\, dt$$

Since $g(t)$ in this case is real, then $g^{*}(t-\tau) = g(t-\tau)$, hence

$$R(\tau) = \int_{-\infty}^{\infty} g(t)\, g(t-\tau)\, dt$$

Consider the 3 cases, $\tau < 0$ and $\tau > 0$ and when $\tau = 0$

$\boxed{\text{case } \tau > 0}$



Break the integral over the 3 regions, $\{-\infty, 0\}, \{0, \tau\}, \{\tau, \infty\}$

$$R(\tau) = \int_{-\infty}^{0} e^{at} e^{a(t-\tau)} dt + \int_{0}^{\tau} e^{-at} e^{a(t-\tau)} dt + \int_{\tau}^{\infty} e^{-at} e^{-a(t-\tau)} dt$$

But $\int_{-\infty}^{0} e^{at} e^{a(t-\tau)} dt = e^{-a\tau} \int_{-\infty}^{0} e^{2at} dt = e^{-a\tau} \frac{\left[e^{2at}\right]_{-\infty}^{0}}{2a} = e^{-a\tau} \frac{[1-0]}{2a} = \frac{e^{-a\tau}}{2a}$

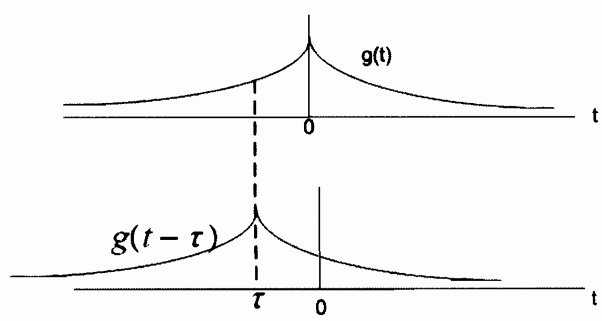
and $\int_0^\tau e^{-at}e^{a(t-\tau)}dt = e^{-a\tau}\int_0^\tau 1dt = \tau e^{-a\tau}$

and $\int_\tau^\infty e^{-at}e^{-a(t-\tau)}dt = e^{a\tau}\int_\tau^\infty e^{-2at}dt = e^{a\tau}\frac{\left[e^{-2at}\right]_\tau^\infty}{-2a} = e^{a\tau}\frac{\left[0-e^{-2a\tau}\right]}{-2a} = \frac{e^{-a\tau}}{2a}$

Hence for $\tau > 0$ we obtain

$$\begin{aligned} R(\tau) &= \frac{e^{-a\tau}}{2a} + \tau e^{-a\tau} + \frac{e^{-a\tau}}{2a} \\ &= \frac{e^{-a\tau}}{a} + \tau e^{-a\tau} \\ &= \boxed{e^{-a\tau}\left(\frac{1}{a} + \tau\right)} \end{aligned}$$

$\boxed{\text{case } \tau < 0}$



Break the integral over the 3 regions, $\{-\infty, \tau\}, \{\tau, 0\}, \{0, \infty\}$

$$R(\tau) = \int_{-\infty}^{\tau} e^{at}e^{a(t-\tau)}dt + \int_\tau^0 e^{-at}e^{a(t-\tau)}dt + \int_0^\infty e^{-at}e^{-a(t-\tau)}dt$$

Now $\int_{-\infty}^\tau e^{at}e^{a(t-\tau)}dt = e^{-a\tau}\int_{-\infty}^\tau e^{2at}dt = e^{-a\tau}\frac{\left[e^{2at}\right]_{-\infty}^\tau}{2a} = e^{-a\tau}\frac{\left[e^{2a\tau}-0\right]}{2a} = \frac{e^{a\tau}}{2a}$

and $\int_\tau^0 e^{-at}e^{a(t-\tau)}dt = e^{-a\tau}\int_\tau^0 1dt = -\tau e^{-a\tau}$

and $\int_0^\infty e^{-at}e^{-a(t-\tau)}dt = e^{a\tau}\frac{\left[e^{-2at}\right]_0^\infty}{-2a} = \frac{e^{a\tau}}{-2a}(0-1) = \frac{e^{a\tau}}{2a}$

Hence

$$\begin{aligned} R(\tau) &= \frac{e^{a\tau}}{2a} - \tau e^{-a\tau} + \frac{e^{a\tau}}{2a} \\ &= \boxed{e^{a\tau}\left(\frac{1}{a} - \tau\right)} \end{aligned}$$

$\boxed{\text{When } \tau = 0}$

$R(0)$ gives the the maximum power in the signal $g(t)$. Now evaluate this

$$
\begin{aligned}
R(\tau) &= \int_{-\infty}^{0} e^{at} e^{at} dt + \int_{0}^{\infty} e^{-at} e^{-at} dt \\
&= \frac{\left[e^{2at}\right]_{-\infty}^{0}}{2a} + \frac{\left[e^{-2at}\right]_{0}^{\infty}}{-2a} \\
&= \frac{1}{a}
\end{aligned}
$$

Hence

$$
R(\tau) = \begin{cases} e^{-a\tau}\left(\frac{1}{a} + \tau\right) & \tau > 0 \\ \frac{1}{a} & \tau = 0 \\ e^{a\tau}\left(\frac{1}{a} - \tau\right) & \tau < 0 \end{cases}
$$

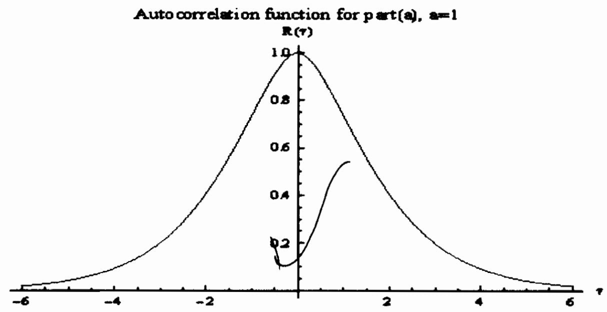
Or we could write

$$
\boxed{R(\tau) = e^{-|\tau|a}\left(\frac{1}{a} - (-|\tau|)\right)}
$$

This is a plot of $R(\tau)$, first plot is for $a = 1$ and the second for $a = 4$

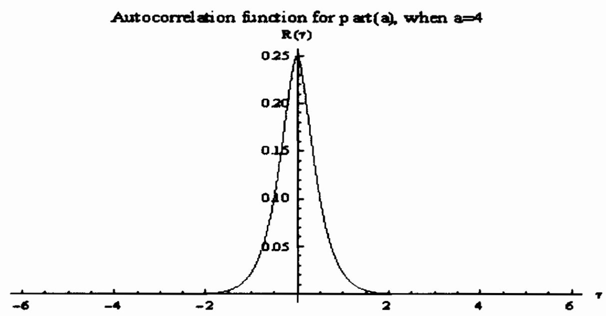
```
f[t_] := Exp[a (-Abs[t])] ((1 + a (-Abs[t]))/a)
Show[{Plot[f[t] /. a → 1, {t, -6, 6}, PlotRange → All, PlotStyle → Red,
    AxesOrigin → {0, 0}]}, AxesLabel -> {"τ", "R(τ)"},
  PlotLabel → "Autocorrelation function for part(a), when a=1"]
```

Out[18]=




```
In[21]:= Show[{Plot[f[t] /. a → 4, {t, -6, 6}, PlotRange → All, PlotStyle → Red,
    AxesOrigin → {0, 0}]}, AxesLabel -> {"τ", "R(τ)"},
  PlotLabel → "Autocorrelation function for part(a), when a=4"]
```

Out[21]=



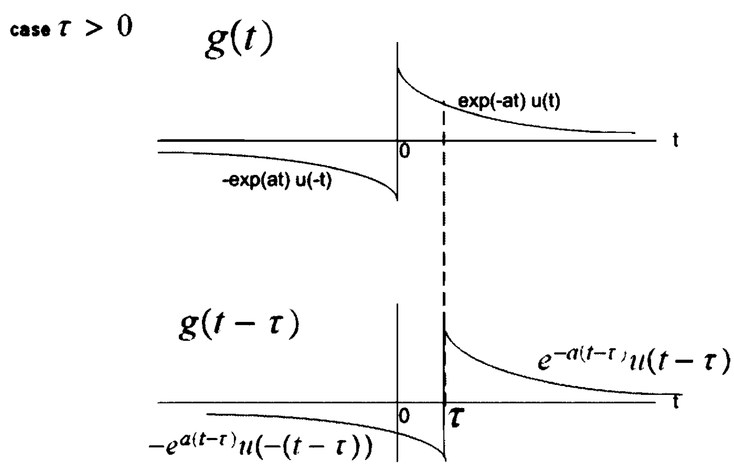
1.2 part(c)

$$g(t) = e^{-at}u(t) - e^{at}u(-t)$$

Assume $a > 0$.

Consider the 3 cases, $\tau < 0$ and $\tau > 0$ and when $\tau = 0$

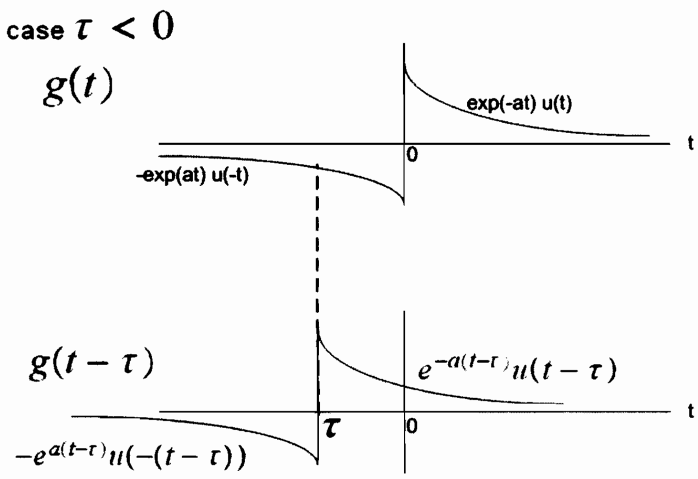
$\boxed{\text{case } \tau > 0}$



Break the integral into 3 parts, $\{-\infty, 0\}, \{0, \tau\}, \{\tau, \infty\}$

$$\begin{aligned}
R(\tau) &= \int_{-\infty}^{0} g(t)\,g(t-\tau)\,dt + \int_{0}^{\tau} g(t)\,g(t-\tau)\,dt + \int_{\tau}^{\infty} g(t)\,g(t-\tau)\,dt \\
&= \int_{-\infty}^{0} -e^{at}\left(-e^{a(t-\tau)}\right)dt + \int_{0}^{\tau} e^{-at}\left(-e^{a(t-\tau)}\right)dt + \int_{\tau}^{\infty} e^{-at}\left(e^{-a(t-\tau)}\right)dt \\
&= e^{-a\tau}\int_{-\infty}^{0} e^{2at}dt - e^{-a\tau}\int_{0}^{\tau} 1dt + e^{a\tau}\int_{\tau}^{\infty} e^{-2at}dt \\
&= e^{-a\tau}\frac{\left[e^{2at}\right]_{-\infty}^{0}}{2a} - \tau e^{-a\tau} + e^{a\tau}\frac{\left[e^{-2at}\right]_{\tau}^{\infty}}{-2a} \\
&= e^{-a\tau}\frac{[1-0]}{2a} - \tau e^{-a\tau} + e^{a\tau}\frac{\left[0-e^{-2a\tau}\right]}{-2a} \\
&= \frac{e^{-a\tau}}{2a} - \tau e^{-a\tau} + \frac{e^{-a\tau}}{2a} \\
&= e^{-a\tau}\left(\frac{1}{2a} - \tau + \frac{1}{2a}\right) \\
&= e^{-a\tau}\left(\frac{1}{a} - \tau\right)
\end{aligned}$$

$\boxed{\text{case } \tau < 0}$



Break the integral into 3 parts, $\{-\infty,\tau\},\{\tau,0\},\{0,\infty\}$

$$\begin{aligned}
R(\tau) &= \int_{-\infty}^{\tau} g(t)\,g(t-\tau)\,dt + \int_{\tau}^{0} g(t)\,g(t-\tau)\,dt + \int_{0}^{\infty} g(t)\,g(t-\tau)\,dt \\
&= \int_{-\infty}^{\tau} -e^{at}\left(-e^{a(t-\tau)}\right)dt + \int_{\tau}^{0} -e^{at}e^{-a(t-\tau)}dt + \int_{0}^{\infty} e^{-at}e^{-a(t-\tau)}dt \\
&= e^{-a\tau}\int_{-\infty}^{\tau} e^{2at}dt - e^{a\tau}\int_{\tau}^{0} 1dt + e^{a\tau}\int_{0}^{\infty} e^{-2at}dt \\
&= e^{-a\tau}\frac{\left[e^{2at}\right]_{-\infty}^{\tau}}{2a} + \tau e^{a\tau} + e^{a\tau}\frac{\left[e^{-2at}\right]_{0}^{\infty}}{-2a} \\
&= e^{-a\tau}\frac{\left[e^{2a\tau}-0\right]}{2a} + \tau e^{a\tau} + e^{a\tau}\frac{\left[0-1\right]}{-2a} \\
&= \frac{e^{a\tau}}{2a} + \tau e^{a\tau} + \frac{e^{a\tau}}{2a} \\
&= e^{a\tau}\left(\frac{1}{a}+\tau\right)
\end{aligned}$$

At $\tau = 0$, we see that $R(0) = \frac{1}{a}$, hence the final answer is

$$R(\tau) = \begin{cases} e^{-a\tau}\left(\frac{1}{a}-\tau\right) & \tau > 0 \\ \frac{1}{a} & \tau = 0 \\ e^{a\tau}\left(\frac{1}{a}+\tau\right) & \tau < 0 \end{cases}$$

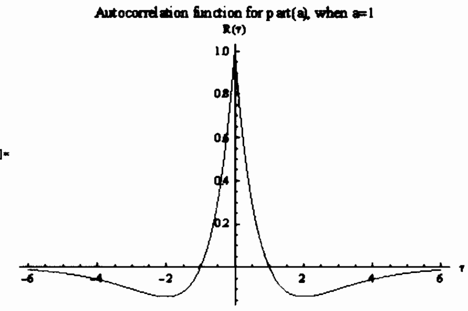
Or we could write

$$\boxed{R(\tau) = e^{-|\tau|a}\left(\frac{1}{a}-|\tau|\right)}$$

This is a plot of $R(\tau)$, first plot is for $a = 1$ and the second for $a = 4$

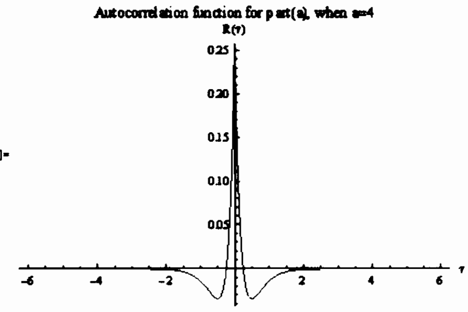
```
In[51]:= f[t_] := Exp[a (-Sign[t] t)] (1/a - Sign[t] t)
         Show[{Plot[f[t] /. a → 1, {t, -6, 6}, PlotRange → All, PlotStyle → Red, AxesOrigin → {0, 0}]},
          AxesLabel -> {"τ", "R(τ)"}, PlotLabel → "Autocorrelation function for part(a), when a=1"]
```

Out[52]=




```
In[53]:= Show[{Plot[f[t] /. a → 4, {t, -6, 6}, PlotRange → All, PlotStyle → Red, AxesOrigin → {0, 0}]},
          AxesLabel -> {"τ", "R(τ)"}, PlotLabel → "Autocorrelation function for part(a), when a=4"]
```

Out[53]=



2 Problem 2.32

problem: Determine the autocorrelation function of $g(t) = A\,\mathrm{sinc}(2Wt)$ and sketch it

solution:

$$R(\tau) = \int_{-\infty}^{\infty} g(t)\, g^{*}(t-\tau)\, dt$$

The above is difficult to do directly, hence we use the second method.

Since the function $g(t)$ is an energy function, hence $R(\tau)$ and the energy spectrum density $\Psi_g(f)$ of $g(t)$make a Fourier transform pairs.

$$R(\tau) \Leftrightarrow \Psi_g(f)$$

Therefore, to find $R(\tau)$, we first find $\Psi_g(f)$, then find the Inverse Fourier Transform of $\Psi_g(f)$, i.e.

$$R(\tau) = F^{-1}(\Psi_g(f)) \tag{1}$$

But

$$\Psi_g(f) = |G(f)|^2 \tag{2}$$

and we know that

$$A\,\mathrm{sinc}(2Wt) \Leftrightarrow \frac{A}{2W} rect\left(\frac{f}{2W}\right)$$

Hence

$$G(f) = \frac{A}{2W} rect\left(\frac{f}{2W}\right)$$

The (2) becomes

$$\begin{aligned}\Psi_g(f) &= \left|\frac{A}{2W} rect\left(\frac{f}{2W}\right)\right|^2 \\ &= \left(\frac{A}{2W}\right)^2 \left|rect\left(\frac{f}{2W}\right)\right|^2\end{aligned}$$

But $\left|rect\left(\frac{f}{2W}\right)\right|^2 = rect\left(\frac{f}{2W}\right)$, since it has height of 1, so

$$\boxed{\Psi_g(f) = \left(\frac{A}{2W}\right)^2 rect\left(\frac{f}{2W}\right)}$$

Hence from (1)

$$R(\tau) = F^{-1}\left(\left(\frac{A}{2W}\right)^2 rect\left(\frac{f}{2W}\right)\right)$$
$$= \left(\frac{A}{2W}\right)^2 F^{-1}\left[rect\left(\frac{f}{2W}\right)\right]$$

Hence

$$\boxed{R(\tau) = \left(\frac{A}{2W}\right)^2 \operatorname{sinc}(2W\tau)}$$

This is a plot of the above function, for $W = 4$, and $A = 1$

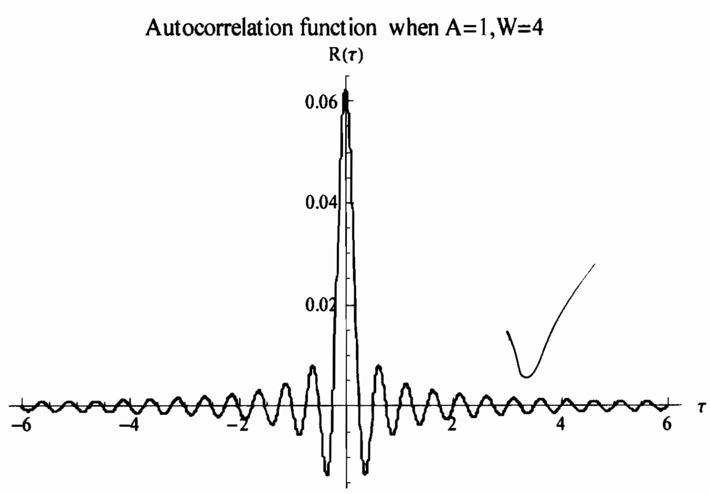
In[9]:=

```
A = 1.
W = 2
f[t_] := (A/(2 W))^2 Sinc[Pi 2 W t]
Show[{Plot[f[t], {t, -6, 6}, PlotRange → All, PlotStyle → Red, AxesOrigin → {0, 0}]},
 AxesLabel -> {"τ", "R(τ)"}, PlotLabel → "Autocorrelation function  when A=1,W=4"]
```

Out[9]= 1.

Out[10]= 2

Out[12]=



3 Problem 2.33

The Fourier transform of a signal is defined by $|\mathrm{sinc}(f)|$. Show that $R(\tau)$ of the signal is triangular in form.

Answer:

Since

$$R(\tau) \Leftrightarrow |G(f)|^2$$

Then

$$\begin{aligned} R(\tau) &\Leftrightarrow |\mathrm{sinc}(f)|^2 \\ &\Leftrightarrow \mathrm{sinc}^2(f) \end{aligned}$$

Hence to find $R(\tau)$ we need to find the inverse Fourier transform of $\mathrm{sinc}^2(f)$

But

$$\begin{aligned} F^{-1}\left(\mathrm{sinc}^2(f)\right) &= F^{-1}(\mathrm{sinc}(f) \times \mathrm{sinc}(f)) \\ &= F^{-1}\{\mathrm{sinc}(f)\} \otimes F^{-1}\{\mathrm{sinc}(f)\} \end{aligned}$$

But $F^{-1}\{\mathrm{sinc}(f)\} = rect(t)$, hence

$$\begin{aligned} F^{-1}\left(\mathrm{sinc}^2(f)\right) &= rect(t) \otimes rect(t) \\ &= \int_{-\infty}^{\infty} rect(\tau)\, rect(t-\tau)\, d\tau \end{aligned}$$

This integral has the value of $tri(t)$ (we also did this in class) Hence

$$tri(\tau) \Leftrightarrow \mathrm{sinc}^2(f)$$

Hence

$$R(\tau) = tri(\tau)$$

Where $tri(\tau)$ is the triangle function, defined as

$$tri(t) = \begin{cases} 1 - |t| & |t| < 0 \\ 0 & otherwise \end{cases}$$

4 Problem 2.35

Consider the signal $g(t)$ defined by

$$g(t) = A_0 + A_1 \cos(2\pi f_1 t + \theta) + A_2 \cos(2\pi f_2 t + \theta)$$

(a) determine $R(\tau)$

(b) what is $R(0)$

(c) has any information been lose in obtaining $R(\tau)$?

Answer:

(a)

Take the Fourier transform of $g(t)$ we obtain

$$G(f) = A_0\delta(f) + \frac{A_1}{2}\left[e^{j\theta}\delta(f-f_1) + e^{-j\theta}\delta(f+f_1)\right] + \frac{A_2}{2}\left[e^{j\theta}\delta(f-f_2) + e^{-j\theta}\delta(f+f_2)\right]$$

Hence $|G(f)|^2 = G(f)G^*(f)$, so we need to find $G^*(f)$

$$G^*(f) = A_0\delta(f) + \frac{A_1}{2}\left[e^{-j\theta}\delta(f-f_1) + e^{j\theta}\delta(f+f_1)\right] + \frac{A_2}{2}\left[e^{-j\theta}\delta(f-f_2) + e^{j\theta}\delta(f+f_2)\right]$$

So

$$G(f)G^*(f) = A_0^2\delta(f) + \frac{A_1^2}{4}\left[\delta(f-f_1) + \delta(f+f_1)\right] + \frac{A_2^2}{4}\left[\delta(f-f_2) + \delta(f+f_2)\right]$$

So

$$S_g(f) = A_0^2\delta(f) + \frac{A_1^2}{4}\left[\delta(f-f_1) + \delta(f+f_1)\right] + \frac{A_2^2}{4}\left[\delta(f-f_2) + \delta(f+f_2)\right]$$

So

$$\begin{aligned} R(\tau) &= F^{-1}(S_g(f)) \\ &= F^{-1}\left(A_0^2\delta(f)\right) + \frac{A_1^2}{4}F^{-1}\left[\delta(f-f_1) + \delta(f+f_1)\right] + \frac{A_2^2}{4}F^{-1}\left[\delta(f-f_2) + \delta(f+f_2)\right] \end{aligned}$$

Hence

$$\boxed{R(\tau) = A_0^2 + \tfrac{A_1^2}{2}\cos 2\pi f_1\tau + \tfrac{A_2^2}{2}\cos 2\pi f_2\tau} \qquad (1)$$

Part (b)

$$R(0) = A_0^2 + \frac{A_1^2}{2} + \frac{A_2^2}{2}$$
$$= \frac{1}{2}\left(2A_0^2 + A_1^2 + A_2^2\right)$$

part(c)

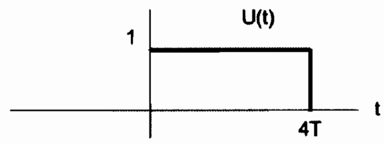
In obtaining $R(\tau)$ we have lost the phase information in the original signal as can be seen from (1) above

5 extra Problem

(a) find $\xi(t) \otimes \xi(t)$ where $\xi(t)$ is unit step function

(b)Find $t\xi(t) \otimes e^{at}\xi(t)$ where $a > 0$

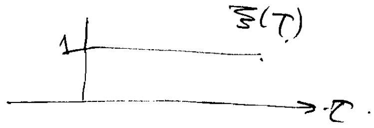
(c)find $u(t) \otimes h(t)$ where $h(t) = e^{-3t}u(t)$ and $u(t)$ is as shown



Answer

$\longrightarrow$

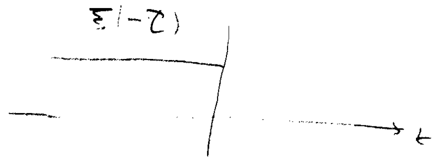
(a)



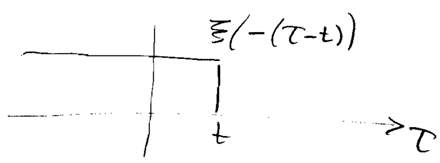
I

$$\xi(t) \otimes \xi(t) = \int_{-\infty}^{\infty} \xi(\tau)\, \xi(t-\tau)\, d\tau .$$

first flip $\xi(\tau)$ to be $\xi(-\tau)$



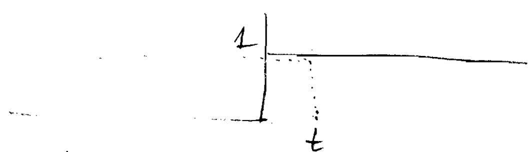
now shift to right.



II

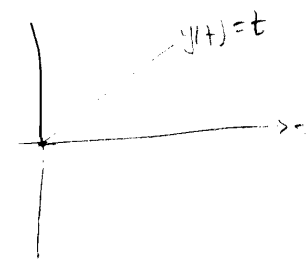
now integrate I with II. we see that we just need to integrate the following.

for $t<0$, product is zero



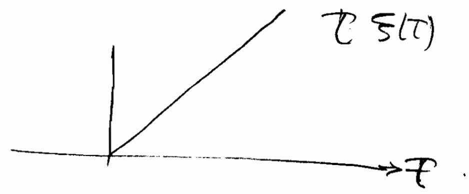
$$\int_0^t \xi(\tau)\,\xi(t-\tau)\, d\tau = \int_0^t 1 \cdot d\tau = \boxed{t}$$

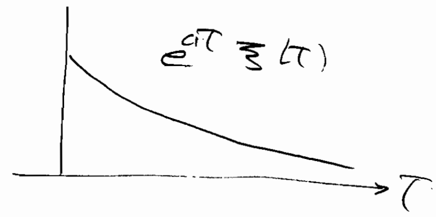
so $y(t) = t$ i.e

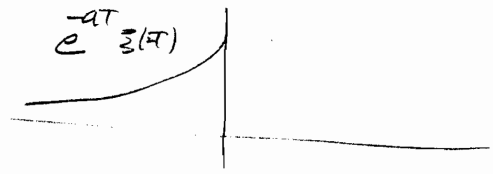


$t>0$ only.

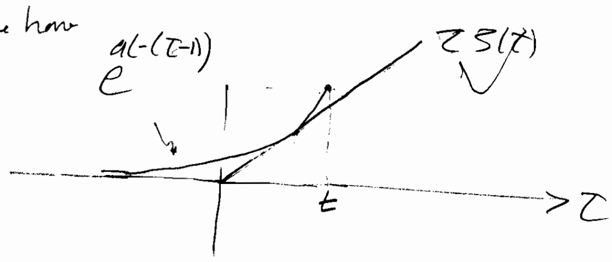
(b) $t\,\xi(t) \otimes e^{at}\,\xi(t)$. $\qquad a<0$







hence we have



only need to do $t>0$

so $$\int_t^\infty \tau e^{a(t-\tau)}\,d\tau = \int_t^\infty \tau e^{at} e^{-a\tau}\,d\tau.$$

$$= e^{at}\int_t^\infty \tau e^{-a\tau}\,d\tau \quad \text{integrate by parts.}$$

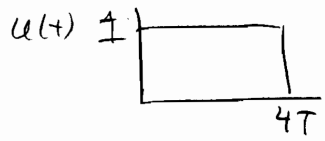
$$\int_t^\infty \tau e^{-a\tau}\,d\tau = \frac{1}{-a}\left[\tau e^{-a\tau}\right]_t^\infty - \int_t^\infty \frac{e^{-a\tau}}{-a}\,d\tau = \frac{\left[0-te^{-at}\right]}{-a} + \frac{1}{a}\,\frac{\left[e^{-a\tau}\right]_t^\infty}{-a}$$

$$= \frac{te^{-at}}{a} - \frac{1}{a^2}\left[0-e^{-at}\right] = \frac{te^{-at}}{a} - \frac{1}{a^2}\left[-e^{-at}\right] = \frac{te^{-at}}{a} + \frac{e^{-at}}{a^2}$$

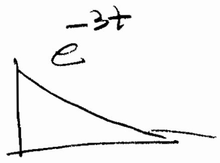
$$= e^{-at}\left(\frac{t}{a}+\frac{1}{a^2}\right) = e^{-at}\left(\frac{at+1}{a^2}\right)$$

so $$y(t) = e^{at}\left[e^{-at}\left(\frac{at+1}{a^2}\right)\right] = \boxed{\frac{at+1}{a^2}}$$ see sol. $t>0$

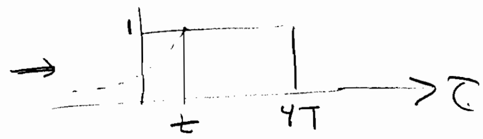
(c) $u(t) \circledast h(t)$



$h(t) =$

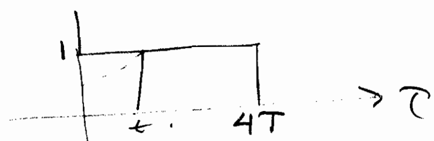


$\rightarrow$



For $t<0$, no product

for $0 \le t \le 4T$

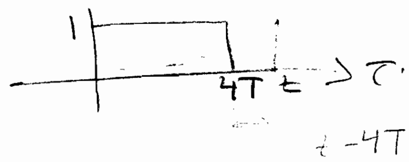


$$\int_0^t 1 \cdot e^{-3(t-\tau)}\,d\tau = e^{-3t}\int_0^t e^{3\tau}\,d\tau = e^{-3t}\,\frac{\left[e^{3\tau}\right]_0^t}{3}$$

$$= \left(\frac{e^{3t}-e^{0}}{3}\right)e^{-3t} = \left(\frac{e^{3t}-1}{3}\right)e^{-3t}$$

$$= \boxed{\frac{1-e^{-3t}}{3}} \quad 0 \le t \le 4T$$

Case $t > 4T$



$$y(t) = \int_0^{4T} 1 \cdot e^{-3(t-\tau)}\,d\tau$$

$$= e^{-3t}\int_0^{4T} e^{3\tau}\,d\tau = e^{-3t}\,\frac{\left[e^{3\tau}\right]_0^{4T}}{3}$$

$$= \frac{e^{-3t}}{3}\left[e^{12T} - e^{0}\right] = \boxed{\frac{e^{-3t}}{3}\left(e^{12T}-1\right)} \quad t > 4T$$