# solving the spin 1 electron problem. Finding $S_x, S_y, S_z$

Nasser M. Abbasi

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Problem statement

Determine $S_x, S_y, S_z$ angular momentum spin matrices for the electron using spin 1. The given is that experiments show that $S_z$ has three possible values (eigenvalues). These are $1, 0, -1$.

Solution

Using eigenbasis for $S_z$ as the following

$$|S_{z=1}\rangle = |1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$
$$|S_{z=0}\rangle = |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$|S_{z=-1}\rangle = |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then the following three equations result from writing $S_z|S_{z=\omega_i}\rangle = \omega_i|S_{z=\omega_i}\rangle$ where $\omega_i$ is the eigenvalue. These three equation are solved to determine $S_z$. Let $S_z = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & m \end{bmatrix}$.

Therefore

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & m \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & m \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & m \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Or

$$\begin{bmatrix} a \\ d \\ g \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} b \\ e \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} c \\ f \\ m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Which gives

$$\begin{aligned} S_z &= \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & m \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned} \tag{1A}$$

Now that $S_z$ is found, the goal is to determine $S_x, S_y$. Let $S_- = S_x - iS_y$ and $S_+ = S_x + iS_y$. We start with $S_+$ (starting with $S_-$ will not work, as it will not be possible to determine $S^2$ that way. So we have to start with $S_+$). We always start with commutator $[S_z, S_+]$

$$\begin{aligned} [S_z, S_+] &= [S_z, S_x + iS_y] \\ &= [S_z, S_x] + i[S_z, S_y] \end{aligned}$$

But $[S_z, S_x] = i\sum_k \epsilon_{ijk} S_k$. Here $i = 3, j = 1$ since $z = 3, x = 1$. Then $[S_z, S_x] = i\epsilon_{312} S_2 = iS_y$ and similarly $[S_z S_y] = i\sum_k \epsilon_{ijk} S_k$. Here $i = 3, j = 2$ since $z = 3, y = 1$. Then $[S_z, S_y] = i\epsilon_{321} S_1 = -iS_x$. The above becomes

$$\begin{aligned}
[S_z, S_+] &= iS_y + i(-iS_x) \\
&= iS_y + S_x \\
&= S_+
\end{aligned}$$

This implies

$$\begin{aligned}
[S_z, S_+] &= S_+ \\
S_z S_+ - S_+ S_z &= S_+ \\
S_z S_+ &= S_+ S_z + S_+ \qquad (2)
\end{aligned}$$

Picking $|1\rangle$ to start with, as it lead to finding $S^2$, which we must find before making any progress. $S^2$ is proportional for the identity matrix $I$. From the above we obtain

$$\begin{aligned}
S_z S_+ |1\rangle &= (S_+ S_z + S_+) |1\rangle \\
&= S_+ S_z |1\rangle + S_+ |1\rangle \qquad (3)
\end{aligned}$$

But $S_z|1\rangle = |1\rangle$ since the eigenvalue is 1 associated with $|1\rangle$ eigenvector. The above becomes

$$\begin{aligned}
S_z S_+ |1\rangle &= S_+ |1\rangle + S_+ |1\rangle \\
&= 2S_+ |1\rangle
\end{aligned}$$

The above shows that $S_+|1\rangle$ is eigenvector of $S_z$ associated with the eigenvalue 2 which is not compatible with experiments. Therefore the only logical result is that

$$S_+ |1\rangle = 0 |1\rangle \qquad (4)$$

Taking the adjoint gives

$$\begin{aligned}
(S_+ |1\rangle)^\dagger &= 0 |1\rangle^\dagger \\
\langle 1| S_+^\dagger &= 0 \langle 1|
\end{aligned}$$

Hence

$$\begin{aligned}
\langle 1| S_+^\dagger S_+ |1\rangle &= 0 \langle 1|1\rangle \\
\langle 1| S_+^\dagger S_+ |1\rangle &= 0 \qquad (4A)
\end{aligned}$$

The above is used to find $S^2$. Since $S_+ = S_x + iS_y$ then the above becomes

$$\langle 1|\left(S_x^\dagger - iS_y^\dagger\right)\left(S_x + iS_y\right)|1\rangle = 0$$
$$\langle 1|S_x^\dagger S_x + iS_x^\dagger S_y - iS_y^\dagger S_x + S_y^\dagger S_y|1\rangle = 0$$

But $S_x, S_y$ are Hermitian. Therefore $S_x^\dagger = S_x, S_y^\dagger = S_y$ and the above reduces to

$$\langle 1|S_x^2 + iS_xS_y - iS_yS_x + S_y^2|1\rangle = 0$$
$$\langle 1|S_x^2 + i(S_xS_y - S_yS_x) + S_y^2|1\rangle = 0$$
$$\langle 1|S_x^2 + i[S_x, S_y] + S_y^2|1\rangle = 0$$

But $[S_x, S_y] = iS_z$, therefore the above becomes

$$\langle 1|S_x^2 - S_z + S_y^2|1\rangle = 0$$

And $S^2 = S_x^2 + S_y^2 + S_z^2$ , therefore $S_x^2 + S_y^2 = S^2 - S_z^2$. Hence

$$S_+^\dagger S_+ = S^2 - S_z^2 - S_z \tag{4B}$$

Therefore (4A) becomes

$$\langle 1|S^2 - S_z^2 - S_z|1\rangle = 0$$

Expanding the above gives

$$\langle 1|S^2|1\rangle - \langle 1|S_z^2|1\rangle - \langle 1|S_z|1\rangle = 0$$
$$\langle 1|S^2|1\rangle = \langle 1|S_z^2|1\rangle + \langle 1|S_z|1\rangle$$

But $\langle 1|S_z|1\rangle = 1$ and $\langle 1|S_z^2|1\rangle = 1$, therefore the above becomes

$$\langle 1|S^2|1\rangle = 2 \tag{5}$$

It is not possible to use the above to solve for a general $S^2$ which is $3 \times 3$ matrix. But since $S^2$ must be proportional to the Identity matrix for all spin numbers, then it must diagonal matrix with same element on the diagonal, then let $S$ be

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

Substituting this in (5) gives

$$\langle 1| \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}^2 |1\rangle = 2$$
$$\langle 1| \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix} |1\rangle = 2 =$$
$$a^2 = 2$$

Hence $a=\sqrt{2}$. Therefore

$$S=\sqrt{2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \tag{6}$$

Now that $S$ is found, the next step is to find $S_{+}|2\rangle$ and $S_{+}|3\rangle$. Starting with (2), but now applying it to $|2\rangle$ gives

$$S_z S_{+}|2\rangle = S_{+}S_z|2\rangle + S_{+}|2\rangle$$

But $S_z|2\rangle = 0|2\rangle$ since $|2\rangle$ is the eigenvector associated with $0$ eigenvalue, the above becomes

$$S_z S_{+}|2\rangle = S_{+}|2\rangle$$

Which means $S_{+}|2\rangle$ is eigenvector of $S_z$ associated with eigenvalue $+1$ which is compatible with experiment. Hence

$$S_{+}|2\rangle = c|1\rangle$$

Where we used $|1\rangle$ since that is the eigenvector of $S_z$ associated with $+1$ eigenvalue. Now we need to find $c$. Taking adjoint of both sides of the above gives

$$\begin{aligned}(S_{+}|2\rangle)^{\dagger} &= (c|1\rangle)^{\dagger}\\ \langle 2|S_{+}^{\dagger} &= c^{*}\langle 1|\end{aligned}$$

Therefore

$$\begin{aligned}\langle 2|S_{+}^{\dagger}S_{+}|2\rangle &= c^{*}c\langle 1|1\rangle\\ &= |c|^2\end{aligned} \tag{7}$$

But $S_{+}^{\dagger}S_{+} = S^2 - S_z^2 - S_z$ which was found earlier above in (4B). Therefore the above equation becomes

$$\langle 2|\left(S^2 - S_z^2 - S_z\right)|2\rangle = |c|^2$$

Using $S^2$ found in (6) the above becomes

$$\begin{aligned}\langle 2|\left(2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix} - \begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}\right)|2\rangle &= |c|^2\\ \langle 2|\begin{bmatrix}0&0&0\\0&2&0\\0&0&2\end{bmatrix}|2\rangle &= |c|^2\\ \begin{bmatrix}0&1&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&2&0\\0&0&2\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} &= |c|^2\\ 2 &= |c|^2\end{aligned}$$

which implies $c = \sqrt{2}$. Therefore

$$S_+|2\rangle = \sqrt{2}|1\rangle \tag{8}$$

Finally, to find $S_+|3\rangle$, starting again with (2), but now applying it to $|3\rangle$ gives

$$S_z S_+|3\rangle = S_+ S_z|3\rangle + S_+|3\rangle$$

But $S_z|3\rangle = -|3\rangle$ since $|3\rangle$ is the eigenvector associated with $-1$ eigenvalue, the above becomes

$$\begin{aligned} S_z S_+|3\rangle &= -S_+|3\rangle + S_+|3\rangle \\ &= 0 S_+|3\rangle \end{aligned}$$

Which means $S_+|3\rangle$ is eigenvector of $S_z$ associated with eigenvalue 0 which is compatible with experiment. Hence

$$S_+|3\rangle = b|2\rangle$$

Where we used $|2\rangle$ since that is the eigenvector of $S_z$ associated with 0 eigenvalue. Now we need to find $b$. Taking adjoint of both sides of the above gives

$$\begin{aligned} (S_+|3\rangle)^\dagger &= (b|2\rangle)^\dagger \\ \langle 3|S_+^\dagger &= b^*\langle 2| \end{aligned}$$

Therefore

$$\begin{aligned} \langle 3|S_+^\dagger S_+|3\rangle &= b^* b\langle 2|2\rangle \\ &= |b|^2 \end{aligned} \tag{9}$$

But $S_+^\dagger S_+ = S^2 - S_z^2 - S_z$ which was found earlier in (4B). Therefore the above equation becomes

$$\langle 3|\left(S^2 - S_z^2 - S_z\right)|3\rangle = |b|^2$$

Using $S^2$ from eq (6) the above becomes

$$\begin{aligned} \langle 3|\left(2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix} - \begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}\right)|3\rangle &= |b|^2 \\ \langle 3|\begin{bmatrix}0&0&0\\0&2&0\\0&0&2\end{bmatrix}|3\rangle &= |b|^2 \\ \begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&2&0\\0&0&2\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} &= |b|^2 \\ 2 &= |b|^2 \end{aligned}$$

which implies $b = \sqrt{2}$. Therefore

$$S_+|3\rangle = \sqrt{2}|2\rangle \tag{10}$$

Now that $S_+|1\rangle, S_+|2\rangle, S_+|3\rangle$ are found, $S_+$ can be calculated. From (4,6,8) the result is

$$\begin{aligned} S_+|1\rangle &= 0|1\rangle \\ S_+|2\rangle &= \sqrt{2}|1\rangle \\ S_+|3\rangle &= \sqrt{2}|2\rangle \end{aligned}$$

Hence

$$\begin{aligned} S_+ &= \begin{bmatrix} \langle 1|S_+|1\rangle & \langle 1|S_+|2\rangle & \langle 1|S_+|3\rangle \\ \langle 2|S_+|1\rangle & \langle 2|S_+|2\rangle & \langle 2|S_+|3\rangle \\ \langle 3|S_+|1\rangle & \langle 3|S_+|2\rangle & \langle 3|S_+|3\rangle \end{bmatrix} \\ &= \begin{bmatrix} 0 & \langle 1|\sqrt{2}|1\rangle & \langle 1|\sqrt{2}|2\rangle \\ 0 & \langle 2|\sqrt{2}|1\rangle & \langle 2|\sqrt{2}|2\rangle \\ 0 & \langle 3|\sqrt{2}|1\rangle & \langle 3|\sqrt{2}|2\rangle \end{bmatrix} \\ &= \sqrt{2} \begin{bmatrix} 0 & \langle 1|1\rangle & \langle 1|2\rangle \\ 0 & \langle 2|1\rangle & \langle 2|2\rangle \\ 0 & \langle 3|1\rangle & \langle 3|2\rangle \end{bmatrix} \end{aligned}$$

Therefore

$$S_+ = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

Now that we know $S_+$ we turn our attention to finding $S_-$. Considering the commutator $[S_z, S_-]$

$$\begin{aligned} [S_z, S_-] &= [S_z, S_x - iS_y] \\ &= [S_z, S_x] - i[S_z, S_y] \end{aligned}$$

But $[S_z, S_x] = iS_y$ and $[S_z S_y] = -iS_x$. The above becomes

$$\begin{aligned} [S_z, S_-] &= iS_y - i(-iS_x) \\ &= iS_y - S_x \\ &= -S_- \end{aligned}$$

This implies

$$\begin{aligned}[S_z, S_-] &= -S_- \\ S_z S_- - S_- S_z &= -S_- \\ S_z S_- &= S_- S_z - S_- \end{aligned} \tag{12}$$

Picking $|1\rangle$ to start with, gives

$$\begin{aligned} S_z S_- |1\rangle &= (S_- S_z - S_-) |1\rangle \\ &= S_- S_z |1\rangle - S_- |1\rangle \end{aligned} \tag{13}$$

But $S_z|1\rangle = |1\rangle$ since the eigenvalue is 1 associated with $|1\rangle$ eigenvector from (1). The above now becomes

$$\begin{aligned} S_z S_- |1\rangle &= S_- |1\rangle - S_- |1\rangle \\ &= 0 S_- |1\rangle \end{aligned}$$

The above shows that $S_+|1\rangle$ is eigenvector of $S_z$ associated with the eigenvalue 0 which is compatible with experiments. This implies

$$S_- |1\rangle = c|2\rangle \tag{14}$$

Where $|2\rangle$ was used above, since that is the eigenvector with 0 eigenvalue. $c$ is constant to be found. Taking adjoint of both sides of the above gives

$$\begin{aligned} (S_- |1\rangle)^\dagger &= (c|2\rangle)^\dagger \\ \langle 1| S_-^\dagger &= c^* \langle 2| \end{aligned}$$

Therefore

$$\begin{aligned} \langle 1| S_-^\dagger S_- |1\rangle &= c^* c \langle 2|2\rangle \\ &= |c|^2 \end{aligned} \tag{15}$$

But

$$\begin{aligned} S_-^\dagger S_- &= (S_x - iS_y)^\dagger (S_x - iS_y) \\ &= \left(S_x^\dagger + iS_y^\dagger\right) (S_x - iS_y) \end{aligned}$$

Since $S_x, S_y$ are Hermitian, then $S_x^\dagger = S_x$ and $S_y^\dagger = S_y$. The above becomes

$$\begin{aligned} S_-^\dagger S_- &= (S_x + iS_y)(S_x - iS_y) \\ &= S_x^2 - iS_x S_y + iS_y S_x + S_y^2 \\ &= S_x^2 + i(S_y S_x - S_x S_y) + S_y^2 \\ &= S_x^2 + i[S_y, S_x] + S_y^2 \end{aligned}$$

But $[S_y, S_x] = -iS_z$. The above becomes

$$\begin{aligned} S_-^\dagger S_- &= S_x^2 + i(-iS_z) + S_y^2 \\ &= S_x^2 + S_y^2 + S_z \end{aligned} \tag{16}$$

Since $S^2 = S_x^2 + S_y^2 + S_z^2$, then $S_x^2 + S_y^2 = S^2 - S_z^2$. This implies

$$S_-^\dagger S_- = S^2 - S_z^2 + S_z \tag{16A}$$

Substituting (16A) in (15) gives

$$\langle 1| \left(S^2 - S_z^2 + S_z\right) |1\rangle = |c|^2 \tag{17}$$

But

$$S_z^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{18}$$

And $S^2$ was found in (6). Substituting (6,18) back in (17) gives an equation to solve for $c$

$$\begin{aligned} \langle 1| \left( \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right) |1\rangle &= |c|^2 \\ \langle 1| \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} |1\rangle &= |c|^2 \\ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= |c|^2 \\ 2 &= |c|^2 \end{aligned}$$

Hence $c = \sqrt{2}$. From (14) this implies

$$S_-|1\rangle = \sqrt{2}|2\rangle \tag{19}$$

Now we pick $|2\rangle$ and using (12) gives

$$\begin{aligned} S_z S_- |2\rangle &= \left(S_- S_z - S_-\right)|2\rangle \\ &= S_- S_z |2\rangle - S_-|2\rangle \end{aligned} \tag{20}$$

But $S_z|2\rangle = 0|2\rangle$ since the eigenvalue is 0 associated with $|2\rangle$ eigenvector. The above now becomes

$$S_z S_- |2\rangle = -S_-|2\rangle$$

The above shows that $S_-|2\rangle$ is eigenvector of $S_z$ associated with the eigenvalue $-1$ which is compatible with experiments. This implies

$$S_-|2\rangle = b|3\rangle \tag{21}$$

Where $|3\rangle$ was used above, since that is the eigenvector with $-1$ eigenvalue. $b$ is constant to be found. Taking adjoint of both sides of the above gives

$$\begin{aligned}(S_-|2\rangle)^\dagger &= (b|3\rangle)^\dagger \\ \langle 2|S_-^\dagger &= b^*\langle 3|\end{aligned}$$

Therefore

$$\begin{aligned}\langle 2|S_-^\dagger S_-|2\rangle &= b^* b\langle 3|3\rangle \\ &= |b|^2\end{aligned}$$

But $S_-^\dagger S_- = S^2 - S_z^2 + S_z$ as calculated earlier in (16A). Hence the above becomes

$$\langle 2|\left(S^2 - S_z^2 + S_z\right)|2\rangle = |b|^2$$

Using $S_z^2, S^2$ calculated earlier in the above gives

$$\begin{aligned}\langle 2|\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix} - \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix} + \begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}|2\rangle &= |b|^2 \\ \langle 2|\begin{bmatrix}2&0&0\\0&2&0\\0&0&0\end{bmatrix}|2\rangle &= |b|^2 \\ \begin{bmatrix}0&1&0\end{bmatrix}\begin{bmatrix}2&0&0\\0&2&0\\0&0&0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} &= |b|^2 \\ 2 &= |b|^2\end{aligned}$$

Hence $b = \sqrt{2}$. From (21) this implies

$$S_-|2\rangle = \sqrt{2}|3\rangle \tag{22}$$

And finally using $|3\rangle$ in (12) results in

$$\begin{aligned}S_z S_-|3\rangle &= (S_- S_z - S_-)|3\rangle \\ &= S_- S_z|3\rangle - S_-|3\rangle\end{aligned}$$

But $S_z|3\rangle = -|3\rangle$ since the eigenvalue is $-1$ associated with $|3\rangle$ eigenvector. The above now becomes

$$S_z S_-|3\rangle = -2S_-|3\rangle$$

The above shows that $S_-|3\rangle$ is eigenvector of $S_z$ associated with the eigenvalue $-2$ which is not compatible with experiments. This implies

$$S_-|3\rangle = 0|1\rangle \tag{23}$$

Now that $S_-|1\rangle, S_-|2\rangle, S_-|3\rangle$ are all known, we are ready to determine $S_-$. From (19,22,23)

$$\begin{aligned} S_-|1\rangle &= \sqrt{2}|2\rangle \\ S_-|2\rangle &= \sqrt{2}|3\rangle \\ S_-|3\rangle &= 0|1\rangle \end{aligned}$$

Therefore

$$\begin{aligned} S_- &= \begin{bmatrix} \langle 1|S_-|1\rangle & \langle 1|S_-|2\rangle & \langle 1|S_-|3\rangle \\ \langle 2|S_-|1\rangle & \langle 2|S_-|2\rangle & \langle 2|S_-|3\rangle \\ \langle 3|S_-|1\rangle & \langle 3|S_-|2\rangle & \langle 3|S_-|3\rangle \end{bmatrix} \\ &= \begin{bmatrix} \langle 1|\sqrt{2}|2\rangle & \langle 1|\sqrt{2}|3\rangle & \langle 1|0|1\rangle \\ \langle 2|\sqrt{2}|2\rangle & \langle 2|\sqrt{2}|3\rangle & \langle 2|0|1\rangle \\ \langle 3|\sqrt{2}|2\rangle & \langle 3|\sqrt{2}|3\rangle & \langle 3|0|1\rangle \end{bmatrix} \\ &= \sqrt{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

Therefore

$$S_- = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \tag{24}$$

Now that $S_+, S_-$ are known, $S_x, S_y$ can be found. Using

$$S_- = S_x - iS_y \tag{25}$$

$$S_+ = S_x + iS_y \tag{26}$$

Adding the above two equations gives, and using (11,24)

$$S_- + S_+ = 2S_x$$

$$\begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} = 2S_x$$

$$S_x = \frac{1}{2}\begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

$$= \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Hence

$$S_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{27}$$

And subtracting (25,26) gives

$$S_- - S_x = -2iS_y$$

$$\begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} = -2iS_y$$

$$\begin{bmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} = -2iS_y$$

$$S_y = \frac{-1}{2i}\begin{bmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

$$= \frac{i}{2}\begin{bmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

$$= \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

Hence

$$S_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \tag{28}$$

This completes the solution. We have found $S_x, S_y$, starting from just knowing the eigenvalues of $S_z$.