Example solving non-linear first order ODE

Nasser M. Abbasi

September 8, 2018 Compiled on January 31, 2024 at 4:15am

$$
\begin{aligned}
\frac{dy}{dt} + y^{\frac{3}{2}}(t) &= a^{\frac{3}{2}} \\
y(0) &= 0
\end{aligned}
$$

Write as

$$
\begin{aligned}
\left(y^{\frac{3}{2}} - a^{\frac{3}{2}}\right) dt + dy &= 0 \\
M(t,y)\,dt + N(t,y)\,dy &= 0 \qquad (1)
\end{aligned}
$$

Where

$$
\begin{aligned}
M &= y^{\frac{3}{2}} - a^{\frac{3}{2}} \\
N &= 1
\end{aligned}
$$

Check if exact

$$
\begin{aligned}
\frac{\partial M(t,y)}{\partial y} &= \frac{3}{2} y^{\frac{1}{2}} \\
\frac{\partial N(t,y)}{\partial t} &= 0
\end{aligned}
$$

Since $\frac{\partial M(t,y)}{\partial y} \neq \frac{\partial N(t,y)}{\partial t}$ then Not exact. Trying integrating factor $A = \frac{\frac{\partial N}{\partial t} - \frac{\partial M}{\partial y}}{M} = \frac{-\frac{3}{2} y^{\frac{1}{2}}}{y^{\frac{3}{2}} - a^{\frac{3}{2}}}$, Since it is a function of y alone, then it (1) can be made exact. The integrating factor is

$$
\begin{aligned}
\mu &= e^{\int A dy} \\
&= e^{\int \frac{-\frac{3}{2} y^{\frac{1}{2}}}{y^{\frac{3}{2}} - a^{\frac{3}{2}}} dy} \\
&= e^{-\ln\left(a^{\frac{3}{2}} - y^{\frac{3}{2}}\right)} \\
&= \frac{1}{a^{\frac{3}{2}} - y^{\frac{3}{2}}}
\end{aligned}
$$

Multiplying (1) by this integrating factor, now it becomes exact

$$\mu M(t,y)\,dt + \mu N(t,y)\,dy = 0$$

Now we follow standard method for solving exact ODE. Let

$$\frac{dU}{dt} = \mu M = \frac{y^{\frac{3}{2}} - a^{\frac{3}{2}}}{a^{\frac{3}{2}} - y^{\frac{3}{2}}} = -1 \tag{2}$$

$$\frac{dU}{dy} = \mu N = \frac{1}{a^{\frac{3}{2}} - y^{\frac{3}{2}}} \tag{3}$$

From (2)

$$\begin{aligned} U &= -\int dt \\ &= -t + f(y) \end{aligned} \tag{4}$$

Substituting this into (3) to solve for $f(y)$

$$f'(y) = \frac{1}{a^{\frac{3}{2}} - y^{\frac{3}{2}}}$$

$$f(y) = \frac{-2\sqrt{3}}{3\sqrt{a}} \arctan\left(\frac{1+2\sqrt{\frac{y}{a}}}{\sqrt{3}}\right) - \frac{2}{3\sqrt{a}} \ln\left(\sqrt{a} - \sqrt{y}\right) + \frac{1}{3\sqrt{a}} \ln\left(a + \sqrt{ay} + y\right) + C$$

Hence the solution from (4) is

$$U = -t + \frac{-2\sqrt{3}}{3\sqrt{a}} \arctan\left(\frac{1+2\sqrt{\frac{y}{a}}}{\sqrt{3}}\right) - \frac{2}{3\sqrt{a}} \ln\left(\sqrt{a} - \sqrt{y}\right) + \frac{1}{3\sqrt{a}} \ln\left(a + \sqrt{ay} + y\right) + C$$

But $\frac{dU}{dt} = 0$, hence $U = C_1$. Therefore, collecting constants into one, the solution is (implicit form)

$$t + \frac{2\sqrt{3}}{3\sqrt{a}} \arctan\left(\frac{1+2\sqrt{\frac{y}{a}}}{\sqrt{3}}\right) + \frac{2}{3\sqrt{a}} \ln\left(\sqrt{a} - \sqrt{y}\right) - \frac{1}{3\sqrt{a}} \ln\left(a + \sqrt{ay} + y\right) = C$$

From initial conditions

$$\frac{2\sqrt{3}}{3\sqrt{a}} \arctan\left(\frac{1}{\sqrt{3}}\right) + \frac{2}{3\sqrt{a}} \ln\left(\sqrt{a}\right) - \frac{1}{3\sqrt{a}} \ln(a) = C$$

$$\begin{aligned} C &= \frac{2\sqrt{3}}{3\sqrt{a}}\frac{\pi}{6} + \frac{2}{3\sqrt{a}} \ln\left(\sqrt{a}\right) - \frac{1}{3\sqrt{a}} \ln(a) \\ C &= \frac{2\sqrt{3}}{3\sqrt{a}}\frac{\pi}{6} \\ C &= \frac{\pi\sqrt{3}}{9\sqrt{a}} \end{aligned}$$

Hence final solution for $y(t)$ in implicit form is

$$
\begin{aligned}
t+\frac{2\sqrt{3}}{3\sqrt{a}}\arctan\left(\frac{1+2\sqrt{\frac{y}{a}}}{\sqrt{3}}\right)+\frac{2}{3\sqrt{a}}\ln\left(\sqrt{a}-\sqrt{y}\right)-\frac{1}{3\sqrt{a}}\ln\left(a+\sqrt{ay}+y\right)&=\frac{\pi\sqrt{3}}{9\sqrt{a}}\\
3t\sqrt{a}+2\sqrt{3}\arctan\left(\frac{\sqrt{a}+2\sqrt{y}}{\sqrt{3}\sqrt{a}}\right)+6\ln\left(\sqrt{a}-\sqrt{y}\right)-\ln\left(a+\sqrt{ay}+y\right)&=\frac{\pi\sqrt{3}}{3}
\end{aligned}
$$