A Solution Manual For

Collection of Kovacic problems

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1.1 problem 1

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Book : Collection of Kovacic problems
Section : section 1
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CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^2-1\right) y''-2xy'+2y=0$$

1.1.1 Solved as second order ode using Kovacic algorithm

Time used: 0.221 (sec)

Writing the ode as

$$\left(x^2-1\right) y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= (x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x+1)^2} + \frac{3}{4(x+1)} - \frac{3}{4(x-1)} + \frac{3}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{\left(x^2 - 1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(x-1)} + \frac{3}{2\,(x+1)} + (-)\,(0) \\
&= -\frac{1}{2\,(x-1)} + \frac{3}{2\,(x+1)} \\
&= \frac{x-2}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\,(x-1)} + \frac{3}{2\,(x+1)}\right)(0) + \left(\left(\frac{1}{2\,(x-1)^2} - \frac{3}{2\,(x+1)^2}\right) + \left(-\frac{1}{2\,(x-1)} + \frac{3}{2\,(x+1)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{3}{2(x+1)}\right) dx} \\ &= \frac{(x+1)^{3/2}}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2-1}\, dx} \\ &= z_1 e^{\frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2}} \\ &= z_1\left(\sqrt{x-1}\,\sqrt{x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (x+1)^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x-1)+\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x\,\mathrm{e}^{\ln(x-1)+\ln(x+1)}}{(x+1)^3 (x-1)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x+1)^2\right) + c_2\left((x+1)^2 \left(-\frac{x\,\mathrm{e}^{\ln(x-1)+\ln(x+1)}}{(x+1)^3 (x-1)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.1.2 Maple step by step solution

Let's solve
$(x^2-1)\left(\frac{d}{dx}y'\right) - 2xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{2y}{x^2-1} + \frac{2xy'}{x^2-1}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{2xy'}{x^2-1} + \frac{2y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2x}{x^2-1}, P_3(x) = \frac{2}{x^2-1}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$
    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = -1$
  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d}{dx}y'\right) - 2xy' + 2y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r-1) + a_k(k+r-1)(k+r-2)\right)u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $\left((-2k-2r-2)\,a_{k+1} + a_k(k+r-2)\right)(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r-2)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{2(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{4}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - u + \frac{1}{4}u^2\right)$

- Revert the change of variables $u = x + 1$

  $\left[y = \frac{a_0(x-1)^2}{4}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k}{2(k+3)}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$

- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y = \frac{a_0(x-1)^2}{4} + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$

1.1.3 Maple trace

```
Methods for second order ODEs:
```

1.1.4 Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 14

```
dsolve((x^2-1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2 x^2 + c_1 x + c_2$$

1.1.5 Mathematica DSolve solution

Solving time : 0.13 (sec)
Leaf size : 39

```
DSolve[{(x^2-1)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x^2-1}(c_1(x-1)^2 + c_2 x)}{\sqrt{1-x^2}}$$

1.2 problem 2

Internal problem ID [8140]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 2
Date solved : `Monday, October 21, 2024 at 04:54:21 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^2-1\right)y''-6xy'+12y=0$$

1.2.1 Solved as second order ode using Kovacic algorithm

Time used: 0.223 (sec)

Writing the ode as

$$\left(x^2-1\right)y''-6xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= -6x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= (x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{(x^2 - 1)^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 3: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{15}{4(x-1)} + \frac{15}{4(x+1)} + \frac{15}{4(x-1)^2} + \frac{15}{4(x+1)^2}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{\left(x^2-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2\,(x-1)} + \frac{5}{2\,(x+1)} + (-)\,(0) \\ &= -\frac{3}{2\,(x-1)} + \frac{5}{2\,(x+1)} \\ &= \frac{x-4}{x^2-1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2(x-1)} + \frac{5}{2(x+1)}\right)(0) + \left(\left(\frac{3}{2(x-1)^2} - \frac{5}{2(x+1)^2}\right) + \left(-\frac{3}{2(x-1)} + \frac{5}{2(x+1)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{3}{2(x-1)} + \frac{5}{2(x+1)}\right) dx} \\
&= \frac{(x+1)^{5/2}}{(x-1)^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6x}{x^2-1}\, dx} \\
&= z_1 e^{\frac{3\ln(x-1)}{2} + \frac{3\ln(x+1)}{2}} \\
&= z_1 \left((x-1)^{3/2} (x+1)^{3/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (x+1)^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x}{x^2-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x-1)+3\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x(x^2+1)\,\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7\,(x-1)^3} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x+1)^4\right) + c_2\left((x+1)^4 \left(-\frac{x(x^2+1)\,\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7\,(x-1)^3} \right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.2.2 Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d}{dx}y'\right) - 6xy' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{12y}{x^2-1} + \frac{6xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{6xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{6x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = -3$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d}{dx}y'\right) - 6xy' + 12y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-6u+6)\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

    $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

    $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0 r(-4+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r-3) + a_k(k+r-3)(k+r-4)\right) u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 4\}$

- Each term in the series must be 0, giving the recursion relation

  $\left((-2k-2r-2)\, a_{k+1} + a_k(k+r-4)\right)(k+r-3) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r-4)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 4$

  $a_{k+1} = \frac{a_k(k-4)}{2(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -2a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{3a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{3a_0}{2}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2}{3}$
- Express in terms of a_0

  $a_3 = -\frac{a_0}{2}$
- Apply recursion relation for $k = 3$

  $a_4 = -\frac{a_3}{8}$
- Express in terms of a_0

  $a_4 = \frac{a_0}{16}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 2u + \frac{3}{2}u^2 - \frac{1}{2}u^3 + \frac{1}{16}u^4\right)$
- Revert the change of variables $u = x + 1$

  $\left[y = \frac{a_0(x-1)^4}{16}\right]$
- Recursion relation for $r = 4$

  $a_{k+1} = \frac{a_k k}{2(k+5)}$
- Solution for $r = 4$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+4}, a_{k+1} = \frac{a_k k}{2(k+5)}\right]$
- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+4}, a_{k+1} = \frac{a_k k}{2(k+5)}\right]$
- Combine solutions and rename parameters

  $\left[y = \frac{a_0(x-1)^4}{16} + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+4}\right), b_{k+1} = \frac{b_k k}{2(k+5)}\right]$

1.2.3 Maple trace

```
Methods for second order ODEs:
```

1.2.4 Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 25

```
dsolve((x^2-1)*diff(diff(y(x),x),x)-6*x*diff(y(x),x)+12*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2 x^4 + c_1 x^3 + 6c_2 x^2 + c_1 x + c_2$$

1.2.5 Mathematica DSolve solution

Solving time : 0.185 (sec)
Leaf size : 45

```
DSolve[{(x^2-1)*D[y[x],{x,2}]-6*x*D[y[x],x]+12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{x^2-1}(c_2 x(x^2+1) + c_1(x-1)^4)}{\sqrt{1-x^2}}$$

1.3 problem 3

Internal problem ID [8141]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 3
Date solved : `Monday, October 21, 2024 at 04:54:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+3\right) y''-7xy'+16y=0$$

1.3.1 Solved as second order ode using Kovacic algorithm

Time used: 0.433 (sec)

Writing the ode as

$$\left(x^2+3\right) y''-7xy'+16y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+3 \\ B &= -7x \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 234}{4\left(x^2 + 3\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 - 234 \\ t &= 4\left(x^2 + 3\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 234}{4\left(x^2 + 3\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 5: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+3)^2$. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{77}{16\left(x - i\sqrt{3}\right)^2} + \frac{77}{16\left(x + i\sqrt{3}\right)^2} + \frac{79i\sqrt{3}}{48\left(x - i\sqrt{3}\right)} - \frac{79i\sqrt{3}}{48\left(x + i\sqrt{3}\right)}
$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4}
\end{aligned}
$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 234}{4\left(x^2+3\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 234}{4\left(x^2+3\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$i\sqrt{3}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$
$-i\sqrt{3}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{7}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{7}{4\left(x - i\sqrt{3}\right)} - \frac{7}{4\left(x + i\sqrt{3}\right)} + (-)(0) \\
&= -\frac{7}{4\left(x - i\sqrt{3}\right)} - \frac{7}{4\left(x + i\sqrt{3}\right)} \\
&= -\frac{7x}{2x^2 + 6}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6x a_3 + 2a_2\right) + 2\left(-\frac{7}{4\left(x - i\sqrt{3}\right)} - \frac{7}{4\left(x + i\sqrt{3}\right)} \right) \left(4x^3 + 3x^2 a_3 + 2x a_2 + a_1\right) + \left(\left(\frac{7}{4\left(x - i\sqrt{3}\right)^2} \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{27}{8}, a_1 = 0, a_2 = -9, a_3 = 0 \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 9x^2 + \frac{27}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 - 9x^2 + \frac{27}{8}\right) \mathrm{e}^{\int \left(-\frac{7}{4\left(x - i\sqrt{3}\right)} - \frac{7}{4\left(x + i\sqrt{3}\right)}\right) dx} \\
&= \left(x^4 - 9x^2 + \frac{27}{8}\right) \frac{1}{\left(\left(i\sqrt{3} - x\right)\left(x + i\sqrt{3}\right)\right)^{7/4}} \\
&= \frac{8x^4 - 72x^2 + 27}{8\left(-x^2 - 3\right)^{7/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-7x}{x^2+3} \, dx} \\
&= z_1 e^{\frac{7 \ln\left(x^2+3\right)}{4}} \\
&= z_1 \left(\left(x^2 + 3\right)^{7/4}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(\frac{1}{2} + \frac{i}{2}\right) \sqrt{2} \left(x^4 - 9x^2 + \frac{27}{8}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-7x}{x^2+3} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{7 \ln\left(x^2+3\right)}{2}}}{(y_1)^2} \, dx \\
&= y_1 (\text{Expression too large to display})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\left(x^4 - 9x^2 + \frac{27}{8}\right)\right) \\
&\quad + c_2\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2}\left(x^4 - 9x^2 + \frac{27}{8}\right)\text{(Expression too large to display)}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.3.2 Maple step by step solution

1.3.3 Maple trace

```
Methods for second order ODEs:
```

1.3.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 65

```
dsolve((x^2+3)*diff(diff(y(x),x),x)-7*x*diff(y(x),x)+16*y(x) = 0,
       y(x),singsol=all)
```

$$
\begin{aligned}
y = {} & 4\left(x^4 - 9x^2 + \frac{27}{8}\right) c_2 \ln\left(\sqrt{x^2+3} - x\right) \\
& + \frac{5(10x^3 - 33x)\, c_2 \sqrt{x^2+3}}{6} + \left(c_1 + \frac{25c_2}{3}\right)\left(x^4 - 9x^2 + \frac{27}{8}\right)
\end{aligned}
$$

1.3.5 Mathematica DSolve solution

Solving time : `0.752` (sec)
Leaf size : 492

```
DSolve[{(x^2+3)*D[y[x],{x,2}]-7*x*D[y[x],x]+16*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{24}c_2\Big(12960x^2\text{RootSum}\Big[7838208000\#1^4-188281584000\#1^2-241544908800\#1$$
$$+18453344881\&,\#1\log\Big(-411757211968704000\#1^3-166063274606980800\#1^2+1013870382516711139$$
$$+5248800x^2\text{RootSum}\Big[2108807205724800000000\#1^4-30882886815600000\#1^2$$
$$+97825688064000\#1$$
$$+18453344881\&,\#1\log\Big(273530830607325028080000000\#1^3-27238528617410025720000\#1^2-410617$$
$$-4860\text{RootSum}\Big[7838208000\#1^4-188281584000\#1^2-241544908800\#1$$
$$+18453344881\&,\#1\log\Big(-411757211968704000\#1^3-166063274606980800\#1^2+1013870382516711139$$
$$-1968300\text{RootSum}\Big[2108807205724800000000\#1^4-30882886815600000\#1^2$$
$$+97825688064000\#1$$
$$+18453344881\&,\#1\log\Big(273530830607325028080000000\#1^3-27238528617410025720000\#1^2-410617$$
$$-1440x^4\text{RootSum}\Big[7838208000\#1^4-188281584000\#1^2-241544908800\#1$$
$$+18453344881\&,\#1\log\Big(-411757211968704000\#1^3-166063274606980800\#1^2+1013870382516711139$$
$$-583200x^4\text{RootSum}\Big[2108807205724800000000\#1^4-30882886815600000\#1^2$$
$$+97825688064000\#1$$
$$+18453344881\&,\#1\log\Big(273530830607325028080000000\#1^3-27238528617410025720000\#1^2-410617$$
$$+165\sqrt{x^2+3}x+216x^2\log\left(\sqrt{x^2+3}-x\right)-81\log\left(\sqrt{x^2+3}-x\right)$$
$$-24x^4\log\left(\sqrt{x^2+3}-x\right)-50\sqrt{x^2+3}x^3\Big)+c_1\left(x^4-9x^2+\frac{27}{8}\right)$$

1.4 problem 4

Internal problem ID [8142]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 4
Date solved : `Monday, October 21, 2024 at 04:54:23 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^{2}-1\right) y''+8 x y'+12 y = 0$$

1.4.1 Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$\left(x^{2}-1\right) y''+8 x y'+12 y = 0 \tag{1}$$
$$A y''+B y'+C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}-1\\ B &= 8 x\\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2 A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 8$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 6: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x-1} + \frac{2}{(x-1)^2} + \frac{2}{(x+1)^2} + \frac{2}{x+1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8}{\left(x^2-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	2	-1
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x-1} + \frac{2}{x+1} + (-)(0) \\ &= -\frac{1}{x-1} + \frac{2}{x+1} \\ &= \frac{x-3}{x^2-1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-1} + \frac{2}{x+1}\right)(0) + \left(\left(\frac{1}{(x-1)^2} - \frac{2}{(x+1)^2}\right) + \left(-\frac{1}{x-1} + \frac{2}{x+1}\right)^2 - \left(\frac{8}{(x^2-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{x-1} + \frac{2}{x+1}\right) dx} \\ &= \frac{(x+1)^2}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x}{x^2-1}\, dx} \\ &= z_1 e^{-2\ln(x-1) - 2\ln(x+1)} \\ &= z_1 \left(\frac{1}{(x-1)^2 (x+1)^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(x-1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x}{x^2-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x-1)-4\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(x+1)\,(3x^2+1)\,(x-1)^4\,\mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{(x-1)^3} \right) + c_2 \left(\frac{1}{(x-1)^3} \left(-\frac{(x+1)\,(3x^2+1)\,(x-1)^4\,\mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.4.2 Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d}{dx}y'\right) + 8xy' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{12y}{x^2-1} - \frac{8xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{8xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{8x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.((x+1)\cdot P_2(x))\right|_{x=-1} = 4$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d}{dx}y'\right) + 8xy' + 12y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (8u-8)\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0 r(3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r+4) + a_k(k+r+4)(k+r+3)\right) u^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+4)\left((-2k-2r-2)\, a_{k+1} + a_k(k+r+3)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+3)}{2(k+1+r)}$

- Recursion relation for $r = -3$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$

- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+3)}{2(k+1)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

1.4.3 Maple trace

```
Methods for second order ODEs:
```

1.4.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 29

```
dsolve((x^2-1)*diff(diff(y(x),x),x)+8*x*diff(y(x),x)+12*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\, x^3 + 3c_1\, x^2 + 3c_2 x + c_1}{(x^2 - 1)^3}$$

1.4.5 Mathematica DSolve solution

Solving time : 0.072 (sec)
Leaf size : 37

```
DSolve[{(x^2-1)*D[y[x],{x,2}]+8*x*D[y[x],x]+12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{3c_1(x-1)^3 - c_2(3x^2+1)}{3\left(x^2-1\right)^3}$$

1.5 problem 5

Internal problem ID [8143]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 5
Date solved : `Monday, October 21, 2024 at 04:54:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3y'' + xy' - 4y = 0$$

1.5.1 Solved as second order ode using Kovacic algorithm

Time used: 0.246 (sec)

Writing the ode as

$$3y'' + xy' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 54}{36} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 54 \\ t &= 36 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{36} + \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 8: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} + \frac{9}{2x} - \frac{243}{4x^3} + \frac{6561}{4x^5} - \frac{885735}{16x^7} + \frac{33480783}{16x^9} - \frac{2711943423}{32x^{11}} + \frac{115063885233}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+54}{36} \\ &= Q + \frac{R}{36} \\ &= \left(\frac{x^2}{36}+\frac{3}{2}\right) + (0) \\ &= \frac{x^2}{36}+\frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{3}{2}\right) - (0) \\ &= \frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{6}} - 1\right) &&= 4 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{6}} - 1\right) &&= -5 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{36}+\frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{6}$	4	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{6}\right) \\ &= \frac{x}{6} \\ &= \frac{x}{6} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{x}{6}\right)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{1}{6}\right) + \left(\frac{x}{6}\right)^2 - \left(\frac{x^2}{36} + \frac{3}{2}\right)\right) = 0$$

$$-\frac{a_3x^3}{3} + \frac{2(18 - a_2)\,x^2}{3} + (-a_1 + 6a_3)\,x - \frac{4a_0}{3} + 2a_2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 27, a_1 = 0, a_2 = 18, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 + 18x^2 + 27$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 + 18x^2 + 27\right) \mathrm{e}^{\int \frac{x}{6} dx} \\
&= \left(x^4 + 18x^2 + 27\right) \mathrm{e}^{\frac{x^2}{12}} \\
&= \left(x^4 + 18x^2 + 27\right) \mathrm{e}^{\frac{x^2}{12}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{3} \, dx} \\
&= z_1 e^{-\frac{x^2}{12}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{12}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^4 + 18x^2 + 27$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{3} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{\mathrm{e}^{-\frac{x^2}{6}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{6}}}{\left(x^4 + 18x^2 + 27\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 + 18x^2 + 27\right) + c_2 \left(x^4 + 18x^2 + 27 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{6}}}{\left(x^4 + 18x^2 + 27\right)^2} dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.5.2 Maple trace

```
Methods for second order ODEs:
```

1.5.3 Maple dsolve solution

Solving time : 0.024 (sec)
Leaf size : 47

```
dsolve(3*diff(diff(y(x),x),x)+x*diff(y(x),x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y = xc_1(x^2 + 15)\sqrt{6}\,\mathrm{e}^{-\frac{x^2}{6}} + (x^4 + 18x^2 + 27)\left(\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{6}\,x}{6}\right)c_1 + c_2\right)$$

1.5.4 Mathematica DSolve solution

Solving time : 0.033 (sec)
Leaf size : 43

```
DSolve[{3*D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{6}}\,\mathrm{HermiteH}\left(-5, \frac{x}{\sqrt{6}}\right) + \frac{1}{27}c_2(x^4 + 18x^2 + 27)$$

1.6 problem 6

Internal problem ID [8144]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 6
Date solved : `Monday, October 21, 2024 at 04:54:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$5y'' - 2xy' + 10y = 0$$

1.6.1 Solved as second order ode using Kovacic algorithm

Time used: 0.288 (sec)

Writing the ode as

$$5y'' - 2xy' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 5 \\ B &= -2x \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 55}{25} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 55 \\ t &= 25 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{25} - \frac{11}{5}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 9: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{5} - \frac{11}{2x} - \frac{605}{8x^3} - \frac{33275}{16x^5} - \frac{9150625}{128x^7} - \frac{704598125}{256x^9} - \frac{116258690625}{1024x^{11}} - \frac{10048072546875}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{5}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{5} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{25}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 55}{25} \\
&= Q + \frac{R}{25} \\
&= \left(\frac{x^2}{25} - \frac{11}{5}\right) + (0) \\
&= \frac{x^2}{25} - \frac{11}{5}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{11}{5}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{11}{5}\right) - (0) \\
&= -\frac{11}{5}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{5} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{11}{5}}{\frac{1}{5}} - 1\right) &&= -6 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{11}{5}}{\frac{1}{5}} - 1\right) &&= 5
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{25} - \frac{11}{5}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{5}$	-6	5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 5$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 5 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{5}\right) \\ &= -\frac{x}{5} \\ &= -\frac{x}{5} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(20x^3 + 12x^2 a_4 + 6x a_3 + 2a_2\right) + 2\left(-\frac{x}{5}\right)\left(5x^4 + 4x^3 a_4 + 3x^2 a_3 + 2x a_2 + a_1\right) + \left(\left(-\frac{1}{5}\right) + \left(-\frac{x}{5}\right)^2 - \left(\frac{x^2}{25}\right.\right.$$

$$\frac{2a_4 x^4}{5} + \frac{4(25 + a_3)\, x^3}{5} + \frac{6(a_2 + 10a_4)\, x^2}{5} + \frac{2(4a_1 + 15a_3)\, x}{5} + 2a_0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = \frac{375}{4}, a_2 = 0, a_3 = -25, a_4 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^5 - 25x^3 + \frac{375}{4}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= \left(x^5 - 25x^3 + \frac{375}{4}x\right) \mathrm{e}^{\int -\frac{x}{5} dx} \\
&= \left(x^5 - 25x^3 + \frac{375}{4}x\right) \mathrm{e}^{-\frac{x^2}{10}} \\
&= \frac{(4x^5 - 100x^3 + 375x)\, \mathrm{e}^{-\frac{x^2}{10}}}{4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{5}\, dx} \\
&= z_1 e^{\frac{x^2}{10}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{10}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^5 - 25x^3 + \frac{375}{4}x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{5}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{5}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{\frac{x^2}{5}}}{\left(x^5 - 25x^3 + \frac{375}{4}x\right)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^5 - 25x^3 + \frac{375}{4}x \right) + c_2 \left(x^5 - 25x^3 + \frac{375}{4}x \left(\int \frac{e^{\frac{x^2}{5}}}{\left(x^5 - 25x^3 + \frac{375}{4}x\right)^2} dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.6.2 Maple step by step solution

Let's solve

$5\frac{d}{dx}y' - 2xy' + 10y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{2xy'}{5} - 2y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2xy'}{5} + 2y = 0$

- Multiply by denominators

  $5\frac{d}{dx}y' - 2xy' + 10y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- ◦ Convert $\frac{d}{dx}y'$ to series expansion

$$\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- ◦ Shift index using $k- >k+2$

$$\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(5a_{k+2}(k+2)\,(k+1) - 2a_k(k-5)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$5(k^2+3k+2)\, a_{k+2} - 2a_k(k-5) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2a_k(k-5)}{5(k^2+3k+2)}\right]$$

1.6.3 Maple trace

```
Methods for second order ODEs:
```

1.6.4 Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 31

```
dsolve(5*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+10*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2 \operatorname{hypergeom}\left(\left[-\frac{5}{2}\right], \left[\frac{1}{2}\right], \frac{x^2}{5}\right) + \frac{4\left(x^4 - 25x^2 + \frac{375}{4}\right) c_1 x}{375}$$

1.6.5 Mathematica DSolve solution

Solving time : `0.197` (sec)
Leaf size : 138

```
DSolve[{5*D[y[x],{x,2}]-2*x*D[y[x],x]+10*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{200}\sqrt{\frac{\pi}{5}}c_2\sqrt{x^2}\left(4x^4-100x^2+375\right)\operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{5}}\right)+\frac{32c_1x^5}{25\sqrt{5}} - \frac{32c_1x^3}{\sqrt{5}}-\frac{9}{20}c_2e^{\frac{x^2}{5}}x^2+c_2e^{\frac{x^2}{5}}+\frac{1}{50}c_2e^{\frac{x^2}{5}}x^4+24\sqrt{5}c_1x$$

1.7 problem 7

Internal problem ID [8145]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 7
Date solved : `Monday, October 21, 2024 at 04:54:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2 y' - 3xy = 0$$

1.7.1 Solved as second order ode using Kovacic algorithm

Time used: 0.298 (sec)

Writing the ode as

$$y'' - x^2 y' - 3xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= -3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3 + 8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x(x^3 + 8)$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3 + 8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 11: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} + \frac{2}{x} - \frac{4}{x^4} + \frac{16}{x^7} - \frac{80}{x^{10}} + \frac{448}{x^{13}} - \frac{2688}{x^{16}} + \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x(x^3+8)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^4 + 2x\right) + (0) \\ &= \frac{1}{4}x^4 + 2x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^2}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 2\right) &= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 2\right) &= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3+8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x^2}{2}\right) \\ &= \frac{x^2}{2} \\ &= \frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{x^2}{2}\right)(1) + \left((x) + \left(\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3+8)}{4}\right)\right) = 0$$

$$-xa_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x)\, e^{\int \frac{x^2}{2} dx} \\ &= (x)\, e^{\frac{x^3}{6}} \\ &= x\, e^{\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{1} \, dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1 \left(e^{\frac{x^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{x^3}{3}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{e^{-\frac{x^3}{3}}}{x^2} dx \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x^3}{3}} x\right) + c_2 \left(\mathrm{e}^{\frac{x^3}{3}} x \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.7.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - x^2 y' - 3xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

- ○ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

- ○ Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

- ○ Shift index using $k- >k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k+2$

$\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\,x^{k}$

Rewrite ODE with series expansions

$2a_{2}+\left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1)-a_{k-1}(k+2)\right) x^{k}\right) = 0$

- Each term must be 0

  $2a_{2} = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\left(k a_{k+2}-a_{k-1}+a_{k+2}\right) = 0$

- Shift index using $k->k+1$

  $(k+3)\left((k+1)\,a_{k+3}-a_{k}+a_{k+3}\right) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_{k} x^{k}, a_{k+3} = \frac{a_{k}}{k+2}, 2a_{2} = 0\right]$

1.7.3 Maple trace

```
Methods for second order ODEs:
```

1.7.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 58

```
dsolve(diff(diff(y(x),x),x)-x^2*diff(y(x),x)-3*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{9\,\mathrm{WhittakerM}\left(\frac{1}{3}, \frac{5}{6}, \frac{x^{3}}{3}\right) \mathrm{e}^{\frac{x^{3}}{6}} c_{2}\, x^{3}+9 c_{1}\, x^{2} \mathrm{e}^{\frac{x^{3}}{3}}+5\, 3^{2/3} c_{2} \left(x^{3}\right)^{1/3} \left(x^{3}+2\right)}{9 x}$$

1.7.5 Mathematica DSolve solution

Solving time : `0.21` (sec)
Leaf size : 51

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]-3*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{9} e^{\frac{x^3}{3}} \left(9 c_1 x - 3^{2/3} c_2 \sqrt[3]{x^3} \Gamma\left(-\frac{1}{3}, \frac{x^3}{3}\right)\right)$$

1.8 problem 8

Internal problem ID [8146]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 8
Date solved : `Monday, October 21, 2024 at 04:54:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''+2 x y'-2 y=0$$

1.8.1 Solved as second order ode using Kovacic algorithm

Time used: 0.300 (sec)

Writing the ode as

$$\left(x^2+1\right) y''+2 x y'-2 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 + 3}{(x^2 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 + 3$$
$$t = (x^2 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 + 3}{(x^2 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 13: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{5i}{4(x-i)} + \frac{5i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+3}{(x^2+1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2+3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x - 2i} + \frac{1}{2x + 2i} + (0) \\
&= \frac{1}{2x - 2i} + \frac{1}{2x + 2i} \\
&= \frac{x}{x^2 + 1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right)(1) + \left(\left(-\frac{1}{2(x - i)^2} - \frac{1}{2(x + i)^2}\right) + \left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right)^2 - \left(\frac{2x^2 + 3}{(x^2 + 1)^2} \right.\right.
$$
$$
- \frac{2(x^2 + 1)\, a_0}{(-x + i)^2 (x + i)^2}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right) dx} \\
&= (x) \sqrt{(-x + i)(x + i)} \\
&= x\sqrt{-x^2 - 1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x^2+1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = ix$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\arctan(x) + \frac{1}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(ix) + c_2\left(ix\left(\arctan(x) + \frac{1}{x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.8.2 Maple step by step solution

1.8.3 Maple trace

```
Methods for second order ODEs:
```

1.8.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+2*x*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + \arctan(x) x c_2 + c_2$$

1.8.5 Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 48

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+2*x*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} i(2c_1 x - c_2 x \log(1 - ix) + c_2 x \log(1 + ix) + 2ic_2)$$

1.9 problem 9

Internal problem ID [8147]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 9
Date solved : `Monday, October 21, 2024 at 04:54:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' - 2y = 0$$

1.9.1 Solved as second order ode using Kovacic algorithm

Time used: 0.225 (sec)

Writing the ode as

$$y'' + xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 10$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 14: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{5}{2x} - \frac{25}{4x^3} + \frac{125}{4x^5} - \frac{3125}{16x^7} + \frac{21875}{16x^9} - \frac{328125}{32x^{11}} + \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} + \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} + \frac{5}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{5}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{5}{2}\right) - (0) \\
&= \frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	2	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 2$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^{+} \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{x}{2}\right)(2x + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{5}{2}\right)\right) = 0$$

$$-a_1 x - 2a_0 + 2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x^2+1)\, \mathrm{e}^{\int \frac{x}{2} dx} \\ &= (x^2+1)\, \mathrm{e}^{\frac{x^2}{4}} \\ &= (x^2+1)\, \mathrm{e}^{\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1} \, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 + 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{(x^2+1)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(x^2+1) + c_2\left(x^2+1\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.9.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions
  - ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$
  - ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - ◦ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(k-2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} + a_k(k-2) = 0$

- Recursion relation; series terminates at $k=2$

  $a_{k+2} = -\frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k=0$

$a_2 = a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

  $y = A_2x^2 + A_1x + a_0$

1.9.3 Maple trace

```
Methods for second order ODEs:
```

1.9.4 Maple dsolve solution

Solving time : 0.079 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{-\frac{x^2}{2}}\sqrt{2}\, c_1 x + \left(x^2+1\right)\left(\sqrt{\pi}\, \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) c_1 + c_2\right)$$

1.9.5 Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 35

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow c_1 e^{-\frac{x^2}{2}} \text{HermiteH}\left(-3, \frac{x}{\sqrt{2}}\right) + c_2\left(x^2+1\right)$$

1.10 problem 10

Internal problem ID [8148]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 10
Date solved : `Monday, October 21, 2024 at 04:54:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-6x+10\right)y''-4(x-3)\,y'+6y=0$$

1.10.1 Solved as second order ode using Kovacic algorithm

Time used: 0.306 (sec)

Writing the ode as

$$\left(x^2-6x+10\right)y''+(-4x+12)\,y'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2-6x+10\\B&=-4x+12\\C&=6\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-8}{(x^2 - 6x + 10)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -8$$
$$t = (x^2 - 6x + 10)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{8}{(x^2 - 6x + 10)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 16: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 6x + 10)^2$. There is a pole at $x = 3 + i$ of order 2. There is a pole at $x = 3 - i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x - 3 - i)^2} + \frac{2}{(x - 3 + i)^2} + \frac{2i}{x - 3 - i} - \frac{2i}{x - 3 + i}$$

For the pole at $x = 3 + i$ let b be the coefficient of $\frac{1}{(x-3-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

For the pole at $x = 3 - i$ let b be the coefficient of $\frac{1}{(x-3+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{8}{(x^2 - 6x + 10)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$3 + i$	2	0	2	-1
$3 - i$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x - 3 - i} + \frac{2}{x - 3 + i} + (-)(0) \\ &= -\frac{1}{x - 3 - i} + \frac{2}{x - 3 + i} \\ &= \frac{x - 3 - 3i}{x^2 - 6x + 10} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right)(0) + \left(\left(\frac{1}{(x-3-i)^2} - \frac{2}{(x-3+i)^2}\right) + \left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right)^2\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right) dx} \\ &= \frac{(x^2 - 6x + 10)^2}{(ix - 3i + 1)^3} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x+12}{x^2-6x+10} \, dx} \\ &= z_1 e^{\ln(x^2-6x+10)} \\ &= z_1 \left(x^2 - 6x + 10\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2 - 6x + 10)^3}{(ix - 3i + 1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-4x+12}{x^2-6x+10}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{2\ln(x^2-6x+10)}}{(y_1)^2}\,dx$$
$$= y_1 \left(\frac{x^2-6x+\frac{26}{3}}{(x-3+i)^3}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{(x^2-6x+10)^3}{(ix-3i+1)^3}\right) + c_2 \left(\frac{(x^2-6x+10)^3}{(ix-3i+1)^3} \left(\frac{x^2-6x+\frac{26}{3}}{(x-3+i)^3}\right)\right)$$

Will add steps showing solving for IC soon.

1.10.2 Maple step by step solution

1.10.3 Maple trace

```
Methods for second order ODEs:
```

1.10.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 31

```
dsolve((x^2-6*x+10)*diff(diff(y(x),x),x)-4*(x-3)*diff(y(x),x)+6*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1\, x^3 + c_2\, x^2 + 6(-5c_1 - c_2)\, x + 60c_1 + \frac{26c_2}{3}$$

1.10.5 Mathematica DSolve solution

Solving time : 0.131 (sec)
Leaf size : 36

```
DSolve[{(x^2-6*x+10)*D[y[x],{x,2}]-4*(x-3)*D[y[x],x]+6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{3}i\left(c_2\left(3x^2 - 18x + 26\right) + 3c_1(x - (3 + i))^3\right)$$

1.11 problem 11

Internal problem ID [8149]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 11
Date solved : `Monday, October 21, 2024 at 04:54:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+6x\right)y''+(3x+9)\,y'-3y=0$$

1.11.1 Solved as second order ode using Kovacic algorithm

Time used: 0.233 (sec)

Writing the ode as

$$\left(x^2+6x\right)y''+(3x+9)\,y'-3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+6x \\ B &= 3x+9 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 + 90x - 27}{4\left(x^2 + 6x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15x^2 + 90x - 27$$
$$t = 4\left(x^2 + 6x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 + 90x - 27}{4\left(x^2 + 6x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 17: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 6x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -6$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} - \frac{3}{16(x+6)^2} + \frac{11}{16x} - \frac{11}{16(x+6)}$$

For the pole at $x = -6$ let b be the coefficient of $\frac{1}{(x+6)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 + 90x - 27}{4(x^2+6x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{15x^2+90x-27}{4\left(x^2+6x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-6	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right)\\
&= \frac{5}{2}-\left(\frac{3}{2}\right)\\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4\left(x+6\right)} + \frac{3}{4x} + (0) \\
&= \frac{3}{4\left(x+6\right)} + \frac{3}{4x} \\
&= \frac{\frac{3x}{2} + \frac{9}{2}}{x\left(x+6\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4\left(x+6\right)} + \frac{3}{4x}\right)(1) + \left(\left(-\frac{3}{4\left(x+6\right)^2} - \frac{3}{4x^2}\right) + \left(\frac{3}{4\left(x+6\right)} + \frac{3}{4x}\right)^2 - \left(\frac{15x^2 + 90x - 27}{4\left(x^2+6x\right)^2}\right)\right) = \frac{9 - 3a_0}{x\left(x+6\right)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 3\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 3
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x+3\right) \mathrm{e}^{\int \left(\frac{3}{4(x+6)} + \frac{3}{4x}\right) dx} \\
&= \left(x+3\right) \left(x(x+6)\right)^{3/4} \\
&= \left(x+3\right) \left(x(x+6)\right)^{3/4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x+9}{x^2+6x}\,dx} \\
&= z_1 e^{-\frac{3\ln(x(x+6))}{4}} \\
&= z_1 \left(\frac{1}{\left(x\left(x+6\right)\right)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x+3
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x+9}{x^2+6x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x(x+6))}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{\left(x+6\right)x(2x^2+12x+9)}{81\left(x+3\right)\left(x\left(x+6\right)\right)^{3/2}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x+3) + c_2\left(x+3\left(-\frac{\left(x+6\right)x(2x^2+12x+9)}{81\left(x+3\right)\left(x\left(x+6\right)\right)^{3/2}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.11.2 Maple step by step solution

Let's solve

$(x^2+6x)\left(\frac{d}{dx}y'\right)+(3x+9)\,y'-3y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{3y}{x(x+6)}-\frac{3(x+3)y'}{x(x+6)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{3(x+3)y'}{x(x+6)}-\frac{3y}{x(x+6)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3(x+3)}{x(x+6)},P_3(x)=-\frac{3}{x(x+6)}\right]$

  ◦ $(x+6)\cdot P_2(x)$ is analytic at $x=-6$

  $\left((x+6)\cdot P_2(x)\right)\Big|_{x=-6}=\frac{3}{2}$

  ◦ $(x+6)^2\cdot P_3(x)$ is analytic at $x=-6$

  $\left((x+6)^2\cdot P_3(x)\right)\Big|_{x=-6}=0$

  ◦ $x=-6$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-6$

- Multiply by denominators

$x(x+6)\left(\frac{d}{dx}y'\right)+(3x+9)\,y'-3y=0$

- Change variables using $x=u-6$ so that the regular singular point is at $u=0$

$(u^2-6u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(3u-9)\left(\frac{d}{du}y(u)\right)-3y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-3a_0r(1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} (-3a_{k+1}(k+1+r)\,(2k+3+2r) + a_k(k+r+3)\,(k+r-1))\, u^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-3r(1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-6(k+1+r)\left(k+\frac{3}{2}+r\right) a_{k+1} + a_k(k+r+3)\,(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r+3)(k+r-1)}{3(k+1+r)(2k+3+2r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$$a_{k+1} = \frac{a_k(k+3)(k-1)}{3(k+1)(2k+3)}$$

- Apply recursion relation for $k = 0$

$$a_1 = -\frac{a_0}{3}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot \left(1 - \frac{u}{3}\right)$$

- Revert the change of variables $u = x + 6$

$$\left[y = a_0\left(-1 - \frac{x}{3}\right)\right]$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

- Revert the change of variables $u = x + 6$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+6)^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k \left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0\left(-1-\frac{x}{3}\right) + \left(\sum_{k=0}^{\infty} b_k (x+6)^{k-\frac{1}{2}}\right), b_{k+1} = \frac{b_k \left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

1.11.3 Maple trace

```
Methods for second order ODEs:
```

1.11.4 Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 30

```
dsolve((x^2+6*x)*diff(diff(y(x),x),x)+(3*x+9)*diff(y(x),x)-3*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1(x+3) + \frac{c_2(2x^2+12x+9)}{\sqrt{x}\,\sqrt{x+6}}$$

1.11.5 Mathematica DSolve solution

Solving time : `0.114` (sec)
Leaf size : 82

```
DSolve[{(x^2+6*x)*D[y[x],{x,2}]+(3*x+9)*D[y[x],x]-3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{9\sqrt{\pi}c_2\sqrt[4]{-x(x+6)}Q_{\frac{3}{2}}^{\frac{1}{2}}\left(\frac{x}{3}+1\right) + \sqrt{6}c_1(2x^2+12x+9)}{9\sqrt{\pi}\sqrt[4]{-x^2}\sqrt{x+6}}$$

1.12 problem 12

Internal problem ID [8150]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 12
Date solved : `Monday, October 21, 2024 at 04:54:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$ty'' + \left(t^2 - 1\right) y' + t^2 y = 0$$

1.12.1 Solved as second order ode using Kovacic algorithm

Time used: 0.322 (sec)

Writing the ode as

$$ty'' + \left(t^2 - 1\right) y' + t^2 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= t^2 - 1 \\ C &= t^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{t^4 - 4t^3 + 3}{4t^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= t^4 - 4t^3 + 3 \\
t &= 4t^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{t^4 - 4t^3 + 3}{4t^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 19: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - t + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{1} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - 1 - \frac{1}{t} - \frac{2}{t^2} - \frac{17}{4t^3} - \frac{25}{2t^4} - \frac{75}{2t^5} - \frac{117}{t^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= -1 + \frac{t}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1 - t + \frac{1}{4}t^2$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^4 - 4t^3 + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}t^2 - t\right) + \left(\frac{3}{4t^2}\right) \\ &= \frac{t^2}{4} - t + \frac{3}{4t^2} \end{aligned}$$

We see that the coefficient of the term t in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (1) \\ &= -1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= -1 + \frac{t}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) &= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^4 - 4t^3 + 3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$-1 + \frac{t}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2t} + (-)\left(-1+\frac{t}{2}\right) \\
&= -\frac{1}{2t} + 1 - \frac{t}{2} \\
&= -\frac{(t-1)^2}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = t + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{2t} + 1 - \frac{t}{2}\right)(1) + \left(\left(\frac{1}{2t^2} - \frac{1}{2}\right) + \left(-\frac{1}{2t} + 1 - \frac{t}{2}\right)^2 - \left(\frac{t^4 - 4t^3 + 3}{4t^2}\right)\right) = 0
$$

$$
\frac{(a_0+1)(t-1)}{t} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -1\}
$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$
p(t) = t - 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= (t-1)\,\mathrm{e}^{\int \left(-\frac{1}{2t} + 1 - \frac{t}{2}\right) dt} \\
&= (t-1)\,\mathrm{e}^{-\frac{t^2}{4} + t - \frac{\ln(t)}{2}} \\
&= \frac{(t-1)\,\mathrm{e}^{-\frac{t(t-4)}{4}}}{\sqrt{t}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2-1}{t}\,dt} \\
&= z_1 e^{-\frac{t^2}{4}+\frac{\ln(t)}{2}} \\
&= z_1\left(\sqrt{t}\,\mathrm{e}^{-\frac{t^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (t-1)\,\mathrm{e}^{-\frac{t(t-2)}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-1}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\frac{t^2}{2}+\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{t^2}{2}+\ln(t)}\mathrm{e}^{t(t-2)}}{(t-1)^2}dt\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left((t-1)\,\mathrm{e}^{-\frac{t(t-2)}{2}}\right) + c_2\left((t-1)\,\mathrm{e}^{-\frac{t(t-2)}{2}}\left(\int \frac{\mathrm{e}^{-\frac{t^2}{2}+\ln(t)}\mathrm{e}^{t(t-2)}}{(t-1)^2}dt\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.12.2 Maple step by step solution

Let's solve

$t\left(\frac{d}{dt}y'\right)+\left(t^{2}-1\right)y'+t^{2}y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dt}y'$

- Isolate 2nd derivative

$\frac{d}{dt}y'=-yt-\frac{\left(t^{2}-1\right)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y'+\frac{\left(t^{2}-1\right)y'}{t}+yt=0$

□ Check to see if $t_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(t)=\frac{t^{2}-1}{t}, P_3(t)=t\right]$

- ◦ $t\cdot P_2(t)$ is analytic at $t=0$

  $\left(t\cdot P_2(t)\right)\Big|_{t=0}=-1$

- ◦ $t^{2}\cdot P_3(t)$ is analytic at $t=0$

  $\left(t^{2}\cdot P_3(t)\right)\Big|_{t=0}=0$

- ◦ $t=0$is a regular singular point

  Check to see if $t_0=0$ is a regular singular point

  $t_0=0$

- Multiply by denominators

$t\left(\frac{d}{dt}y'\right)+\left(t^{2}-1\right)y'+t^{2}y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k t^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $t^{2}\cdot y$ to series expansion

  $t^{2}\cdot y=\sum_{k=0}^{\infty}a_k t^{k+r+2}$

- ◦ Shift index using $k->k-2$

  $t^{2}\cdot y=\sum_{k=2}^{\infty}a_{k-2} t^{k+r}$

- ◦ Convert $t^{m}\cdot y'$ to series expansion for $m=0..2$

$$t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- Convert $t \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

  $$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r-1}$$

- Shift index using $k- >k+1$

  $$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r)\, t^{-1+r} + a_1(1+r)\,(-1+r)\, t^r + (a_2(2+r)\, r + a_0 r)\, t^{1+r} + \left(\sum_{k=2}^{\infty} (a_{k+1}(k+1+r)\,(k+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- The coefficients of each power of t must be 0

  $[a_1(1+r)\,(-1+r) = 0, a_2(2+r)\, r + a_0 r = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = -\frac{a_0}{2+r}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r-1) + a_{k-1}(k+r-1) + a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+3}(k+3+r)\,(k+1+r) + a_{k+1}(k+1+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+3} = -\frac{k a_{k+1} + r a_{k+1} + a_k + a_{k+1}}{(k+3+r)(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+3} = -\frac{k a_{k+1} + a_k + a_{k+1}}{(k+3)(k+1)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+3} = -\frac{k a_{k+1} + a_k + a_{k+1}}{(k+3)(k+1)}, a_1 = 0, a_2 = -\frac{a_0}{2}\right]$

- Recursion relation for $r = 2$

  $a_{k+3} = -\frac{k a_{k+1} + a_k + 3 a_{k+1}}{(k+5)(k+3)}$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+3} = -\frac{k a_{k+1} + a_k + 3 a_{k+1}}{(k+5)(k+3)}, a_1 = 0, a_2 = -\frac{a_0}{4}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+3} = -\frac{k a_{k+1} + a_k + a_{k+1}}{(k+3)(k+1)}, a_1 = 0, a_2 = -\frac{a_0}{2}, b_{k+3} = -\frac{k b_{k+1} + b_k + 3 b_k}{(k+5)(k+3)}$$

1.12.3 Maple trace

```
Methods for second order ODEs:
```

1.12.4 Maple dsolve solution

Solving time : `0.014` (sec)
Leaf size : 82

```
dsolve(t*diff(diff(y(t),t),t)+(t^2-1)*diff(y(t),t)+t^2*y(t) = 0,
       y(t),singsol=all)
```

$$y = -\frac{\sqrt{2}\,\mathrm{e}^{-\frac{t(t-2)}{2}}\left(-c_2\sqrt{\pi}\,(t-1)\,(t-2)\,\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{-(t-2)^2}}{2}\right) + \sqrt{2}\,\sqrt{-(t-2)^2}\,\left(c_2\,\mathrm{e}^{\frac{(t-2)^2}{2}} - t c_1 + c_1\right)\right)}{2\sqrt{-(t-2)^2}}$$

1.12.5 Mathematica DSolve solution

Solving time : `0.721` (sec)
Leaf size : 70

```
DSolve[{t*D[y[t],{t,2}]+(t^2-1)*D[y[t],t]+t^2*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{1}{2} e^{-\frac{t^2}{2}+t-2}\left(\sqrt{2\pi} c_2 (t-1) \mathrm{erfi}\left(\frac{t-2}{\sqrt{2}}\right) + 2e^2 c_1 (t-1) - 2 c_2 e^{\frac{1}{2}(t-2)^2}\right)$$

1.13 problem 13

Internal problem ID [8151]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 13
Date solved : `Monday, October 21, 2024 at 04:54:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2 y'' - t(t+2)\, y' + (t+2)\, y = 0$$

1.13.1 Solved as second order ode using Kovacic algorithm

Time used: 0.102 (sec)

Writing the ode as

$$t^2 y'' + \left(-t^2 - 2t\right) y' + (t+2)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 - 2t \\ C &= t + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 21: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-\frac{t}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t^2-2t}{t^2}\,dt} \\
&= z_1 e^{\frac{t}{2}+\ln(t)} \\
&= z_1 \left(t\,\mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-2t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{t+2\ln(t)}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{\mathrm{e}^{t+2\ln(t)}}{t^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(t) + c_2 \left(t \left(\frac{e^{t+2\ln(t)}}{t^2} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.13.2 Maple step by step solution

Let's solve
$t^2\left(\frac{d}{dt}y'\right) - t(t+2)\,y' + (t+2)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dt}y'$
- Isolate 2nd derivative
  $\frac{d}{dt}y' = -\frac{(t+2)y}{t^2} + \frac{(t+2)y'}{t}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dt}y' - \frac{(t+2)y'}{t} + \frac{(t+2)y}{t^2} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = -\frac{t+2}{t}, P_3(t) = \frac{t+2}{t^2}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$
    $\left.(t \cdot P_2(t))\right|_{t=0} = -2$
  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$
    $\left.(t^2 \cdot P_3(t))\right|_{t=0} = 2$
  - $t = 0$is a regular singular point
    Check to see if $t_0 = 0$ is a regular singular point
    $t_0 = 0$

- Multiply by denominators
  $t^2\left(\frac{d}{dt}y'\right) - t(t+2)\,y' + (t+2)\,y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $t^m \cdot y$ to series expansion for $m = 0..1$

  $t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$

- Shift index using $k- >k - m$

  $t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$

- Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t^2 \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

  $t^2 \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-2) - a_{k-1}(k+r-2)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_k(k+r-1) - a_{k-1}) = 0$

- Shift index using $k- >k + 1$

  $(k+r-1)\,(a_{k+1}(k+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+r}$

- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+2}$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+2}\right]$$

1.13.3 Maple trace

```
Methods for second order ODEs:
```

1.13.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve(t^2*diff(diff(y(t),t),t)-t*(t+2)*diff(y(t),t)+(t+2)*y(t) = 0,
       y(t),singsol=all)
```

$$y = t\left(c_1 + c_2\, e^{t}\right)$$

1.13.5 Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 16

```
DSolve[{t^2*D[y[t],{t,2}]-t*(t+2)*D[y[t],t]+(t+2)*y[t] == 0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to t\left(c_2 e^t + c_1\right)$$

1.14 problem 14

Internal problem ID [8152]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 14
Date solved : `Monday, October 21, 2024 at 04:54:32 PM`
CAS classification : `[_Laguerre]`

Solve

$$ty'' - (1+t)\,y' + y = 0$$

1.14.1 Solved as second order ode using Kovacic algorithm

Time used: 0.240 (sec)

Writing the ode as

$$ty'' + (-1-t)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -1-t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 2t + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 23: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2t} \\ &= \frac{t-1}{2t}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2-2t+3}{4t^2}\right) \right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\ &= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1-t}{t}\, dt} \\ &= z_1 e^{\frac{t}{2} + \frac{\ln(t)}{2}} \\ &= z_1 \left(\sqrt{t}\, \mathrm{e}^{\frac{t}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1-t}{t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\, dt \\ &= y_1 \left(-\frac{(1+t)\,\mathrm{e}^{t+\ln(t)}\mathrm{e}^{-2t}}{t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^t\right) + c_2 \left(\mathrm{e}^t \left(-\frac{(1+t)\,\mathrm{e}^{t+\ln(t)}\mathrm{e}^{-2t}}{t}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.14.2 Maple step by step solution

Let's solve

$t\left(\frac{d}{dt}y'\right) - (1+t)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{y}{t} + \frac{(1+t)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' - \frac{(1+t)y'}{t} + \frac{y}{t} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

- Define functions
  $\left[P_2(t) = -\frac{1+t}{t}, P_3(t) = \frac{1}{t}\right]$
- $t \cdot P_2(t)$ is analytic at $t = 0$
  $(t \cdot P_2(t))\bigg|_{t=0} = -1$
- $t^2 \cdot P_3(t)$ is analytic at $t = 0$
  $(t^2 \cdot P_3(t))\bigg|_{t=0} = 0$
- $t = 0$is a regular singular point
  Check to see if $t_0 = 0$ is a regular singular point
  $t_0 = 0$

- Multiply by denominators
  $t\left(\frac{d}{dt}y'\right) + (-1 - t)\, y' + y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $t^m \cdot y'$ to series expansion for $m = 0..1$
  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, t^{k+r-1+m}$
- Shift index using $k- >k + 1 - m$
  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, t^{k+r}$
- Convert $t \cdot \left(\frac{d}{dt}y'\right)$ to series expansion
  $t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, t^{k+r-1}$
- Shift index using $k- >k + 1$
  $t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\,(k + r)\, t^{k+r}$

  Rewrite ODE with series expansions
  $a_0 r(-2 + r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k + 1 + r)\,(k + r - 1) - a_k(k + r - 1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-2 + r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y=\sum_{k=0}^{\infty} a_k t^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$
  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k t^k\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

1.14.3 Maple trace

```
Methods for second order ODEs:
```

1.14.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 13

```
dsolve(t*diff(diff(y(t),t),t)-(1+t)*diff(y(t),t)+y(t) = 0,
       y(t),singsol=all)
```

$$y = c_2\,\mathrm{e}^{t}+c_1 t+c_1$$

1.14.5 Mathematica DSolve solution

Solving time : 0.049 (sec)
Leaf size : 19

```
DSolve[{t*D[y[t],{t,2}]-(1+t)*D[y[t],t]+y[t] == 0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_1 e^t - c_2(t+1)$$

1.15 problem 15

Internal problem ID [8153]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 15
Date solved : `Monday, October 21, 2024 at 04:54:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-t)\,y''+ty'-y=0$$

1.15.1 Solved as second order ode using Kovacic algorithm

Time used: 0.256 (sec)

Writing the ode as

$$(1-t)\,y''+ty'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=1-t\\B&=t\\C&=-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 4t + 6}{4(-1+t)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 4t + 6$$
$$t = 4(-1+t)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 4t + 6}{4(-1+t)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 25: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+t)^2$. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(-1+t)} + \frac{3}{4(-1+t)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(-1+t)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{t^3} + \frac{11}{4t^4} + \frac{21}{4t^5} + \frac{15}{2t^6} + \frac{6}{t^7} - \frac{117}{16t^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 4t + 6}{4t^2 - 8t + 4} \\ &= Q + \frac{R}{4t^2 - 8t + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 5}{4t^2 - 8t + 4}\right) \\ &= \frac{1}{4} + \frac{-2t + 5}{4t^2 - 8t + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 4t + 6}{4(-1+t)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(-1+t)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(-1+t)} + \frac{1}{2} \\
&= \frac{t-2}{2t-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(-1+t)^2} \right) + \left(-\frac{1}{2(-1+t)} + \frac{1}{2} \right)^2 - \left(\frac{t^2-4t+6}{4(-1+t)^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right) dt} \\
&= \frac{e^{\frac{t}{2}}}{\sqrt{-1+t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t}{1-t} \, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(-1+t)}{2}} \\
&= z_1 \left(\sqrt{-1+t}\, e^{\frac{t}{2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{1-t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t+\ln(-1+t)}}{(y_1)^2}\, dt \\ &= y_1 \left(-\frac{t\, \mathrm{e}^{t+\ln(-1+t)} \mathrm{e}^{-2t}}{-1+t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^t\right) + c_2 \left(\mathrm{e}^t \left(-\frac{t\, \mathrm{e}^{t+\ln(-1+t)} \mathrm{e}^{-2t}}{-1+t}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.15.2 Maple step by step solution

Let's solve

$(1-t)\left(\frac{d}{dt}y'\right) + ty' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{y}{-1+t} + \frac{ty'}{-1+t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' - \frac{ty'}{-1+t} + \frac{y}{-1+t} = 0$

□ Check to see if $t_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(t)=-\frac{t}{-1+t}, P_3(t)=\frac{1}{-1+t}\right]$
- $(-1+t)\cdot P_2(t)$ is analytic at $t=1$
  $$\left((-1+t)\cdot P_2(t)\right)\Big|_{t=1}=-1$$
- $(-1+t)^2\cdot P_3(t)$ is analytic at $t=1$
  $$\left((-1+t)^2\cdot P_3(t)\right)\Big|_{t=1}=0$$
- $t=1$is a regular singular point
  Check to see if $t_0=1$ is a regular singular point
  $t_0=1$

- Multiply by denominators
  $(-1+t)\left(\frac{d}{dt}y'\right)-ty'+y=0$
- Change variables using $t=u+1$ so that the regular singular point is at $u=0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$
- Assume series solution for $y(u)$
  $$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

☐ Rewrite ODE with series expansions
- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$$
- Shift index using $k->k+1$
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation
  $r \in \{0,2\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$$
- Revert the change of variables $u=-1+t$
  $$\left[y=\sum_{k=0}^{\infty} a_k(-1+t)^k, a_{k+1}=\frac{a_k}{k+1}\right]$$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$
  $$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$$
- Revert the change of variables $u=-1+t$
  $$\left[y=\sum_{k=0}^{\infty} a_k(-1+t)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$$
- Combine solutions and rename parameters
  $$\left[y=\left(\sum_{k=0}^{\infty} a_k(-1+t)^k\right)+\left(\sum_{k=0}^{\infty} b_k(-1+t)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$$

1.15.3 Maple trace

```
Methods for second order ODEs:
```

1.15.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((1-t)*diff(diff(y(t),t),t)+t*diff(y(t),t)-y(t) = 0,
       y(t),singsol=all)
```

$$y = c_1 t + c_2 \,\mathrm{e}^{t}$$

1.15.5 Mathematica DSolve solution

Solving time : 0.074 (sec)
Leaf size : 17

```
DSolve[{(1-t)*D[y[t],{t,2}]+t*D[y[t],t]-y[t] == 0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_1 e^t - c_2 t$$

1.16 problem 16

Internal problem ID [8154]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 16
Date solved : `Monday, October 21, 2024 at 04:54:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

1.16.1 Solved as second order ode using Kovacic algorithm

Time used: 0.167 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 27: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos (x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos (x)}{\sqrt{x}} (\tan (x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.16.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 \left(\frac{d}{dx} y'\right) + 4x y' + (4x^2 - 1) y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

1.16.3 Maple trace

```
Methods for second order ODEs:
```

1.16.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1 \sin(x)+c_2 \cos(x)}{\sqrt{x}}$$

1.16.5 Mathematica DSolve solution

Solving time : 0.049 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-25/100)*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{e^{-ix}(2c_1-ic_2e^{2ix})}{2\sqrt{x}}$$

1.17 problem 17

Internal problem ID [8155]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 17
Date solved : `Monday, October 21, 2024 at 04:54:35 PM`
CAS classification : `[_Laguerre]`

Solve

$$ty'' - (1+t)\,y' + y = 0$$

1.17.1 Solved as second order ode using Kovacic algorithm

Time used: 0.277 (sec)

Writing the ode as

$$ty'' + (-1-t)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -1-t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 2t + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 29: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4t^2} - \frac{1}{2t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 - 2t + 3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2t} \\
&= \frac{t-1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2 - 2t + 3}{4t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\
&= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-1-t}{t} \, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(t)}{2}} \\
&= z_1 \left(\sqrt{t}\, \mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1-t}{t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\, dt \\ &= y_1 \left(-\frac{(1+t)\,\mathrm{e}^{t+\ln(t)}\mathrm{e}^{-2t}}{t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^t\right) + c_2 \left(\mathrm{e}^t \left(-\frac{(1+t)\,\mathrm{e}^{t+\ln(t)}\mathrm{e}^{-2t}}{t}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.17.2 Maple step by step solution

Let's solve

$t\left(\frac{d}{dt}y'\right) - (1+t)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{y}{t} + \frac{(1+t)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' - \frac{(1+t)y'}{t} + \frac{y}{t} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

- Define functions
  $\left[P_2(t) = -\frac{1+t}{t}, P_3(t) = \frac{1}{t}\right]$
- $t \cdot P_2(t)$ is analytic at $t = 0$
  $$\left(t \cdot P_2(t)\right)\Big|_{t=0} = -1$$
- $t^2 \cdot P_3(t)$ is analytic at $t = 0$
  $$\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 0$$
- $t = 0$is a regular singular point
  Check to see if $t_0 = 0$ is a regular singular point
  $t_0 = 0$

- Multiply by denominators
  $t\left(\frac{d}{dt}y'\right) + (-1-t)\, y' + y = 0$
- Assume series solution for y
  $$y = \sum_{k=0}^{\infty} a_k t^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $t^m \cdot y'$ to series expansion for $m = 0..1$
  $$t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$$
- Shift index using $k- >k + 1 - m$
  $$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$
- Convert $t \cdot \left(\frac{d}{dt}y'\right)$ to series expansion
  $$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r-1}$$
- Shift index using $k- >k + 1$
  $$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0 r(-2+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) - a_k(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y=\sum_{k=0}^{\infty} a_k t^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$
  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k t^k\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

1.17.3 Maple trace

```
Methods for second order ODEs:
```

1.17.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 13

```
dsolve(t*diff(diff(y(t),t),t)-(1+t)*diff(y(t),t)+y(t) = 0,
       y(t),singsol=all)
```

$$y = c_2\,\mathrm{e}^{t}+c_1 t+c_1$$

1.17.5 Mathematica DSolve solution

Solving time : 0.041 (sec)
Leaf size : 19

```
DSolve[{t*D[y[t],{t,2}]-(1+t)*D[y[t],t]+y[t] ==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_1 e^t - c_2(t+1)$$

1.18 problem 18

Internal problem ID [8156]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 18
Date solved : `Monday, October 21, 2024 at 04:54:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-t)\,y''+ty'-y=0$$

1.18.1 Solved as second order ode using Kovacic algorithm

Time used: 0.279 (sec)

Writing the ode as

$$(1-t)\,y''+ty'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-t \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 4t + 6}{4\left(-1+t\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 4t + 6 \\ t &= 4(-1+t)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 4t + 6}{4\left(-1+t\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 31: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+t)^2$. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(-1+t)} + \frac{3}{4(-1+t)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(-1+t)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{t^3} + \frac{11}{4t^4} + \frac{21}{4t^5} + \frac{15}{2t^6} + \frac{6}{t^7} - \frac{117}{16t^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 4t + 6}{4t^2 - 8t + 4} \\ &= Q + \frac{R}{4t^2 - 8t + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 5}{4t^2 - 8t + 4}\right) \\ &= \frac{1}{4} + \frac{-2t + 5}{4t^2 - 8t + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 4t + 6}{4\left(-1 + t\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(-1+t)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(-1+t)} + \frac{1}{2} \\
&= \frac{t-2}{2t-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(-1+t)^2}\right) + \left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right)^2 - \left(\frac{t^2-4t+6}{4(-1+t)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right) dt} \\
&= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{-1+t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t}{1-t} \, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(-1+t)}{2}} \\
&= z_1 \left(\sqrt{-1+t}\, \mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{1-t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t+\ln(-1+t)}}{(y_1)^2}\, dt \\ &= y_1 \left(-\frac{t\, e^{t+\ln(-1+t)} e^{-2t}}{-1+t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^t\right) + c_2 \left(e^t \left(-\frac{t\, e^{t+\ln(-1+t)} e^{-2t}}{-1+t}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.18.2 Maple step by step solution

Let's solve

$(1-t)\left(\frac{d}{dt}y'\right) + ty' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{y}{-1+t} + \frac{ty'}{-1+t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' - \frac{ty'}{-1+t} + \frac{y}{-1+t} = 0$

□ Check to see if $t_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(t) = -\frac{t}{-1+t}, P_3(t) = \frac{1}{-1+t}\right]$
- $(-1+t) \cdot P_2(t)$ is analytic at $t = 1$
  $$\left((-1+t) \cdot P_2(t)\right)\bigg|_{t=1} = -1$$
- $(-1+t)^2 \cdot P_3(t)$ is analytic at $t = 1$
  $$\left((-1+t)^2 \cdot P_3(t)\right)\bigg|_{t=1} = 0$$
- $t = 1$is a regular singular point
  Check to see if $t_0 = 1$ is a regular singular point
  $t_0 = 1$

- Multiply by denominators
  $$(-1+t)\left(\frac{d}{dt}y'\right) - ty' + y = 0$$
- Change variables using $t = u+1$ so that the regular singular point is at $u = 0$
  $$u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions
- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$
- Shift index using $k- > k+1-m$
  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$
- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$$
- Shift index using $k- > k+1$
  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

  Rewrite ODE with series expansions
  $$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)(a_{k+1}(k+1+r) - a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$
- Revert the change of variables $u = -1+t$

  $\left[y = \sum_{k=0}^{\infty} a_k (-1+t)^k, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Revert the change of variables $u = -1+t$

  $\left[y = \sum_{k=0}^{\infty} a_k (-1+t)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (-1+t)^k\right) + \left(\sum_{k=0}^{\infty} b_k (-1+t)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

1.18.3 Maple trace

```
Methods for second order ODEs:
```

1.18.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 12

```
dsolve((1-t)*diff(diff(y(t),t),t)+t*diff(y(t),t)-y(t) = 0,
       y(t),singsol=all)
```

$$y = c_1 t + c_2 \, \mathrm{e}^{t}$$

1.18.5 Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 17

```
DSolve[{(1-t)*D[y[t],{t,2}]+t*D[y[t],t]-y[t] ==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_1 e^t - c_2 t$$

1.19 problem 19

Internal problem ID [8157]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 19
Date solved : `Monday, October 21, 2024 at 04:54:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

1.19.1 Solved as second order ode using Kovacic algorithm

Time used: 0.280 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 33: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 6}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{3}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) = 0$$

$$a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \,\mathrm{e}^{\int -\frac{x}{2} dx} \\ &= (x) \,\mathrm{e}^{-\frac{x^2}{4}} \\ &= x \,\mathrm{e}^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{2}} x\left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.19.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite DE with series expansions
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

1.19.3 Maple trace

```
Methods for second order ODEs:
```

1.19.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -x\left(c_2\pi \,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)-c_1\right)\mathrm{e}^{-\frac{x^2}{2}}+i\sqrt{\pi}\,\sqrt{2}\,c_2$$

1.19.5 Mathematica DSolve solution

Solving time : 0.076 (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\sqrt{\frac{\pi}{2}}c_2e^{-\frac{x^2}{2}}\sqrt{x^2}\text{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right)+\sqrt{2}c_1e^{-\frac{x^2}{2}}x+c_2$$

1.20 problem 20

Internal problem ID [8158]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 20
Date solved : `Monday, October 21, 2024 at 04:54:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^{2}+1\right) y''-4 x y'+6 y=0$$

1.20.1 Solved as second order ode using Kovacic algorithm

Time used: 0.336 (sec)

Writing the ode as

$$\left(x^{2}+1\right) y''-4 x y'+6 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}+1 \\ B &= -4 x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A} d x}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-8}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -8$$
$$t = (x^2+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{8}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 35: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x-i)^2} + \frac{2}{(x+i)^2} + \frac{2i}{x-i} - \frac{2i}{x+i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{8}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	2	-1
$-i$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x-i} + \frac{2}{x+i} + (-)\,(0) \\ &= -\frac{1}{x-i} + \frac{2}{x+i} \\ &= \frac{x-3i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-i} + \frac{2}{x+i}\right)(0) + \left(\left(\frac{1}{(x-i)^2} - \frac{2}{(x+i)^2}\right) + \left(-\frac{1}{x-i} + \frac{2}{x+i}\right)^2 - \left(-\frac{8}{(x^2+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{x-i} + \frac{2}{x+i}\right) dx} \\ &= \frac{(x^2+1)^2}{(ix+1)^3} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2+1}\, dx} \\ &= z_1 e^{\ln(x^2+1)} \\ &= z_1 (x^2+1) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^3}{(ix+1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^2-\frac{1}{3}}{(x+i)^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^3}{(ix+1)^3}\right) + c_2 \left(\frac{(x^2+1)^3}{(ix+1)^3}\left(\frac{x^2-\frac{1}{3}}{(x+i)^3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.20.2 Maple step by step solution

1.20.3 Maple trace

```
Methods for second order ODEs:
```

1.20.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 21

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-4*x*diff(y(x),x)+6*y(x) = 0,
       y(x),singsol=all)
```

$$
y = c_2\, x^3 - 3c_1\, x^2 - 3c_2 x + c_1
$$

1.20.5 Mathematica DSolve solution

Solving time : `0.1` (sec)
Leaf size : 33

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-4*x*D[y[x],x]+6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{3} i \left(c_2 \left(3x^2 - 1\right) + 3c_1 (x - i)^3\right)$$

1.21 problem 21

Internal problem ID [8159]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 21
Date solved : `Monday, October 21, 2024 at 04:54:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-x)\,y''+xy'-y=0$$

1.21.1 Solved as second order ode using Kovacic algorithm

Time used: 0.283 (sec)

Writing the ode as

$$(1-x)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4\left(-1+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(-1+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4\left(-1+x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 36: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+x)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(-1+x)^2} - \frac{1}{2(-1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4\left(-1 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(-1+x)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(-1+x)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(-1+x)^2}\right) + \left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(-1+x)^2}\right) \right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{-1+x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1-x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(-1+x)}{2}} \\
&= z_1 \left(\sqrt{-1+x}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1-x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(-1+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{x\, e^{x+\ln(-1+x)} e^{-2x}}{-1+x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2\left(e^x \left(-\frac{x\, e^{x+\ln(-1+x)} e^{-2x}}{-1+x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.21.2 Maple step by step solution

Let's solve

$(1-x)\left(\frac{d}{dx}y'\right) + xy' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{-1+x} + \frac{xy'}{-1+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{xy'}{-1+x} + \frac{y}{-1+x} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(x)=-\frac{x}{-1+x}, P_3(x)=\frac{1}{-1+x}\right]$
- $(-1+x)\cdot P_2(x)$ is analytic at $x=1$
  $$\left((-1+x)\cdot P_2(x)\right)\Big|_{x=1}=-1$$
- $(-1+x)^2\cdot P_3(x)$ is analytic at $x=1$
  $$\left((-1+x)^2\cdot P_3(x)\right)\Big|_{x=1}=0$$
- $x=1$is a regular singular point
  Check to see if $x_0=1$ is a regular singular point
  $x_0=1$

- Multiply by denominators
  $$(-1+x)\left(\frac{d}{dx}y'\right)-xy'+y=0$$
- Change variables using $x=u+1$ so that the regular singular point is at $u=0$
  $$u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$$
- Assume series solution for $y(u)$
  $$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$$
- Shift index using $k->k+1$
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

  Rewrite ODE with series expansions
  $$a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = -1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (-1+x)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = -1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (-1+x)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (-1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (-1+x)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

1.21.3 Maple trace

```
Methods for second order ODEs:
```

1.21.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 12

```
dsolve((1-x)*diff(diff(y(x),x),x)+x*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2 \, \mathrm{e}^{x}$$

1.21.5 Mathematica DSolve solution

Solving time : `0.054` (sec)
Leaf size : 17

```
DSolve[{(1-x)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2 x$$

1.22 problem 22

Internal problem ID [8160]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 22
Date solved : `Monday, October 21, 2024 at 04:54:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2y'' + xy' + 3y = 0$$

1.22.1 Solved as second order ode using Kovacic algorithm

Time used: 0.302 (sec)

Writing the ode as

$$2y'' + xy' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 20}{16} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 20$$
$$t = 16$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{16} - \frac{5}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 38: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{5}{2x} - \frac{25}{2x^3} - \frac{125}{x^5} - \frac{3125}{2x^7} - \frac{21875}{x^9} - \frac{328125}{x^{11}} - \frac{5156250}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 20}{16} \\ &= Q + \frac{R}{16} \\ &= \left(\frac{x^2}{16} - \frac{5}{4}\right) + (0) \\ &= \frac{x^2}{16} - \frac{5}{4} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{4}\right) - (0) \\ &= -\frac{5}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{4}}{\frac{1}{4}} - 1\right) &&= -3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{4}}{\frac{1}{4}} - 1\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{16} - \frac{5}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{4}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{4}\right) \\ &= -\frac{x}{4} \\ &= -\frac{x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{x}{4}\right)(2x + a_1) + \left(\left(-\frac{1}{4}\right) + \left(-\frac{x}{4}\right)^2 - \left(\frac{x^2}{16} - \frac{5}{4}\right)\right) = 0$$

$$2 + \frac{a_1 x}{2} + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= \left(x^2 - 2\right) \mathrm{e}^{\int -\frac{x}{4} dx} \\
&= \left(x^2 - 2\right) \mathrm{e}^{-\frac{x^2}{8}} \\
&= \left(x^2 - 2\right) \mathrm{e}^{-\frac{x^2}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{2} \, dx} \\
&= z_1 e^{-\frac{x^2}{8}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{4}} \left(x^2 - 2\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{4}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{(x^2 - 2)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^2}{4}} \left(x^2 - 2\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{4}} \left(x^2 - 2\right) \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{\left(x^2 - 2\right)^2} dx \right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.22.2 Maple step by step solution

Let's solve

$2\frac{d}{dx}y' + xy' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{xy'}{2} - \frac{3y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{xy'}{2} + \frac{3y}{2} = 0$

- Multiply by denominators

  $2\frac{d}{dx}y' + xy' + 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(2a_{k+2}(k+2)(k+1) + a_k(k+3)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation
  $(2k^2 + 6k + 4)\, a_{k+2} + a_k(k+3) = 0$
- Recursion relation that defines the series solution to the ODE
  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+3)}{2(k^2+3k+2)}\right]$

1.22.3 Maple trace

```
Methods for second order ODEs:
```

1.22.4 Maple dsolve solution

Solving time : `0.091` (sec)
Leaf size : 32

```
dsolve(2*diff(diff(y(x),x),x)+x*diff(y(x),x)+3*y(x) = 0,
       y(x),singsol=all)
```

$$y = (x^2 - 2)\left(c_1 \operatorname{erfi}\left(\frac{x}{2}\right)\sqrt{\pi} + c_2\right) \mathrm{e}^{-\frac{x^2}{4}} - 2c_1 x$$

1.22.5 Mathematica DSolve solution

Solving time : `0.434` (sec)
Leaf size : 61

```
DSolve[{2*D[y[x],{x,2}]+x*D[y[x],x]+3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{8} e^{-\frac{x^2}{4}} \left(\sqrt{\pi} c_2 (x^2 - 2) \operatorname{erfi}\left(\frac{x}{2}\right) + 8c_1 (x^2 - 2) - 2c_2 e^{\frac{x^2}{4}} x\right)$$

1.23 problem 23

Internal problem ID [8161]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 23
Date solved : `Monday, October 21, 2024 at 04:54:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

1.23.1 Solved as second order ode using Kovacic algorithm

Time used: 0.245 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 40: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= (x) \, \mathrm{e}^{-\frac{x^2}{4}} \\
&= x \, \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{2}} x\left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.23.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- > k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

1.23.3 Maple trace

```
Methods for second order ODEs:
```

1.23.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -x\left(c_2\pi\ \mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) - c_1\right)\mathrm{e}^{-\frac{x^2}{2}} + i\sqrt{\pi}\,\sqrt{2}\,c_2$$

1.23.5 Mathematica DSolve solution

Solving time : 0.072 (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}}c_2e^{-\frac{x^2}{2}}\sqrt{x^2}\mathrm{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2}c_1e^{-\frac{x^2}{2}}x + c_2$$

1.24 problem 24

Internal problem ID [8162]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 24
Date solved : `Monday, October 21, 2024 at 04:54:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-x)\,y''+xy'-y=0$$

1.24.1 Solved as second order ode using Kovacic algorithm

Time used: 0.244 (sec)

Writing the ode as

$$(1-x)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(-1+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 6 \\ t &= 4(-1+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(-1+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 42: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+x)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\,(-1+x)^2} - \frac{1}{2\,(-1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{2}\right) - (0)\\ &= -\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2}\\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2}\\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4\left(-1+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right)\\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(-1+x)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(-1+x)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(-1+x)^2}\right) + \left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(-1+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right) dx} \\
&= \frac{e^{\frac{x}{2}}}{\sqrt{-1+x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1-x}\, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(-1+x)}{2}} \\
&= z_1 \left(\sqrt{-1+x}\, e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1-x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(-1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{x\, e^{x+\ln(-1+x)} e^{-2x}}{-1+x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (e^x) + c_2 \left(e^x \left(-\frac{x\, e^{x+\ln(-1+x)} e^{-2x}}{-1+x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.24.2 Maple step by step solution

Let's solve

$(1-x)\left(\frac{d}{dx}y'\right) + xy' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{-1+x} + \frac{xy'}{-1+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{xy'}{-1+x} + \frac{y}{-1+x} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x}{-1+x}, P_3(x) = \frac{1}{-1+x}\right]$
- $(-1+x) \cdot P_2(x)$ is analytic at $x = 1$
  $$\left((-1+x) \cdot P_2(x)\right)\Big|_{x=1} = -1$$
- $(-1+x)^2 \cdot P_3(x)$ is analytic at $x = 1$
  $$\left((-1+x)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$$
- $x = 1$is a regular singular point
  Check to see if $x_0 = 1$ is a regular singular point
  $x_0 = 1$

• Multiply by denominators
  $(-1+x)\left(\frac{d}{dx}y'\right) - xy' + y = 0$

• Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

• Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions
- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$
- Shift index using $k- >k+1-m$
  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$
- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$
- Shift index using $k- >k+1$
  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

  Rewrite ODE with series expansions
  $$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

• $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = -1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (-1+x)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = -1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (-1+x)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (-1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (-1+x)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

1.24.3 Maple trace

```
Methods for second order ODEs:
```

1.24.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 12

```
dsolve((1-x)*diff(diff(y(x),x),x)+x*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2 \,\mathrm{e}^{x}$$

1.24.5 Mathematica DSolve solution

Solving time : `0.047` (sec)
Leaf size : 17

```
DSolve[{(1-x)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2 x$$

1.25 problem 25

Internal problem ID [8163]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 25
Date solved : `Monday, October 21, 2024 at 04:54:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

1.25.1 Solved as second order ode using Kovacic algorithm

Time used: 0.238 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 44: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 6}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{3}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, e^{\int -\frac{x}{2} dx} \\ &= (x) \, e^{-\frac{x^2}{4}} \\ &= x \, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(e^{-\frac{x^2}{4}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{2}} x\left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.25.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

1.25.3 Maple trace

```
Methods for second order ODEs:
```

1.25.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -x\left(c_2\pi \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)-c_1\right)\mathrm{e}^{-\frac{x^2}{2}}+i\sqrt{\pi}\,\sqrt{2}\,c_2$$

1.25.5 Mathematica DSolve solution

Solving time : `0.068` (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\sqrt{\frac{\pi}{2}}c_2e^{-\frac{x^2}{2}}\sqrt{x^2}\text{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right)+\sqrt{2}c_1e^{-\frac{x^2}{2}}x+c_2$$

1.26 problem 26

Internal problem ID [8164]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 26
Date solved : `Monday, October 21, 2024 at 04:54:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(-x^2+4\right) y''+x y'+2 y=0$$

1.26.1 Solved as second order ode using Kovacic algorithm

Time used: 0.985 (sec)

Writing the ode as

$$\left(-x^2+4\right) y''+x y'+2 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+4 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{11x^2 - 24}{4\left(x^2 - 4\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 11x^2 - 24$$
$$t = 4\left(x^2 - 4\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{11x^2 - 24}{4\left(x^2 - 4\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 46: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 4)^2$. There is a pole at $x = 2$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{17}{32(x+2)} + \frac{17}{32(x-2)} + \frac{5}{16(x-2)^2} + \frac{5}{16(x+2)^2}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{11x^2 - 24}{4(x^2 - 4)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{11}{4}$. Hence

$$
\begin{aligned}
E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\
&= \{2\}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
2	2	$\{-1,2,5\}$
-2	2	$\{-1,2,5\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$
e_1 = -1,\ e_2 = -1,\ e_\infty = 2
$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{1}{2}(2-(-1+(-1))) \\
&= 2
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\
&= \frac{1}{2}\left(\frac{-1}{(x-(2))}+\frac{-1}{(x-(-2))}\right) \\
&= -\frac{1}{2(x-2)} - \frac{1}{2(x+2)}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d=2$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}
$$

Since $d=2$, then letting

$$
p = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$\frac{11x^2 a_1 + 16(6 + a_0)\,x + 36a_1}{\left(x^2 - 4\right)^2} = 0$$

And solving for p gives

$$p = x^2 - 6$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{2x}{x^2 - 6} - \frac{1}{2\,(x - 2)} - \frac{1}{2\,(x + 2)}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{2x}{x^2 - 6} - \frac{1}{2\,(x - 2)} - \frac{1}{2\,(x + 2)}\right) w + \frac{-11x^4 + 74x^2 - 128}{4x^6 - 56x^4 + 256x^2 - 384} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{3}\,x^2\sqrt{x^2 - 4} + x^3 - 8\sqrt{3}\,\sqrt{x^2 - 4} - 2x}{2\left(x^2 - 6\right)(x - 2)(x + 2)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{2\sqrt{3}\,x^2\sqrt{x^2-4}+x^3-8\sqrt{3}\,\sqrt{x^2-4}-2x}{2\left(x^2-6\right)(x-2)(x+2)} dx} \\ &= \frac{\sqrt{x^2 - 6}\,\left(x + \sqrt{x^2 - 4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\left(\sqrt{2}\,\sqrt{3}\,x-4\right)\sqrt{2}}{2\sqrt{x^2-4}}\right)}{2} - \frac{\operatorname{arctanh}\left(\frac{\left(4+\sqrt{2}\,\sqrt{3}\,x\right)\sqrt{2}}{2\sqrt{x^2-4}}\right)}{2}}}{(x + 2)^{1/4}\,(x - 2)^{1/4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{-x^2+4}\, dx} \\ &= z_1 e^{\frac{\ln\left(x^2-4\right)}{4}} \\ &= z_1\left(\left(x^2 - 4\right)^{1/4}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{-x^2+4}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{\ln(x^2-4)}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{\sqrt{x^2-4}\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}} \mathrm{e}^{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}}{x^2-6}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}{2}}\right) + c_2\Bigg(\sqrt{x^2-6}\,\Big(x \\ &\quad +\sqrt{x^2-4}\Big)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}{2}} \left(\int \frac{\sqrt{x^2-4}\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}} \mathrm{e}^{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}}}{x^2-6}\right. \end{aligned}$$

Will add steps showing solving for IC soon.

1.26.2 Maple step by step solution

Let's solve

$(-x^2+4)\left(\frac{d}{dx}y'\right)+xy'+2y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{2y}{x^2-4}+\frac{xy'}{x^2-4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{xy'}{x^2-4}-\frac{2y}{x^2-4}=0$

□ Check to see if x_0 is a regular singular point

 ◦ Define functions

 $\left[P_2(x)=-\frac{x}{x^2-4}, P_3(x)=-\frac{2}{x^2-4}\right]$

 ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

 $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=-\frac{1}{2}$

 ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

 $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

 ◦ $x=-2$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-2$

- Multiply by denominators

$(x^2-4)\left(\frac{d}{dx}y'\right)-xy'-2y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$(u^2-4u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-u+2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

 ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

 $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

 ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- ○ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- ○ Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r(-3+2r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(2k-1+2r) + a_k(k^2+2kr+r^2-2k-2r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-4\left(k-\frac{1}{2}+r\right)(k+1+r)\,a_{k+1} + \left(k^2+(2r-2)\,k+r^2-2r-2\right)a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(k^2+2kr+r^2-2k-2r-2)\,a_k}{2(2k-1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{(k^2-2k-2)\,a_k}{2(2k-1)(k+1)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{(k^2-2k-2)\,a_k}{2(2k-1)(k+1)}\right]$$

- Revert the change of variables $u = x+2$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+2)^k, a_{k+1} = \frac{(k^2-2k-2)\,a_k}{2(2k-1)(k+1)}\right]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = \frac{\left(k^2+k-\frac{11}{4}\right)a_k}{2(2k+2)\left(k+\frac{5}{2}\right)}$$

- Solution for $r = \frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1} = \frac{\left(k^2+k-\frac{11}{4}\right)a_k}{2(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

- Revert the change of variables $u = x+2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+\frac{3}{2}}, a_{k+1} = \frac{\left(k^2+k-\frac{11}{4}\right) a_k}{2(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+\frac{3}{2}}\right), a_{k+1} = \frac{(k^2-2k-2) a_k}{2(2k-1)(k+1)}, b_{k+1} = \frac{\left(k^2+k-\frac{11}{4}\right) b_k}{2(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

1.26.3 Maple trace

```
Methods for second order ODEs:
```

1.26.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 37

```
dsolve((-x^2+4)*diff(diff(y(x),x),x)+x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = (x^2-4)^{3/4} \left(\mathrm{LegendreQ}\left(-\frac{1}{2}+\sqrt{3}, \frac{3}{2}, \frac{x}{2}\right) c_2 + \mathrm{LegendreP}\left(-\frac{1}{2}+\sqrt{3}, \frac{3}{2}, \frac{x}{2}\right) c_1\right)$$

1.26.5 Mathematica DSolve solution

Solving time : `0.077` (sec)
Leaf size : 58

```
DSolve[{(4-x^2)*D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x^2-4)^{3/4} \left(c_1 P_{-\frac{1}{2}+\sqrt{3}}^{\frac{3}{2}}\left(\frac{x}{2}\right) + c_2 Q_{-\frac{1}{2}+\sqrt{3}}^{\frac{3}{2}}\left(\frac{x}{2}\right)\right)$$

1.27 problem 27

Internal problem ID [8165]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 27
Date solved : `Monday, October 21, 2024 at 04:54:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4xy' + \left(-16x^2+3\right) y = 0$$

1.27.1 Solved as second order ode using Kovacic algorithm

Time used: 0.112 (sec)

Writing the ode as

$$4x^2y'' - 4xy' + \left(-16x^2+3\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 48: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-2x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x}\,\mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,\mathrm{e}^{-2x}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.27.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) - 4xy' + \left(-16x^2 + 3\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(16x^2-3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'}{x} - \frac{(16x^2-3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4xy' + \left(-16x^2 + 3\right)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)\,(-3+2r)\, x^r + a_1(1+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-3) - 1$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$$

- Each term must be 0

  $$a_1(1+2r)\,(-1+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$4\left(k+r-\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right) a_k - 16a_{k-2} = 0$$

- Shift index using $k- >k+2$

  $$4\left(k+\frac{3}{2}+r\right)\left(k+\frac{1}{2}+r\right) a_{k+2} - 16a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{16a_k}{(2k+3+2r)(2k+1+2r)}$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_{k} x^{k+\frac{1}{2}}, a_{k+2}=\frac{16 a_{k}}{(2 k+4)(2 k+2)}, a_{1}=0\right]$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+2}=\frac{16 a_{k}}{(2 k+6)(2 k+4)}$$

- Solution for $r=\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_{k} x^{k+\frac{3}{2}}, a_{k+2}=\frac{16 a_{k}}{(2 k+6)(2 k+4)}, a_{1}=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_{k} x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_{k} x^{k+\frac{3}{2}}\right), a_{k+2}=\frac{16 a_{k}}{(2 k+4)(2 k+2)}, a_{1}=0, b_{k+2}=\frac{16 b_{k}}{(2 k+6)(2 k+4)}, b_{1}=0\right]$$

1.27.3 Maple trace

```
Methods for second order ODEs:
```

1.27.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 21

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(-16*x^2+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\sqrt{x}\left(c_{1} \sinh (2 x)+c_{2} \cosh (2 x)\right)$$

1.27.5 Mathematica DSolve solution

Solving time : 0.052 (sec)
Leaf size : 32

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(3-16*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{1}{4} e^{-2 x} \sqrt{x}\left(c_{2} e^{4 x}+4 c_{1}\right)$$

1.28 problem 28

Internal problem ID [8166]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 28
Date solved : `Monday, October 21, 2024 at 04:54:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\, y'' - x y' + y = 0$$

1.28.1 Solved as second order ode using Kovacic algorithm

Time used: 0.249 (sec)

Writing the ode as

$$(x-1)\, y'' - x y' + y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 50: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(x-1)} + \frac{3}{4(x-1)^2}$$

For the <u>pole at $x = 1$</u> let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{e^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{x\, e^{x+\ln(x-1)} e^{-2x}}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(e^x) + c_2 \left(e^x \left(-\frac{x\, e^{x+\ln(x-1)} e^{-2x}}{x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.28.2 Maple step by step solution

Let's solve

$(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$
- $(x-1)\cdot P_2(x)$ is analytic at $x=1$
  $$\left((x-1)\cdot P_2(x)\right)\bigg|_{x=1} = -1$$
- $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$
  $$\left((x-1)^2\cdot P_3(x)\right)\bigg|_{x=1} = 0$$
- $x=1$is a regular singular point
  Check to see if $x_0 = 1$ is a regular singular point
  $x_0 = 1$

- Multiply by denominators
  $(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$
- Change variables using $x = u+1$ so that the regular singular point is at $u=0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$
- Shift index using $k->k+1$
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Revert the change of variables $u=x-1$
  $\left[y=\sum_{k=0}^{\infty} a_k(x-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Revert the change of variables $u=x-1$
  $\left[y=\sum_{k=0}^{\infty} a_k(x-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k(x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

1.28.3 Maple trace

```
Methods for second order ODEs:
```

1.28.4 Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2 \, \mathrm{e}^{x}$$

1.28.5 Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 17

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2 x$$

1.29 problem 29

Internal problem ID [8167]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 29
Date solved : `Monday, October 21, 2024 at 04:54:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + (x^2+2)\,y = 0$$

1.29.1 Solved as second order ode using Kovacic algorithm

Time used: 0.145 (sec)

Writing the ode as

$$x^2y'' - 2xy' + (x^2+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 52: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x \cos(x)) + c_2 (x \cos(x) (\tan(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.29.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) - 2xy' + (x^2 + 2) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2 \left(\frac{d}{dx} y'\right) - 2xy' + (x^2 + 2) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) + a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}(k+1+r)\,(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 2$

$$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$$

1.29.3 Maple trace

```
Methods for second order ODEs:
```

1.29.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+(x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x(c_1 \sin(x) + c_2 \cos(x))$$

1.29.5 Mathematica DSolve solution

Solving time : `0.043` (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

1.30 problem 31

Internal problem ID [8168]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 31
Date solved : `Monday, October 21, 2024 at 04:54:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-2x\right)y''+\left(-x^2+2\right)y'+\left(2x-2\right)y=0$$

1.30.1 Solved as second order ode using Kovacic algorithm

Time used: 0.316 (sec)

Writing the ode as

$$\left(x^2-2x\right)y''+\left(-x^2+2\right)y'+\left(2x-2\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-2x \\ B &= -x^2+2 \\ C &= 2x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4(x^2 - 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 8x^3 + 24x^2 - 24x + 12$$
$$t = 4(x^2 - 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4(x^2 - 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 54: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4(x-2)} + \frac{3}{4(x-2)^2} - \frac{3}{4x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{2}{x^3} + \frac{11}{x^4} + \frac{42}{x^5} + \frac{132}{x^6} + \frac{348}{x^7} + \frac{711}{x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \\
&= Q + \frac{R}{4x^4 - 16x^3 + 16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}\right) \\
&= \frac{1}{4} + \frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to

determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{+} = -1$ then

$$\begin{aligned} d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + (+)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{2x} - \frac{1}{2\left(x-2\right)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2x} - \frac{1}{2\left(x-2\right)} + \frac{1}{2} \\ &= -\frac{1}{2x} - \frac{1}{2x-4} + \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{1}{2\left(x-2\right)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2x^2} + \frac{1}{2\left(x-2\right)^2}\right) + \left(-\frac{1}{2x} - \frac{1}{2\left(x-2\right)} + \frac{1}{2}\right)^2 - \left(\frac{x^4 - 8x^3 +}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-2}\,\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+2}{x^2-2x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-2)}{2} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{x}}{\sqrt{x\,(x-2)}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^2-2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x+\ln(x-2)+\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{x\,\mathrm{e}^{x+\ln(x-2)+\ln(x)}\mathrm{e}^{-2x}}{x-2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{x}}{\sqrt{x\,(x-2)}}\right) + c_2 \left(\frac{\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{x}}{\sqrt{x\,(x-2)}} \left(-\frac{x\,\mathrm{e}^{x+\ln(x-2)+\ln(x)}\mathrm{e}^{-2x}}{x-2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.30.2 Maple step by step solution

Let's solve

$(x^2-2x)\left(\frac{d}{dx}y'\right)+(-x^2+2)\,y'+(2x-2)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{2(x-1)y}{x(x-2)}+\frac{(x^2-2)y'}{x(x-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{(x^2-2)y'}{x(x-2)}+\frac{2(x-1)y}{x(x-2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x^2-2}{x(x-2)},P_3(x)=\frac{2(x-1)}{x(x-2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x-2)\left(\frac{d}{dx}y'\right)+(-x^2+2)\,y'+(2x-2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx} y'\right)$ to series expansion for $m = 1..2$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r(-2+r)\, x^{-1+r} + \left(-2a_1(1+r)\,(-1+r) + a_0(1+r)\,(-2+r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(-2a_{k+1}(k+r+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term must be 0
  $-2a_1(1+r)\,(-1+r) + a_0(1+r)\,(-2+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+1)\,(k+r-2) - 2k^2 a_{k+1} + (-4r a_{k+1} - a_{k-1})\, k - 2r^2 a_{k+1} - a_{k-1} r + 3a_{k-1} + 2a_{k+1} =$
- Shift index using $k- >k+1$
  $a_{k+1}(k+2+r)\,(k+r-1) - 2(k+1)^2 a_{k+2} + (-4r a_{k+2} - a_k)\,(k+1) - 2r^2 a_{k+2} - r a_k + 3a_k +$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{k^2 a_{k+1} + 2kr a_{k+1} + r^2 a_{k+1} - k a_k + k a_{k+1} - r a_k + r a_{k+1} + 2a_k - 2a_{k+1}}{2(k^2 + 2kr + r^2 + 2k + 2r)}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{k^2 a_{k+1} - k a_k + k a_{k+1} + 2a_k - 2a_{k+1}}{2(k^2 + 2k)}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$
  $a_{k+2} = \frac{k^2 a_{k+1} - k a_k + k a_{k+1} + 2a_k - 2a_{k+1}}{2(k^2 + 2k)}$
- Recursion relation for $r = 2$

$$a_{k+2} = \frac{k^2 a_{k+1} - k a_k + 5k a_{k+1} + 4a_{k+1}}{2(k^2+6k+8)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{k^2 a_{k+1} - k a_k + 5k a_{k+1} + 4a_{k+1}}{2(k^2+6k+8)}, -6a_1 = 0\right]$$

1.30.3 Maple trace

```
Methods for second order ODEs:
```

1.30.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 14

```
dsolve((x^2-2*x)*diff(diff(y(x),x),x)+(-x^2+2)*diff(y(x),x)+(2*x-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x^2 + c_2 e^x$$

1.30.5 Mathematica DSolve solution

Solving time : `0.151` (sec)
Leaf size : 18

```
DSolve[{(x^2-2*x)*D[y[x],{x,2}]+(2-x^2)*D[y[x],x]+(2*x-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x^2 + c_1 e^x$$

1.31 problem 32

Internal problem ID [8169]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 32
Date solved : `Monday, October 21, 2024 at 04:54:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + \left(-8x^2 + 4x\right) y' + \left(4x^2 - 4x - 1\right) y = 0$$

1.31.1 Solved as second order ode using Kovacic algorithm

Time used: 0.112 (sec)

Writing the ode as

$$4x^2y'' + \left(-8x^2 + 4x\right) y' + \left(4x^2 - 4x - 1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2 + 4x \\ C &= 4x^2 - 4x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 56: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\,dx} \\ &= z_1 e^{x-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{\sqrt{x}}\right) + c_2 \left(\frac{e^x}{\sqrt{x}}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.31.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-4x-1)y}{4x^2} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(2x-1)y'}{x} + \frac{(4x^2-4x-1)y}{4x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2-4x-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4x(2x - 1)\, y' + (4x^2 - 4x - 1)\, y = 0$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\, x^r+\left(a_1(3+2r)\,(1+2r)-4a_0(1+2r)\right) x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(2k+2r+1)\,(2k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)\,(-1+2r)=0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0
  $a_1(3+2r)\,(1+2r)-4a_0(1+2r)=0$
- Solve for the dependent coefficient(s)
  $a_1=\frac{4a_0}{3+2r}$
- Each term in the series must be 0, giving the recursion relation
  $a_k(4k^2+8kr+4r^2-1)+(-8k-8r+4)\,a_{k-1}+4a_{k-2}=0$
- Shift index using $k->k+2$
  $a_{k+2}\left(4(k+2)^2+8(k+2)\,r+4r^2-1\right)+(-8k-12-8r)\,a_{k+1}+4a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{4(2ka_{k+1}+2ra_{k+1}-a_k+3a_{k+1})}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1 = 2a_0\right]$$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}, a_1 = a_0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1 = 2a_0, b_{k+2} = \frac{4(2kb_{k+1}-b_k+4b_{k+1}}{4k^2+20k+24}\right.$$

1.31.3 Maple trace

```
Methods for second order ODEs:
```

1.31.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 15

```
dsolve(4*x^2*diff(diff(y(x),x),x)+(-8*x^2+4*x)*diff(y(x),x)+(4*x^2-4*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^x (c_2 x + c_1)}{\sqrt{x}}$$

1.31.5 Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 21

```
DSolve[{4*x^2*D[y[x],{x,2}]+(4*x-8*x^2)*D[y[x],x]+(4*x^2-4*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^x (c_2 x + c_1)}{\sqrt{x}}$$

1.32 problem 33

Internal problem ID [8170]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 33
Date solved : `Monday, October 21, 2024 at 04:54:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0$$

1.32.1 Solved as second order ode using Kovacic algorithm

Time used: 0.085 (sec)

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 58: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\
&= z_1 e^{-x^2} \\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.32.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 4xy' + (4x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k + 2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2})\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2 = -a_0, a_3 = -a_1\right]$

1.32.3 Maple trace

```
Methods for second order ODEs:
```

1.32.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(4*x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-x^2}(c_2 x + c_1)$$

1.32.5 Mathematica DSolve solution

Solving time : `0.037` (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x^2}(c_2 x + c_1)$$

1.33 problem 34

Internal problem ID [8171]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 34
Date solved : `Monday, October 21, 2024 at 04:54:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2x+1)\,y''-2y'-(2x+3)\,y=0$$

1.33.1 Solved as second order ode using Kovacic algorithm

Time used: 0.236 (sec)

Writing the ode as

$$(2x+1)\,y''-2y'+(-2x-3)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x+1\\B&=-2\\C&=-2x-3\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 8x + 6}{(2x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 + 8x + 6$$
$$t = (2x+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 8x + 6}{(2x+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 60: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x+1)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x + \frac{1}{2}\right)^2} + \frac{1}{x + \frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 + 8x + 6}{(2x + 1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(x + \frac{1}{2}\right)} + (-)(1) \\
&= -\frac{1}{2\left(x + \frac{1}{2}\right)} - 1 \\
&= -\frac{2(x+1)}{2x+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(x + \frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x + \frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x + \frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2 + 8x + 6}{(2x+1)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x + \frac{1}{2}\right)} - 1\right) dx} \\
&= \frac{\mathrm{e}^{-x}}{\sqrt{2x+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2}{2x+1} \, dx} \\
&= z_1 e^{\frac{\ln(2x+1)}{2}} \\
&= z_1 \left(\sqrt{2x+1}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{2x+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(2x+1)}}{(y_1)^2}\, dx \\ &= y_1 \left(x\,\mathrm{e}^{2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(x\,\mathrm{e}^{2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.33.2 Maple step by step solution

Let's solve
$(2x+1)\left(\frac{d}{dx}y'\right) - 2y' - (2x+3)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = \frac{(2x+3)y}{2x+1} + \frac{2y'}{2x+1}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{2y'}{2x+1} - \frac{(2x+3)y}{2x+1} = 0$

□ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2}{2x+1}, P_3(x) = -\frac{2x+3}{2x+1}\right]$

- $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}}=-1$

- $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}}=0$

- $x=-\frac{1}{2}$is a regular singular point

  Check to see if $x_0=-\frac{1}{2}$ is a regular singular point

  $x_0=-\frac{1}{2}$

- Multiply by denominators

  $(2x+1)\left(\frac{d}{dx}y'\right)-2y'+(-2x-3)\,y=0$

- Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

  $2u\left(\frac{d}{du}\frac{d}{du}y(u)\right)-2\frac{d}{du}y(u)+(-2u-2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{du}y(u)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,u^{k+r}$

- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- Shift index using $k->k+1$

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$2a_0 r(-2+r)\,u^{-1+r} + (2a_1(1+r)\,(-1+r) - 2a_0)\,u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+r)\,(k+r-1) - 2a_k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term must be 0

  $2a_1(1+r)\,(-1+r) - 2a_0 = 0$
- Each term in the series must be 0, giving the recursion relation

  $2a_{k+1}(k+1+r)\,(k+r-1) - 2a_k - 2a_{k-1} = 0$
- Shift index using $k- >k+1$

  $2a_{k+2}(k+2+r)\,(k+r) - 2a_{k+1} - 2a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2+r)(k+r)}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$
- Recursion relation for $r = 2$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$
- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$

1.33.3 Maple trace

```
Methods for second order ODEs:
```

1.33.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 16

```
dsolve((2*x+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)-(2*x+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \,\mathrm{e}^{-x} + c_2 x \,\mathrm{e}^{x}$$

1.33.5 Mathematica DSolve solution

Solving time : `0.128` (sec)
Leaf size : 29

```
DSolve[{(2*x+1)*D[y[x],{x,2}]-2*D[y[x],x]-(2*x+3)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x-\frac{1}{2}}\left(c_2 e^{2x+1} x + c_1\right)$$

1.34 problem 35

Internal problem ID [8172]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 35
Date solved : `Monday, October 21, 2024 at 04:54:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (2x+2)\,y' + (x+2)\,y = 0$$

1.34.1 Solved as second order ode using Kovacic algorithm

Time used: 0.154 (sec)

Writing the ode as

$$xy'' + (-2x-2)\,y' + (x+2)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 62: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)\,(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1(x \, \mathrm{e}^x) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x \, \mathrm{e}^{2x+2\ln(x)} \mathrm{e}^{-2x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x\,\mathrm{e}^{2x+2\ln(x)}\mathrm{e}^{-2x}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.34.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) - (2x+2)\,y' + (x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x+2)y}{x} + \frac{2(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(x+1)y'}{x} + \frac{(x+2)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{x+2}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + (-2x-2)\,y' + (x+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\, x^{-1+r} + (a_1(1+r)\,(-2+r) - 2a_0(-1+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-2+r)\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$
- Each term must be 0

  $a_1(1+r)\,(-2+r) - 2a_0(-1+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k-2+r) - 2a_k k - 2a_k r + 2a_k + a_{k-1} = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+r-1) - 2a_{k+1}(k+1) - 2r a_{k+1} + 2a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2k a_{k+1} + 2r a_{k+1} - a_k}{(k+2+r)(k+r-1)}$
- Recursion relation for $r = 0$

$$a_{k+2} = \frac{2ka_{k+1} - a_k}{(k+2)(k-1)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

$$a_{k+2} = \frac{2ka_{k+1} - a_k}{(k+2)(k-1)}$$

- Recursion relation for $r = 3$

$$a_{k+2} = \frac{2ka_{k+1} - a_k + 6a_{k+1}}{(k+5)(k+2)}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{2ka_{k+1} - a_k + 6a_{k+1}}{(k+5)(k+2)}, 4a_1 - 4a_0 = 0\right]$$

1.34.3 Maple trace

```
Methods for second order ODEs:
```

1.34.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 14

```
dsolve(x*diff(diff(y(x),x),x)-(2*x+2)*diff(y(x),x)+(x+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{x} \left(c_2 x^3 + c_1\right)$$

1.34.5 Mathematica DSolve solution

Solving time : `0.038` (sec)
Leaf size : 23

```
DSolve[{x*D[y[x],{x,2}]-(2*x+2)*D[y[x],x]+(x+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3} e^{x} \left(c_2 x^3 + 3c_1\right)$$

1.35 problem 36

Internal problem ID [8173]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 36
Date solved : `Monday, October 21, 2024 at 04:54:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0$$

1.35.1 Solved as second order ode using Kovacic algorithm

Time used: 0.149 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 64: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.35.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k+r-1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term must be 0

  $a_1 r(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k (k+r-1)\,(k+r-2) + a_{k-2} = 0$
- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)\,(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$

$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

1.35.3 Maple trace

```
Methods for second order ODEs:
```

1.35.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+(x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x(c_1 \sin(x) + c_2 \cos(x))$$

1.35.5 Mathematica DSolve solution

Solving time : `0.04` (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

1.36 problem 38

Internal problem ID [8174]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 38
Date solved : `Monday, October 21, 2024 at 04:54:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4xy' + \left(-16x^2 + 3\right) y = 0$$

1.36.1 Solved as second order ode using Kovacic algorithm

Time used: 0.113 (sec)

Writing the ode as

$$4x^2y'' - 4xy' + \left(-16x^2 + 3\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2 + 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 66: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\, \mathrm{e}^{-2x}\right) + c_2\left(\sqrt{x}\, \mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.36.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) - 4xy' + \left(-16x^2+3\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(16x^2-3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'}{x} - \frac{(16x^2-3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4xy' + \left(-16x^2+3\right)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+2r)\,(-3+2r)\, x^r + a_1(1+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-3) - 1$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$$

- Each term must be 0

  $$a_1(1+2r)\,(-1+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$4\left(k+r-\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right) a_k - 16a_{k-2} = 0$$

- Shift index using $k->k+2$

  $$4\left(k+\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right) a_{k+2} - 16a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{16a_k}{(2k+1+2r)(2k+3+2r)}$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+2} = \frac{16a_k}{(2k+4)(2k+6)}$$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = \frac{16a_k}{(2k+4)(2k+6)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}, a_1 = 0, b_{k+2} = \frac{16b_k}{(2k+4)(2k+6)}, b_1 = 0\right]$$

1.36.3 Maple trace

```
Methods for second order ODEs:
```

1.36.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 21

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(-16*x^2+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\,(c_1 \sinh(2x) + c_2 \cosh(2x))$$

1.36.5 Mathematica DSolve solution

Solving time : `0.044` (sec)
Leaf size : 32

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(3-16*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4} e^{-2x} \sqrt{x} \left(c_2 e^{4x} + 4c_1\right)$$

1.37 problem 39

Internal problem ID [8175]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 39
Date solved : `Monday, October 21, 2024 at 04:54:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4xy' + \left(4x^2+3\right)y = 0$$

1.37.1 Solved as second order ode using Kovacic algorithm

Time used: 0.191 (sec)

Writing the ode as

$$4x^2y'' - 4xy' + \left(4x^2+3\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= 4x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 68: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\, \cos(x)\right) + c_2\left(\sqrt{x}\, \cos(x)\,(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.37.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) - 4xy' + (4x^2+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2+3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'}{x} + \frac{(4x^2+3)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{4x^2+3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4xy' + (4x^2+3)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (-1+2r)\,(-3+2r)\, x^r + a_1 (1+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k (2k+2r-1)\,(2k+2r-3) + 4$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$$

- Each term must be 0

$$a_1 (1+2r)\,(-1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right) a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k + 2$

$$4\left(k+\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right) a_{k+2} + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4a_k}{(2k+1+2r)(2k+3+2r)}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{4a_k}{(2k+2)(2k+4)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{(2k+2)(2k+4)}, a_1 = 0\right]$$

- Recursion relation for $r=\frac{3}{2}$

  $a_{k+2}=-\frac{4a_k}{(2k+4)(2k+6)}$
- Solution for $r=\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2}=-\frac{4a_k}{(2k+4)(2k+6)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2}=-\frac{4a_k}{(2k+2)(2k+4)}, a_1=0, b_{k+2}=-\frac{4b_k}{(2k+4)(2k+6)}, b_1=0\right]$

1.37.3 Maple trace

```
Methods for second order ODEs:
```

1.37.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 17

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(4*x^2+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\sqrt{x}\,(c_1\sin(x)+c_2\cos(x))$$

1.37.5 Mathematica DSolve solution

Solving time : `0.048` (sec)
Leaf size : 39

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2+3)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{1}{2}e^{-ix}\sqrt{x}\left(2c_1-ic_2e^{2ix}\right)$$

1.38 problem 40

Internal problem ID [8176]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 40
Date solved : `Monday, October 21, 2024 at 04:54:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' - \left(x^2 - 2\right) y = 0$$

1.38.1 Solved as second order ode using Kovacic algorithm

Time used: 0.091 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(-x^2 + 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= -x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 70: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-x}\right) + c_2 \left(x\,\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.38.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) - 2xy' - \left(x^2 - 2\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{(x^2-2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' - \frac{2y'}{x} - \frac{(x^2-2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = -\frac{x^2-2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 \left(\frac{d}{dx} y'\right) - 2xy' + \left(-x^2 + 2\right) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-2+r)\,x^r + a_1 r(-1+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) - a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) - a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)\,(k+r) - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$$

- Recursion relation for $r = 2$

$a_{k+2} = \frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = \frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$$

1.38.3 Maple trace

```
Methods for second order ODEs:
```

1.38.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*diff(y(x),x)-(x^2-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x(c_1 \sinh(x) + c_2 \cosh(x))$$

1.38.5 Mathematica DSolve solution

Solving time : `0.04` (sec)
Leaf size : 25

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]-(x^2-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-x} x + \frac{1}{2} c_2 e^x x$$

1.39 problem 41

Internal problem ID [8177]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 41
Date solved : `Monday, October 21, 2024 at 04:54:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2x(x+1)\,y' + \left(x^2+2x+2\right)y = 0$$

1.39.1 Solved as second order ode using Kovacic algorithm

Time used: 0.109 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^2-2x\right)y' + \left(x^2+2x+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2-2x \\ C &= x^2+2x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 72: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-2x}{x^2}\, dx} \\
&= z_1 e^{x+\ln(x)} \\
&= z_1 (x\, \mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\, \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\,\mathrm{e}^x) + c_2(x\,\mathrm{e}^x(x))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.39.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - 2x(x+1)\,y' + (x^2+2x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+2x+2)y}{x^2} + \frac{2(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(x+1)y'}{x} + \frac{(x^2+2x+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{x^2+2x+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - 2x(x+1)\,y' + (x^2+2x+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + (a_1 r(-1+r) - 2a_0(-1+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)\,(k+r-2) -\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term must be 0

  $a_1 r(-1+r) - 2a_0(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = \frac{2a_0}{r}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) - 2a_{k-1}k - 2a_{k-1}r + a_{k-2} + 4a_{k-1} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}(k+1+r)\,(k+r) - 2a_{k+1}(k+2) - 2a_{k+1}r + a_k + 4a_{k+1} = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1} + 2a_{k+1}r - a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 1$

  $a_{k+2} = \frac{2ka_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{2k a_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0\right]$$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{2k a_{k+1} - a_k + 4a_{k+1}}{(k+3)(k+2)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2k a_{k+1} - a_k + 4a_{k+1}}{(k+3)(k+2)}, a_1 = a_0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = \frac{2k a_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0, b_{k+2} = \frac{2k b_{k+1} - b_k + 4b_{k+1}}{(k+3)(k+2)}, b_1 =$$

1.39.3 Maple trace

```
Methods for second order ODEs:
```

1.39.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 13

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*(x+1)*diff(y(x),x)+(x^2+2*x+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^x x(c_2 x + c_1)$$

1.39.5 Mathematica DSolve solution

Solving time : `0.038` (sec)
Leaf size : 17

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*(x+1)*D[y[x],x]+(x^2+2*x+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^x x(c_2 x + c_1)$$

1.40 problem 42

Internal problem ID [8178]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 42
Date solved : `Monday, October 21, 2024 at 04:54:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2x(x+2)\,y' + \left(x^2+4x+6\right)y = 0$$

1.40.1 Solved as second order ode using Kovacic algorithm

Time used: 0.097 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^2-4x\right)y' + \left(x^2+4x+6\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2-4x \\ C &= x^2+4x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 74: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-4x}{x^2}\,dx} \\ &= z_1 e^{x+2\ln(x)} \\ &= z_1\left(x^2 e^x\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x+4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2 \mathrm{e}^x\right) + c_2 \left(x^2 \mathrm{e}^x (x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.40.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) - 2x(x+2)\, y' + \left(x^2+4x+6\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{\left(x^2+4x+6\right) y}{x^2} + \frac{2(x+2) y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' - \frac{2(x+2) y'}{x} + \frac{\left(x^2+4x+6\right) y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(x+2)}{x}, P_3(x) = \frac{x^2+4x+6}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.\left(x \cdot P_2(x)\right)\right|_{x=0} = -4$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.\left(x^2 \cdot P_3(x)\right)\right|_{x=0} = 6$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 \left(\frac{d}{dx} y'\right) - 2x(x+2)\, y' + \left(x^2+4x+6\right) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k- > k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-3+r)\, x^r + (a_1(-1+r)\,(-2+r) - 2a_0(-2+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{2, 3\}$
- Each term must be 0

  $a_1(-1+r)\,(-2+r) - 2a_0(-2+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = \frac{2a_0}{-1+r}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)\,(k+r-3) - 2a_{k-1}k - 2a_{k-1}r + a_{k-2} + 6a_{k-1} = 0$
- Shift index using $k- > k + 2$

  $a_{k+2}(k+r)\,(k+r-1) - 2a_{k+1}(k+2) - 2a_{k+1}r + a_k + 6a_{k+1} = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1} + 2a_{k+1}r - a_k - 2a_{k+1}}{(k+r)(k+r-1)}$
- Recursion relation for $r = 2$

  $a_{k+2} = \frac{2ka_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2k a_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0\right]$$

- Recursion relation for $r = 3$

$$a_{k+2} = \frac{2k a_{k+1} - a_k + 4a_{k+1}}{(k+3)(k+2)}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{2k a_{k+1} - a_k + 4a_{k+1}}{(k+3)(k+2)}, a_1 = a_0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = \frac{2k a_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0, b_{k+2} = \frac{2k b_{k+1} - b_k + 4b_{k+1}}{(k+3)(k+2)}, b_1 =$$

1.40.3 Maple trace

```
Methods for second order ODEs:
```

1.40.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*(x+2)*diff(y(x),x)+(x^2+4*x+6)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^x x^2 (c_2 x + c_1)$$

1.40.5 Mathematica DSolve solution

Solving time : `0.041` (sec)
Leaf size : 19

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*(x+2)*D[y[x],x]+(x^2+4*x+6)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^x x^2 (c_2 x + c_1)$$

1.41 problem 43

Internal problem ID [8179]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 43
Date solved : `Monday, October 21, 2024 at 04:54:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 4xy' + (x^2+6)\, y = 0$$

1.41.1 Solved as second order ode using Kovacic algorithm

Time used: 0.149 (sec)

Writing the ode as

$$x^2y'' - 4xy' + (x^2+6)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= x^2+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 76: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\
&= z_1 e^{2\ln(x)} \\
&= z_1 \left(x^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2 \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^2 \cos(x)\right) + c_2\left(x^2 \cos(x)\left(\tan(x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.41.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - 4xy' + \left(x^2+6\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$
- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+6)y}{x^2} + \frac{4y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{4y'}{x} + \frac{(x^2+6)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{4}{x}, P_3(x) = \frac{x^2+6}{x^2}\right]$
- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -4$
- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 6$
- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - 4xy' + \left(x^2+6\right)y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-3+r)\, x^r + a_1(-1+r)\,(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-3) + a_{k-2})\, x^{k}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 3\}$

- Each term must be 0

  $a_1(-1+r)\,(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)\,(k+r-3) + a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+r)\,(k+r-1) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+r)(k+r-1)}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 3$

$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

1.41.3 Maple trace

```
Methods for second order ODEs:
```

1.41.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(x^2+6)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^2(c_1 \sin(x) + c_2 \cos(x))$$

1.41.5 Mathematica DSolve solution

Solving time : `0.043` (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(x^2+6)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-ix} x^2 \left(2c_1 - i c_2 e^{2ix}\right)$$

1.42 problem 44

Internal problem ID [8180]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 44
Date solved : `Monday, October 21, 2024 at 04:54:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\,y''-xy'+y=0$$

1.42.1 Solved as second order ode using Kovacic algorithm

Time used: 0.245 (sec)

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1\\ B &= -x\\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 78: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(x-1)} + \frac{3}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{e^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{x\, \mathrm{e}^{x+\ln(x-1)} \mathrm{e}^{-2x}}{x-1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-\frac{x\, \mathrm{e}^{x+\ln(x-1)} \mathrm{e}^{-2x}}{x-1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.42.2 Maple step by step solution

Let's solve

$(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$
- $(x-1)\cdot P_2(x)$ is analytic at $x=1$
  $$\left((x-1)\cdot P_2(x)\right)\Bigg|_{x=1} = -1$$
- $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$
  $$\left((x-1)^2\cdot P_3(x)\right)\Bigg|_{x=1} = 0$$
- $x=1$is a regular singular point
  Check to see if $x_0 = 1$ is a regular singular point
  $x_0 = 1$

- Multiply by denominators
  $(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$
- Change variables using $x = u+1$ so that the regular singular point is at $u=0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions
- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$$
- Shift index using $k->k+1$
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r-1) - a_k(k+r-1)\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0,2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+1+r}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{k+1}$

- Solution for $r=0$

  $$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$$

- Revert the change of variables $u=x-1$

  $$\left[y=\sum_{k=0}^{\infty} a_k(x-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$$

- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+3}$

- Solution for $r=2$

  $$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$$

- Revert the change of variables $u=x-1$

  $$\left[y=\sum_{k=0}^{\infty} a_k(x-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty} a_k(x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$$

1.42.3 Maple trace

```
Methods for second order ODEs:
```

1.42.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2 \,\mathrm{e}^{x}$$

1.42.5 Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 17

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{x} - c_2 x$$

1.43 problem 45

Internal problem ID [8181]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 45
Date solved : `Monday, October 21, 2024 at 04:54:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4x(x+1)\,y' + (2x+3)\,y = 0$$

1.43.1 Solved as second order ode using Kovacic algorithm

Time used: 0.120 (sec)

Writing the ode as

$$4x^2y'' + \left(-4x^2 - 4x\right) y' + (2x+3)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 - 4x \\ C &= 2x + 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 80: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x^2-4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\,\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{x+\ln(x)}}{x}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{x}\right) + c_2 \left(\sqrt{x}\left(\frac{e^{x+\ln(x)}}{x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.43.2 Maple step by step solution

Let's solve
$4x^2\left(\frac{d}{dx}y'\right) - 4x(x+1)\,y' + (2x+3)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{(2x+3)y}{4x^2} + \frac{(x+1)y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{(x+1)y'}{x} + \frac{(2x+3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{x+1}{x}, P_3(x) = \frac{2x+3}{4x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{4}$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$

- Multiply by denominators
  $4x^2\left(\frac{d}{dx}y'\right) - 4x(x+1)\,y' + (2x+3)\,y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

◦ Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)\,(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(2k+2r-3) - 2a_{k-1}(2k+2r-3)\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$(-1+2r)\,(-3+2r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$$
- Each term in the series must be 0, giving the recursion relation
$$4\left(k+r-\frac{3}{2}\right)\left(\left(k+r-\frac{1}{2}\right) a_k - a_{k-1}\right) = 0$$
- Shift index using $k- >k + 1$
$$4\left(k+r-\frac{1}{2}\right)\left(\left(k+\frac{1}{2}+r\right) a_{k+1} - a_k\right) = 0$$
- Recursion relation that defines series solution to ODE
$$a_{k+1} = \frac{2a_k}{2k+1+2r}$$
- Recursion relation for $r = \frac{1}{2}$
$$a_{k+1} = \frac{2a_k}{2k+2}$$
- Solution for $r = \frac{1}{2}$
$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k}{2k+2}\right]$$
- Recursion relation for $r = \frac{3}{2}$

$a_{k+1} = \frac{2a_k}{2k+4}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+4}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{2a_k}{2k+2}, b_{k+1} = \frac{2b_k}{2k+4}\right]$

1.43.3 Maple trace

```
Methods for second order ODEs:
```

1.43.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x*(x+1)*diff(y(x),x)+(2*x+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\,(c_2\,\mathrm{e}^{x} + c_1)$$

1.43.5 Mathematica DSolve solution

Solving time : 0.038 (sec)
Leaf size : 20

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*(x+1)*D[y[x],x]+(2*x+3)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{x}(c_2 e^x + c_1)$$

1.44 problem 46

Internal problem ID [8182]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 46
Date solved : `Monday, October 21, 2024 at 04:54:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(3x-1)\,y''-(3x+2)\,y'-(6x-8)\,y=0$$

1.44.1 Solved as second order ode using Kovacic algorithm

Time used: 0.270 (sec)

Writing the ode as

$$(3x-1)\,y''+(-3x-2)\,y'+(-6x+8)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x-1 \\ B &= -3x-2 \\ C &= -6x+8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{81x^2 - 108x + 54}{4\left(3x - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 81x^2 - 108x + 54$$
$$t = 4(3x - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{81x^2 - 108x + 54}{4\left(3x - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 82: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x-1)^2$. There is a pole at $x = \frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{3}{4\left(x - \frac{1}{3}\right)^2} - \frac{3}{2\left(x - \frac{1}{3}\right)}$$

For the pole at $x = \frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{1}{2x} + \frac{1}{9x^3} + \frac{11}{108x^4} + \frac{7}{108x^5} + \frac{5}{162x^6} + \frac{2}{243x^7} - \frac{13}{3888x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{81x^2 - 108x + 54}{36x^2 - 24x + 4} \\ &= Q + \frac{R}{36x^2 - 24x + 4} \\ &= \left(\frac{9}{4}\right) + \left(\frac{-54x + 45}{36x^2 - 24x + 4}\right) \\ &= \frac{9}{4} + \frac{-54x + 45}{36x^2 - 24x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -54. Dividing this by leading coefficient in t which is 36 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{3}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{81x^2 - 108x + 54}{4(3x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{1}{3}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x - \frac{1}{3}\right)} + \left(\frac{3}{2}\right) \\
&= -\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2} \\
&= \frac{9x - 6}{6x - 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right)(0) + \left(\left(\frac{1}{2\left(x - \frac{1}{3}\right)^2}\right) + \left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right)^2 - \left(\frac{81x^2 - 108x + 54}{4\left(3x - 1\right)^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right) dx} \\
&= \frac{e^{\frac{3x}{2}}}{\sqrt{3x - 1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x-2}{3x-1}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(3x-1)}{2}} \\
&= z_1\left(\sqrt{3x-1}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-2}{3x-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+\ln(3x-1)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(3x-1)}\mathrm{e}^{-4x}}{3x-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}\left(-\frac{x\,\mathrm{e}^{x+\ln(3x-1)}\mathrm{e}^{-4x}}{3x-1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.44.2 Maple step by step solution

Let's solve

$(3x-1)\left(\frac{d}{dx}y'\right)-(3x+2)\,y'-(6x-8)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{2(3x-4)y}{3x-1}+\frac{(3x+2)y'}{3x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{(3x+2)y'}{3x-1}-\frac{2(3x-4)y}{3x-1}=0$

□ Check to see if $x_0=\frac{1}{3}$ is a regular singular point

- ○ Define functions

$\left[P_2(x)=-\frac{3x+2}{3x-1}, P_3(x)=-\frac{2(3x-4)}{3x-1}\right]$

- ○ $\left(x-\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=\frac{1}{3}$

$\left(\left(x-\frac{1}{3}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{1}{3}}=-1$

- ○ $\left(x-\frac{1}{3}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{1}{3}$

$\left(\left(x-\frac{1}{3}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{1}{3}}=0$

- ○ $x=\frac{1}{3}$is a regular singular point

Check to see if $x_0=\frac{1}{3}$ is a regular singular point

$x_0=\frac{1}{3}$

- Multiply by denominators

$(3x-1)\left(\frac{d}{dx}y'\right)+(-3x-2)\,y'+(-6x+8)\,y=0$

- Change variables using $x=u+\frac{1}{3}$ so that the regular singular point is at $u=0$

$3u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-3u-3)\left(\frac{d}{du}y(u)\right)+(-6u+6)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d}{du} \frac{d}{du} y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d}{du} \frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-1}$

- Shift index using $k->k+1$

  $u \cdot \left(\frac{d}{du} \frac{d}{du} y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $3a_0 r(-2+r)\, u^{-1+r} + \left(3a_1(1+r)\,(-1+r) - 3a_0(-2+r)\right) u^r + \left(\sum_{k=1}^{\infty} (3a_{k+1}(k+1+r)\,(k+r-\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $3r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term must be 0

  $3a_1(1+r)\,(-1+r) - 3a_0(-2+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $3a_{k+1}(k+1+r)\,(k+r-1) + a_k(-3k-3r+6) - 6a_{k-1} = 0$
- Shift index using $k->k+1$

  $3a_{k+2}(k+2+r)\,(k+r) + a_{k+1}(-3k+3-3r) - 6a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{k a_{k+1} + r a_{k+1} + 2a_k - a_{k+1}}{(k+2+r)(k+r)}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{k a_{k+1} + 2a_k - a_{k+1}}{(k+2)k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{k a_{k+1} + 2a_k - a_{k+1}}{(k+2)k}$
- Recursion relation for $r = 2$

$a_{k+2} = \frac{k a_{k+1}+2 a_k+a_{k+1}}{(k+4)(k+2)}$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{k a_{k+1}+2 a_k+a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$$

- Revert the change of variables $u = x - \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{1}{3}\right)^{k+2}, a_{k+2} = \frac{k a_{k+1}+2 a_k+a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$$

1.44.3 Maple trace

```
Methods for second order ODEs:
```

1.44.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 18

```
dsolve((3*x-1)*diff(diff(y(x),x),x)-(3*x+2)*diff(y(x),x)-(6*x-8)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \,\mathrm{e}^{2x} + c_2 x \,\mathrm{e}^{-x}$$

1.44.5 Mathematica DSolve solution

Solving time : 0.236 (sec)
Leaf size : 35

```
DSolve[{(3*x-1)*D[y[x],{x,2}]-(3*x+2)*D[y[x],x]-(6*x-8)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x-\frac{1}{2}}(c_1 e^{3x} + 2ec_2 x)}{\sqrt{2}}$$

1.45 problem 47

Internal problem ID [8183]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 47
Date solved : `Monday, October 21, 2024 at 04:54:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2+x)\,y''+xy'+3y=0$$

1.45.1 Solved as second order ode using Kovacic algorithm

Time used: 0.322 (sec)

Writing the ode as

$$(2+x)\,y''+xy'+3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2+x\\ B&=x\\ C&=3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 12x - 20}{4\left(2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 12x - 20$$
$$t = 4(2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 12x - 20}{4\left(2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 84: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2+x)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(2+x)^2} - \frac{4}{2+x}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{4}{x} - \frac{6}{x^2} - \frac{72}{x^3} - \frac{556}{x^4} - \frac{5440}{x^5} - \frac{55088}{x^6} - \frac{586688}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 12x - 20}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-16x - 24}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{-16x - 24}{4x^2 + 16x + 16} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -16. Dividing this by leading coefficient in t which is 4 gives -4. Now b can be found.

$$\begin{aligned}b &= (-4) - (0)\\ &= -4\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2}\\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-4}{\frac{1}{2}} - 0\right) &&= -4\\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-4}{\frac{1}{2}} - 0\right) &&= 4\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 12x - 20}{4\,(2+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-4	4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 4$ then

$$\begin{aligned}d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right)\\ &= 4 - (2)\\ &= 2\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{2}{2+x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{2}{2+x} - \frac{1}{2} \\
&= -\frac{x-2}{2\,(2+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{2}{2+x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{2}{(2+x)^2}\right) + \left(\frac{2}{2+x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 12x - 20}{4\,(2+x)^2}\right)\right) = 0$$

$$\frac{(a_1 + 6)\,x + 2a_0 + 2a_1 + 4}{2+x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4, a_1 = -6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 6x + 4$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x^2 - 6x + 4\right) e^{\int \left(\frac{2}{2+x} - \frac{1}{2}\right) dx} \\
&= \left(x^2 - 6x + 4\right) e^{-\frac{x}{2} + 2\ln(2+x)} \\
&= \left(x^2 - 6x + 4\right) (2+x)^2\, e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{2+x}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\ln(2+x)} \\
&= z_1\left((2+x)\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (2+x)^3\,\mathrm{e}^{-x}\left(x^2-6x+4\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{2+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+2\ln(2+x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{x}(x^4-x^3-18x^2-22x+8)}{240\,(x^2-6x+4)\,(2+x)^3}-\frac{\mathrm{e}^{-2}\,\mathrm{Ei}_1\left(-2-x\right)}{240}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)^3\,\mathrm{e}^{-x}\left(x^2-6x+4\right)\right) + c_2\Bigg((2+x)^3\,\mathrm{e}^{-x}\left(x^2-6x\right. \\
&\qquad \left.+4\right)\left(-\frac{\mathrm{e}^{x}(x^4-x^3-18x^2-22x+8)}{240\,(x^2-6x+4)\,(2+x)^3}-\frac{\mathrm{e}^{-2}\,\mathrm{Ei}_1\left(-2-x\right)}{240}\right)\Bigg)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.45.2 Maple step by step solution

Let's solve

$(2+x)\left(\frac{d}{dx}y'\right)+xy'+3y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{3y}{2+x}-\frac{xy'}{2+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{xy'}{2+x}+\frac{3y}{2+x}=0$

□ Check to see if $x_0=-2$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{x}{2+x}, P_3(x)=\frac{3}{2+x}\right]$

  ◦ $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((2+x)\cdot P_2(x)\right)\Big|_{x=-2}=-2$

  ◦ $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((2+x)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $(2+x)\left(\frac{d}{dx}y'\right)+xy'+3y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)+3y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$$

- Shift index using $k->k+1$

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0r(-3+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r+3)\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r+3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r+3)}{(k+1+r)(k-2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-2)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

$$a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-2)}$$

- Recursion relation for $r = 3$

$$a_{k+1} = -\frac{a_k(k+6)}{(k+4)(k+1)}$$

- Solution for $r = 3$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+6)}{(k+4)(k+1)}\right]$$

- Revert the change of variables $u = 2 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(2+x)^{k+3}, a_{k+1} = -\frac{a_k(k+6)}{(k+4)(k+1)}\right]$$

1.45.3 Maple trace

```
Methods for second order ODEs:
```

1.45.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 72

```
dsolve((2+x)*diff(diff(y(x),x),x)+x*diff(y(x),x)+3*y(x) = 0,
       y(x),singsol=all)
```

$$
\begin{aligned}
y = {} & c_2\,\mathrm{e}^{-2-x}(2+x)^3\left(x^2-6x+4\right)\mathrm{Ei}_1\left(-2-x\right) \\
& + c_1(2+x)^3\,\mathrm{e}^{-x}\left(x^2-6x+4\right)+c_2\left(x^4-x^3-18x^2-22x+8\right)
\end{aligned}
$$

1.45.5 Mathematica DSolve solution

Solving time : 1.095 (sec)
Leaf size : 81

```
DSolve[{(2+x)*D[y[x],{x,2}]+x*D[y[x],x]+3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to \frac{1}{960}e^{-x-1}\big(& c_2\left(x^2-6x+4\right)(x+2)^3\,\mathrm{ExpIntegralEi}(x+2) \\
& + 3840c_1\left(x^2-6x+4\right)(x+2)^3 - c_2e^{x+2}\left(x^4-x^3-18x^2-22x+8\right)\big)
\end{aligned}
$$

1.46 problem 48

Internal problem ID [8184]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 48
Date solved : `Monday, October 21, 2024 at 04:55:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-x)\,y''+x(4+x)\,y'+(2-x)\,y=0$$

1.46.1 Solved as second order ode using Kovacic algorithm

Time used: 0.244 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(x^2+4x\right)y'+(2-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^3+x^2\\B&=x^2+4x\\C&=2-x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-x + 36}{4x\left(-1 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -x + 36 \\
t &= 4x(-1 + x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-x + 36}{4x\left(-1 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 86: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(-1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{9}{-1+x} + \frac{35}{4\left(-1+x\right)^2} + \frac{9}{x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+36}{4x\left(-1+x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-x+36}{4x\left(-1+x\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
1	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{1}{2}-\left(-\frac{3}{2}\right)\\
&= 2
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} - \frac{5}{2\left(-1+x\right)} + (-)\left(0\right) \\
&= \frac{1}{x} - \frac{5}{2\left(-1+x\right)} \\
&= \frac{1}{x} - \frac{5}{-2+2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{1}{x} - \frac{5}{2\left(-1+x\right)}\right)\left(2x + a_1\right) + \left(\left(-\frac{1}{x^2} + \frac{5}{2\left(-1+x\right)^2}\right) + \left(\frac{1}{x} - \frac{5}{2\left(-1+x\right)}\right)^2 - \left(\frac{-x+36}{4x\left(-1+x\right)^2}\right.\right.
$$

$$
\frac{\left(a_1-6\right)x + 4a_0 - 2a_1}{x\left(-1+x\right)}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 3, a_1 = 6\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + 6x + 3
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + 6x + 3\right) e^{\int \left(\frac{1}{x} - \frac{5}{2(-1+x)}\right) dx} \\
&= \left(x^2 + 6x + 3\right) e^{\ln(x) - \frac{5\ln(-1+x)}{2}} \\
&= \frac{\left(x^2 + 6x + 3\right) x}{\left(-1+x\right)^{5/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+4x}{-x^3+x^2}\,dx} \\
&= z_1 e^{-2\ln(x)+\frac{5\ln(-1+x)}{2}} \\
&= z_1\left(\frac{(-1+x)^{5/2}}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2+6x+3}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+4x}{-x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x)+5\ln(-1+x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{4\left(-38x-\frac{69}{2}\right)}{9\left(x^2+6x+3\right)}+\ln\left(x\right)+\frac{1}{9x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2+6x+3}{x}\right) + c_2\left(\frac{x^2+6x+3}{x}\left(-\frac{4\left(-38x-\frac{69}{2}\right)}{9\left(x^2+6x+3\right)}+\ln\left(x\right)+\frac{1}{9x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.46.2 Maple step by step solution

1.46.3 Maple trace

```
Methods for second order ODEs:
```

1.46.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 49

```
dsolve(x^2*(1-x)*diff(diff(y(x),x),x)+x*(4+x)*diff(y(x),x)+(2-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{3xc_2(x^2+6x+3)\ln(x)+c_1x^3+(6c_1+51c_2)x^2+(3c_1+48c_2)x+c_2}{x^2}$$

1.46.5 Mathematica DSolve solution

Solving time : 0.124 (sec)
Leaf size : 53

```
DSolve[{x^2*(1-x)*D[y[x],{x,2}]+x*(4+x)*D[y[x],x]+(2-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3c_1x(x^2+6x+3)-c_2(51x^2+3(x^2+6x+3)x\log(x)+48x+1)}{3x^2}$$

1.47 problem 49

Internal problem ID [8185]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 49
Date solved : `Monday, October 21, 2024 at 04:55:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+x(1+2x)\,y'-(4+6x)\,y=0$$

1.47.1 Solved as second order ode using Kovacic algorithm

Time used: 0.254 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(2x^2+x\right)y'+(-6x-4)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= 2x^2+x \\ C &= -6x-4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{24x^2 + 40x + 15}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 24x^2 + 40x + 15$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{24x^2 + 40x + 15}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 87: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{5}{2x} - \frac{5}{2(1+x)} + \frac{15}{4x^2} - \frac{1}{4(1+x)^2}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{24x^2 + 40x + 15}{4(x^2 + x)^2}
$$

Since the gcd$(s,t)=1$. This gives $b=6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=3 \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{24x^2+40x+15}{4\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=3$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{+}\right) \\
&= 3-(3) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2 + 2x} + \frac{5}{2x} + (0) \\
&= \frac{1}{2 + 2x} + \frac{5}{2x} \\
&= \frac{6x + 5}{2x\,(1 + x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2 + 2x} + \frac{5}{2x}\right)(0) + \left(\left(-\frac{1}{2\,(1 + x)^2} - \frac{5}{2x^2}\right) + \left(\frac{1}{2 + 2x} + \frac{5}{2x}\right)^2 - \left(\frac{24x^2 + 40x + 15}{4\,(x^2 + x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2+2x} + \frac{5}{2x}\right) dx} \\
&= \sqrt{1 + x}\, x^{5/2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^2 + x}{x^2 (1+x)} \, dx} \\
&= z_1 e^{-\frac{\ln(x(1+x))}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x\,(1 + x)}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\sqrt{1+x}\, x^{5/2}}{\sqrt{x\,(1+x)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+x}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x(1+x))}}{(y_1)^2}\, dx \\
&= y_1 \left(-\ln(1+x) - \frac{1}{4x^4} - \frac{1}{2x^2} + \ln(x) + \frac{1}{3x^3} + \frac{1}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{1+x}\, x^{5/2}}{\sqrt{x\,(1+x)}}\right) + c_2 \left(\frac{\sqrt{1+x}\, x^{5/2}}{\sqrt{x\,(1+x)}} \left(-\ln(1+x) - \frac{1}{4x^4} - \frac{1}{2x^2} + \ln(x) + \frac{1}{3x^3} + \frac{1}{x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.47.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right) + x(1+2x)\,y' - (4+6x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{2(2+3x)y}{x^2(1+x)} - \frac{(1+2x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(1+2x)y'}{x(1+x)} - \frac{2(2+3x)y}{x^2(1+x)} = 0$

- □ Check to see if x_0 is a regular singular point
  - ○ Define functions

    $\left[P_2(x) = \frac{1+2x}{x(1+x)}, P_3(x) = -\frac{2(2+3x)}{x^2(1+x)}\right]$
  - ○ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $$\left((1+x) \cdot P_2(x)\right)\Bigg|_{x=-1} = 1$$
  - ○ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $$\left((1+x)^2 \cdot P_3(x)\right)\Bigg|_{x=-1} = 0$$
  - ○ $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$
- • Multiply by denominators

  $x^2(1+x)\left(\frac{d}{dx}y'\right) + x(1+2x)\,y' + (-6x-4)\,y = 0$
- • Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - 2u^2 + u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (2u^2 - 3u + 1)\left(\frac{d}{du}y(u)\right) + (-6u + 2)\,y(u) = 0$
- • Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$
- □ Rewrite ODE with series expansions
  - ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$
  - ○ Shift index using $k- >k - m$

    $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$
  - ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$
  - ○ Shift index using $k- >k + 1 - m$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$
  - ○ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 u^{-1+r} + \left(a_1(1+r)^2 - a_0(2r^2+r-2)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k^2+4kr+2r^2+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term must be 0

$$a_1(1+r)^2 - a_0(2r^2+r-2) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2-k-6)\, a_{k-1} + (-2k^2-k+2)\, a_k + a_{k+1}(k+1)^2 = 0$$

- Shift index using $k->k+1$

$$\left((k+1)^2-k-7\right) a_k + \left(-2(k+1)^2-k+1\right) a_{k+1} + a_{k+2}(k+2)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}, a_1 + 2a_0 = 0\right]$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}, a_1 + 2a_0 = 0\right]$$

1.47.3 Maple trace

```
Methods for second order ODEs:
```

1.47.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 46

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)+x*(1+2*x)*diff(y(x),x)-(4+6*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x^2 + \frac{c_2(12 \ln(1+x) x^4 - 12 \ln(x) x^4 - 12x^3 + 6x^2 - 4x + 3)}{x^2}$$

1.47.5 Mathematica DSolve solution

Solving time : 0.082 (sec)
Leaf size : 52

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+x*(1+2*x)*D[y[x],x]-(4+6*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x^2 + \frac{c_2(12x^4 \log(x) - 12x^4 \log(x+1) + 12x^3 - 6x^2 + 4x - 3)}{12x^2}$$

1.48 problem 50

Internal problem ID [8186]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 50
Date solved : `Monday, October 21, 2024 at 04:55:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(2x^2+1\right)y''+x\left(2x^2+4\right)y'+2\left(-x^2+1\right)y=0$$

1.48.1 Solved as second order ode using Kovacic algorithm

Time used: 0.373 (sec)

Writing the ode as

$$\left(2x^4+x^2\right)y''+\left(2x^3+4x\right)y'+\left(-2x^2+2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+x^2 \\ B &= 2x^3+4x \\ C &= -2x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3x^2 - 9}{(2x^2 + 1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3x^2 - 9 \\
t &= (2x^2 + 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3x^2 - 9}{(2x^2 + 1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 89: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{21}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} + \frac{15i\sqrt{2}}{16\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{15i\sqrt{2}}{16\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 9}{\left(2x^2 + 1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 9}{\left(2x^2 + 1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{4\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x + \frac{i\sqrt{2}}{2}\right)} + (-)(0) \\
&= -\frac{3}{4\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x + \frac{i\sqrt{2}}{2}\right)} \\
&= -\frac{3x}{2x^2 + 1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x + \frac{i\sqrt{2}}{2}\right)} \right)(1) + \left(\left(\frac{3}{4\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{3}{4\left(x + \frac{i\sqrt{2}}{2}\right)^2} \right) + \left(-\frac{3}{4\left(x - \frac{i\sqrt{2}}{2}\right)} - \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)}-\frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)}\right)dx} \\
&= (x)\, \frac{1}{\left(\left(i\sqrt{2}-2x\right)\left(2x+i\sqrt{2}\right)\right)^{3/4}} \\
&= \frac{x}{\left(-4x^2-2\right)^{3/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+4x}{2x^4+x^2}\, dx} \\
&= z_1 e^{-2\ln(x)+\frac{3\ln\left(2x^2+1\right)}{4}} \\
&= z_1 \left(\frac{\left(2x^2+1\right)^{3/4}}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{2^{1/4}\left(4x^2+2\right)^{3/4}}{2x\left(-4x^2-2\right)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+4x}{2x^4+x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-4\ln(x)+\frac{3\ln\left(2x^2+1\right)}{2}}}{\left(y_1\right)^2}\, dx
\end{aligned}$$

$$
= y_1 \left(-\frac{2i(2x^4 - x^2 - 1)\sqrt{2x^2+1}\,\sqrt{2}\,\sqrt{\frac{(-4x^2-2)(4x^2+2)}{(2x^2+1)^2}}}{x\sqrt{-4x^2-2}\,\sqrt{4x^2+2}} - \frac{6i\,\mathrm{arcsinh}\left(\sqrt{2}\,x\right)\sqrt{\frac{(-4x^2-2)(4x^2+2)}{(2x^2+1)^2}}\,(2x^2+1)}{\sqrt{-4x^2-2}\,\sqrt{4x^2+2}} \right)
$$

Therefore the solution is

$$
y = c_1 y_1 + c_2 y_2
$$

$$
= c_1 \left(\frac{2^{1/4}(4x^2+2)^{3/4}}{2x\left(-4x^2-2\right)^{3/4}} \right) + c_2 \left(\frac{2^{1/4}(4x^2+2)^{3/4}}{2x\left(-4x^2-2\right)^{3/4}} \left(-\frac{2i(2x^4 - x^2 - 1)\sqrt{2x^2+1}\,\sqrt{2}\,\sqrt{\frac{(-4x^2-2)(4x^2+2)}{(2x^2+1)^2}}}{x\sqrt{-4x^2-2}\,\sqrt{4x^2+2}} - \frac{6i
$$

Will add steps showing solving for IC soon.

1.48.2 Maple step by step solution

Let's solve

$x^2(2x^2+1)\left(\frac{d}{dx}y'\right) + x(2x^2+4)\,y' + 2(-x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{2(x^2-1)y}{x^2(2x^2+1)} - \frac{2(x^2+2)y'}{x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2(x^2+2)y'}{x(2x^2+1)} - \frac{2(x^2-1)y}{x^2(2x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2(x^2+2)}{x(2x^2+1)}, P_3(x) = -\frac{2(x^2-1)}{x^2(2x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 2$$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(2x^2+1)\left(\frac{d}{dx}y'\right) + 2x(x^2+2)\,y' + (-2x^2+2)\,y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ○ Shift index using $k- >k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - ○ Shift index using $k- >k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - ○ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

    $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - ○ Shift index using $k- >k+2-m$

    $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(1+r)\,x^r + a_1(3+r)\,(2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)\,(k+r+1) + 2a_{k-2}(k+r-1\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$

- Each term must be 0
  $a_1(3+r)(2+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+2)(k+r+1) + 2a_{k-2}(k+r-1)(k-3+r) = 0$
- Shift index using $k->k+2$
  $a_{k+2}(k+4+r)(k+3+r) + 2a_k(k+r+1)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2a_k(k+r+1)(k+r-1)}{(k+4+r)(k+3+r)}$
- Recursion relation for $r = -2$
  $a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}$
- Solution for $r = -2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = -1$; series terminates at $k = 2$
  $a_{k+2} = -\frac{2a_k k(k-2)}{(k+3)(k+2)}$
- Solution for $r = -1$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k k(k-2)}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{2b_k k(k-2)}{(k+3)(k+2)}, b_1 = 0\right]$

1.48.3 Maple trace

```
Methods for second order ODEs:
```

1.48.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 42

```
dsolve(x^2*(2*x^2+1)*diff(diff(y(x),x),x)+x*(2*x^2+4)*diff(y(x),x)+2*(-x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\sqrt{2}\,(x-1)\,(x+1)\,\sqrt{2x^2+1}+x\left(3c_2 \operatorname{arcsinh}\left(\sqrt{2}\,x\right)+c_1\right)}{x^2}$$

1.48.5 Mathematica DSolve solution

Solving time : `0.333` (sec)
Leaf size : 70

```
DSolve[{x^2*(1+2*x^2)*D[y[x],{x,2}]+x*(4+2*x^2)*D[y[x],x]+2*(1-x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3c_2\text{arctanh}\left(\frac{x}{\sqrt{x^2+\frac{1}{2}}}\right)}{\sqrt{2}x} - \frac{c_2\sqrt{2x^2+1}}{x^2} + c_2\sqrt{2x^2+1} + \frac{c_1}{x}$$

1.49 problem 51

Internal problem ID [8187]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 51
Date solved : `Monday, October 21, 2024 at 04:55:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+2)\,y''+2x(x^2+5)\,y'+2(-x^2+3)\,y=0$$

1.49.1 Solved as second order ode using Kovacic algorithm

Time used: 0.508 (sec)

Writing the ode as

$$\left(x^4+2x^2\right)y''+\left(2x^3+10x\right)y'+\left(-2x^2+6\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+2x^2\\ B &= 2x^3+10x\\ C &= -2x^2+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^4 - 5x^2 + 3}{(x^3 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^4 - 5x^2 + 3$$
$$t = (x^3 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^4 - 5x^2 + 3}{(x^3 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 91: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = \left(x^3 + 2x\right)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2} + \frac{21}{16\left(x - i\sqrt{2}\right)^2} + \frac{21}{16\left(x + i\sqrt{2}\right)^2} + \frac{11i\sqrt{2}}{32\left(x - i\sqrt{2}\right)} - \frac{11i\sqrt{2}}{32\left(x + i\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x=-i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{4}\\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{2x^4-5x^2+3}{(x^3+2x)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=2\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{2x^4-5x^2+3}{(x^3+2x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$i\sqrt{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i\sqrt{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= 2 - (0) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{3}{2x} - \frac{3}{4\left(x-i\sqrt{2}\right)} - \frac{3}{4\left(x+i\sqrt{2}\right)} + (0) \\ &= \frac{3}{2x} - \frac{3}{4\left(x-i\sqrt{2}\right)} - \frac{3}{4\left(x+i\sqrt{2}\right)} \\ &= \frac{3}{x^3+2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2x} - \frac{3}{4\left(x-i\sqrt{2}\right)} - \frac{3}{4\left(x+i\sqrt{2}\right)}\right)(2x+a_1) + \left(\left(-\frac{3}{2x^2} + \frac{3}{4\left(x-i\sqrt{2}\right)^2} + \frac{3}{4\left(x+i\sqrt{2}\right)^2}\right) + \right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2+8\right) \mathrm{e}^{\int \left(\frac{3}{2x} - \frac{3}{4\left(x - i\sqrt{2}\right)} - \frac{3}{4\left(x+i\sqrt{2}\right)}\right) dx} \\
&= \left(x^2+8\right) \mathrm{e}^{-\frac{3\ln\left(x^2+2\right)}{4} + \frac{3\ln(x)}{2}} \\
&= \frac{\left(x^2+8\right) x^{3/2}}{\left(x^2+2\right)^{3/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^3+10x}{x^4+2x^2} \, dx} \\
&= z_1 e^{\frac{3\ln\left(x^2+2\right)}{4} - \frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{\left(x^2+2\right)^{3/4}}{x^{5/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2+8}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+10x}{x^4+2x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{3\ln\left(x^2+2\right)}{2} - 5\ln(x)}}{\left(y_1\right)^2} \, dx
\end{aligned}$$

$$= y_1 \left(-\frac{(x^2+2)^{5/2}}{256x^2} + \frac{(x^2+2)^{3/2}}{384} + \frac{\sqrt{x^2+2}}{96} - \frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)}{64} + \frac{3\sqrt{x^2+2}}{64\,(x^2+8)} + \frac{x^2\sqrt{x^2+2}}{768} \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x^2+8}{x} \right) + c_2 \left(\frac{x^2+8}{x} \left(-\frac{(x^2+2)^{5/2}}{256x^2} + \frac{(x^2+2)^{3/2}}{384} + \frac{\sqrt{x^2+2}}{96} - \frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)}{64} + \frac{3\sqrt{x^2+2}}{64\,(x^2+8)} + \frac{x^2\sqrt{x^2+2}}{768} \right) \right)$$

Will add steps showing solving for IC soon.

1.49.2 Maple step by step solution

Let's solve

$x^2(x^2+2)\left(\frac{d}{dx}y'\right) + 2x(x^2+5)\,y' + 2(-x^2+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{2(x^2-3)y}{x^2(x^2+2)} - \frac{2(x^2+5)y'}{x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2(x^2+5)y'}{x(x^2+2)} - \frac{2(x^2-3)y}{x^2(x^2+2)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2(x^2+5)}{x(x^2+2)}, P_3(x) = -\frac{2(x^2-3)}{x^2(x^2+2)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 5$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 3$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2(x^2+2)\left(\frac{d}{dx}y'\right) + 2x(x^2+5)\,y' + (-2x^2+6)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$2a_0(3+r)\,(1+r)\,x^r + 2a_1(4+r)\,(2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(2a_k(k+r+3)\,(k+r+1) + a_{k-2}(k+r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2(3+r)\,(1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-3, -1\}$

- Each term must be 0

  $2a_1(4+r)(2+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $2a_k(k+r+3)(k+r+1) + a_{k-2}(k+r)(k-3+r) = 0$
- Shift index using $k- >k+2$

  $2a_{k+2}(k+5+r)(k+r+3) + a_k(k+r+2)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+2)(k+r-1)}{2(k+5+r)(k+r+3)}$
- Recursion relation for $r = -3$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}$
- Solution for $r = -3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}, a_1 = 0\right]$
- Recursion relation for $r = -1$; series terminates at $k = 2$

  $a_{k+2} = -\frac{a_k(k+1)(k-2)}{2(k+4)(k+2)}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k(k+1)(k-2)}{2(k+4)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}, a_1 = 0, b_{k+2} = -\frac{b_k(k+1)(k-2)}{2(k+4)(k+2)}, b_1 = 0\right]$

1.49.3 Maple trace

```
Methods for second order ODEs:
```

1.49.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 53

```
dsolve(x^2*(x^2+2)*diff(diff(y(x),x),x)+2*x*(x^2+5)*diff(y(x),x)+2*(-x^2+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-(x-2)\, c_2 (x+2) \sqrt{x^2+2}\, \sqrt{2} + (x^2+8) \left(\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right) c_2 + c_1\right) x^2}{x^3}$$

1.49.5 Mathematica DSolve solution

Solving time : 0.391 (sec)
Leaf size : 88

```
DSolve[{x^2*(2+x^2)*D[y[x],{x,2}]+2*x*(x^2+5)*D[y[x],x]+2*(3-x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-\sqrt{2} c_2 (x^2+8)\, x^2 \operatorname{arctanh}\left(\frac{\sqrt{x^2+2}}{\sqrt{2}}\right) + 64 c_1 x^4 + 2x^2 \left(c_2 \sqrt{x^2+2} + 256 c_1\right) - 8 c_2 \sqrt{x^2+2}}{64 x^3}$$

1.50 problem 52

Internal problem ID [8188]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 52
Date solved : `Monday, October 21, 2024 at 04:55:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right)y''+6xy'+6y=0$$

1.50.1 Solved as second order ode using Kovacic algorithm

Time used: 0.283 (sec)

Writing the ode as

$$\left(x^2+1\right)y''+6xy'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 6x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 93: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 0\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-i)} + \frac{3}{2(x+i)} + (-)(0) \\
&= -\frac{1}{2(x-i)} + \frac{3}{2(x+i)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{\left(x^2+1\right)^{3/2}}{\left(ix+1\right)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{6x}{x^2+1} \, dx} \\
&= z_1 e^{-\frac{3 \ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{1}{\left(x^2+1\right)^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\left(ix+1\right)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x}{(x+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{(ix+1)^2}\right) + c_2 \left(\frac{1}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.50.2 Maple step by step solution

1.50.3 Maple trace

```
Methods for second order ODEs:
```

1.50.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 24

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+6*x*diff(y(x),x)+6*y(x) = 0,
       y(x),singsol=all)
```

$$
y = \frac{c_2\, x^2 + c_1 x - c_2}{(x^2+1)^2}
$$

1.50.5 Mathematica DSolve solution

Solving time : 0.073 (sec)
Leaf size : 29

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+6*x*D[y[x],x]+6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x - c_1 (x - i)^2}{(x^2 + 1)^2}$$

1.51 problem 53

Internal problem ID [8189]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 53
Date solved : `Monday, October 21, 2024 at 04:55:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''+2xy'-2y=0$$

1.51.1 Solved as second order ode using Kovacic algorithm

Time used: 0.287 (sec)

Writing the ode as

$$\left(x^2+1\right) y''+2xy'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 + 3}{(x^2 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 + 3$$
$$t = (x^2 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 + 3}{(x^2 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 94: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{5i}{4(x-i)} + \frac{5i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+3}{(x^2+1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{2x^2+3}{(x^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= 2-(1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} + (0) \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} \\
&= \frac{x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)(1) + \left(\left(-\frac{1}{2(x-i)^2} - \frac{1}{2(x+i)^2}\right) + \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)^2 - \left(\frac{2x^2+3}{(x^2+1)^2}\right.\right.
$$

$$
-\frac{2(x^2+1)\,a_0}{(-x+i)^2\,(x+i)^2}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x)\, e^{\int \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right) dx} \\
&= (x)\sqrt{(-x+i)(x+i)} \\
&= x\sqrt{-x^2-1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x^2+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = ix
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x^2+1)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{1}{x} + \arctan\left(x\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(ix) + c_2\left(ix\left(\frac{1}{x} + \arctan\left(x\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.51.2 Maple step by step solution

1.51.3 Maple trace

```
Methods for second order ODEs:
```

1.51.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 14

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+2*x*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1x + \arctan(x)\,xc_2 + c_2$$

1.51.5 Mathematica DSolve solution

Solving time : `0.032` (sec)
Leaf size : 48

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+2*x*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}i(2c_1x - c_2x\log(1 - ix) + c_2x\log(1 + ix) + 2ic_2)$$

1.52 problem 54

Internal problem ID [8190]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 54
Date solved : `Monday, October 21, 2024 at 04:55:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y'' - 8xy' + 20y = 0$$

1.52.1 Solved as second order ode using Kovacic algorithm

Time used: 0.316 (sec)

Writing the ode as

$$\left(x^2+1\right) y'' - 8xy' + 20y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -8x \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-24}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -24$$
$$t = (x^2+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{24}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 95: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{(x-i)^2} + \frac{6}{(x+i)^2} + \frac{6i}{x-i} - \frac{6i}{x+i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{24}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	3	-2
$-i$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x-i} + \frac{3}{x+i} + (-)(0) \\ &= -\frac{2}{x-i} + \frac{3}{x+i} \\ &= \frac{x-5i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x-i} + \frac{3}{x+i}\right)(0) + \left(\left(\frac{2}{(x-i)^2} - \frac{3}{(x+i)^2}\right) + \left(-\frac{2}{x-i} + \frac{3}{x+i}\right)^2 - \left(-\frac{24}{(x^2+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{2}{x-i} + \frac{3}{x+i}\right) dx} \\
&= \frac{(x^2+1)^3}{(ix+1)^5}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-8x}{x^2+1} \, dx} \\
&= z_1 e^{2\ln(x^2+1)} \\
&= z_1\left(\left(x^2+1\right)^2\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^5}{(ix+1)^5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-8x}{x^2+1}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{4\ln(x^2+1)}}{(y_1)^2}\,dx$$
$$= y_1 \left(\frac{x^4 - 2x^2 + \frac{1}{5}}{(x+i)^5}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{(x^2+1)^5}{(ix+1)^5}\right) + c_2 \left(\frac{(x^2+1)^5}{(ix+1)^5} \left(\frac{x^4 - 2x^2 + \frac{1}{5}}{(x+i)^5}\right)\right)$$

Will add steps showing solving for IC soon.

1.52.2 Maple step by step solution

1.52.3 Maple trace

```
Methods for second order ODEs:
```

1.52.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 33

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-8*x*diff(y(x),x)+20*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2\, x^5 + 5c_1\, x^4 - 10c_2\, x^3 - 10c_1\, x^2 + 5c_2 x + c_1$$

1.52.5 Mathematica DSolve solution

Solving time : `0.114` (sec)
Leaf size : 38

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-8*x*D[y[x],x]+20*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{5} i c_2 (5x^4 - 10x^2 + 1) + c_1 (1 + ix)^5$$

1.53 problem 55

Internal problem ID [8191]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 55
Date solved : `Monday, October 21, 2024 at 04:55:07 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-8xy'-12y=0$$

1.53.1 Solved as second order ode using Kovacic algorithm

Time used: 0.266 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-8xy'-12y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -8x \\ C &= -12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8}{(x^2-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 8$$
$$t = (x^2-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8}{(x^2-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 96: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x+1} - \frac{2}{x-1} + \frac{2}{(x+1)^2} + \frac{2}{(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	2	-1
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x-1} + \frac{2}{x+1} + (-)\,(0) \\
&= -\frac{1}{x-1} + \frac{2}{x+1} \\
&= \frac{x-3}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-1} + \frac{2}{x+1}\right)(0) + \left(\left(\frac{1}{(x-1)^2} - \frac{2}{(x+1)^2}\right) + \left(-\frac{1}{x-1} + \frac{2}{x+1}\right)^2 - \left(\frac{8}{(x^2-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{x-1} + \frac{2}{x+1}\right) dx} \\ &= \frac{(x+1)^2}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x}{-x^2+1}\, dx} \\ &= z_1 e^{-2\ln(x-1) - 2\ln(x+1)} \\ &= z_1 \left(\frac{1}{(x-1)^2 (x+1)^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(x-1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x-1)-4\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(x+1)\,(3x^2+1)\,(x-1)^4\,\mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{(x-1)^3}\right) + c_2 \left(\frac{1}{(x-1)^3} \left(-\frac{(x+1)\,(3x^2+1)\,(x-1)^4\,\mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.53.2 Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d}{dx}y'\right) - 8xy' - 12y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{12y}{x^2-1} - \frac{8xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{8xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{8x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = 4$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$
- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d}{dx}y'\right) + 8xy' + 12y = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (8u-8)\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0 r(3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r+4) + a_k(k+r+4)(k+r+3)\right) u^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r+4)\left((-2k-2r-2)\, a_{k+1} + a_k(k+r+3)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+3)}{2(k+1+r)}$
- Recursion relation for $r = -3$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$
- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+3)}{2(k+1)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$
- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

1.53.3 Maple trace

```
Methods for second order ODEs:
```

1.53.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 29

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-8*x*diff(y(x),x)-12*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 x^3 + 3c_1 x^2 + 3c_2 x + c_1}{(x^2 - 1)^3}$$

1.53.5 Mathematica DSolve solution

Solving time : 0.068 (sec)
Leaf size : 37

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-8*x*D[y[x],x]-12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3c_1(x-1)^3 - c_2(3x^2+1)}{3\left(x^2-1\right)^3}$$

1.54 problem 56

Internal problem ID [8192]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 56
Date solved : `Monday, October 21, 2024 at 04:55:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+1\right)y''+7xy'+2y=0$$

1.54.1 Solved as second order ode using Kovacic algorithm

Time used: 0.434 (sec)

Writing the ode as

$$\left(2x^2+1\right)y''+7xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^2+1\\ B&=7x\\ C&=2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 6}{4\left(2x^2 + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 + 6$$
$$t = 4\left(2x^2 + 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 6}{4\left(2x^2 + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 98: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{7}{64\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{17i\sqrt{2}}{64\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{17i\sqrt{2}}{64\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2+6}{4\left(2x^2+1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2+6}{4\left(2x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{5}{4} - \left(\frac{1}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} + (0) \\
&= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} \\
&= \frac{x}{4x^2 + 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}\right)(1) + \left(\left(-\frac{1}{8\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{8\left(x + \frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, e^{\int \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}\right) dx} \\
&= (x) \left(\left(i\sqrt{2} - 2x\right) \left(2x + i\sqrt{2}\right) \right)^{1/8} \\
&= x \left(-4x^2 - 2\right)^{1/8}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x}{2x^2+1}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^2+1\right)}{8}} \\
&= z_1 \left(\frac{1}{\left(2x^2+1\right)^{7/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{7/8} x(-4x^2-2)^{1/8}}{\left(4x^2+2\right)^{7/8}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x}{2x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^2+1\right)}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\left(2x^2+1\right)^{7/4} x^2 \left(-4x^2-2\right)^{1/4}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2^{7/8} x(-4x^2-2)^{1/8}}{\left(4x^2+2\right)^{7/8}}\right) + c_2 \left(\frac{2^{7/8} x(-4x^2-2)^{1/8}}{\left(4x^2+2\right)^{7/8}}\left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\left(2x^2+1\right)^{7/4} x^2 \left(-4x^2-2\right)^{1/4}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.54.2 Maple step by step solution

1.54.3 Maple trace

```
Methods for second order ODEs:
```

1.54.4 Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 37

```
dsolve((2*x^2+1)*diff(diff(y(x),x),x)+7*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \operatorname{LegendreP}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\, x\right) + c_2 \operatorname{LegendreQ}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\, x\right)}{\left(2x^2+1\right)^{3/8}}$$

1.54.5 Mathematica DSolve solution

Solving time : 0.096 (sec)
Leaf size : 66

```
DSolve[{(1+2*x^2)*D[y[x],{x,2}]+7*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 Q_{\frac{1}{4}}^{\frac{3}{4}}\left(i\sqrt{2}x\right)}{\left(2x^2+1\right)^{3/8}} + \frac{2i\sqrt[4]{2}c_1 x}{\left(2x^2+1\right)^{3/4} \operatorname{Gamma}\left(\frac{1}{4}\right)}$$

1.55 problem 57

Internal problem ID [8193]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 57
Date solved : `Monday, October 21, 2024 at 04:55:09 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-5 x y'-4 y=0$$

1.55.1 Solved as second order ode using Kovacic algorithm

Time used: 0.277 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-5 x y'-4 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -5x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-x^2 + 6}{4\left(x^2 - 1\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -x^2 + 6 \\
t &= 4\left(x^2 - 1\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-x^2 + 6}{4\left(x^2 - 1\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 99: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{16(x-1)} + \frac{5}{16(x+1)^2} + \frac{7}{16(x+1)} + \frac{5}{16(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2+6}{4(x^2-1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^2+6}{4\left(x^2-1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
-1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(x+1\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(x+1\right)} \\
&= -\frac{x}{2x^{2}-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^{2} - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(x+1\right)}\right)(1) + \left(\left(\frac{1}{4\left(x-1\right)^{2}} + \frac{1}{4\left(x+1\right)^{2}}\right) + \left(-\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(x+1\right)}\right)^{2} - \Big(
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x\right) \mathrm{e}^{\int \left(-\frac{1}{4(x-1)} - \frac{1}{4(x+1)}\right) dx} \\
&= \left(x\right) \frac{1}{\left(\left(x-1\right)\left(x+1\right)\right)^{1/4}} \\
&= \frac{x}{\left(x^{2}-1\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x}{-x^2+1}\,dx} \\
&= z_1 e^{-\frac{5\ln(x-1)}{4}-\frac{5\ln(x+1)}{4}} \\
&= z_1 \left(\frac{1}{(x-1)^{5/4}(x+1)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{(x-1)^{5/4}(x+1)^{5/4}(x^2-1)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x-1)}{2}-\frac{5\ln(x+1)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(x^2-1)^{3/2}}{x} - x\sqrt{x^2-1} + \ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{(x-1)^{5/4}(x+1)^{5/4}(x^2-1)^{1/4}}\right) + c_2 \left(\frac{x}{(x-1)^{5/4}(x+1)^{5/4}(x^2-1)^{1/4}}\left(\frac{(x^2-1)^{3/2}}{x} - x\sqrt{x^2-1} +
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.55.2 Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d}{dx}y'\right)-5xy'-4y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{4y}{x^2-1}-\frac{5xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{5xy'}{x^2-1}+\frac{4y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{5x}{x^2-1}, P_3(x)=\frac{4}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=\frac{5}{2}$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d}{dx}y'\right)+5xy'+4y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(5u-5)\left(\frac{d}{du}y(u)\right)+4y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(3+2r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+5+2r) + a_k(k+r+2)^2\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+2)^2 - 2\left(k+\frac{5}{2}+r\right) a_{k+1}(k+1+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r+2)^2}{(2k+5+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k(k+2)^2}{(2k+5)(k+1)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+2)^2}{(2k+5)(k+1)}\right]$$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = \frac{a_k(k+2)^2}{(2k+5)(k+1)}\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k \left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k-\frac{3}{2}}\right), a_{k+1} = \frac{a_k (k+2)^2}{(2k+5)(k+1)}, b_{k+1} = \frac{b_k \left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$$

1.55.3 Maple trace

```
Methods for second order ODEs:
```

1.55.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 39

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-5*x*diff(y(x),x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \ln\left(x + \sqrt{x^2-1}\right) x - \sqrt{x^2-1}\, c_2 + c_1 x}{\left(x^2-1\right)^{3/2}}$$

1.55.5 Mathematica DSolve solution

Solving time : 0.164 (sec)
Leaf size : 49

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-5*x*D[y[x],x]-4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x \operatorname{arctanh}\left(\frac{x}{\sqrt{x^2-1}}\right) - c_2 \sqrt{x^2-1} + c_1 x}{\left(x^2-1\right)^{3/2}}$$

1.56 problem 58

Internal problem ID [8194]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 58
Date solved : `Monday, October 21, 2024 at 04:55:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-10 x y'+28 y=0$$

1.56.1 Solved as second order ode using Kovacic algorithm

Time used: 0.366 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-10 x y'+28 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -10 x \\ C &= 28 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{2x^2 - 33}{(x^2 + 1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 2x^2 - 33 \\
t &= (x^2 + 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{2x^2 - 33}{(x^2 + 1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 101: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4(x-i)^2} + \frac{35}{4(x+i)^2} + \frac{31i}{4(x-i)} - \frac{31i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 33}{(x^2+1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{2x^2-33}{(x^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-i$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= 2-(1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{5}{2\left(x - i\right)} + \frac{7}{2\left(x + i\right)} + (0) \\
&= -\frac{5}{2\left(x - i\right)} + \frac{7}{2\left(x + i\right)} \\
&= \frac{x - 6i}{x^2 + 1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{5}{2\left(x - i\right)} + \frac{7}{2\left(x + i\right)}\right)(1) + \left(\left(\frac{5}{2\left(x - i\right)^2} - \frac{7}{2\left(x + i\right)^2}\right) + \left(-\frac{5}{2\left(x - i\right)} + \frac{7}{2\left(x + i\right)}\right)^2 - \left(\frac{2x^{\cdots}}{(x^{2\cdots}}\right.\right.
$$

$$
-\frac{2(x^2 + 1)\left(6i \cdots\right.}{\left(-x + i\right)^2 \left(x \cdots\right.}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -6i\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 6i
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - 6i\right) e^{\int \left(-\frac{5}{2(x-i)} + \frac{7}{2(x+i)}\right) dx} \\
&= \left(x - 6i\right) e^{\frac{\ln\left(x^2+1\right)}{2} - 6i \arctan(x)} \\
&= \frac{\left(-x + 6i\right)\left(x^2 + 1\right)^{7/2}}{\left(-x + i\right)^6}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10x}{x^2+1}\,dx} \\
&= z_1 e^{\frac{5\ln\left(x^2+1\right)}{2}} \\
&= z_1\left(\left(x^2+1\right)^{5/2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+1\right)^6 (-x+6i)}{(-x+i)^6}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{3125}{823543\,(x-6i)} + \frac{724i}{2401\,(x+i)^4} - \frac{16i}{147\,(x+i)^6} - \frac{3125i}{117649\,(x+i)^2} \right. \\
&\quad \left. + \frac{496}{1715\,(x+i)^5} - \frac{7432}{50421\,(x+i)^3} - \frac{3125}{823543\,(x+i)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x^2+1\right)^6 (-x+6i)}{(-x+i)^6}\right) + c_2\left(\frac{\left(x^2+1\right)^6 (-x+6i)}{(-x+i)^6}\left(\frac{3125}{823543\,(x-6i)} + \frac{724i}{2401\,(x+i)^4} \right.\right. \\
&\quad - \frac{16i}{147\,(x+i)^6} - \frac{3125i}{117649\,(x+i)^2} + \frac{496}{1715\,(x+i)^5} - \frac{7432}{50421\,(x+i)^3} \\
&\quad \left.\left. - \frac{3125}{823543\,(x+i)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.56.2 Maple step by step solution

1.56.3 Maple trace

```
Methods for second order ODEs:
```

1.56.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 39

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-10*x*diff(y(x),x)+28*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 + \frac{35}{3}c_1 x^4 - 14c_1 x^2 + c_2 x^7 + 21c_2 x^5 - 105c_2 x^3 + 35c_2 x$$

1.56.5 Mathematica DSolve solution

Solving time : `0.112` (sec)
Leaf size : 40

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-10*x*D[y[x],x]+28*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{105}c_2\left(35x^4 - 42x^2 + 3\right) - c_1(x - i)^6(x + 6i)$$

1.57 problem 59

Internal problem ID [8195]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 59
Date solved : `Monday, October 21, 2024 at 04:55:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

1.57.1 Solved as second order ode using Kovacic algorithm

Time used: 0.257 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 102: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= (x)\, \mathrm{e}^{\int -\frac{x}{2} dx} \\ &= (x)\, \mathrm{e}^{-\frac{x^2}{4}} \\ &= x\, \mathrm{e}^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} x \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.57.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions

  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - ○ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - ○ Shift index using $k- > k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

1.57.3 Maple trace

```
Methods for second order ODEs:
```

1.57.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -x \left(c_2 \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)\pi - c_1\right) \mathrm{e}^{-\frac{x^2}{2}} + i\sqrt{\pi}\,\sqrt{2}\,c_2$$

1.57.5 Mathematica DSolve solution

Solving time : 0.068 (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}} \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2} c_1 e^{-\frac{x^2}{2}} x + c_2$$

1.58 problem 60

Internal problem ID [8196]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 60
Date solved : `Monday, October 21, 2024 at 04:55:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2x^2+1)\,y''-9xy'-6y=0$$

1.58.1 Solved as second order ode using Kovacic algorithm

Time used: 0.507 (sec)

Writing the ode as

$$(2x^2+1)\,y''-9xy'-6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+1 \\ B &= -9x \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{165x^2 + 6}{4\left(2x^2 + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 165x^2 + 6$$
$$t = 4\left(2x^2 + 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{165x^2 + 6}{4\left(2x^2 + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 104: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{153}{64\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{153}{64\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{177i\sqrt{2}}{64\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{177i\sqrt{2}}{64\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{153}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{8} \end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{153}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{165x^2 + 6}{4\left(2x^2+1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{165}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{15}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{11}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{165x^2 + 6}{4\left(2x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{17}{8}$	$-\frac{9}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{17}{8}$	$-\frac{9}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{15}{4}$	$-\frac{11}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{15}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{15}{4} - \left(-\frac{9}{4}\right) \\
&= 6
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{9}{8\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{9}{8\left(x + \frac{i\sqrt{2}}{2}\right)} + (0) \\
&= -\frac{9}{8\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{9}{8\left(x + \frac{i\sqrt{2}}{2}\right)} \\
&= -\frac{9x}{4x^2 + 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(30x^4 + 20x^3 a_5 + 12x^2 a_4 + 6x a_3 + 2a_2\right) + 2\left(-\frac{9}{8\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{9}{8\left(x + \frac{i\sqrt{2}}{2}\right)} \right) \left(6x^5 + 5x^4 a_5 + 4x^3 a_4 + 3x\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{3}, a_1 = 0, a_2 = 1, a_3 = 0, a_4 = \frac{5}{3}, a_5 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 + \frac{5}{3}x^4 + x^2 + \frac{1}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^6 + \frac{5}{3}x^4 + x^2 + \frac{1}{3}\right) \mathrm{e}^{\int \left(-\frac{9}{8\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{9}{8\left(x+\frac{i\sqrt{2}}{2}\right)}\right) dx} \\
&= \left(x^6 + \frac{5}{3}x^4 + x^2 + \frac{1}{3}\right) \frac{1}{\left(\left(i\sqrt{2} - 2x\right)\left(2x + i\sqrt{2}\right)\right)^{9/8}} \\
&= \frac{-3x^6 - 5x^4 - 3x^2 - 1}{\left(-4x^2 - 2\right)^{1/8}\left(12x^2 + 6\right)}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-9x}{2x^2+1} \, dx} \\
&= z_1 e^{\frac{9\ln\left(2x^2+1\right)}{8}} \\
&= z_1 \left(\left(2x^2 + 1\right)^{9/8}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = -\frac{2^{7/8}\left(4x^2 + 2\right)^{1/8}\left(3x^6 + 5x^4 + 3x^2 + 1\right)}{12\left(-4x^2 - 2\right)^{1/8}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-9x}{2x^2+1} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{9\ln\left(2x^2+1\right)}{4}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{36\left(2x^2 + 1\right)^{9/4} 2^{1/4}\left(-4x^2 - 2\right)^{1/4}}{\left(4x^2 + 2\right)^{1/4}\left(3x^6 + 5x^4 + 3x^2 + 1\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(-\frac{2^{7/8}(4x^2+2)^{1/8}\,(3x^6+5x^4+3x^2+1)}{12\,(-4x^2-2)^{1/8}}\right) + c_2\left(-\frac{2^{7/8}(4x^2+2)^{1/8}\,(3x^6+5x^4+3x^2+1)}{12\,(-4x^2-2)^{1/8}}\left(\int \frac{36}{(4x^2}\right.\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.58.2 Maple step by step solution

1.58.3 Maple trace

```
Methods for second order ODEs:
```

1.58.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 37

```
dsolve((2*x^2+1)*diff(diff(y(x),x),x)-9*x*diff(y(x),x)-6*y(x) = 0,
       y(x),singsol=all)
```

$$
y = (2x^2+1)^{13/8}\left(\text{LegendreP}\left(\frac{11}{4}, \frac{13}{4}, i\sqrt{2}\,x\right)c_1 + \text{LegendreQ}\left(\frac{11}{4}, \frac{13}{4}, i\sqrt{2}\,x\right)c_2\right)
$$

1.58.5 Mathematica DSolve solution

Solving time : `0.373` (sec)
Leaf size : 71

```
DSolve[{(1+2*x^2)*D[y[x],{x,2}]-9*x*D[y[x],x]-6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to c_2(2x^2+1)^{13/8}\,Q_{\frac{11}{4}}^{\frac{13}{4}}\left(i\sqrt{2}x\right) + \frac{64\sqrt[4]{2}c_1(3x^6+5x^4+3x^2+1)}{3\,\text{Gamma}\left(-\frac{9}{4}\right)}
$$

1.59 problem 61

Internal problem ID [8197]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 61
Date solved : `Monday, October 21, 2024 at 04:55:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2-8x+11\right)y''-16(x-2)\,y'+36y=0$$

1.59.1 Solved as second order ode using Kovacic algorithm

Time used: 0.627 (sec)

Writing the ode as

$$\left(2x^2-8x+11\right)y''+(-16x+32)\,y'+36y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2-8x+11 \\ B &= -16x+32 \\ C &= 36 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 8x^2 - 32x - 100$$
$$t = (2x^2 - 8x + 11)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 105: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 - 8x + 11)^2$. There is a pole at $x = 2 + \frac{i\sqrt{6}}{2}$ of order 2. There is a pole at $x = 2 - \frac{i\sqrt{6}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{\left(x - 2 - \frac{i\sqrt{6}}{2}\right)^2} + \frac{6}{\left(x - 2 + \frac{i\sqrt{6}}{2}\right)^2} + \frac{5i\sqrt{6}}{3\left(x - 2 - \frac{i\sqrt{6}}{2}\right)} - \frac{5i\sqrt{6}}{3\left(x - 2 + \frac{i\sqrt{6}}{2}\right)}$$

For the pole at $x = 2 + \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x - 2 - \frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

For the pole at $x = 2 - \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x - 2 + \frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$2 + \frac{i\sqrt{6}}{2}$	2	0	3	-2
$2 - \frac{i\sqrt{6}}{2}$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{2}{x - 2 - \frac{i\sqrt{6}}{2}} + \frac{3}{x - 2 + \frac{i\sqrt{6}}{2}} + (0) \\
&= -\frac{2}{x - 2 - \frac{i\sqrt{6}}{2}} + \frac{3}{x - 2 + \frac{i\sqrt{6}}{2}} \\
&= \frac{-5i\sqrt{6} + 2x - 4}{2x^2 - 8x + 11}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x - 2 - \frac{i\sqrt{6}}{2}} + \frac{3}{x - 2 + \frac{i\sqrt{6}}{2}} \right)(1) + \left(\left(\frac{2}{\left(x - 2 - \frac{i\sqrt{6}}{2}\right)^2} - \frac{3}{\left(x - 2 + \frac{i\sqrt{6}}{2}\right)^2} \right) + \left(-\frac{2}{x - 2 - \frac{i\sqrt{6}}{2}}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = -\frac{5i\sqrt{6}}{2} - 2 \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 2 - \frac{5i\sqrt{6}}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \left(x-2-\frac{5i\sqrt{6}}{2}\right) \mathrm{e}^{\int \left(-\frac{2}{x-2-\frac{i\sqrt{6}}{2}}+\frac{3}{x-2+\frac{i\sqrt{6}}{2}}\right)dx} \\
&= \left(x-2-\frac{5i\sqrt{6}}{2}\right) \mathrm{e}^{\frac{\ln\left(4x^2-16x+22\right)}{2}-5i\arctan\left(\frac{(2x-4)\sqrt{6}}{6}\right)} \\
&= \frac{9\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^3\sqrt{6}}{2\left(-x\sqrt{6}+2\sqrt{6}+3i\right)^5}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-16x+32}{2x^2-8x+11}\,dx} \\
&= z_1 e^{2\ln\left(2x^2-8x+11\right)} \\
&= z_1\left(\left(2x^2-8x+11\right)^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{9\left(2x^2-8x+11\right)^5\left(5\sqrt{6}+2ix-4i\right)\sqrt{6}}{2\left(x\sqrt{6}-2\sqrt{6}-3i\right)^5}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-16x+32}{2x^2-8x+11}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln\left(2x^2-8x+11\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{10i\sqrt{6}}{27\left(2x-4+i\sqrt{6}\right)^4}+\frac{8i\sqrt{6}}{729\left(2x-4+i\sqrt{6}\right)^2}-\frac{16}{15\left(2x-4+i\sqrt{6}\right)^5}\right. \\
&\quad\left.+\frac{22}{81\left(2x-4+i\sqrt{6}\right)^3}+\frac{4}{2187\left(2x-4+i\sqrt{6}\right)}-\frac{4}{2187\left(-5i\sqrt{6}+2x-4\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{9(2x^2-8x+11)^5\left(5\sqrt{6}+2ix-4i\right)\sqrt{6}}{2\left(x\sqrt{6}-2\sqrt{6}-3i\right)^5}\right) \\
&\quad + c_2\left(\frac{9(2x^2-8x+11)^5\left(5\sqrt{6}+2ix-4i\right)\sqrt{6}}{2\left(x\sqrt{6}-2\sqrt{6}-3i\right)^5}\left(-\frac{10i\sqrt{6}}{27\left(2x-4+i\sqrt{6}\right)^4}\right.\right. \\
&\quad + \frac{8i\sqrt{6}}{729\left(2x-4+i\sqrt{6}\right)^2} - \frac{16}{15\left(2x-4+i\sqrt{6}\right)^5} + \frac{22}{81\left(2x-4+i\sqrt{6}\right)^3} \\
&\quad \left.\left. + \frac{4}{2187\left(2x-4+i\sqrt{6}\right)} - \frac{4}{2187\left(-5i\sqrt{6}+2x-4\right)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.59.2 Maple step by step solution

Let's solve

$(2x^2-8x+11)\left(\frac{d}{dx}y'\right)-16(x-2)\,y'+36y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{36y}{2x^2-8x+11}+\frac{16(x-2)y'}{2x^2-8x+11}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{16(x-2)y'}{2x^2-8x+11}+\frac{36y}{2x^2-8x+11}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{16(x-2)}{2x^2-8x+11}, P_3(x)=\frac{36}{2x^2-8x+11}\right]$

- ◦ $\left(x-2+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)$ is analytic at $x=2-\frac{\mathrm{I}\sqrt{6}}{2}$

  $\left(\left(x-2+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)\right)\bigg|_{x=2-\frac{\mathrm{I}\sqrt{6}}{2}}=0$

- ◦ $\left(x-2+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2\cdot P_3(x)$ is analytic at $x=2-\frac{\mathrm{I}\sqrt{6}}{2}$

$$\left(\left(x-2+\frac{I\sqrt{6}}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=2-\frac{I\sqrt{6}}{2}}=0$$

- $x=2-\frac{I\sqrt{6}}{2}$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0=2-\frac{I\sqrt{6}}{2}$

- Multiply by denominators
  $(2x^2-8x+11)\left(\frac{d}{dx}y'\right)+(-16x+32)\,y'+36y=0$
- Change variables using $x=u+2-\frac{I\sqrt{6}}{2}$ so that the regular singular point is at $u=0$
  $(2u^2-2\,Iu\sqrt{6})\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\left(-16u+8\,I\sqrt{6}\right)\left(\frac{d}{du}y(u)\right)+36y(u)=0$
- Assume series solution for $y(u)$
  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$
  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
- Shift index using $k->k+1-m$
  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
- Convert $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$
  $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
- Shift index using $k->k+2-m$
  $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions
  $-2\,I\sqrt{6}\,r(r-5)\,a_0u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2\,I\sqrt{6}\,(k+1+r)\,(k-4+r)\,a_{k+1}+2a_k(k+r-3)\,(k+r-\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2\,I\sqrt{6}\,r(r-5)=0$
- Values of r that satisfy the indicial equation
  $r\in\{0,5\}$
- Each term in the series must be 0, giving the recursion relation
  $-2\,I\sqrt{6}\,(k+1+r)\,(k-4+r)\,a_{k+1}+2a_k(k+r-3)\,(k+r-6)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k (k^2+2kr+r^2-9k-9r+18)\sqrt{6}}{k^2+2kr+r^2-3k-3r-4}$

- Recursion relation for $r = 0$; series terminates at $k = 3$

$a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k (k^2-9k+18)\sqrt{6}}{k^2-3k-4}$

- Apply recursion relation for $k = 0$

$a_1 = \frac{3\,\mathrm{I}}{4} a_0 \sqrt{6}$

- Apply recursion relation for $k = 1$

$a_2 = \frac{5\,\mathrm{I}}{18} a_1 \sqrt{6}$

- Express in terms of a_0

$a_2 = -\frac{5a_0}{4}$

- Apply recursion relation for $k = 2$

$a_3 = \frac{\mathrm{I}}{9} a_2 \sqrt{6}$

- Express in terms of a_0

$a_3 = -\frac{5\,\mathrm{I}}{36} a_0 \sqrt{6}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$y(u) = a_0 \cdot \left(1 + \frac{3\,\mathrm{I}\sqrt{6}\,u}{4} - \frac{5u^2}{4} - \frac{5\,\mathrm{I}\sqrt{6}\,u^3}{36}\right)$

- Revert the change of variables $u = x - 2 + \frac{\mathrm{I}\sqrt{6}}{2}$

$\left[y = -\frac{\mathrm{I}}{72} a_0 \sqrt{6}\,(10x^3 - 60x^2 + 111x - 62)\right]$

- Recursion relation for $r = 5$; series terminates at $k = 1$

$a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k (k^2+k-2)\sqrt{6}}{k^2+7k+6}$

- Apply recursion relation for $k = 0$

$a_1 = \frac{\mathrm{I}}{18} a_0 \sqrt{6}$

- Terminating series solution of the ODE for $r = 5$. Use reduction of order to find the second li

$y(u) = a_0 \cdot \left(1 + \frac{\mathrm{I}\sqrt{6}\,u}{18}\right)$

- Revert the change of variables $u = x - 2 + \frac{\mathrm{I}\sqrt{6}}{2}$

$\left[y = a_0 \left(\frac{5}{6} + \frac{\mathrm{I}(x-2)\sqrt{6}}{18}\right)\right]$

- Combine solutions and rename parameters

$\left[y = -\frac{\mathrm{I} a_0 \sqrt{6}\,(10x^3-60x^2+111x-62)}{72} + b_0 \left(\frac{5}{6} + \frac{\mathrm{I}(x-2)\sqrt{6}}{18}\right)\right]$

1.59.3 Maple trace

```
Methods for second order ODEs:
```

1.59.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 55

```
dsolve((2*x^2-8*x+11)*diff(diff(y(x),x),x)-16*(x-2)*diff(y(x),x)+36*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2 x^6 - 12c_2 x^5 + \frac{165c_2 x^4}{2} + c_1 x^3 + \frac{3(-8c_1 - 1815c_2) x^2}{4} + \frac{3(37c_1 + 10890c_2) x}{10} - \frac{31c_1}{5} - \frac{16577c_2}{8}$$

1.59.5 Mathematica DSolve solution

Solving time : 1.501 (sec)
Leaf size : 91

```
DSolve[{(11-8*x+2*x^2)*D[y[x],{x,2}]-16*(x-2)*D[y[x],x]+36*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{15} i c_2 \left(10x^3 - 60x^2 + 111x - 62\right) + \frac{c_1 \left(2x + 5i\sqrt{6} - 4\right) \left(2(x-4)x + 11\right)^2 \left(2ix + \sqrt{6} - 4i\right)^3}{2\left(-2ix + \sqrt{6} + 4i\right)^2}$$

1.60 problem 62

Internal problem ID [8198]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 62
Date solved : `Monday, October 21, 2024 at 04:55:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + (x-3)\,y' + 3y = 0$$

1.60.1 Solved as second order ode using Kovacic algorithm

Time used: 0.315 (sec)

Writing the ode as

$$y'' + (x-3)\,y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x-3 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6x - 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 107: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 2\\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2} - \frac{5}{2x} - \frac{15}{2x^2} - \frac{115}{4x^3} - \frac{495}{4x^4} - \frac{2285}{4x^5} - \frac{11055}{4x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i\\ &= \frac{x}{2} - \frac{3}{2} \qquad (10)\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 - \frac{3}{2}x + \frac{9}{4}$$

This shows that the coefficient of 1 in the above is $\frac{9}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6x - 1}{4} \\ &= Q + \frac{R}{4} \\ &= \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right) + (0) \\ &= -\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{4}\right) - \left(\frac{9}{4}\right) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} - \frac{3}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}-\frac{3}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} - \frac{3}{2}\right) \\ &= \frac{3}{2} - \frac{x}{2} \\ &= \frac{3}{2} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2} - \frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(\frac{3}{2} - \frac{x}{2}\right)^2 - \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right)\right) = 0$$

$$(x+3)\, a_1 + 6x + 2a_0 + 2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = -6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 6x + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x^2 - 6x + 8\right) \mathrm{e}^{\int \left(\frac{3}{2} - \frac{x}{2}\right) dx} \\
&= \left(x^2 - 6x + 8\right) \mathrm{e}^{\frac{3}{2}x - \frac{1}{4}x^2} \\
&= \left(x^2 - 6x + 8\right) \mathrm{e}^{-\frac{x(-6+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x-3}{1}\, dx} \\
&= z_1 e^{\frac{3}{2}x - \frac{1}{4}x^2} \\
&= z_1 \left(\mathrm{e}^{-\frac{x(-6+x)}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x(-6+x)}{2}} \left(x^2 - 6x + 8\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x-3}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{1}{2}x^2 + 3x}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{1}{2}x^2 + 3x} \mathrm{e}^{x(-6+x)}}{\left(x^2 - 6x + 8\right)^2}\, dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x(-6+x)}{2}}\left(x^2-6x+8\right)\right) + c_2\left(\mathrm{e}^{-\frac{x(-6+x)}{2}}\left(x^2-6x+8\right)\left(\int \frac{\mathrm{e}^{-\frac{1}{2}x^2+3x}\mathrm{e}^{x(-6+x)}}{\left(x^2-6x+8\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.60.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + (x-3)\,y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 3a_{k+1}(k+1) + a_k(k+3)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $k^2 a_{k+2} + \left(a_k - 3a_{k+1} + 3a_{k+2}\right) k + 3a_k - 3a_{k+1} + 2a_{k+2} = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k k - 3a_{k+1}k + 3a_k - 3a_{k+1}}{k^2 + 3k + 2}\right]$$

1.60.3 Maple trace

```
Methods for second order ODEs:
```

1.60.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 73

```
dsolve(diff(diff(y(x),x),x)+(x-3)*diff(y(x),x)+3*y(x) = 0,
       y(x),singsol=all)
```

$$y = (x-2)\,\mathrm{e}^{-\frac{(x-3)^2}{2}}\left(\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{-(x-3)^2}}{2}\right)-1\right)c_2(x-4)\sqrt{\pi} - \sqrt{2}\,\sqrt{-(x-3)^2}\,c_2 - c_1\,\mathrm{e}^{-\frac{(x-3)^2}{2}}(x-2)(x-4)$$

1.60.5 Mathematica DSolve solution

Solving time : `0.974` (sec)
Leaf size : 90

```
DSolve[{D[y[x],{x,2}]+(x-3)*D[y[x],x]+3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}e^{-\frac{1}{2}(x-6)x-8}\left(e^{7/2}\sqrt{2\pi}c_2(x^2-6x+8)\,\mathrm{erfi}\left(\frac{x-3}{\sqrt{2}}\right)+4e^8c_1(x^2-6x+8) - 2c_2e^{\frac{1}{2}(x-4)^2+x}(x-3)\right)$$

1.61 problem 63

Internal problem ID [8199]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 63
Date solved : `Monday, October 21, 2024 at 04:55:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-8x+14\right)y''-8(x-4)\,y'+20y=0$$

1.61.1 Solved as second order ode using Kovacic algorithm

Time used: 0.282 (sec)

Writing the ode as

$$\left(x^2-8x+14\right)y''+(-8x+32)\,y'+20y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-8x+14 \\ B &= -8x+32 \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{48}{(x^2 - 8x + 14)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 48$$
$$t = (x^2 - 8x + 14)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{48}{(x^2 - 8x + 14)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 109: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 8x + 14)^2$. There is a pole at $x = 4 + \sqrt{2}$ of order 2. There is a pole at $x = 4 - \sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{\left(x - 4 + \sqrt{2}\right)^2} + \frac{6}{\left(x - 4 - \sqrt{2}\right)^2} + \frac{3\sqrt{2}}{x - 4 + \sqrt{2}} - \frac{3\sqrt{2}}{x - 4 - \sqrt{2}}$$

For the pole at $x = 4 + \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-4-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

For the pole at $x = 4 - \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-4+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{48}{\left(x^2 - 8x + 14\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$4+\sqrt{2}$	2	0	3	-2
$4-\sqrt{2}$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}} + (-)(0) \\
&= -\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}} \\
&= \frac{x-4-5\sqrt{2}}{x^2-8x+14}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}}\right)(0) + \left(\left(\frac{2}{\left(x-4-\sqrt{2}\right)^2} - \frac{3}{\left(x-4+\sqrt{2}\right)^2}\right) + \left(-\frac{2}{x-4-\sqrt{2}} + \right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}}\right) dx} \\
&= \frac{\left(x-4+\sqrt{2}\right)^3}{\left(-x+4+\sqrt{2}\right)^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-8x+32}{x^2-8x+14}\, dx} \\
&= z_1 e^{2\ln(x^2-8x+14)} \\
&= z_1\left(\left(x^2-8x+14\right)^2\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{(x^2 - 8x + 14)^2 \left(x - 4 + \sqrt{2}\right)^3}{\left(-x + 4 + \sqrt{2}\right)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x+32}{x^2-8x+14}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{4\ln(x^2-8x+14)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{4\sqrt{2}}{\left(x - 4 + \sqrt{2}\right)^2} + \frac{16\sqrt{2}}{\left(x - 4 + \sqrt{2}\right)^4} - \frac{64}{5\left(x - 4 + \sqrt{2}\right)^5} - \frac{16}{\left(x - 4 + \sqrt{2}\right)^3} \right. \\
&\qquad \left. - \frac{1}{x - 4 + \sqrt{2}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2 - 8x + 14)^2 \left(x - 4 + \sqrt{2}\right)^3}{\left(-x + 4 + \sqrt{2}\right)^2}\right) \\
&\quad + c_2 \left(\frac{(x^2 - 8x + 14)^2 \left(x - 4 + \sqrt{2}\right)^3}{\left(-x + 4 + \sqrt{2}\right)^2} \left(\frac{4\sqrt{2}}{\left(x - 4 + \sqrt{2}\right)^2} + \frac{16\sqrt{2}}{\left(x - 4 + \sqrt{2}\right)^4} \right.\right. \\
&\qquad \left.\left. - \frac{64}{5\left(x - 4 + \sqrt{2}\right)^5} - \frac{16}{\left(x - 4 + \sqrt{2}\right)^3} - \frac{1}{x - 4 + \sqrt{2}}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.61.2 Maple step by step solution

Let's solve

$(x^2-8x+14)\left(\frac{d}{dx}y'\right)-8(x-4)\,y'+20y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{20y}{x^2-8x+14}+\frac{8(x-4)y'}{x^2-8x+14}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{8(x-4)y'}{x^2-8x+14}+\frac{20y}{x^2-8x+14}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{8(x-4)}{x^2-8x+14},P_3(x)=\frac{20}{x^2-8x+14}\right]$

- ○ $(x-4+\sqrt{2})\cdot P_2(x)$ is analytic at $x=4-\sqrt{2}$

  $\left((x-4+\sqrt{2})\cdot P_2(x)\right)\Big|_{x=4-\sqrt{2}}=0$

- ○ $(x-4+\sqrt{2})^2\cdot P_3(x)$ is analytic at $x=4-\sqrt{2}$

  $\left((x-4+\sqrt{2})^2\cdot P_3(x)\right)\Big|_{x=4-\sqrt{2}}=0$

- ○ $x=4-\sqrt{2}$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=4-\sqrt{2}$

- Multiply by denominators

$(x^2-8x+14)\left(\frac{d}{dx}y'\right)+(-8x+32)\,y'+20y=0$

- Change variables using $x=u+4-\sqrt{2}$ so that the regular singular point is at $u=0$

$(u^2-2u\sqrt{2})\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-8u+8\sqrt{2})\left(\frac{d}{du}y(u)\right)+20y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2\sqrt{2}\,(r-5)\, r a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2\sqrt{2}\,(k+r-4)\,(k+1+r)\, a_{k+1} + a_k(k+r-4)\,(k+r-5)\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2\sqrt{2}\,(r-5)\, r = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 5\}$$

- Each term in the series must be 0, giving the recursion relation

$$\left(-2a_{k+1}(k+1+r)\sqrt{2} + a_k(k+r-5)\right)(k+r-4) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r-5)\sqrt{2}}{4(k+1+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 5$

$$a_{k+1} = \frac{a_k(k-5)\sqrt{2}}{4(k+1)}$$

- Apply recursion relation for $k = 0$

$$a_1 = -\frac{5a_0\sqrt{2}}{4}$$

- Apply recursion relation for $k = 1$

$$a_2 = -\frac{a_1\sqrt{2}}{2}$$

- Express in terms of a_0

$$a_2 = \frac{5a_0}{4}$$

- Apply recursion relation for $k = 2$

$$a_3 = -\frac{a_2\sqrt{2}}{4}$$

- Express in terms of a_0

$$a_3 = -\frac{5a_0\sqrt{2}}{16}$$

- Apply recursion relation for $k = 3$

$$a_4 = -\frac{a_3\sqrt{2}}{8}$$

- Express in terms of a_0

  $a_4 = \frac{5a_0}{64}$
- Apply recursion relation for $k = 4$

  $a_5 = -\frac{a_4\sqrt{2}}{20}$
- Express in terms of a_0

  $a_5 = -\frac{a_0\sqrt{2}}{256}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{5u\sqrt{2}}{4} + \frac{5u^2}{4} - \frac{5\sqrt{2}\,u^3}{16} + \frac{5u^4}{64} - \frac{\sqrt{2}\,u^5}{256}\right)$
- Revert the change of variables $u = x - 4 + \sqrt{2}$

  $\left[y = a_0\left(\frac{(-x^5+20x^4-180x^3+880x^2-2260x+2384)\sqrt{2}}{256} + \frac{5x^4}{128} - \frac{5x^3}{8} + \frac{125x^2}{32} - \frac{45x}{4} + \frac{401}{32}\right)\right]$
- Recursion relation for $r = 5$

  $a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}$
- Solution for $r = 5$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+5}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}\right]$
- Revert the change of variables $u = x - 4 + \sqrt{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k\left(x - 4 + \sqrt{2}\right)^{k+5}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0\left(\frac{(-x^5+20x^4-180x^3+880x^2-2260x+2384)\sqrt{2}}{256} + \frac{5x^4}{128} - \frac{5x^3}{8} + \frac{125x^2}{32} - \frac{45x}{4} + \frac{401}{32}\right) + \left(\sum_{k=0}^{\infty} b_k\left(x - 4 + \right.\right.\right.$

1.61.3 Maple trace

```
Methods for second order ODEs:
```

1.61.4 Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 55

```
dsolve((x^2-8*x+14)*diff(diff(y(x),x),x)-8*(x-4)*diff(y(x),x)+20*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1\, x^5 + c_2\, x^4 + 4(-35c_1 - 4c_2)\, x^3 + 20(56c_1 + 5c_2)\, x^2 + 4(-875c_1 - 72c_2)\, x + 4032c_1 + \frac{1604c_2}{5}$$

1.61.5 Mathematica DSolve solution

Solving time : `0.155` (sec)
Leaf size : 77

```
DSolve[{(x^2-8*x+14)*D[y[x],{x,2}]+8*(x-4)*D[y[x],x]+20*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1 P^3_{\frac{1}{2}i\left(i+\sqrt{31}\right)}\left(\frac{x-4}{\sqrt{2}}\right) + c_2 Q^3_{\frac{1}{2}i\left(i+\sqrt{31}\right)}\left(\frac{x-4}{\sqrt{2}}\right)}{(x^2-8x+14)^{3/2}}$$

1.62 problem 64

Internal problem ID [8200]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 64
Date solved : `Monday, October 21, 2024 at 04:55:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+4x+5\right)y''-20(x+1)\,y'+60y=0$$

1.62.1 Solved as second order ode using Kovacic algorithm

Time used: 0.597 (sec)

Writing the ode as

$$\left(2x^2+4x+5\right)y''+(-20x-20)\,y'+60y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^2+4x+5\\B&=-20x-20\\C&=60\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-210}{(2x^2 + 4x + 5)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -210$$
$$t = (2x^2 + 4x + 5)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{210}{(2x^2 + 4x + 5)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 111: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + 4x + 5)^2$. There is a pole at $x = -1 + \frac{i\sqrt{6}}{2}$ of order 2. There is a pole at $x = -1 - \frac{i\sqrt{6}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4\left(x+1-\frac{i\sqrt{6}}{2}\right)^2} + \frac{35}{4\left(x+1+\frac{i\sqrt{6}}{2}\right)^2} + \frac{35i\sqrt{6}}{12\left(x+1-\frac{i\sqrt{6}}{2}\right)} - \frac{35i\sqrt{6}}{12\left(x+1+\frac{i\sqrt{6}}{2}\right)}$$

For the pole at $x = -1 + \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x+1-\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = -1 - \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x+1+\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{210}{(2x^2 + 4x + 5)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-1 + \frac{i\sqrt{6}}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-1 - \frac{i\sqrt{6}}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)} + (-)(0) \\
&= -\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)} \\
&= \frac{-6i\sqrt{6}+2x+2}{2x^2+4x+5}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)}\right)(0) + \left(\left(\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)^2} - \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)^2}\right) + \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)}\right) dx} \\
&= \frac{27\sqrt{2}\,(2x^2+4x+5)^{7/2}}{\left(3+i\,(x+1)\sqrt{6}\right)^6}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-20x-20}{2x^2+4x+5}\, dx} \\
&= z_1 e^{\frac{5\ln\left(2x^2+4x+5\right)}{2}} \\
&= z_1 \left(\left(2x^2+4x+5\right)^{5/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = -\frac{\left(2x^2+4x+5\right)^6 \sqrt{2}}{27 \left(i - \frac{(x+1)\sqrt{2}\,\sqrt{3}}{3}\right)^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-20x-20}{2x^2+4x+5}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(2x^2+4x+5)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\frac{1}{2}x^5 + \frac{5}{2}x^4 + \frac{5}{2}x^3 - \frac{5}{2}x^2 - \frac{31}{8}x - \frac{7}{8}}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)^6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(-\frac{\left(2x^2+4x+5\right)^6 \sqrt{2}}{27 \left(i - \frac{(x+1)\sqrt{2}\,\sqrt{3}}{3}\right)^6}\right) \\
&\quad + c_2 \left(-\frac{\left(2x^2+4x+5\right)^6 \sqrt{2}}{27 \left(i - \frac{(x+1)\sqrt{2}\,\sqrt{3}}{3}\right)^6} \left(-\frac{\frac{1}{2}x^5 + \frac{5}{2}x^4 + \frac{5}{2}x^3 - \frac{5}{2}x^2 - \frac{31}{8}x - \frac{7}{8}}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)^6}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.62.2 Maple step by step solution

Let's solve

$(2x^2+4x+5)\left(\frac{d}{dx}y'\right)-20(x+1)\,y'+60y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{60y}{2x^2+4x+5}+\frac{20(x+1)y'}{2x^2+4x+5}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{20(x+1)y'}{2x^2+4x+5}+\frac{60y}{2x^2+4x+5}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=-\frac{20(x+1)}{2x^2+4x+5},P_3(x)=\frac{60}{2x^2+4x+5}\right]$

- ○ $\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)$ is analytic at $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

$\left(\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-1-\frac{\mathrm{I}\sqrt{6}}{2}}=0$

- ○ $\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

$\left(\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-1-\frac{\mathrm{I}\sqrt{6}}{2}}=0$

- ○ $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

- Multiply by denominators

$(2x^2+4x+5)\left(\frac{d}{dx}y'\right)+(-20x-20)\,y'+60y=0$

- Change variables using $x=u-1-\frac{\mathrm{I}\sqrt{6}}{2}$ so that the regular singular point is at $u=0$

$\left(2u^2-2\,\mathrm{I}u\sqrt{6}\right)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\left(-20u+10\,\mathrm{I}\sqrt{6}\right)\left(\frac{d}{du}y(u)\right)+60y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2\,\mathrm{I}\sqrt{6}\, r(r-6)\, a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2\,\mathrm{I}\sqrt{6}\,(k+1+r)\,(k+r-5)\, a_{k+1} + 2a_k(k+r-5)\,(k+r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2\,\mathrm{I}\sqrt{6}\, r(r-6) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 6\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2(k+r-5)\left(\mathrm{I}(k+1+r)\, a_{k+1}\sqrt{6} - a_k(k+r-6)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k(k+r-6)\sqrt{6}}{k+1+r}$$

- Recursion relation for $r = 0$; series terminates at $k = 6$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k(k-6)\sqrt{6}}{k+1}$$

- Recursion relation that defines the terminating series solution of the ODE for $r = 0$

$$\left[y(u) = \sum_{k=0}^{5} a_k u^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k(k-6)\sqrt{6}}{k+1}\right]$$

- Revert the change of variables $u = x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}$

$$\left[y = \sum_{k=0}^{5} a_k \left(x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}\right)^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k(k-6)\sqrt{6}}{k+1}\right]$$

- Recursion relation for $r = 6$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k k\sqrt{6}}{k+7}$$

- Solution for $r = 6$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+1}=\frac{-\frac{I}{6} a_k k\sqrt{6}}{k+7}\right]$$

- Revert the change of variables $u=x+1+\frac{I\sqrt{6}}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(x+1+\frac{I\sqrt{6}}{2}\right)^{k+6}, a_{k+1}=\frac{-\frac{I}{6} a_k k\sqrt{6}}{k+7}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{5} a_k \left(x+1+\frac{I\sqrt{6}}{2}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k \left(x+1+\frac{I\sqrt{6}}{2}\right)^{k+6}\right), a_{k+1}=\frac{-\frac{I}{6} a_k (k-6)\sqrt{6}}{k+1}, b_{k+1}=\frac{-\frac{I}{6} b_k k\sqrt{6}}{k+7}\right]$$

1.62.3 Maple trace

```
Methods for second order ODEs:
```

1.62.4 Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 65

```
dsolve((2*x^2+4*x+5)*diff(diff(y(x),x),x)-20*(x+1)*diff(y(x),x)+60*y(x) = 0,
       y(x),singsol=all)
```

$$y=c_2 x^6+c_1 x^5+\frac{5(2c_1-15c_2)\,x^4}{2}+5(c_1-20c_2)\,x^3 +\frac{5(-4c_1-45c_2)\,x^2}{4}+\frac{(-31c_1+120c_2)\,x}{4}-\frac{7c_1}{4}+\frac{155c_2}{8}$$

1.62.5 Mathematica DSolve solution

Solving time : `1.468` (sec)
Leaf size : 83

```
DSolve[{(2*x^2+4*x+5)*D[y[x],{x,2}]-20*(x+1)*D[y[x],x]+60*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{(2x^2+4x+5)^{5/2}\left(4c_2(4x^5+20x^4+20x^3-20x^2-31x-7)+c_1\left(2ix+\sqrt{6}+2i\right)^6\right)}{(4x^2+8x+10)^{5/2}}$$

1.63 problem 65

Internal problem ID [8201]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 65
Date solved : `Monday, October 21, 2024 at 04:55:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^3+1\right) y''+7x^2y'+9xy=0$$

1.63.1 Solved as second order ode using Kovacic algorithm

Time used: 0.431 (sec)

Writing the ode as

$$\left(x^3+1\right) y''+7x^2y'+9xy=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3+1 \\ B &= 7x^2 \\ C &= 9x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x(x^3 + 8)}{4\left(x^3 + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x\left(x^3 + 8\right)$$
$$t = 4\left(x^3 + 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{x(x^3 + 8)}{4\left(x^3 + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 113: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 6 - 4\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+1)^2$. There is a pole at $x = -1$ of order 2. There is a pole at $x = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned}r = {}& \frac{5}{18(x+1)} + \frac{7}{36(x+1)^2} + \frac{7}{36\left(x - \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2}\\ &+ \frac{7}{36\left(x - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{36} + \frac{5i\sqrt{3}}{36}}{x - \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{5}{36} - \frac{5i\sqrt{3}}{36}}{x - \frac{1}{2} + \frac{i\sqrt{3}}{2}}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}\end{aligned}$$

For the pole at $x = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}\end{aligned}$$

For the pole at $x = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{x(x^3+8)}{4\left(x^3+1\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{x(x^3+8)}{4\left(x^3+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
$\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{6(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)} + (-)(0) \\
&= -\frac{1}{6(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)} \\
&= -\frac{x^2}{2x^3+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}\right)(1) + \left(\left(\frac{1}{6(x+1)^2} + \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2} + \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{1}{6(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}\right) dx} \\
&= (x) \frac{1}{\left((x+1)\left(2x-1+i\sqrt{3}\right)\left(i\sqrt{3}-2x+1\right)\right)^{1/6}} \\
&= \frac{x}{\left((x+1)\left(2x-1+i\sqrt{3}\right)\left(i\sqrt{3}-2x+1\right)\right)^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^2}{x^3+1} \, dx} \\
&= z_1 e^{-\frac{7\ln\left(x^3+1\right)}{6}} \\
&= z_1 \left(\frac{1}{\left(x^3+1\right)^{7/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{\left(x^3+1\right)^{7/6}\left(-4x^3-4\right)^{1/6}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{7x^2}{x^3+1}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{-\frac{7\ln\left(x^3+1\right)}{3}}}{(y_1)^2}\,dx$$

$$= y_1 \left(\int \frac{\left(-4x^3-4\right)^{1/3}}{x^2}dx\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x}{\left(x^3+1\right)^{7/6}\left(-4x^3-4\right)^{1/6}}\right) + c_2 \left(\frac{x}{\left(x^3+1\right)^{7/6}\left(-4x^3-4\right)^{1/6}}\left(\int \frac{\left(-4x^3-4\right)^{1/3}}{x^2}dx\right)\right)$$

Will add steps showing solving for IC soon.

1.63.2 Maple step by step solution

Let's solve

$\left(x^3+1\right)\left(\frac{d}{dx}y'\right) + 7x^2y' + 9xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{9xy}{x^3+1} - \frac{7x^2y'}{x^3+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{7x^2y'}{x^3+1} + \frac{9xy}{x^3+1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{7x^2}{x^3+1}, P_3(x) = \frac{9x}{x^3+1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{7}{3}$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^3+1)\left(\frac{d}{dx}y'\right) + 7x^2y' + 9xy = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^3 - 3u^2 + 3u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (7u^2 - 14u + 7)\left(\frac{d}{du}y(u)\right) + (9u-9)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(4+3r)\, u^{-1+r} + \left(a_1(1+r)(7+3r) - a_0(3r^2+11r+9)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)(3k+7\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(4+3r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{4}{3}\right\}$
- Each term must be 0

  $a_1(1+r)(7+3r) - a_0(3r^2+11r+9) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(3k+7+3r) - a_k(3k^2+6kr+3r^2+11k+11r+9) + a_{k-1}(k+2+r)^2 = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(3k+10+3r) - a_{k+1}\left(3(k+1)^2+6(k+1)r+3r^2+11k+20+11r\right) + a_k(k+$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+2kra_k-6kra_{k+1}+r^2a_k-3r^2a_{k+1}+6ka_k-17ka_{k+1}+6ra_k-17ra_{k+1}+9a_k-23a_{k+1}}{(k+2+r)(3k+10+3r)}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+6ka_k-17ka_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+6ka_k-17ka_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)}, 7a_1 - 9a_0 = 0\right]$
- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+6ka_k-17ka_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)}, 7a_1 - 9a_0 = 0\right]$
- Recursion relation for $r = -\frac{4}{3}$

  $a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+\frac{10}{3}ka_k-9ka_{k+1}+\frac{25}{9}a_k-\frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}$
- Solution for $r = -\frac{4}{3}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+\frac{10}{3}ka_k-9ka_{k+1}+\frac{25}{9}a_k-\frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}, -a_1 + \frac{a_0}{3} = 0\right]$
- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{4}{3}}, a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+\frac{10}{3}ka_k-9ka_{k+1}+\frac{25}{9}a_k-\frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}, -a_1 + \frac{a_0}{3} = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{4}{3}}\right), a_{k+2} = -\frac{k^2a_k-3k^2a_{k+1}+6ka_k-17ka_{k+1}+9a_k-23a_{k+1}}{(k+2)(3k+10)}, 7a_1\right.$

1.63.3 Maple trace

```
Methods for second order ODEs:
```

1.63.4 Maple dsolve solution

Solving time : `0.025` (sec)
Leaf size : 28

```
dsolve((x^3+1)*diff(diff(y(x),x),x)+7*x^2*diff(y(x),x)+9*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \,\text{hypergeom}\left(\left[1, 1\right], \left[\frac{2}{3}\right], -x^3\right) + \frac{c_2 x}{\left(x^3+1\right)^{4/3}}$$

1.63.5 Mathematica DSolve solution

Solving time : `1.163` (sec)
Leaf size : 118

```
DSolve[{(1+x^3)*D[y[x],{x,2}]+7*x^2*D[y[x],x]+9*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-2\sqrt{3}c_2 x \arctan\left(\frac{\sqrt{3}x}{2\sqrt[3]{x^3+1}+x}\right) - 6c_2\sqrt[3]{x^3+1} - 2c_2 x \log\left(\sqrt[3]{x^3+1}-x\right) + c_2 x \log\left(\sqrt[3]{x^3+1}x + (x^3 -}{6\left(x^3+1\right)^{4/3}}$$

1.64 problem 66

Internal problem ID [8202]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 66
Date solved : `Monday, October 21, 2024 at 04:55:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^5+1\right)y''+14x^4y'+10x^3y=0$$

1.64.1 Solved as second order ode using Kovacic algorithm

Time used: 1.118 (sec)

Writing the ode as

$$\left(2x^5+1\right)y''+14x^4y'+10x^3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^5+1 \\ B &= 14x^4 \\ C &= 10x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^3(5x^5 + 6)}{(2x^5 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^3(5x^5 + 6)$$
$$t = (2x^5 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^3(5x^5 + 6)}{(2x^5 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 115: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 10 - 8 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^5+1)^2$. There is a pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ of order 2. There is a pole at $x = \frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}$ of order 2. There is a pole at $x = -\frac{2^{4/5}}{2}$ of order 2. There is a pole at $x = \frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}$ of order 2. There is a pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \text{Expression too large to display}$$

For the pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{2^{4/5}\sqrt{5}}{8} - \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10} \end{aligned}$$

For the pole at $x = \frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}$ let b be the coefficient of $\frac{1}{\left(x - \frac{2^{4/5}}{8} + \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10} \end{aligned}$$

For the pole at $x = -\frac{2^{4/5}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2^{4/5}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}
$$

For the pole at $x = \frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{4/5}}{8}+\frac{2^{4/5}\sqrt{5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}
$$

For the pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{4/5}\sqrt{5}}{8}-\frac{2^{4/5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^3(5x^5+6)}{(2x^5+1)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^3(5x^5+6)}{(2x^5+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{2^{4/5}\sqrt{5}}{8}+\frac{2^{4/5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{4/5}}{8}-\frac{2^{4/5}\sqrt{5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$-\frac{2^{4/5}}{2}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{4/5}}{8}-\frac{2^{4/5}\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{4/5}\sqrt{5}}{8}+\frac{2^{4/5}}{8}-\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^- + \alpha_{c_4}^- + \alpha_{c_5}^-\right) \\ &= \frac{5}{2} - \left(\frac{3}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\omega = \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + \left((-)[\sqrt{r}]_{c_4} + \frac{\alpha_{c_4}^-}{x-c_4}\right)$$
$$= \frac{3}{10\left(x - \frac{2^{4/5}\sqrt{5}}{8} - \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)} + \frac{3}{10\left(x - \frac{2^{4/5}}{8} + \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8}\right)} + \frac{}{10(}$$
$$= \frac{3}{10\left(x - \frac{2^{4/5}\sqrt{5}}{8} - \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)} + \frac{3}{10\left(x - \frac{2^{4/5}}{8} + \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8}\right)} + \frac{}{10(}$$
$$= \frac{3x^4}{2x^5+1}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) and Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$z_1(x) = pe^{\int \omega\, dx}$$
$$= (x)\, \mathrm{e}^{\int \left(\frac{3}{10\left(x - \frac{2^{4/5}\sqrt{5}}{8} - \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)} + \frac{3}{10\left(x - \frac{2^{4/5}}{8} + \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8}\right)} + \frac{3}{10\left(x + \frac{2^{4/5}}{2}\right)} + \frac{}{10\left(x - \frac{2^{4/5}}{8}\right.}\right.}$$
$$= (x)\left(\left(2^{4/5}\sqrt{5} + 2i2^{3/10}\sqrt{5-\sqrt{5}} + 2^{4/5} - 8x\right)\left(-i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5} + 2^{4/5}\sqrt{5} - i2^{3/10}\sqrt{5-\sqrt{\right.\right.$$
$$= x8^{3/10}\left(\left(x + \frac{2^{4/5}}{2}\right)\left(i\left(\sqrt{5}+1\right)2^{3/10}\sqrt{5-\sqrt{5}} + \left(\sqrt{5}-1\right)2^{4/5} + 8x\right)\left(i2^{3/10}\sqrt{5-\sqrt{5}} + \frac{(-\sqrt{}}{}\right.\right.$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{14x^4}{2x^5+1}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^5+1\right)}{10}} \\
&= z_1 \left(\frac{1}{\left(2x^5+1\right)^{7/10}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x 8^{3/10} \left(1024x^5+512\right)^{3/10}}{\left(2x^5+1\right)^{7/10}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{14x^4}{2x^5+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^5+1\right)}{5}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{8^{2/5}}{8x^2\left(1024x^5+512\right)^{3/5}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x 8^{3/10}\left(1024x^5+512\right)^{3/10}}{\left(2x^5+1\right)^{7/10}}\right) + c_2 \left(\frac{x 8^{3/10}\left(1024x^5+512\right)^{3/10}}{\left(2x^5+1\right)^{7/10}} \left(\int \frac{8^{2/5}}{8x^2\left(1024x^5+512\right)^{3/5}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.64.2 Maple step by step solution

1.64.3 Maple trace

```
Methods for second order ODEs:
```

1.64.4 Maple dsolve solution

Solving time : 0.087 (sec)
Leaf size : 30

```
dsolve((2*x^5+1)*diff(diff(y(x),x),x)+14*x^4*diff(y(x),x)+10*x^3*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 x}{(2x^5+1)^{2/5}} + c_2 \,\text{hypergeom}\left(\left[\frac{1}{5}, 1\right], \left[\frac{4}{5}\right], -2x^5\right)$$

1.64.5 Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{(1+2*x^5)*D[y[x],{x,2}]+14*x^4*D[y[x],x]+10*x^3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Timed out

1.65 problem 67

Internal problem ID [8203]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 67
Date solved : `Monday, October 21, 2024 at 04:55:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + x^6 y' + 7x^5 y = 0$$

1.65.1 Solved as second order ode using Kovacic algorithm

Time used: 0.438 (sec)

Writing the ode as

$$y'' + x^6 y' + 7x^5 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x^6 \\ C &= 7x^5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^5(x^7 - 16)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^5(x^7 - 16)$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^5(x^7 - 16)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 116: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 12 \\ &= -12 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -12 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -12$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{12}{2} = 6$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{6} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^6$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^6}{2} - \frac{4}{x} - \frac{16}{x^8} - \frac{128}{x^{15}} - \frac{1280}{x^{22}} - \frac{14336}{x^{29}} - \frac{172032}{x^{36}} - \frac{2162688}{x^{43}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 6$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{6} a_i x^i \\ &= \frac{x^6}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^5 = x^5$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^{12}}{4}$$

This shows that the coefficient of x^5 in the above is 0. Now we need to find the coefficient of x^5 in r. How this is done depends on if $v = 0$ or not. Since $v = 6$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^5 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^5(x^7 - 16)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^{12} - 4x^5\right) + (0) \\
&= \frac{1}{4}x^{12} - 4x^5
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -4. Now b can be found.

$$\begin{aligned}
b &= (-4) - (0) \\
&= -4
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^6}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-4}{\frac{1}{2}} - 6\right) &&= -7 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-4}{\frac{1}{2}} - 6\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^5(x^7 - 16)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-12	$\frac{x^6}{2}$	-7	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^6}{2}\right) \\ &= -\frac{x^6}{2} \\ &= -\frac{x^6}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x^6}{2}\right)(1) + \left((-3x^5) + \left(-\frac{x^6}{2}\right)^2 - \left(\frac{x^5(x^7-16)}{4}\right)\right) = 0$$

$$x^5 a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, \mathrm{e}^{\int -\frac{x^6}{2} dx} \\
&= (x) \, \mathrm{e}^{-\frac{x^7}{14}} \\
&= x \, \mathrm{e}^{-\frac{x^7}{14}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^6}{1} \, dx} \\
&= z_1 e^{-\frac{x^7}{14}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^7}{14}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^7}{7}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^6}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^7}{7}}}{(y_1)^2} \, dx
\end{aligned}$$

$$= y_1 \left(\frac{7^{6/7}(-1)^{1/7}\left(-\frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}\right)}{(-x^7)^{6/7}} + \frac{7\,7^{1/7}(-1)^{6/7}\mathrm{e}^{\frac{x^7}{7}}}{x} + \frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)}{(-x^7)^{6/7}}\right)}{49} \right)$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{x^7}{7}} x\right) \\ &\quad + c_2 \left(\mathrm{e}^{-\frac{x^7}{7}} x \left(\frac{7^{6/7}(-1)^{1/7}\left(-\frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}\right)}{(-x^7)^{6/7}} + \frac{7\,7^{1/7}(-1)^{6/7}\mathrm{e}^{\frac{x^7}{7}}}{x} + \frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)}{(-x^7)^{6/7}}\right)}{49} \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.65.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + x^6 y' + 7x^5 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^5 \cdot y$ to series expansion

  $x^5 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+5}$

- ◦ Shift index using $k- >k-5$

  $x^5 \cdot y = \sum_{k=5}^{\infty} a_{k-5} x^k$

- ◦ Convert $x^6 \cdot y'$ to series expansion

  $x^6 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+5}$

- Shift index using $k->k-5$

  $$x^6 \cdot y' = \sum_{k=5}^{\infty} a_{k-5}(k-5)\, x^k$$

- Convert $\frac{d}{dx}y'$ to series expansion

  $$\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$30a_6x^4 + 20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=5}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-5}(k+2)\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $[2a_2 = 0, 6a_3 = 0, 12a_4 = 0, 20a_5 = 0, 30a_6 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2} + a_{k-5} + a_{k+2}) = 0$

- Shift index using $k->k+5$

  $(k+7)\,((k+5)\, a_{k+7} + a_k + a_{k+7}) = 0$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+7} = -\frac{a_k}{k+6}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0\right]$$

1.65.3 Maple trace

```
Methods for second order ODEs:
```

1.65.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 62

```
dsolve(diff(diff(y(x),x),x)+x^6*diff(y(x),x)+7*x^5*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{\left(-c_1\,\mathrm{e}^{-\frac{x^7}{7}}x - c_2 7^{1/7}\right)\left(-x^7\right)^{6/7} + x^7 c_2\,\mathrm{e}^{-\frac{x^7}{7}}\left(\Gamma\left(\frac{6}{7}\right) - \Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)\right)}{\left(-x^7\right)^{6/7}}$$

1.65.5 Mathematica DSolve solution

Solving time : 0.391 (sec)
Leaf size : 53

```
DSolve[{D[y[x],{x,2}]+x^6*D[y[x],x]+7*x^5*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{49}e^{-\frac{x^7}{7}}\left(49c_1x - 7^{6/7}c_2\sqrt[7]{-x^7}\,\Gamma\left(-\frac{1}{7}, -\frac{x^7}{7}\right)\right)$$

1.66 problem 68

Internal problem ID [8204]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 68
Date solved : `Tuesday, October 22, 2024 at 03:00:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^{8}+1\right) y''-16x^{7}y'+72x^{6}y=0$$

1.66.1 Solved as second order ode using Kovacic algorithm

Time used: 434.082 (sec)

Writing the ode as

$$\left(x^{8}+1\right) y''-16x^{7}y'+72x^{6}y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{8}+1\\ B &= -16x^{7}\\ C &= 72x^{6} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-128x^6}{(x^8+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -128x^6$$
$$t = (x^8+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{128x^6}{(x^8+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 118: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 16 - 6 \\
&= 10
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^8+1)^2$. There is a pole at $x = \frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 10 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 10 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned}
r = {} & \frac{2}{\left(x - \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} \\
& + \frac{2}{\left(x + \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x + \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} \\
& + \frac{2}{\left(x + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} \\
& + \frac{2}{\left(x - \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} + \frac{2\left(\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} + \frac{2\left(\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}} \\
& + \frac{2\left(-\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}} + \frac{2\left(-\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} + \frac{2\left(-\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}} \\
& + \frac{2\left(-\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}} + \frac{2\left(\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}} + \frac{2\left(\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}}
\end{aligned}$$

For the pole at $x = \frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$

in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x = \frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is $10 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{128x^6}{(x^8+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
10	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=1$ then

$$\begin{aligned}d&=\alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^- + \alpha_{c_4}^- + \alpha_{c_5}^- + \alpha_{c_6}^- + \alpha_{c_7}^- + \alpha_{c_8}^+\right)\\&=1-(-5)\\&=6\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\omega = \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + \left((-)[\sqrt{r}]_{c_4} + \frac{\alpha_{c_4}^{-}}{x-c_4}\right)$$

$$= -\frac{1}{x - \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} -}$$

$$= -\frac{1}{x - \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} -}$$

$$= \frac{((3x^6 - 3ix^4 - 3ix^2 - 3)\sqrt{2} - 3((-1+i)x^4 + 1 + i)(x^2+1))\sqrt{2-\sqrt{2}} - 3x\left(((-1+i)x^4 + 1 + i\right.}{2\left(x^2 - x\sqrt{2-\sqrt{2}} + 1\right)\left(-x(1+\sqrt{2})\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(x\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(x(1+\sqrt{2})\right.}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + x^5 a_5 + x^4 a_4 + a_3 x^3 + a_2 x^2 + x a_1 + a_0 \tag{2A}$$

Substituting the above in eq. (1A) and Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{i\sqrt{2} - 1 + i}{i\sqrt{2} + 1 + i}, a_1 = \frac{\left(\frac{12}{7} - \frac{12i}{7}\right)\sqrt{2}}{(i\sqrt{2} + 1 + i)\sqrt{2-\sqrt{2}}}, a_2 = -\frac{15(-\sqrt{2} - 1 + i)}{7(i\sqrt{2} + 1 + i)}, a_3 = \frac{32}{7\sqrt{2-\sqrt{2}}\,(i\sqrt{2} + 1\right.$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)x^5\sqrt{2}}{(i\sqrt{2} + 1 + i)\sqrt{2-\sqrt{2}}} + \frac{15x^4(\sqrt{2} + 1 + i)}{7(i\sqrt{2} + 1 + i)} + \frac{32x^3}{7\sqrt{2-\sqrt{2}}\,(i\sqrt{2} + 1 + i)} - \frac{15(-\sqrt{2} - 1 +}{7(i\sqrt{2} + 1 +}$$

Therefore the first solution to the ode $z'' = rz$ is

$$z_1(x) = pe^{\int \omega\, dx}$$

$$= \left(x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)x^5\sqrt{2}}{(i\sqrt{2} + 1 + i)\sqrt{2-\sqrt{2}}} + \frac{15x^4(\sqrt{2} + 1 + i)}{7(i\sqrt{2} + 1 + i)} + \frac{32x^3}{7\sqrt{2-\sqrt{2}}\,(i\sqrt{2} + 1 + i)} - \frac{15(-\sqrt{2} - 1}{7(i\sqrt{2} +}\right.$$

$$= \left(x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)x^5\sqrt{2}}{(i\sqrt{2} + 1 + i)\sqrt{2-\sqrt{2}}} + \frac{15x^4(\sqrt{2} + 1 + i)}{7(i\sqrt{2} + 1 + i)} + \frac{32x^3}{7\sqrt{2-\sqrt{2}}\,(i\sqrt{2} + 1 + i)} - \frac{15(-\sqrt{2} - 1}{7(i\sqrt{2} +}\right.$$

$= \text{Expression too large to display}$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-16x^7}{x^8+1}\,dx} \\
&= z_1 e^{\ln(x^8+1)} \\
&= z_1 \left(x^8+1\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \text{Expression too large to display}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-16x^7}{x^8+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x^8+1)}}{(y_1)^2}\,dx \\
&= y_1(\text{Expression too large to display})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\text{Expression too large to display}) \\
&\quad + c_2(\text{Expression too large to display}(\text{Expression too large to display}))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.66.2 Maple step by step solution

1.66.3 Maple trace

```
Methods for second order ODEs:
```

1.66.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 22

```
dsolve((x^8+1)*diff(diff(y(x),x),x)-16*x^7*diff(y(x),x)+72*x^6*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{7}{9}c_1 + c_1\,x^8 + c_2\,x^9 - \frac{9}{7}c_2 x$$

1.66.5 Mathematica DSolve solution

Solving time : `0.0` (sec)
Leaf size : 0

```
DSolve[{(1+x^8)*D[y[x],{x,2}]-16*x^7*D[y[x],x]+72*x^6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Timed out

1.67 problem 69

Internal problem ID [8205]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 69
Date solved : `Monday, October 21, 2024 at 05:02:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + x^5 y' + 6x^4 y = 0$$

1.67.1 Solved as second order ode using Kovacic algorithm

Time used: 0.464 (sec)

Writing the ode as

$$y'' + x^5 y' + 6x^4 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x^5 \\ C &= 6x^4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4(x^6 - 14)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4(x^6 - 14)$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4(x^6 - 14)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 119: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 10 \\
&= -10
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -10 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -10$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{10}{2} = 5
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{5} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^5$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{x^5}{2} - \frac{7}{2x} - \frac{49}{4x^7} - \frac{343}{4x^{13}} - \frac{12005}{16x^{19}} - \frac{117649}{16x^{25}} - \frac{2470629}{32x^{31}} - \frac{27176919}{32x^{37}} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 5$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{5} a_i x^i \\
&= \frac{x^5}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^4 = x^4$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^{10}}{4}$$

This shows that the coefficient of x^4 in the above is 0. Now we need to find the coefficient of x^4 in r. How this is done depends on if $v = 0$ or not. Since $v = 5$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^4 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4(x^6 - 14)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^{10} - \frac{7}{2}x^4\right) + (0) \\
&= \frac{1}{4}x^{10} - \frac{7}{2}x^4
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{7}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{7}{2}\right) - (0) \\
&= -\frac{7}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^5}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{2}}{\frac{1}{2}} - 5\right) &&= -6 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{2}}{\frac{1}{2}} - 5\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4(x^6 - 14)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-10	$\frac{x^5}{2}$	-6	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^5}{2}\right) \\ &= -\frac{x^5}{2} \\ &= -\frac{x^5}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x^5}{2}\right)(1) + \left(\left(-\frac{5x^4}{2}\right) + \left(-\frac{x^5}{2}\right)^2 - \left(\frac{x^4(x^6-14)}{4}\right)\right) = 0$$

$$x^4 a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (x)\, \mathrm{e}^{\int -\frac{x^5}{2} dx} \\ &= (x)\, \mathrm{e}^{-\frac{x^6}{12}} \\ &= x\, \mathrm{e}^{-\frac{x^6}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^5}{1}\, dx} \\ &= z_1 e^{-\frac{x^6}{12}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^6}{12}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^6}{6}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^5}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^6}{6}}}{(y_1)^2}\, dx \end{aligned}$$

$$
= y_1 \left(\frac{6^{5/6}(-1)^{1/6}\left(-\frac{6x^5(-1)^{5/6}\Gamma\left(\frac{5}{6}\right)}{(-x^6)^{5/6}} + \frac{6\,6^{1/6}(-1)^{5/6}\mathrm{e}^{\frac{x^6}{6}}}{x} + \frac{6x^5(-1)^{5/6}\Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)}{(-x^6)^{5/6}}\right)}{36}\right)
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^6}{6}} x\right) \\
&\quad + c_2\left(\mathrm{e}^{-\frac{x^6}{6}} x \left(\frac{6^{5/6}(-1)^{1/6}\left(-\frac{6x^5(-1)^{5/6}\Gamma\left(\frac{5}{6}\right)}{(-x^6)^{5/6}} + \frac{6\,6^{1/6}(-1)^{5/6}\mathrm{e}^{\frac{x^6}{6}}}{x} + \frac{6x^5(-1)^{5/6}\Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)}{(-x^6)^{5/6}}\right)}{36}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.67.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + x^5 y' + 6x^4 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^4 \cdot y$ to series expansion

    $x^4 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+4}$

  - Shift index using $k- >k-4$

    $x^4 \cdot y = \sum_{k=4}^{\infty} a_{k-4} x^k$

  - Convert $x^5 \cdot y'$ to series expansion

    $x^5 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+4}$

- Shift index using $k->k-4$

  $$x^5 \cdot y' = \sum_{k=4}^{\infty} a_{k-4}(k-4)\, x^k$$

- Convert $\frac{d}{dx}y'$ to series expansion

  $$\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=4}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-4}(k+2)\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 = 0, 6a_3 = 0, 12a_4 = 0, 20a_5 = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+2)\,(ka_{k+2} + a_{k-4} + a_{k+2}) = 0$$

- Shift index using $k->k+4$

  $$(k+6)\,((k+4)\,a_{k+6} + a_k + a_{k+6}) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{a_k}{k+5}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\right]$$

1.67.3 Maple trace

```
Methods for second order ODEs:
```

1.67.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 62

```
dsolve(diff(diff(y(x),x),x)+x^5*diff(y(x),x)+6*x^4*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{\left(-c_1\,\mathrm{e}^{-\frac{x^6}{6}}x - c_2 6^{1/6}\right)\left(-x^6\right)^{5/6} + x^6 c_2\,\mathrm{e}^{-\frac{x^6}{6}}\left(\Gamma\left(\frac{5}{6}\right) - \Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)\right)}{\left(-x^6\right)^{5/6}}$$

1.67.5 Mathematica DSolve solution

Solving time : `0.353` (sec)
Leaf size : 53

```
DSolve[{D[y[x],{x,2}]+x^5*D[y[x],x]+6*x^4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{36}e^{-\frac{x^6}{6}}\left(36c_1x - 6^{5/6}c_2\sqrt[6]{-x^6}\Gamma\left(-\frac{1}{6}, -\frac{x^6}{6}\right)\right)$$

1.68 problem 70

Internal problem ID [8206]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 70
Date solved : `Monday, October 21, 2024 at 05:02:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1+3x)\,y''+xy'+2y=0$$

1.68.1 Solved as second order ode using Kovacic algorithm

Time used: 31.984 (sec)

Writing the ode as

$$(1+3x)\,y''+xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1+3x\\ B&=x\\ C&=2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 24x - 6}{4(1+3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 24x - 6$$
$$t = 4(1+3x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 24x - 6}{4(1+3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 121: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(1+3x)^2$. There is a pole at $x = -\frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{19}{324\left(x+\frac{1}{3}\right)^2} - \frac{37}{54\left(x+\frac{1}{3}\right)}$$

For the pole at $x = -\frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{19}{324}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{18} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{18}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} - \frac{37}{18x} - \frac{319}{27x^2} - \frac{11831}{81x^3} - \frac{2157901}{972x^4} - \frac{110035199}{2916x^5} - \frac{1501983319}{2187x^6} - \frac{85889060456}{6561x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 24x - 6}{36x^2 + 24x + 4} \\ &= Q + \frac{R}{36x^2 + 24x + 4} \\ &= \left(\frac{1}{36}\right) + \left(\frac{-\frac{74x}{3} - \frac{55}{9}}{36x^2 + 24x + 4}\right) \\ &= \frac{1}{36} + \frac{-\frac{74x}{3} - \frac{55}{9}}{36x^2 + 24x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is $-\frac{74}{3}$. Dividing this by leading coefficient in t which is 36 gives $-\frac{37}{54}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{37}{54}\right) - (0) \\ &= -\frac{37}{54} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{37}{54}}{\frac{1}{6}} - 0\right) &&= -\frac{37}{18} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{37}{54}}{\frac{1}{6}} - 0\right) &&= \frac{37}{18} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 24x - 6}{4(1+3x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{1}{3}$	2	0	$\frac{19}{18}$	$-\frac{1}{18}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{6}$	$-\frac{37}{18}$	$\frac{37}{18}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{37}{18}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{37}{18} - \left(\frac{19}{18}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{19}{18\left(x+\frac{1}{3}\right)} + (-)\left(\frac{1}{6}\right) \\
&= \frac{19}{18\left(x+\frac{1}{3}\right)} - \frac{1}{6} \\
&= -\frac{-6+x}{2\,(1+3x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{19}{18\left(x+\frac{1}{3}\right)} - \frac{1}{6}\right)(1) + \left(\left(-\frac{19}{18\left(x+\frac{1}{3}\right)^2}\right) + \left(\frac{19}{18\left(x+\frac{1}{3}\right)} - \frac{1}{6}\right)^2 - \left(\frac{x^2-24x-6}{4\,(1+3x)^2}\right)\right) = 0$$

$$\frac{a_0+6}{1+3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -6 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= (-6+x)\,\mathrm{e}^{\int\left(\frac{19}{18\left(x+\frac{1}{3}\right)} - \frac{1}{6}\right)dx} \\
&= (-6+x)\,\mathrm{e}^{-\frac{x}{6} + \frac{19\ln(1+3x)}{18}} \\
&= (-6+x)\,(1+3x)^{19/18}\,\mathrm{e}^{-\frac{x}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1+3x}\,dx} \\
&= z_1 e^{-\frac{x}{6}+\frac{\ln(1+3x)}{18}} \\
&= z_1\left((1+3x)^{1/18}\,\mathrm{e}^{-\frac{x}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (1+3x)^{10/9}\,\mathrm{e}^{-\frac{x}{3}}(-6+x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1+3x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{3}+\frac{\ln(1+3x)}{9}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{x}{3}+\frac{\ln(1+3x)}{9}}\,\mathrm{e}^{\frac{2x}{3}}}{(1+3x)^{20/9}\,(-6+x)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left((1+3x)^{10/9}\,\mathrm{e}^{-\frac{x}{3}}(-6+x)\right) \\
&\quad + c_2\left((1+3x)^{10/9}\,\mathrm{e}^{-\frac{x}{3}}(-6+x)\left(\int \frac{\mathrm{e}^{-\frac{x}{3}+\frac{\ln(1+3x)}{9}}\,\mathrm{e}^{\frac{2x}{3}}}{(1+3x)^{20/9}\,(-6+x)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.68.2 Maple step by step solution

Let's solve

$(1+3x)\left(\frac{d}{dx}y'\right)+xy'+2y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{2y}{1+3x}-\frac{xy'}{1+3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{xy'}{1+3x}+\frac{2y}{1+3x}=0$

□ Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

- ∘ Define functions

  $\left[P_2(x)=\frac{x}{1+3x}, P_3(x)=\frac{2}{1+3x}\right]$

- ∘ $\left(x+\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{3}$

  $\left(\left(x+\frac{1}{3}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{1}{3}}=-\frac{1}{9}$

- ∘ $\left(x+\frac{1}{3}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{3}$

  $\left(\left(x+\frac{1}{3}\right)^2\cdot P_3(x)\right)\Big|_{x=-\frac{1}{3}}=0$

- ∘ $x=-\frac{1}{3}$is a regular singular point

Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

$x_0=-\frac{1}{3}$

- Multiply by denominators

$(1+3x)\left(\frac{d}{dx}y'\right)+xy'+2y=0$

- Change variables using $x=u-\frac{1}{3}$ so that the regular singular point is at $u=0$

$3u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\left(u-\frac{1}{3}\right)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ∘ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^{m}\cdot \left(\frac{d}{du}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty}a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$

- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y \left(u \right)\right)$ to series expansion

  $u \cdot \left(\frac{d}{du}\frac{d}{du}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -1}$

- Shift index using $k->k+1$

  $u \cdot \left(\frac{d}{du}\frac{d}{du}y \left(u \right)\right)=\sum_{k =-1}^{\infty}a_{k +1} \left(k +1+r \right) \left(k +r \right) u^{k +r}$

  Rewrite ODE with series expansions

  $\frac{a_{0} r \left(-10+9 r \right) u^{-1+r}}{3}+\left(\sum_{k =0}^{\infty}\left(\frac{a_{k +1} \left(k +1+r \right) \left(9 k -1+9 r \right)}{3}+a_{k} \left(k +r +2\right)\right) u^{k +r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $\frac{r \left(-10+9 r \right)}{3}=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{10}{9}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3 \left(k +1+r \right) \left(k -\frac{1}{9}+r \right) a_{k +1}+a_{k} \left(k +r +2\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k +1}=-\frac{3 a_{k} \left(k +r +2\right)}{\left(k +1+r \right) \left(9 k -1+9 r \right)}$

- Recursion relation for $r =0$

  $a_{k +1}=-\frac{3 a_{k} \left(k +2\right)}{\left(k +1\right) \left(9 k -1\right)}$

- Solution for $r =0$

  $\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k}, a_{k +1}=-\frac{3 a_{k} \left(k +2\right)}{\left(k +1\right) \left(9 k -1\right)}\right]$

- Revert the change of variables $u =x +\frac{1}{3}$

  $\left[y=\sum_{k =0}^{\infty}a_{k} \left(x +\frac{1}{3}\right)^{k}, a_{k +1}=-\frac{3 a_{k} \left(k +2\right)}{\left(k +1\right) \left(9 k -1\right)}\right]$

- Recursion relation for $r =\frac{10}{9}$

  $a_{k +1}=-\frac{3 a_{k} \left(k +\frac{28}{9}\right)}{\left(k +\frac{19}{9}\right) \left(9 k +9\right)}$

- Solution for $r =\frac{10}{9}$

  $\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k +\frac{10}{9}}, a_{k +1}=-\frac{3 a_{k} \left(k +\frac{28}{9}\right)}{\left(k +\frac{19}{9}\right) \left(9 k +9\right)}\right]$

- Revert the change of variables $u = x + \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{3}\right)^{k+\frac{10}{9}}, a_{k+1} = -\frac{3a_k \left(k+\frac{28}{9}\right)}{\left(k+\frac{19}{9}\right)(9k+9)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x + \frac{1}{3}\right)^{k}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + \frac{1}{3}\right)^{k+\frac{10}{9}}\right), a_{k+1} = -\frac{3a_k(k+2)}{(k+1)(9k-1)}, b_{k+1} = -\frac{3b_k \left(k+\frac{28}{9}\right)}{\left(k+\frac{19}{9}\right)(9k+9)}\right]$$

1.68.3 Maple trace

```
Methods for second order ODEs:
```

1.68.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 62

```
dsolve((1+3*x)*diff(diff(y(x),x),x)+x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \left(\Gamma\left(-\frac{1}{9}\right) + \frac{10\Gamma\left(-\frac{10}{9}, -\frac{1}{9}-\frac{x}{3}\right)}{9}\right)(-6+x)\left(x+\frac{1}{3}\right) \mathrm{e}^{-\frac{x}{3}} \left(-\frac{1}{9}-\frac{x}{3}\right)^{1/9}}{9} + 3c_2(-6+x)\left(x+\frac{1}{3}\right) \mathrm{e}^{-\frac{x}{3}} \left(\frac{1}{9}+\frac{x}{3}\right)^{1/9} - \frac{10c_1 \,\mathrm{e}^{\frac{1}{9}}}{9}$$

1.68.5 Mathematica DSolve solution

Solving time : `4.083` (sec)
Leaf size : 124

```
DSolve[{(1+3*x)*D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{x}{3}-\frac{1}{9}} \left(1520c_1 \sqrt[9]{3x+1}(3x^2-17x-6) - 2^{8/9} c_2 e^{\frac{x}{3}+\frac{1}{9}}(9x^2-48x-26) + 2^{8/9}3^{7/9} c_2 \sqrt[9]{-3x-1}(3x^2-17\right.}{380\ 2^{17/18}}$$

1.69 problem 71

Internal problem ID [8207]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 71
Date solved : `Monday, October 21, 2024 at 05:03:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(3x^2+x+1\right)y''+(2+15x)\,y'+12y=0$$

1.69.1 Solved as second order ode using Kovacic algorithm

Time used: 0.829 (sec)

Writing the ode as

$$\left(3x^2+x+1\right)y''+(2+15x)\,y'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=3x^2+x+1\\B&=2+15x\\C&=12\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -9x^2 - 12x - 18 \\
t &= 4\left(3x^2 + x + 1\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 123: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\&= 4 - 2\\&= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x^2 + x + 1)^2$. There is a pole at $x = -\frac{1}{6} + \frac{i\sqrt{11}}{6}$ of order 2. There is a pole at $x = -\frac{1}{6} - \frac{i\sqrt{11}}{6}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{\frac{27}{88} + \frac{3i\sqrt{11}}{88}}{\left(x + \frac{1}{6} - \frac{i\sqrt{11}}{6}\right)^2} + \frac{\frac{27}{88} - \frac{3i\sqrt{11}}{88}}{\left(x + \frac{1}{6} + \frac{i\sqrt{11}}{6}\right)^2} + \frac{57i\sqrt{11}}{242\left(x + \frac{1}{6} - \frac{i\sqrt{11}}{6}\right)} - \frac{57i\sqrt{11}}{242\left(x + \frac{1}{6} + \frac{i\sqrt{11}}{6}\right)}$$

For the pole at $x = -\frac{1}{6} + \frac{i\sqrt{11}}{6}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{6}-\frac{i\sqrt{11}}{6}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{27}{88} + \frac{3i\sqrt{11}}{88}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{1078 + 66i\sqrt{11}}}{44}\\\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{1078 + 66i\sqrt{11}}}{44}\end{aligned}$$

For the pole at $x = -\frac{1}{6} - \frac{i\sqrt{11}}{6}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{6}+\frac{i\sqrt{11}}{6}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{27}{88} - \frac{3i\sqrt{11}}{88}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{1078 - 66i\sqrt{11}}}{44}\\\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{1078 - 66i\sqrt{11}}}{44}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{6} + \frac{i\sqrt{11}}{6}$	2	0	$\frac{1}{2} + \frac{\sqrt{1078+66i\sqrt{11}}}{44}$	$\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}$
$-\frac{1}{6} - \frac{i\sqrt{11}}{6}$	2	0	$\frac{1}{2} + \frac{\sqrt{1078-66i\sqrt{11}}}{44}$	$\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x + \frac{1}{6} - \frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x + \frac{1}{6} + \frac{i\sqrt{11}}{6}} + (-)(0) \\
&= \frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x + \frac{1}{6} - \frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x + \frac{1}{6} + \frac{i\sqrt{11}}{6}} \\
&= -\frac{3x}{6x^2 + 2x + 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x + \frac{1}{6} - \frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x + \frac{1}{6} + \frac{i\sqrt{11}}{6}} \right)(1) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{\left(x + \frac{1}{6} - \frac{i\sqrt{11}}{6}\right)^2} - \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{\left(x + \frac{1}{6} + \frac{i\sqrt{11}}{6}\right)^2} \right) + \left(\frac{1}{2}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(\frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x + \frac{1}{6} - \frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x + \frac{1}{6} + \frac{i\sqrt{11}}{6}} \right) dx} \\
&= (x)\, \mathrm{e}^{-\frac{\ln\left(36x^2+12x+12\right)}{4} + \frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}} \\
&= \frac{x\sqrt{2}\, 3^{3/4} \mathrm{e}^{\frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}}}{6\, (3x^2 + x + 1)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2+15x}{3x^2+x+1} \, dx} \\
&= z_1 e^{-\frac{5\ln\left(3x^2+x+1\right)}{4} + \frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}}}{(3x^2 + x + 1)^{5/4}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{\frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} x\sqrt{2}\, 3^{3/4}}{6\, (3x^2 + x + 1)^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2+15x}{3x^2+x+1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(3x^2+x+1\right)}{2} + \frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}}}{(y_1)^2} \, dx
\end{aligned}
$$

$$= y_1 \left(\int \frac{2\,\mathrm{e}^{-\frac{5\ln\left(3x^2+x+1\right)}{2}+\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} \left(3x^2+x+1\right)^3 \mathrm{e}^{-\frac{2\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} \sqrt{3}}{x^2} dx \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} x\sqrt{2}\,3^{3/4}}{6\left(3x^2+x+1\right)^{3/2}} \right)$$
$$+ c_2 \left(\frac{\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} x\sqrt{2}\,3^{3/4}}{6\left(3x^2+x+1\right)^{3/2}} \left(\int \frac{2\,\mathrm{e}^{-\frac{5\ln\left(3x^2+x+1\right)}{2}+\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} \left(3x^2+x+1\right)^3 \mathrm{e}^{-\frac{2\sqrt{11}\,\arctan\left(\frac{(6x-}{}\right)}{11}}}{x^2} \right.\right.$$

Will add steps showing solving for IC soon.

1.69.2 Maple step by step solution

Let's solve

$\left(3x^2+x+1\right)\left(\frac{d}{dx}y'\right) + (2+15x)\,y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{12y}{3x^2+x+1} - \frac{(2+15x)y'}{3x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(2+15x)y'}{3x^2+x+1} + \frac{12y}{3x^2+x+1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2+15x}{3x^2+x+1}, P_3(x) = \frac{12}{3x^2+x+1}\right]$

  ◦ $\left(x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}\right)\cdot P_2(x)$ is analytic at $x = -\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}$

  $\left(\left(x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}} = 0$

  ◦ $\left(x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}\right)^2\cdot P_3(x)$ is analytic at $x = -\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}$

$$\left(\left(x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}}=0$$

- $x=-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}$

- Multiply by denominators

  $(3x^2+x+1)\left(\frac{d}{dx}y'\right)+(2+15x)\,y'+12y=0$

- Change variables using $x=u-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}$ so that the regular singular point is at $u=0$

  $(3u^2-\mathrm{I}u\sqrt{11})\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\left(-\frac{1}{2}+15u-\frac{5\,\mathrm{I}\sqrt{11}}{2}\right)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $\frac{\mathrm{I}\sqrt{11}\,r\left(\mathrm{I}\sqrt{11}-33-22r\right)a_0u^{-1+r}}{22}+\left(\sum_{k=0}^{\infty}\left(\frac{\mathrm{I}\sqrt{11}\,(k+1+r)\left(\mathrm{I}\sqrt{11}-22k-55-22r\right)a_{k+1}}{22}+3a_k(k+r+2)^2\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\frac{\mathrm{I}}{22}\sqrt{11}\,r\left(\mathrm{I}\sqrt{11}-33-22r\right)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right\}$

- Each term in the series must be 0, giving the recursion relation

$$3a_k(k+r+2)^2-(k+1+r)\,a_{k+1}\left(\frac{1}{2}+\mathrm{I}\left(k+r+\frac{5}{2}\right)\sqrt{11}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{6a_k\left(k^2+2kr+r^2+4k+4r+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}kr\sqrt{11}+2\,\mathrm{I}\sqrt{11}\,r^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}r\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k+r+1}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}\right]$$

- Revert the change of variables $u=x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}$

$$\left[y=\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}\right)^k, a_{k+1}=\frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}\right]$$

- Recursion relation for $r=-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}$

$$a_{k+1}=\frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2+\frac{2\,\mathrm{I}\sqrt{11}}{11}\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\,\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+\frac{111\,\mathrm{I}\sqrt{11}}{22}+k-\frac{1}{2}}$$

- Solution for $r=-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}, a_{k+1}=\frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2+\frac{2\,\mathrm{I}\sqrt{11}}{11}\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\,\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}}$$

- Revert the change of variables $u=x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}$

$$\left[y=\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}\right)^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}, a_{k+1}=\frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2\right.}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\,\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}\left(\right.}$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}\right)^k\right)+\left(\sum_{k=0}^{\infty}b_k\left(x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}\right)^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}\right), a_{k+1}=\frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\right.}$$

1.69.3 Maple trace

```
Methods for second order ODEs:
```

1.69.4 Maple dsolve solution

Solving time : `0.170` (sec)
Leaf size : 163

```
dsolve((3*x^2+x+1)*diff(diff(y(x),x),x)+(2+15*x)*diff(y(x),x)+12*y(x) = 0,
      y(x),singsol=all)
```

$$y = \frac{e^{\frac{\sqrt{11}\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}}\left(\left(i\sqrt{11}-6x-1\right)^{3/2} c_1\left(-36x^2-12x-12\right)^{-\frac{1}{4}+\frac{i\sqrt{11}}{44}} \operatorname{hypergeom}\left(\left[\frac{1}{2}+\frac{i\sqrt{11}}{22}, \frac{1}{2}+\frac{i\sqrt{11}}{22}\right.\right.\right.}{}$$

1.69.5 Mathematica DSolve solution

Solving time : `5.481` (sec)
Leaf size : 93

```
DSolve[{(1+x+3*x^2)*D[y[x],{x,2}]+(2+15*x)*D[y[x],x]+12*y[x]==0,{}},
      y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{xe^{\frac{\arctan\left(\frac{6x+1}{\sqrt{11}}\right)}{\sqrt{11}}}\left(c_2\int_1^x \frac{e^{-\frac{\arctan\left(\frac{6K[1]+1}{\sqrt{11}}\right)}{\sqrt{11}}}\sqrt{3K[1]^2+K[1]+1}}{K[1]^2}dK[1]+c_1\right)}{(3x^2+x+1)^{3/2}}$$

1.70 problem 72

Internal problem ID [8208]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 72
Date solved : `Monday, October 21, 2024 at 05:03:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2+x)\,y''+(1+x)\,y'+3y=0$$

1.70.1 Solved as second order ode using Kovacic algorithm

Time used: 0.393 (sec)

Writing the ode as

$$(2+x)\,y''+(1+x)\,y'+3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2+x\\B&=1+x\\C&=3\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10x - 21}{4\left(2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 10x - 21$$
$$t = 4(2+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 10x - 21}{4\left(2+x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 125: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2+x)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{7}{2(2+x)} + \frac{3}{4(2+x)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{7}{2x} - \frac{9}{2x^2} - \frac{97}{2x^3} - \frac{1291}{4x^4} - \frac{11103}{4x^5} - \frac{98061}{4x^6} - \frac{913053}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 10x - 21}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-14x - 25}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{-14x - 25}{4x^2 + 16x + 16}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -14. Dividing this by leading coefficient in t which is 4 gives $-\frac{7}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{2}\right) - (0) \\ &= -\frac{7}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{2}}{\frac{1}{2}} - 0\right) &&= \frac{7}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 10x - 21}{4\left(2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{7}{2}$	$\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{7}{2} - \left(\frac{3}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{3}{2\,(2+x)} + (-)\left(\frac{1}{2}\right) \\ &= \frac{3}{2\,(2+x)} - \frac{1}{2} \\ &= -\frac{-1+x}{2\,(2+x)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2\,(2+x)} - \frac{1}{2}\right)(2x+a_1) + \left(\left(-\frac{3}{2\,(2+x)^2}\right) + \left(\frac{3}{2\,(2+x)} - \frac{1}{2}\right)^2 - \left(\frac{x^2-10x-21}{4\,(2+x)^2}\right)\right) = 0$$

$$\frac{(a_1+4)\,x + 2a_0 + a_1 + 4}{2+x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = -4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 4x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^2 - 4x\right) e^{\int \left(\frac{3}{2(2+x)} - \frac{1}{2}\right) dx} \\ &= \left(x^2 - 4x\right) e^{-\frac{x}{2} + \frac{3\ln(2+x)}{2}} \\ &= x(x-4)\,(2+x)^{3/2}\, e^{-\frac{x}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+x}{2+x}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\frac{\ln(2+x)}{2}} \\
&= z_1\left(\sqrt{2+x}\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (2+x)^2\,\mathrm{e}^{-x} x(x-4)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+x}{2+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+\ln(2+x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{x}}{128x} - \frac{\mathrm{e}^{-2}\,\mathrm{Ei}_1(-2-x)}{48} - \frac{\mathrm{e}^{x}}{3456\,(x-4)} - \frac{11\,\mathrm{e}^{x}}{864\,(2+x)} - \frac{\mathrm{e}^{x}}{288\,(2+x)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)^2\,\mathrm{e}^{-x} x(x-4)\right) + c_2\left((2+x)^2\,\mathrm{e}^{-x} x(x-4)\left(-\frac{\mathrm{e}^{x}}{128x} - \frac{\mathrm{e}^{-2}\,\mathrm{Ei}_1(-2-x)}{48}\right.\right. \\
&\qquad \left.\left. - \frac{\mathrm{e}^{x}}{3456\,(x-4)} - \frac{11\,\mathrm{e}^{x}}{864\,(2+x)} - \frac{\mathrm{e}^{x}}{288\,(2+x)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.70.2 Maple step by step solution

Let's solve

$(2+x)\left(\frac{d}{dx}y'\right)+(1+x)\,y'+3y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{3y}{2+x}-\frac{(1+x)y'}{2+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(1+x)y'}{2+x}+\frac{3y}{2+x}=0$

□ Check to see if $x_0=-2$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1+x}{2+x}, P_3(x)=\frac{3}{2+x}\right]$

  - $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((2+x)\cdot P_2(x)\right)\Big|_{x=-2}=-1$

  - $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((2+x)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  - $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $(2+x)\left(\frac{d}{dx}y'\right)+(1+x)\,y'+3y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-1+u)\left(\frac{d}{du}y(u)\right)+3y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

$$u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k->k+1$

  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) + a_k(k+r+3)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)\,(k+r-1) + a_k(k+r+3) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k(k+r+3)}{(k+1+r)(k+r-1)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-1)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

  $$a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-1)}$$

- Recursion relation for $r = 2$

  $$a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}$$

- Solution for $r = 2$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}\right]$$

- Revert the change of variables $u = 2+x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(2+x)^{k+2}, a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}\right]$$

1.70.3 Maple trace

```
Methods for second order ODEs:
```

1.70.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 59

```
dsolve((2+x)*diff(diff(y(x),x),x)+(1+x)*diff(y(x),x)+3*y(x) = 0,
       y(x),singsol=all)
```

$$y = xc_2 \, \mathrm{e}^{-2-x}(x-4)\,(2+x)^2 \, \mathrm{Ei}_1\,(-2-x)+c_1(2+x)^2 \, \mathrm{e}^{-x}x(x-4)+c_2\left(x^3-x^2-10x-6\right)$$

1.70.5 Mathematica DSolve solution

Solving time : 0.618 (sec)
Leaf size : 99

```
DSolve[{(2+x)*D[y[x],{x,2}]+(1+x)*D[y[x],x]+3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x-1}(c_2(x-4)(x+2)^2 x\,\text{ExpIntegralEi}(x+2)+384c_1x^4-c_2e^{x+2}x^3+x^2(c_2e^{x+2}-4608c_1)+x(10c_2e}{96\sqrt{2}}$$

1.71 problem 73

Internal problem ID [8209]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 73
Date solved : `Monday, October 21, 2024 at 05:03:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(4+x)\,y''+(2+x)\,y'+2y=0$$

1.71.1 Solved as second order ode using Kovacic algorithm

Time used: 0.326 (sec)

Writing the ode as

$$(4+x)\,y''+(2+x)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4+x \\ B &= 2+x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x - 24}{4\left(4 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x - 24$$
$$t = 4(4 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x - 24}{4\left(4 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 127: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(4+x)^2$. There is a pole at $x = -4$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(4+x)^2} - \frac{3}{4+x}$$

For the pole at $x = -4$ let b be the coefficient of $\frac{1}{(4+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{x} + \frac{5}{x^2} - \frac{34}{x^3} + \frac{59}{x^4} - \frac{586}{x^5} + \frac{370}{x^6} - \frac{12484}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x - 24}{4x^2 + 32x + 64} \\ &= Q + \frac{R}{4x^2 + 32x + 64} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-12x - 40}{4x^2 + 32x + 64}\right) \\ &= \frac{1}{4} + \frac{-12x - 40}{4x^2 + 32x + 64} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -12. Dividing this by leading coefficient in t which is 4 gives -3. Now b can be found.

$$\begin{aligned} b &= (-3) - (0) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-3}{\frac{1}{2}} - 0\right) &= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-3}{\frac{1}{2}} - 0\right) &= 3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x - 24}{4\left(4+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-4	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-3	3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 3$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= 3 - (2) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{2}{4+x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{2}{4+x} - \frac{1}{2} \\
&= -\frac{x}{2\,(4+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{4+x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{2}{(4+x)^2}\right) + \left(\frac{2}{4+x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-4x-24}{4\,(4+x)^2}\right)\right) = 0$$

$$\frac{a_0}{4+x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x)\, e^{\int \left(\frac{2}{4+x} - \frac{1}{2}\right) dx} \\
&= (x)\, e^{-\frac{x}{2} + 2\ln(4+x)} \\
&= x(4+x)^2\, e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2+x}{4+x}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\ln(4+x)} \\
&= z_1\left((4+x)\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (4+x)^3\,\mathrm{e}^{-x} x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2+x}{4+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+2\ln(4+x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{x}}{48\,(4+x)^3} - \frac{5\,\mathrm{e}^{x}}{192\,(4+x)^2} - \frac{29\,\mathrm{e}^{x}}{768\,(4+x)} - \frac{\mathrm{e}^{-4}\,\mathrm{Ei}_1(-4-x)}{24} - \frac{\mathrm{e}^{x}}{256x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((4+x)^3\,\mathrm{e}^{-x} x\right) + c_2\left((4+x)^3\,\mathrm{e}^{-x} x\left(-\frac{\mathrm{e}^{x}}{48\,(4+x)^3} - \frac{5\,\mathrm{e}^{x}}{192\,(4+x)^2} - \frac{29\,\mathrm{e}^{x}}{768\,(4+x)} \right.\right. \\
&\qquad \left.\left. - \frac{\mathrm{e}^{-4}\,\mathrm{Ei}_1(-4-x)}{24} - \frac{\mathrm{e}^{x}}{256x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.71.2 Maple step by step solution

Let's solve

$(4+x)\left(\frac{d}{dx}y'\right)+(2+x)\,y'+2y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{2y}{4+x}-\frac{(2+x)y'}{4+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(2+x)y'}{4+x}+\frac{2y}{4+x}=0$

☐ Check to see if $x_0=-4$ is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{2+x}{4+x}, P_3(x)=\frac{2}{4+x}\right]$

- ○ $(4+x)\cdot P_2(x)$ is analytic at $x=-4$

$\left((4+x)\cdot P_2(x)\right)\bigg|_{x=-4}=-2$

- ○ $(4+x)^2\cdot P_3(x)$ is analytic at $x=-4$

$\left((4+x)^2\cdot P_3(x)\right)\bigg|_{x=-4}=0$

- ○ $x=-4$is a regular singular point

Check to see if $x_0=-4$ is a regular singular point

$x_0=-4$

- Multiply by denominators

$(4+x)\left(\frac{d}{dx}y'\right)+(2+x)\,y'+2y=0$

- Change variables using $x=u-4$ so that the regular singular point is at $u=0$

$u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-2+u)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- >k+1$

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r+2)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r+2)}{(k+1+r)(k-2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-2)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

$$a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-2)}$$

- Recursion relation for $r = 3$

$$a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}$$

- Solution for $r = 3$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}\right]$$

- Revert the change of variables $u = 4 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(4+x)^{k+3}, a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}\right]$$

1.71.3 Maple trace

```
Methods for second order ODEs:
```

1.71.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 53

```
dsolve((4+x)*diff(diff(y(x),x),x)+(2+x)*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = xc_2 \,\mathrm{e}^{-4-x}(4+x)^3 \,\mathrm{Ei}_1\left(-4-x\right) + c_1(4+x)^3 \,\mathrm{e}^{-x} x + c_2\left(x^3+9x^2+22x+6\right)$$

1.71.5 Mathematica DSolve solution

Solving time : 0.317 (sec)
Leaf size : 97

```
DSolve[{(4+x)*D[y[x],{x,2}]+(2+x)*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{24}e^{-x-4}\left(c_2x(x+4)^3 \,\mathrm{ExpIntegralEi}(x+4) + e^4\left(24c_1x^4+x^3(288c_1-c_2e^x)+9x^2(128c_1-c_2e^x)+2x(768c_1-11c_2e^x)-6c_2e^x\right)\right)$$

1.72 problem 74

Internal problem ID [8210]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 74
Date solved : `Monday, October 21, 2024 at 05:03:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+3x\right)y''+10(1+x)\,y'+8y=0$$

1.72.1 Solved as second order ode using Kovacic algorithm

Time used: 0.326 (sec)

Writing the ode as

$$\left(2x^2+3x\right)y''+(10x+10)\,y'+8y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+3x \\ B &= 10x+10 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6x + 10}{(2x^2 + 3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 6x + 10$$
$$t = (2x^2 + 3x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6x + 10}{(2x^2 + 3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 129: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36\left(x+\frac{3}{2}\right)^2} + \frac{22}{27\left(x+\frac{3}{2}\right)} - \frac{22}{27x} + \frac{10}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{10}{9}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{3}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{2}{3}\end{aligned}$$

For the pole at $x = -\frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 6x + 10}{\left(2x^2 + 3x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 6x + 10}{\left(2x^2 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{3}$	$-\frac{2}{3}$
$-\frac{3}{2}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{3x} + \frac{1}{6x+9} + (-)(0) \\
&= -\frac{2}{3x} + \frac{1}{6x+9} \\
&= -\frac{x+2}{x(2x+3)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{2}{3x} + \frac{1}{6x+9}\right)(1) + \left(\left(\frac{2}{3x^2} - \frac{1}{6\left(x+\frac{3}{2}\right)^2}\right) + \left(-\frac{2}{3x} + \frac{1}{6x+9}\right)^2 - \left(\frac{-x^2+6x+10}{(2x^2+3x)^2}\right)\right) = 0
$$

$$
\frac{-4+2a_0}{x(2x+3)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, e^{\int \left(-\frac{2}{3x} + \frac{1}{6x+9}\right) dx} \\
&= (x+2)\, e^{-\frac{2\ln(x)}{3} + \frac{\ln(2x+3)}{6}} \\
&= \frac{(x+2)(2x+3)^{1/6}}{x^{2/3}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{10x+10}{2x^2+3x}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{3}-\frac{5\ln(2x+3)}{6}} \\
&= z_1\left(\frac{1}{x^{5/3}\left(2x+3\right)^{5/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x+2}{x^{7/3}\left(2x+3\right)^{2/3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{10x+10}{2x^2+3x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{10\ln(x)}{3}-\frac{5\ln(2x+3)}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{10\ln(x)}{3}-\frac{5\ln(2x+3)}{3}}x^{14/3}\left(2x+3\right)^{4/3}}{\left(x+2\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x+2}{x^{7/3}\left(2x+3\right)^{2/3}}\right) + c_2\left(\frac{x+2}{x^{7/3}\left(2x+3\right)^{2/3}}\left(\int \frac{\mathrm{e}^{-\frac{10\ln(x)}{3}-\frac{5\ln(2x+3)}{3}}x^{14/3}\left(2x+3\right)^{4/3}}{\left(x+2\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.72.2 Maple step by step solution

Let's solve

$(2x^2+3x)\left(\frac{d}{dx}y'\right)+10(1+x)\,y'+8y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{8y}{(2x+3)x}-\frac{10(1+x)y'}{x(2x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{10(1+x)y'}{x(2x+3)}+\frac{8y}{(2x+3)x}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{10(1+x)}{x(2x+3)},P_3(x)=\frac{8}{(2x+3)x}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$(x\cdot P_2(x))\bigg|_{x=0}=\frac{10}{3}$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2\cdot P_3(x))\bigg|_{x=0}=0$

- ○ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x(2x+3)\left(\frac{d}{dx}y'\right)+(10x+10)\,y'+8y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

$x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=1..2$

$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(7+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(3k+10+3r) + 2a_k(k+r+2)^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(7+3r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{7}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(3k+10+3r) + 2a_k(k+r+2)^2 = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k(k+r+2)^2}{(k+1+r)(3k+10+3r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}\right]$
- Recursion relation for $r = -\frac{7}{3}$

  $a_{k+1} = -\frac{2a_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}$
- Solution for $r = -\frac{7}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{7}{3}}, a_{k+1} = -\frac{2a_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{7}{3}}\right), a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}, b_{k+1} = -\frac{2b_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$

1.72.3 Maple trace

```
Methods for second order ODEs:
```

1.72.4 Maple dsolve solution

Solving time : 0.030 (sec)
Leaf size : 31

```
dsolve((2*x^2+3*x)*diff(diff(y(x),x),x)+10*(1+x)*diff(y(x),x)+8*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1(x+2)}{\left(1+\frac{2x}{3}\right)^{2/3} x^{7/3}} + c_2 \,\mathrm{hypergeom}\left([2,2], \left[\frac{10}{3}\right], -\frac{2x}{3}\right)$$

1.72.5 Mathematica DSolve solution

Solving time : 0.967 (sec)
Leaf size : 245

```
DSolve[{(3*x+2*x^2)*D[y[x],{x,2}]+10*(1+x)*D[y[x],x]+8*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{-2\ 2^{2/3}\sqrt{3}c_2(x+2)\arctan\left(\frac{\sqrt{3}\sqrt[3]{2x+3}}{2\sqrt[3]{2}\sqrt[3]{x}+\sqrt[3]{2x+3}}\right) + 2^{2/3}c_2 x\log\left(2^{2/3}x^{2/3}+\sqrt[3]{2}\sqrt[3]{2x+3}\sqrt[3]{x}+(2x+3)^{2}\right.}{}$$

1.73 problem 75

Internal problem ID [8211]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 75
Date solved : `Monday, October 21, 2024 at 05:03:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - (6 - 7x)\, y' + 8y = 0$$

1.73.1 Solved as second order ode using Kovacic algorithm

Time used: 0.316 (sec)

Writing the ode as

$$x^2y'' + (-6 + 7x)\, y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -6 + 7x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 60x + 36}{4x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2 - 60x + 36 \\ t &= 4x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 60x + 36}{4x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 131: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{15}{x^3} + \frac{9}{x^4} + \frac{3}{4x^2}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{3}{x^2} - \frac{5}{2x} - \frac{11}{12} - \frac{55x}{72} - \frac{671x^2}{864} - \frac{4565x^3}{5184} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{3}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = 3$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -15. Therefore

$$\begin{aligned} b &= (-15) - (0) \\ &= -15 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{3}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{-15}{3} + 2\right) &= -\frac{3}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{-15}{3} + 2\right) &= \frac{7}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 60x + 36}{4x^4}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 60x + 36}{4x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{3}{x^2}$	$-\frac{3}{2}$	$\frac{7}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{x^2} - \frac{3}{2x} + (-)\,(0) \\
&= \frac{3}{x^2} - \frac{3}{2x} \\
&= -\frac{3(-2+x)}{2x^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{3}{x^2}-\frac{3}{2x}\right)(1)+\left(\left(-\frac{6}{x^3}+\frac{3}{2x^2}\right)+\left(\frac{3}{x^2}-\frac{3}{2x}\right)^2-\left(\frac{3x^2-60x+36}{4x^4}\right)\right)=0$$
$$\frac{6+3a_0}{x^2}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=-2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=-2+x$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= (-2+x)\,e^{\int \left(\frac{3}{x^2}-\frac{3}{2x}\right)dx} \\
&= (-2+x)\,e^{-\frac{3}{x}-\frac{3\ln(x)}{2}} \\
&= \frac{(-2+x)\,e^{-\frac{3}{x}}}{x^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6+7x}{x^2}\, dx} \\
&= z_1 e^{-\frac{3}{x}-\frac{7\ln(x)}{2}} \\
&= z_1\left(\frac{e^{-\frac{3}{x}}}{x^{7/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{e^{-\frac{6}{x}}(-2+x)}{x^5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6+7x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{6}{x}-7\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(7x\,\mathrm{e}^{\frac{6}{x}} + 54\,\mathrm{Ei}_1\left(-\frac{6}{x}\right) + \frac{12\,\mathrm{e}^{\frac{6}{x}}}{\frac{6}{x}-3} + \frac{x^2\mathrm{e}^{\frac{6}{x}}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\frac{6}{x}}(-2+x)}{x^5}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{6}{x}}(-2+x)}{x^5}\left(7x\,\mathrm{e}^{\frac{6}{x}} + 54\,\mathrm{Ei}_1\left(-\frac{6}{x}\right) + \frac{12\,\mathrm{e}^{\frac{6}{x}}}{\frac{6}{x}-3} + \frac{x^2\mathrm{e}^{\frac{6}{x}}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.73.2 Maple step by step solution

1.73.3 Maple trace

```
Methods for second order ODEs:
```

1.73.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 50

```
dsolve(x^2*diff(diff(y(x),x),x)-(6-7*x)*diff(y(x),x)+8*y(x) = 0,
       y(x),singsol=all)
```

$$
y = \frac{108c_2\,\mathrm{e}^{-\frac{6}{x}}(-2+x)\,\mathrm{Ei}_1\left(-\frac{6}{x}\right) + c_1\,\mathrm{e}^{-\frac{6}{x}}(-2+x) + c_2 x(x^2+12x-36)}{x^5}
$$

1.73.5 Mathematica DSolve solution

Solving time : `0.265` (sec)
Leaf size : 59

```
DSolve[{x^2*D[y[x],{x,2}]-(6-7*x)*D[y[x],x]+8*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-6/x}\left(-108c_2(x-2)\,\text{ExpIntegralEi}\left(\frac{6}{x}\right)+c_2e^{6/x}x(x^2+12x-36)+2c_1(x-2)\right)}{2x^5}$$

1.74 problem 76

Internal problem ID [8212]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 76
Date solved : `Monday, October 21, 2024 at 05:03:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+x+1\right)y''+(1+7x)\,y'+2y=0$$

1.74.1 Solved as second order ode using Kovacic algorithm

Time used: 1.066 (sec)

Writing the ode as

$$\left(2x^2+x+1\right)y''+(1+7x)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+x+1 \\ B &= 1+7x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 - 2x + 5}{4\left(2x^2 + x + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 - 2x + 5$$
$$t = 4\left(2x^2 + x + 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 - 2x + 5}{4\left(2x^2 + x + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 132: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x + 1)^2$. There is a pole at $x = -\frac{1}{4} + \frac{i\sqrt{7}}{4}$ of order 2. There is a pole at $x = -\frac{1}{4} - \frac{i\sqrt{7}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{29}{224} + \frac{9i\sqrt{7}}{224}}{\left(x + \frac{1}{4} - \frac{i\sqrt{7}}{4}\right)^2} + \frac{-\frac{29}{224} - \frac{9i\sqrt{7}}{224}}{\left(x + \frac{1}{4} + \frac{i\sqrt{7}}{4}\right)^2} - \frac{8i\sqrt{7}}{49\left(x + \frac{1}{4} - \frac{i\sqrt{7}}{4}\right)} + \frac{8i\sqrt{7}}{49\left(x + \frac{1}{4} + \frac{i\sqrt{7}}{4}\right)}$$

For the pole at $x = -\frac{1}{4} + \frac{i\sqrt{7}}{4}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{4} - \frac{i\sqrt{7}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{224} + \frac{9i\sqrt{7}}{224}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{3\sqrt{42 + 14i\sqrt{7}}}{56} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{3\sqrt{42 + 14i\sqrt{7}}}{56} \end{aligned}$$

For the pole at $x = -\frac{1}{4} - \frac{i\sqrt{7}}{4}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{4} + \frac{i\sqrt{7}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{224} - \frac{9i\sqrt{7}}{224}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{3\sqrt{42 - 14i\sqrt{7}}}{56} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{3\sqrt{42 - 14i\sqrt{7}}}{56} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 - 2x + 5}{4\left(2x^2 + x + 1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 - 2x + 5}{4\left(2x^2 + x + 1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{4} + \frac{i\sqrt{7}}{4}$	2	0	$\frac{1}{2} + \frac{3\sqrt{42+14i\sqrt{7}}}{56}$	$\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}$
$-\frac{1}{4} - \frac{i\sqrt{7}}{4}$	2	0	$\frac{1}{2} + \frac{3\sqrt{42-14i\sqrt{7}}}{56}$	$\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{5}{4} - \left(\frac{1}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x + \frac{1}{4} - \frac{i\sqrt{7}}{4}} + \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x + \frac{1}{4} + \frac{i\sqrt{7}}{4}} + (0) \\
&= \frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x + \frac{1}{4} - \frac{i\sqrt{7}}{4}} + \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x + \frac{1}{4} + \frac{i\sqrt{7}}{4}} \\
&= \frac{x + 1}{4x^2 + 2x + 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x + \frac{1}{4} - \frac{i\sqrt{7}}{4}} + \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x + \frac{1}{4} + \frac{i\sqrt{7}}{4}} \right)(1) + \left(\left(-\frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{\left(x + \frac{1}{4} - \frac{i\sqrt{7}}{4}\right)^2} - \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{\left(x + \frac{1}{4} + \frac{i\sqrt{7}}{4}\right)^2} \right) + \left(\frac{\frac{1}{2} - 3\sqrt{}}{x + \frac{1}{4}} \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+1)\, \mathrm{e}^{\int \left(\frac{\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x+\frac{1}{4}-\frac{i\sqrt{7}}{4}}+\frac{\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x+\frac{1}{4}+\frac{i\sqrt{7}}{4}}\right)dx} \\
&= (x+1)\, \mathrm{e}^{\frac{\ln\left(16x^2+8x+8\right)}{8}+\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}} \\
&= (x+1)\, 2^{3/8} \left(2x^2+x+1\right)^{1/8} \mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+7x}{2x^2+x+1}\, dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^2+x+1\right)}{8}+\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}}}{\left(2x^2+x+1\right)^{7/8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}} (x+1)\, 2^{3/8}}{\left(2x^2+x+1\right)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+7x}{2x^2+x+1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^2+x+1\right)}{4}+\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{7\ln\left(2x^2+x+1\right)}{4}+\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}} \left(2x^2+x+1\right)^{3/2} \mathrm{e}^{-\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{7}}\, 2^{1/4}}{2\,(x+1)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}} (x+1)\, 2^{3/8}}{(2x^2+x+1)^{3/4}} \right) \\
&\quad + c_2 \left(\frac{\mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}} (x+1)\, 2^{3/8}}{(2x^2+x+1)^{3/4}} \left(\int \frac{\mathrm{e}^{-\frac{7\ln\left(2x^2+x+1\right)}{4}+\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}} (2x^2+x+1)^{3/2}\, \mathrm{e}^{-\frac{3\sqrt{7}\, \arctan\left(\frac{(4x}{7}}}{2\,(x+1)^2} \right.\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.74.2 Maple step by step solution

Let's solve

$(2x^2+x+1)\left(\frac{d}{dx}y'\right) + (1+7x)\,y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2y}{2x^2+x+1} - \frac{(1+7x)y'}{2x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(1+7x)y'}{2x^2+x+1} + \frac{2y}{2x^2+x+1} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1+7x}{2x^2+x+1}, P_3(x) = \frac{2}{2x^2+x+1}\right]$

  ◦ $\left(x+\frac{1}{4}+\frac{\mathrm{I}\sqrt{7}}{4}\right)\cdot P_2(x)$ is analytic at $x = -\frac{1}{4}-\frac{\mathrm{I}\sqrt{7}}{4}$

    $\left(\left(x+\frac{1}{4}+\frac{\mathrm{I}\sqrt{7}}{4}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{1}{4}-\frac{\mathrm{I}\sqrt{7}}{4}} = 0$

  ◦ $\left(x+\frac{1}{4}+\frac{\mathrm{I}\sqrt{7}}{4}\right)^2\cdot P_3(x)$ is analytic at $x = -\frac{1}{4}-\frac{\mathrm{I}\sqrt{7}}{4}$

    $\left(\left(x+\frac{1}{4}+\frac{\mathrm{I}\sqrt{7}}{4}\right)^2\cdot P_3(x)\right)\Big|_{x=-\frac{1}{4}-\frac{\mathrm{I}\sqrt{7}}{4}} = 0$

  ◦ $x = -\frac{1}{4}-\frac{\mathrm{I}\sqrt{7}}{4}$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -\frac{1}{4} - \frac{I\sqrt{7}}{4}$

- Multiply by denominators

  $(2x^2 + x + 1)\left(\frac{d}{dx}y'\right) + (1 + 7x)\,y' + 2y = 0$

- Change variables using $x = u - \frac{1}{4} - \frac{I\sqrt{7}}{4}$ so that the regular singular point is at $u = 0$

  $(2u^2 - Iu\sqrt{7})\left(\frac{d}{du}\frac{d}{du}y(u)\right) + \left(-\frac{3}{4} + 7u - \frac{7\,I\sqrt{7}}{4}\right)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,u^{k+r-1+m}$

  - Shift index using $k- > k + 1 - m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\,u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\,u^{k+r-2+m}$

  - Shift index using $k- > k + 2 - m$

    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $\frac{I\sqrt{7}\,r\left(3\,I\sqrt{7}-21-28r\right)a_0 u^{-1+r}}{28} + \left(\sum_{k=0}^{\infty}\left(\frac{I\sqrt{7}\,(k+1+r)\left(3\,I\sqrt{7}-28k-49-28r\right)a_{k+1}}{28} + a_k(k + r + 2)\,(2k + 2r + 1)\right)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\frac{I}{28}\sqrt{7}\,r\left(3\,I\sqrt{7} - 21 - 28r\right) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3\,I\sqrt{7}}{28} - \frac{3}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-I\left(k + r + \frac{7}{4}\right) a_{k+1}(k + 1 + r)\,\sqrt{7} + \frac{(-3k-3r-3)a_{k+1}}{4} + 2\left(k + r + \frac{1}{2}\right)(k + r + 2)\,a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{4a_k\,(2k^2+4kr+2r^2+5k+5r+2)}{3+4\,I\sqrt{7}\,k^2+8\,I\sqrt{7}\,kr+4\,I\sqrt{7}\,r^2+11\,I\sqrt{7}\,k+11\,I\sqrt{7}\,r+7\,I\sqrt{7}+3k+3r}$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{4a_k(2k^2+5k+2)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}+3k}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{4a_k(2k^2+5k+2)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}+3k}\right]$$

- Revert the change of variables $u = x + \frac{1}{4} + \frac{\mathrm{I}\sqrt{7}}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{4} + \frac{\mathrm{I}\sqrt{7}}{4}\right)^k, a_{k+1} = \frac{4a_k(2k^2+5k+2)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}+3k}\right]$$

- Recursion relation for $r = \frac{3\,\mathrm{I}\sqrt{7}}{28} - \frac{3}{4}$

$$a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\,\mathrm{I}\sqrt{7}}{28}-\frac{7}{4}\right)}{\frac{3}{4}+4\,\mathrm{I}\sqrt{7}\,k^2+8\,\mathrm{I}\sqrt{7}\,k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+4\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\,\mathrm{I}\sqrt{7}\,k+11\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+\frac{205\,\mathrm{I}\sqrt{7}}{28}+3k}$$

- Solution for $r = \frac{3\,\mathrm{I}\sqrt{7}}{28} - \frac{3}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}}, a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\,\mathrm{I}\sqrt{7}}{28}-\frac{7}{4}\right)}{\frac{3}{4}+4\,\mathrm{I}\sqrt{7}\,k^2+8\,\mathrm{I}\sqrt{7}\,k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+4\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\,\mathrm{I}\sqrt{7}\,k+11\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+$$

- Revert the change of variables $u = x + \frac{1}{4} + \frac{\mathrm{I}\sqrt{7}}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{4} + \frac{\mathrm{I}\sqrt{7}}{4}\right)^{k+\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}}, a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\,\mathrm{I}\sqrt{7}}{28}\right.}{\frac{3}{4}+4\,\mathrm{I}\sqrt{7}\,k^2+8\,\mathrm{I}\sqrt{7}\,k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+4\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\,\mathrm{I}\sqrt{7}\,k+11\,\mathrm{I}\sqrt{7}}$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x + \frac{1}{4} + \frac{\mathrm{I}\sqrt{7}}{4}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + \frac{1}{4} + \frac{\mathrm{I}\sqrt{7}}{4}\right)^{k+\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}}\right), a_{k+1} = \frac{4a_k(2k^2+5k+2)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}}\right.$$

1.74.3 Maple trace

```
Methods for second order ODEs:
```

1.74.4 Maple dsolve solution

Solving time : `0.031` (sec)
Leaf size : 77

```
dsolve((2*x^2+x+1)*diff(diff(y(x),x),x)+(1+7*x)*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \,\mathrm{hypergeom}\left(\left[\frac{1}{2}, 2\right], \left[-\frac{\left(-7\sqrt{7}+3i\right)\sqrt{7}}{28}\right], \frac{1}{2}+\frac{i(-4x-1)\sqrt{7}}{14}\right) + c_2\left(4x+1+i\sqrt{7}\right)^{\frac{3i\sqrt{7}}{28}-\frac{3}{4}}\left(i\sqrt{7}-4x-1\right)^{-\frac{3i\sqrt{7}}{28}-\frac{3}{4}}(x+1)$$

1.74.5 Mathematica DSolve solution

Solving time : `3.707` (sec)
Leaf size : 102

```
DSolve[{(1+x+2*x^2)*D[y[x],{x,2}]+(1+7*x)*D[y[x],x]+2*y[x]==0,{}},
      y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{(x+1)e^{\frac{3\arctan\left(\frac{4x+1}{\sqrt{7}}\right)}{2\sqrt{7}}}\left(c_2\int_1^x \frac{e^{-\frac{3\arctan\left(\frac{4K[1]+1}{\sqrt{7}}\right)}{2\sqrt{7}}}}{(K[1]+1)^2\sqrt[4]{2K[1]^2+K[1]+1}}dK[1]+c_1\right)}{(2x^2+x+1)^{3/4}}$$

1.75 problem 77

Internal problem ID [8213]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 77
Date solved : `Monday, October 21, 2024 at 05:03:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(3+x)\,y''+(1+2x)\,y'-(2-x)\,y=0$$

1.75.1 Solved as second order ode using Kovacic algorithm

Time used: 0.202 (sec)

Writing the ode as

$$(3+x)\,y''+(1+2x)\,y'+(x-2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3+x \\ B &= 1+2x \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35}{4(3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35$$
$$t = 4(3+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35}{4(3+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 134: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3+x)^2$. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{35}{4\left(3+x\right)^2}
$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(3+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{35}{4\left(3+x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{35}{4 \left(3+x \right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-3	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{5}{2} - \left(-\frac{5}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{5}{2 \left(3+x \right)} + \left(-\right) \left(0\right) \\ &= -\frac{5}{2 \left(3+x \right)} \\ &= -\frac{5}{2 \left(3+x \right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2(3+x)}\right)(0) + \left(\left(\frac{5}{2(3+x)^2}\right) + \left(-\frac{5}{2(3+x)}\right)^2 - \left(\frac{35}{4(3+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{5}{2(3+x)} dx} \\
&= \frac{1}{(3+x)^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+2x}{3+x} \, dx} \\
&= z_1 e^{-x + \frac{5\ln(3+x)}{2}} \\
&= z_1 \left((3+x)^{5/2} e^{-x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{1+2x}{3+x}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-2x+5\ln(3+x)}}{(y_1)^2}\,dx$$
$$= y_1 \left(\frac{x(x^5 + 18x^4 + 135x^3 + 540x^2 + 1215x + 1458)\,\mathrm{e}^{-2x+5\ln(3+x)}\mathrm{e}^{2x}}{6\,(3+x)^5}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x}\left(\frac{x(x^5 + 18x^4 + 135x^3 + 540x^2 + 1215x + 1458)\,\mathrm{e}^{-2x+5\ln(3+x)}\mathrm{e}^{2x}}{6\,(3+x)^5}\right)\right)$$

Will add steps showing solving for IC soon.

1.75.2 Maple step by step solution

Let's solve

$(3+x)\left(\frac{d}{dx}y'\right) + (1+2x)\,y' - (2-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x-2)y}{3+x} - \frac{(1+2x)y'}{3+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(1+2x)y'}{3+x} + \frac{(x-2)y}{3+x} = 0$

□ Check to see if $x_0 = -3$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1+2x}{3+x}, P_3(x) = \frac{x-2}{3+x}\right]$

  ◦ $(3+x)\cdot P_2(x)$ is analytic at $x = -3$

    $\left((3+x)\cdot P_2(x)\right)\Big|_{x=-3} = -5$

  ◦ $(3+x)^2\cdot P_3(x)$ is analytic at $x = -3$

$\left((3+x)^2 \cdot P_3(x)\right)\Big|_{x=-3} = 0$

- $x=-3$is a regular singular point
  Check to see if $x_0=-3$ is a regular singular point
  $x_0=-3$

- Multiply by denominators
  $(3+x)\left(\frac{d}{dx}y'\right)+(1+2x)\,y'+(x-2)\,y=0$
- Change variables using $x=u-3$ so that the regular singular point is at $u=0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-5+2u)\left(\frac{d}{du}y(u)\right)+(u-5)\,y(u)=0$
- Assume series solution for $y(u)$
  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m=0..1$
  $u^m \cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$
- Shift index using $k->k-m$
  $u^m \cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$
  $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
- Shift index using $k->k+1-m$
  $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$
- Shift index using $k->k+1$
  $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions
  $a_0r(-6+r)\,u^{-1+r}+\left(a_1(1+r)\,(-5+r)+a_0(-5+2r)\right)u^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)\,(k-5+r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-6+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 6\}$
- Each term must be 0

  $a_1(1+r)(-5+r) + a_0(-5+2r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-5+r) + 2a_k k + 2a_k r - 5a_k + a_{k-1} = 0$
- Shift index using $k- > k+1$

  $a_{k+2}(k+2+r)(k-4+r) + 2a_{k+1}(k+1) + 2ra_{k+1} - 5a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2ka_{k+1}+2ra_{k+1}+a_k-3a_{k+1}}{(k+2+r)(k-4+r)}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{2ka_{k+1}+a_k-3a_{k+1}}{(k+2)(k-4)}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 4$

  $a_{k+2} = -\frac{2ka_{k+1}+a_k-3a_{k+1}}{(k+2)(k-4)}$
- Recursion relation for $r = 6$

  $a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}$
- Solution for $r = 6$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}, 7a_1 + 7a_0 = 0\right]$
- Revert the change of variables $u = 3 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (3+x)^{k+6}, a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}, 7a_1 + 7a_0 = 0\right]$

1.75.3 Maple trace

```
Methods for second order ODEs:
```

1.75.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 33

```
dsolve((3+x)*diff(diff(y(x),x),x)+(1+2*x)*diff(y(x),x)-(2-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \left(\left(x^2+3x+9\right)\left(x^2+9x+27\right)(x+6)\, c_2 x + c_1\right) e^{-x}$$

1.75.5 Mathematica DSolve solution

Solving time : 0.074 (sec)
Leaf size : 29

```
DSolve[{(3+x)*D[y[x],{x,2}]+(1+2*x)*D[y[x],x]-(2-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{6} e^{-x-3} \left(c_2 (x+3)^6 + 6 c_1\right)$$

1.76 problem 78

Internal problem ID [8214]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 78
Date solved : `Monday, October 21, 2024 at 05:03:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 3xy' + (2x^2 + 4)\, y = 0$$

1.76.1 Solved as second order ode using Kovacic algorithm

Time used: 0.300 (sec)

Writing the ode as

$$y'' + 3xy' + (2x^2 + 4)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3x \\ C &= 2x^2 + 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 10}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 10 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 136: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{5}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\ a_1 x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^2 - 1) \, e^{\int -\frac{x}{2} dx} \\
&= (x^2 - 1) \, e^{-\frac{x^2}{4}} \\
&= (x^2 - 1) \, e^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x}{1} \, dx} \\
&= z_1 e^{-\frac{3x^2}{4}} \\
&= z_1 \left(e^{-\frac{3x^2}{4}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = (x^2 - 1) \, e^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{3x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{-\frac{3x^2}{2}} e^{2x^2}}{(x^2 - 1)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2-1\right) e^{-x^2}\right) + c_2\left(\left(x^2-1\right) e^{-x^2}\left(\int \frac{e^{-\frac{3x^2}{2}} e^{2x^2}}{\left(x^2-1\right)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.76.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 3xy' + \left(2x^2+4\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- > k + 2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 + 4a_0 + (6a_3 + 7a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(3k+4) + 2a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $[2a_2 + 4a_0 = 0, 6a_3 + 7a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = -2a_0, a_3 = -\frac{7a_1}{6}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + 3a_k k + 4a_k + 2a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} + 3a_{k+2}(k+2) + 4a_{k+2} + 2a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{3ka_{k+2}+2a_k+10a_{k+2}}{k^2+7k+12}, a_2 = -2a_0, a_3 = -\frac{7a_1}{6}\right]$

1.76.3 Maple trace

```
Methods for second order ODEs:
```

1.76.4 Maple dsolve solution

Solving time : `0.012` (sec)
Leaf size : 48

```
dsolve(diff(diff(y(x),x),x)+3*x*diff(y(x),x)+(2*x^2+4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = 2\,\mathrm{e}^{-\frac{x^2}{2}} c_1 x - \mathrm{e}^{-x^2}(x-1)\,(x+1)\left(c_1\sqrt{\pi}\,\sqrt{2}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right) - c_2\right)$$

1.76.5 Mathematica DSolve solution

Solving time : `0.499` (sec)
Leaf size : 63

```
DSolve[{D[y[x],{x,2}]+3*x*D[y[x],x]+(4+2*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{1}{4}e^{-x^2}\left(\sqrt{2\pi}c_2\left(x^2-1\right)\operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right)+4c_1\left(x^2-1\right)-2c_2e^{\frac{x^2}{2}}x\right)$$

1.77 problem 79

Internal problem ID [8215]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 79
Date solved : `Monday, October 21, 2024 at 05:03:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2+4x)\, y'' - 4y' - (6+4x)\, y = 0$$

1.77.1 Solved as second order ode using Kovacic algorithm

Time used: 0.321 (sec)

Writing the ode as

$$(2+4x)\, y'' - 4y' + (-4x-6)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2+4x \\ B &= -4 \\ C &= -4x-6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 8x + 6}{(1 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 + 8x + 6$$
$$t = (1 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 8x + 6}{(1 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 138: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+2x)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x+\frac{1}{2}\right)^2} + \frac{1}{x+\frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 + 8x + 6}{(1 + 2x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x + \frac{1}{2}\right)} + (-)(1) \\
&= -\frac{1}{2\left(x + \frac{1}{2}\right)} - 1 \\
&= -\frac{2(x+1)}{1+2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(x + \frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x + \frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x + \frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2 + 8x + 6}{(1+2x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2\left(x + \frac{1}{2}\right)} - 1\right) dx} \\
&= \frac{e^{-x}}{\sqrt{1+2x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{2+4x} \, dx} \\
&= z_1 e^{\frac{\ln(1+2x)}{2}} \\
&= z_1\left(\sqrt{1+2x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{2+4x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\ln(1+2x)}}{(y_1)^2} \, dx \\
&= y_1 \left(x \, \mathrm{e}^{2x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(x \, \mathrm{e}^{2x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.77.2 Maple step by step solution

Let's solve

$(2+4x)\left(\frac{d}{dx}y'\right) - 4y' - (6+4x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(2x+3)y}{1+2x} + \frac{2y'}{1+2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y'}{1+2x} - \frac{(2x+3)y}{1+2x} = 0$

□ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2}{1+2x}, P_3(x) = -\frac{2x+3}{1+2x}\right]$

- $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}}=-1$

- $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}}=0$

- $x=-\frac{1}{2}$is a regular singular point

  Check to see if $x_0=-\frac{1}{2}$ is a regular singular point

  $x_0=-\frac{1}{2}$

- Multiply by denominators

  $(1+2x)\left(\frac{d}{dx}y'\right)-2y'+(-2x-3)\,y=0$

- Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

  $2u\left(\frac{d}{du}\frac{d}{du}y(u)\right)-2\frac{d}{du}y(u)+(-2u-2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{du}y(u)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,u^{k+r}$

- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- Shift index using $k->k+1$

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$2a_0r(-2+r)\,u^{-1+r} + (2a_1(1+r)\,(-1+r) - 2a_0)\,u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+r)\,(k+r-1) - 2a_k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $2r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term must be 0
  $2a_1(1+r)\,(-1+r) - 2a_0 = 0$
- Each term in the series must be 0, giving the recursion relation
  $2a_{k+1}(k+1+r)\,(k+r-1) - 2a_k - 2a_{k-1} = 0$
- Shift index using $k- >k+1$
  $2a_{k+2}(k+2+r)\,(k+r) - 2a_{k+1} - 2a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2+r)(k+r)}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$
  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$
- Recursion relation for $r = 2$
  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}$
- Solution for $r = 2$
  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$$
- Revert the change of variables $u = x + \frac{1}{2}$
  $$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$$

1.77.3 Maple trace

```
Methods for second order ODEs:
```

1.77.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve((2+4*x)*diff(diff(y(x),x),x)-4*diff(y(x),x)-(6+4*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1\,\mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{x}$$

1.77.5 Mathematica DSolve solution

Solving time : 0.081 (sec)
Leaf size : 29

```
DSolve[{(2+4*x)*D[y[x],{x,2}]-4*D[y[x],x]-(6+4*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x-\frac{1}{2}}\left(c_2 e^{2x+1} x + c_1\right)$$

1.78 problem 80

Internal problem ID [8216]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 80
Date solved : `Monday, October 21, 2024 at 05:03:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 3xy' + (2x^2 + 5)\,y = 0$$

1.78.1 Solved as second order ode using Kovacic algorithm

Time used: 0.321 (sec)

Writing the ode as

$$y'' - 3xy' + (2x^2 + 5)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3x \\ C &= 2x^2 + 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 26}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 26$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{13}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 140: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{13}{2x} - \frac{169}{4x^3} - \frac{2197}{4x^5} - \frac{142805}{16x^7} - \frac{2599051}{16x^9} - \frac{101362989}{32x^{11}} - \frac{2070701061}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 26}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{13}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{13}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{13}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{13}{2}\right) - (0) \\
&= -\frac{13}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{13}{2}}{\frac{1}{2}} - 1\right) &&= -7 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{13}{2}}{\frac{1}{2}} - 1\right) &&= 6
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{13}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-7	6

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 6$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 6 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(30x^4 + 20x^3a_5 + 12x^2a_4 + 6xa_3 + 2a_2) + 2\left(-\frac{x}{2}\right)(6x^5 + 5x^4a_5 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1) + \left(\left(-\frac{1}{2}\right.\right.$$

$$a_5x^5 + 2(15 + a_4)\,x^4 + (3a_3 + 20a_5)\,x^3 + 4(a_2 + 3a_4)\,x^2 + (5$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -15, a_1 = 0, a_2 = 45, a_3 = 0, a_4 = -15, a_5 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 - 15x^4 + 45x^2 - 15$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^6 - 15x^4 + 45x^2 - 15\right) \mathrm{e}^{\int -\frac{x}{2} dx} \\ &= \left(x^6 - 15x^4 + 45x^2 - 15\right) \mathrm{e}^{-\frac{x^2}{4}} \\ &= \left(x^6 - 15x^4 + 45x^2 - 15\right) \mathrm{e}^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-3x}{1} \, dx} \\ &= z_1 e^{\frac{3x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{3x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x^2}{2}} \left(x^6 - 15x^4 + 45x^2 - 15\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{3x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}} \mathrm{e}^{-x^2}}{\left(x^6 - 15x^4 + 45x^2 - 15\right)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right)\right) \\
&\quad + c_2\left(\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right)\left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}}\mathrm{e}^{-x^2}}{\left(x^6-15x^4+45x^2-15\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.78.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - 3xy' + (2x^2+5)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2 + 5a_0 + (6a_3 + 2a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - a_k(3k-5) + 2a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $[2a_2 + 5a_0 = 0, 6a_3 + 2a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = -\frac{5a_0}{2}, a_3 = -\frac{a_1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - 3a_k k + 5a_k + 2a_{k-2} = 0$
- Shift index using $k- >k+2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 3a_{k+2}(k+2) + 5a_{k+2} + 2a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{3k a_{k+2} - 2a_k + a_{k+2}}{k^2+7k+12}, a_2 = -\frac{5a_0}{2}, a_3 = -\frac{a_1}{3}\right]$

1.78.3 Maple trace

```
Methods for second order ODEs:
```

1.78.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 62

```
dsolve(diff(diff(y(x),x),x)-3*x*diff(y(x),x)+(2*x^2+5)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \left(x^6 - 15x^4 + 45x^2 - 15\right)\left(c_1\sqrt{\pi}\,\sqrt{2}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right) + c_2\right) \mathrm{e}^{\frac{x^2}{2}} - 2\,\mathrm{e}^{x^2} c_1 x\left(x^2 - 11\right)\left(x^2 - 3\right)$$

1.78.5 Mathematica DSolve solution

Solving time : `1.324` (sec)
Leaf size : 95

```
DSolve[{D[y[x],{x,2}]-3*x*D[y[x],x]+(5+2*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{\frac{x^2}{2}}\left(\sqrt{2\pi}c_2(x^6-15x^4+45x^2-15)\operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right)-2c_2e^{\frac{x^2}{2}}x(x^4-14x^2+33)+1440c_1(x^6-15x^4+45x^2-}{1440}$$

1.79 problem 81

Internal problem ID [8217]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 81
Date solved : `Monday, October 21, 2024 at 05:03:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2y'' + 5xy' + \left(2x^2 + 4\right) y = 0$$

1.79.1 Solved as second order ode using Kovacic algorithm

Time used: 0.238 (sec)

Writing the ode as

$$2y'' + 5xy' + \left(2x^2 + 4\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 5x \\ C &= 2x^2 + 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 - 12}{16} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9x^2 - 12$$
$$t = 16$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2}{16} - \frac{3}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 142: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3x}{4} - \frac{1}{2x} - \frac{1}{6x^3} - \frac{1}{9x^5} - \frac{5}{54x^7} - \frac{7}{81x^9} - \frac{7}{81x^{11}} - \frac{22}{243x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{3x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^2 - 12}{16} \\ &= Q + \frac{R}{16} \\ &= \left(\frac{9x^2}{16} - \frac{3}{4}\right) + (0) \\ &= \frac{9x^2}{16} - \frac{3}{4} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{4}\right) - (0) \\ &= -\frac{3}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{3x}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{4}}{\frac{3}{4}} - 1\right) &&= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{4}}{\frac{3}{4}} - 1\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^2}{16} - \frac{3}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{3x}{4}$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{3x}{4}\right) \\ &= -\frac{3x}{4} \\ &= -\frac{3x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{3x}{4}\right)(0) + \left(\left(-\frac{3}{4}\right) + \left(-\frac{3x}{4}\right)^2 - \left(\frac{9x^2}{16} - \frac{3}{4}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{3x}{4} dx} \\
&= \mathrm{e}^{-\frac{3x^2}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{5x}{2} dx} \\
&= z_1 e^{-\frac{5x^2}{8}} \\
&= z_1 \left(\mathrm{e}^{-\frac{5x^2}{8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{2} dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{\mathrm{e}^{-\frac{5x^2}{4}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{i\sqrt{\pi}\,\sqrt{3}\,\mathrm{erf}\left(\frac{i\sqrt{3}\,x}{2}\right)}{3} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2} \left(-\frac{i\sqrt{\pi}\,\sqrt{3}\,\mathrm{erf}\left(\frac{i\sqrt{3}\,x}{2}\right)}{3} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.79.2 Maple step by step solution

Let's solve

$2\frac{d}{dx}y' + 5xy' + (2x^2 + 4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = (-x^2 - 2)\, y - \frac{5xy'}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{5xy'}{2} + (x^2 + 2)\, y = 0$

- Multiply by denominators

  $2\frac{d}{dx}y' + 5xy' + (2x^2 + 4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k + 2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$4a_2 + 4a_0 + (12a_3 + 9a_1)\, x + \left(\sum_{k=2}^{\infty} (2a_{k+2}(k+2)\,(k+1) + a_k(5k+4) + 2a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $[4a_2 + 4a_0 = 0, 12a_3 + 9a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = -a_0, a_3 = -\frac{3a_1}{4}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(2k^2 + 6k + 4)\, a_{k+2} + 5a_k k + 4a_k + 2a_{k-2} = 0$
- Shift index using $k- >k+2$

  $\left(2(k+2)^2 + 6k + 16\right) a_{k+4} + 5a_{k+2}(k+2) + 4a_{k+2} + 2a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{5ka_{k+2}+2a_k+14a_{k+2}}{2(k^2+7k+12)}, a_2 = -a_0, a_3 = -\frac{3a_1}{4}\right]$

1.79.3 Maple trace

```
Methods for second order ODEs:
```

1.79.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 22

```
dsolve(2*diff(diff(y(x),x),x)+5*x*diff(y(x),x)+(2*x^2+4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-x^2}\left(c_1 + \mathrm{erf}\left(\frac{i\sqrt{3}\, x}{2}\right) c_2\right)$$

1.79.5 Mathematica DSolve solution

Solving time : `0.146` (sec)
Leaf size : 42

```
DSolve[{2*D[y[x],{x,2}]+5*x*D[y[x],x]+(4+2*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3}e^{-x^2}\left(\sqrt{3\pi}c_2\operatorname{erfi}\left(\frac{\sqrt{3}x}{2}\right) + 3c_1\right)$$

1.80 problem 82

Internal problem ID [8218]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 82
Date solved : `Monday, October 21, 2024 at 05:03:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + (4x^2 + 2)\, y = 0$$

1.80.1 Solved as second order ode using Kovacic algorithm

Time used: 0.097 (sec)

Writing the ode as

$$y'' + 4xy' + (4x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 144: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4x}{1} \, dx} \\
&= z_1 e^{-x^2} \\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-2x^2}}{\left(y_1\right)^2} \, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.80.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 4xy' + (4x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k + 2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2 = -a_0, a_3 = -a_1\right]$

1.80.3 Maple trace

```
Methods for second order ODEs:
```

1.80.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(4*x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-x^2}(c_2 x + c_1)$$

1.80.5 Mathematica DSolve solution

Solving time : `0.036` (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(2+4*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x^2}(c_2 x + c_1)$$

1.81 problem 83

Internal problem ID [8219]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 83
Date solved : `Monday, October 21, 2024 at 05:03:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0$$

1.81.1 Solved as second order ode using Kovacic algorithm

Time used: 0.102 (sec)

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 146: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\
&= z_1 e^{-x^2} \\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.81.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 4xy' + (4x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + 2a_0 + (6a_3 + 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$
- Shift index using $k- > k+2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2 = -a_0, a_3 = -a_1\right]$

1.81.3 Maple trace

```
Methods for second order ODEs:
```

1.81.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(4*x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-x^2}(c_2 x + c_1)$$

1.81.5 Mathematica DSolve solution

Solving time : `0.032` (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(2+4*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x^2}(c_2 x + c_1)$$

1.82 problem 84

Internal problem ID [8220]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 84
Date solved : `Monday, October 21, 2024 at 05:03:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+x+1\right)y''+x\left(11x^2+11x+9\right)y'+\left(7x^2+10x+6\right)y=0$$

1.82.1 Solved as second order ode using Kovacic algorithm

Time used: 1.190 (sec)

Writing the ode as

$$\left(2x^4+2x^3+2x^2\right)y''+\left(11x^3+11x^2+9x\right)y'+\left(7x^2+10x+6\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+2x^3+2x^2 \\ B &= 11x^3+11x^2+9x \\ C &= 7x^2+10x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 21x^4 + 18x^3 + 27x^2 - 2x - 3 \\
t &= 16\left(x^3 + x^2 + x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 148: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 4 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{-\frac{5}{24} + \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{24} - \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{1}{8} - \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{1}{8} + \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} - \frac{3}{16x^2} + \frac{1}{4x}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4}
\end{aligned}
$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} + \frac{i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{6 + 6i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{6 + 6i\sqrt{3}}}{12}
\end{aligned}
$$

For the pole at $x=-\frac{1}{2}-\frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{5}{24}-\frac{i\sqrt{3}}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}-\frac{\sqrt{6-6i\sqrt{3}}}{12}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r=\frac{s}{t}=\frac{21x^4+18x^3+27x^2-2x-3}{16\left(x^3+x^2+x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{21x^4+18x^3+27x^2-2x-3}{16\left(x^3+x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}$	$\frac{1}{2}-\frac{\sqrt{6+6i\sqrt{3}}}{12}$
$-\frac{1}{2}-\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}$	$\frac{1}{2}-\frac{\sqrt{6-6i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{7}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+} + \alpha_{c_3}^{+}\right) \\ &= \frac{7}{4} - \left(\frac{7}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} + \frac{\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} + (0) \\ &= \frac{1}{4x} + \frac{\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\ &= \frac{7x^2+3x+1}{4x\left(x^2+x+1\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} + \frac{\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}\right) + \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right) dx} \\
&= 2\left(x^2+x+1\right)^{3/4} \sqrt{2}\, x^{1/4} \mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2} \, dx} \\
&= z_1 e^{-\frac{9\ln(x)}{4} - \frac{\ln\left(x^2+x+1\right)}{4} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{x^{9/4} \left(x^2+x+1\right)^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2\sqrt{x^2+x+1}\, \mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \sqrt{2}}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{9\ln(x)}{2} - \frac{\ln\left(x^2+x+1\right)}{2} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{9\ln(x)}{2} - \frac{\ln\left(x^2+x+1\right)}{2} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} x^4 \mathrm{e}^{\frac{2\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2+8x+8} dx\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1\left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\right)$$
$$+c_2\left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\left(\int \frac{\mathrm{e}^{-\frac{9\ln(x)}{2}-\frac{\ln\left(x^2+x+1\right)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}x^4\mathrm{e}^{\frac{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2+8x+8}\right.\right.$$

Will add steps showing solving for IC soon.

1.82.2 Maple step by step solution

Let's solve
$2x^2(x^2+x+1)\left(\frac{d}{dx}y'\right) + x(11x^2+11x+9)\,y' + (7x^2+10x+6)\,y = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d}{dx}y'$
- Isolate 2nd derivative
$\frac{d}{dx}y' = -\frac{(7x^2+10x+6)y}{2x^2(x^2+x+1)} - \frac{(11x^2+11x+9)y'}{2x(x^2+x+1)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
$\frac{d}{dx}y' + \frac{(11x^2+11x+9)y'}{2x(x^2+x+1)} + \frac{(7x^2+10x+6)y}{2x^2(x^2+x+1)} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
  $\left[P_2(x) = \frac{11x^2+11x+9}{2x(x^2+x+1)}, P_3(x) = \frac{7x^2+10x+6}{2x^2(x^2+x+1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{9}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 3$
  - $x = 0$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0 = 0$

- Multiply by denominators

  $2x^2(x^2+x+1)\left(\frac{d}{dx}y'\right)+x(11x^2+11x+9)\,y'+(7x^2+10x+6)\,y=0$
- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..2$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m=1..3$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m=2..4$

    $x^m \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(3+2r)\,x^r+\left(a_1(3+r)\,(5+2r)+a_0(5+2r)\,(2+r)\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r+2)\,(2k+\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(3+2r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{-2,-\frac{3}{2}\right\}$
- Each term must be 0

  $a_1(3+r)\,(5+2r)+a_0(5+2r)\,(2+r)=0$
- Solve for the dependent coefficient(s)

  $a_1=-\frac{(2+r)a_0}{3+r}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{3}{2}\right)\left(\left(a_k+a_{k-2}+a_{k-1}\right)k+\left(a_k+a_{k-2}+a_{k-1}\right)r+2a_k-a_{k-2}+a_{k-1}\right)=0$
- Shift index using $k->k+2$

  $2\left(k+\frac{7}{2}+r\right)\left(\left(a_{k+2}+a_k+a_{k+1}\right)(k+2)+\left(a_{k+2}+a_k+a_{k+1}\right)r+2a_{k+2}-a_k+a_{k+1}\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{ka_k+ka_{k+1}+ra_k+ra_{k+1}+a_k+3a_{k+1}}{k+4+r}$
- Recursion relation for $r=-2$

  $a_{k+2}=-\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2}$
- Solution for $r=-2$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k-2}, a_{k+2}=-\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2}, a_1=0\right]$
- Recursion relation for $r=-\frac{3}{2}$

  $a_{k+2}=-\frac{ka_k+ka_{k+1}-\frac{1}{2}a_k+\frac{3}{2}a_{k+1}}{k+\frac{5}{2}}$
- Solution for $r=-\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k-\frac{3}{2}}, a_{k+2}=-\frac{ka_k+ka_{k+1}-\frac{1}{2}a_k+\frac{3}{2}a_{k+1}}{k+\frac{5}{2}}, a_1=-\frac{a_0}{3}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-2}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k-\frac{3}{2}}\right), a_{k+2}=-\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2}, a_1=0, b_{k+2}=-\frac{kb_k+kb_{k+1}-\frac{1}{2}b_k+}{k+\frac{5}{2}}\right.$

1.82.3 Maple trace

```
Methods for second order ODEs:
```

1.82.4 Maple dsolve solution

Solving time : `0.123` (sec)
Leaf size : 231

```
dsolve(2*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+x*(11*x^2+11*x+9)*diff(y(x),x)+(7*x^2+10*x
       y(x),singsol=all)
```

$$y=\frac{\left(2x+1+i\sqrt{3}\right)^{\frac{5\sqrt{3}+3i}{6\sqrt{3}+6i}}\left(i\sqrt{3}-2x-1\right)^{\frac{64i\sqrt{3}+2368}{\left(\sqrt{3}+i\right)^3\left(i-\sqrt{3}\right)^4\left(13\sqrt{3}+9i\right)}}\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}\left(\operatorname{HeunG}\left(\frac{\sqrt{3}+i}{i-\sqrt{3}},0,0,\frac{5}{2},\frac{1}{2},\right.\right.}{x^{5/2}\left(x^2+x+1\right)}$$

1.82.5 Mathematica DSolve solution

Solving time : `1.879` (sec)
Leaf size : 93

```
DSolve[{2*x^2*(1+x+x^2)*D[y[x],{x,2}]+x*(9+11*x+11*x^2)*D[y[x],x]+(6+10*x+7*x^2)*y[x]==0,{}}
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x^2+x+1}e^{-\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}}\left(c_2\int_1^x \frac{e^{\frac{\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}{\sqrt{3}}}}{\sqrt{K[1]}(K[1]^2+K[1]+1)^{3/2}}dK[1]+c_1\right)}{x^2}$$

1.83 problem 85

Internal problem ID [8221]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 85
Date solved : `Monday, October 21, 2024 at 05:03:40 PM`
CAS classification :
`[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]`

Solve

$$3x^2y'' + 2x\left(-2x^2 + x + 1\right)y' + \left(-8x^2 + 2x\right)y = 0$$

1.83.1 Solved as second order ode using Kovacic algorithm

Time used: 0.424 (sec)

Writing the ode as

$$3x^2y'' + \left(-4x^3 + 2x^2 + 2x\right)y' + \left(-8x^2 + 2x\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= -4x^3 + 2x^2 + 2x \\ C &= -8x^2 + 2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^4 - 4x^3 + 15x^2 - 4x - 2$$
$$t = 9x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 150: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 9x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{4x^2}{9} - \frac{4x}{9} + \frac{5}{3} - \frac{4}{9x} - \frac{2}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{3} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{2x}{3} - \frac{1}{3} + \frac{7}{6x} + \frac{1}{4x^2} - \frac{17}{16x^3} - \frac{31}{32x^4} + \frac{85}{64x^5} + \frac{353}{128x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{2}{3}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= -\frac{1}{3} + \frac{2x}{3} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{9} - \frac{4}{9}x + \frac{4}{9}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{9}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2} \\ &= Q + \frac{R}{9x^2} \\ &= \left(\frac{4}{9}x^2 - \frac{4}{9}x + \frac{5}{3}\right) + \left(\frac{-4x - 2}{9x^2}\right) \\ &= \frac{4x^2}{9} - \frac{4x}{9} + \frac{5}{3} + \frac{-4x - 2}{9x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $\frac{5}{3}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{3}\right) - \left(\frac{1}{9}\right) \\ &= \frac{14}{9} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= -\frac{1}{3} + \frac{2x}{3} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{14}{9}}{\frac{2}{3}} - 1\right) &&= \frac{2}{3} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{14}{9}}{\frac{2}{3}} - 1\right) &&= -\frac{5}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$-\frac{1}{3} + \frac{2x}{3}$	$\frac{2}{3}$	$-\frac{5}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{2}{3}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{2}{3} - \left(\frac{2}{3}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{2}{3x} + \left(-\frac{1}{3} + \frac{2x}{3}\right) \\
&= \frac{2}{3x} - \frac{1}{3} + \frac{2x}{3} \\
&= \frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right)(0) + \left(\left(-\frac{2}{3x^2} + \frac{2}{3}\right) + \left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right)^2 - \left(\frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}\right)\right) = 0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right) dx} \\
&= x^{2/3} \mathrm{e}^{\frac{x(x-1)}{3}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x^3 + 2x^2 + 2x}{3x^2} \, dx} \\
&= z_1 e^{\frac{x^2}{3} - \frac{x}{3} - \frac{\ln(x)}{3}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x(x-1)}{3}}}{x^{1/3}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^{1/3} \mathrm{e}^{\frac{2x(x-1)}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3+2x^2+2x}{3x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{2x^2}{3}-\frac{2x}{3}-\frac{2\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{2x^2}{3}-\frac{2x}{3}-\frac{2\ln(x)}{3}} \mathrm{e}^{-\frac{4x(x-1)}{3}}}{x^{2/3}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{1/3} \mathrm{e}^{\frac{2x(x-1)}{3}}\right) + c_2 \left(x^{1/3} \mathrm{e}^{\frac{2x(x-1)}{3}} \left(\int \frac{\mathrm{e}^{\frac{2x^2}{3}-\frac{2x}{3}-\frac{2\ln(x)}{3}} \mathrm{e}^{-\frac{4x(x-1)}{3}}}{x^{2/3}} dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.83.2 Maple step by step solution

Let's solve

$3x^2 \left(\frac{d}{dx} y'\right) + 2x(-2x^2 + x + 1)\, y' + (-8x^2 + 2x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$
- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{2(4x-1)y}{3x} + \frac{2(2x^2-x-1)y'}{3x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' - \frac{2(2x^2-x-1)y'}{3x} - \frac{2(4x-1)y}{3x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{2(2x^2-x-1)}{3x}, P_3(x) = -\frac{2(4x-1)}{3x}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{2}{3}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3\left(\frac{d}{dx}y'\right)x + \left(-4x^2 + 2x + 2\right)y' + \left(-8x + 2\right)y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- ○ Shift index using $k- > k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ○ Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- ○ Shift index using $k- > k + 1$

$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-1+3r)\,x^{-1+r} + \left(a_1(1+r)\,(2+3r) + 2a_0(1+r)\right)x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(3k+2+3r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-1+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{1}{3}\right\}$
- Each term must be 0
  $a_1(1+r)\,(2+3r) + 2a_0(1+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+1+r)\,(3ka_{k+1} + 3ra_{k+1} + 2a_k - 4a_{k-1} + 2a_{k+1}) = 0$
- Shift index using $k- >k+1$
  $(k+r+2)\,(3(k+1)\,a_{k+2} + 3ra_{k+2} + 2a_{k+1} - 4a_k + 2a_{k+2}) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5+3r}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}, 2a_1 + 2a_0 = 0\right]$
- Recursion relation for $r = \frac{1}{3}$
  $a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+6}$
- Solution for $r = \frac{1}{3}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+6}, 4a_1 + \frac{8a_0}{3} = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}, 2a_1 + 2a_0 = 0, b_{k+2} = \frac{2(-b_{k+1}+2b_k)}{3k+6}, 4b_1 +\right.$

1.83.3 Maple trace

```
Methods for second order ODEs:
```

1.83.4 Maple dsolve solution

Solving time : 0.036 (sec)
Leaf size : 38

```
dsolve(3*x^2*diff(diff(y(x),x),x)+2*x*(-2*x^2+x+1)*diff(y(x),x)+(-8*x^2+2*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x^{1/3} e^{\frac{2x(x-1)}{3}} + c_2 \,\mathrm{HeunB}\left(-\frac{1}{3}, \frac{\sqrt{6}}{3}, -\frac{7}{3}, \frac{4\sqrt{6}}{9}, -\frac{\sqrt{6}\,x}{3}\right)$$

1.83.5 Mathematica DSolve solution

Solving time : 4.768 (sec)
Leaf size : 53

```
DSolve[{3*x^2*D[y[x],{x,2}]+2*x*(1+x-2*x^2)*D[y[x],x]+(2*x-8*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{\frac{2}{3}(x-1)x} \sqrt[3]{x} \left(c_2 \int_1^x \frac{e^{-\frac{2}{3}(K[1]-1)K[1]}}{K[1]^{4/3}} dK[1] + c_1\right)$$

1.84 problem 86

Internal problem ID [8222]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 86
Date solved : `Monday, October 21, 2024 at 05:03:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$12x^2(1+x)\,y''+x\left(3x^2+35x+11\right)y'-\left(-5x^2-10x+1\right)y=0$$

1.84.1 Solved as second order ode using Kovacic algorithm

Time used: 0.403 (sec)

Writing the ode as

$$\left(12x^3+12x^2\right)y''+\left(3x^3+35x^2+11x\right)y'+\left(5x^2+10x-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=12x^3+12x^2\\ B&=3x^3+35x^2+11x\\ C&=5x^2+10x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576 (x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9x^4 - 30x^3 - 197x^2 - 190x - 95$$
$$t = 576(x^2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576 (x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 152: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 576(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{64} - \frac{7}{64\,(1+x)^2} - \frac{95}{576x^2} - \frac{1}{12\,(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{95}{576}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{24} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{24} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{8} - \frac{1}{3x} - \frac{29}{24x^2} - \frac{193}{72x^3} - \frac{3017}{216x^4} - \frac{40009}{648x^5} - \frac{642029}{1944x^6} - \frac{10350493}{5832x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{8}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t}\\
&= \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2}\\
&= Q + \frac{R}{576x^4 + 1152x^3 + 576x^2}\\
&= \left(\frac{1}{64}\right) + \left(\frac{-48x^3 - 206x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2}\right)\\
&= \frac{1}{64} + \frac{-48x^3 - 206x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -48. Dividing this by leading coefficient in t which is 576 gives $-\frac{1}{12}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{12}\right) - (0)\\
&= -\frac{1}{12}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{8}\\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{12}}{\frac{1}{8}} - 0\right) &&= -\frac{1}{3}\\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{12}}{\frac{1}{8}} - 0\right) &&= \frac{1}{3}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{8}$	$\frac{1}{8}$
0	2	0	$\frac{19}{24}$	$\frac{5}{24}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{8}$	$-\frac{1}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= \frac{1}{3} - \left(\frac{1}{3}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{8+8x} + \frac{5}{24x} + (-)\left(\frac{1}{8}\right) \\ &= \frac{1}{8+8x} + \frac{5}{24x} - \frac{1}{8} \\ &= \frac{1}{8+8x} + \frac{5}{24x} - \frac{1}{8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8+8x} + \frac{5}{24x} - \frac{1}{8}\right)(0) + \left(\left(-\frac{1}{8(1+x)^2} - \frac{5}{24x^2}\right) + \left(\frac{1}{8+8x} + \frac{5}{24x} - \frac{1}{8}\right)^2 - \left(\frac{9x^4 - 30x^3 -}{576}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{8+8x} + \frac{5}{24x} - \frac{1}{8}\right) dx} \\
&= x^{5/24} (1+x)^{1/8} \, \mathrm{e}^{-\frac{x}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x^3+35x^2+11x}{12x^3+12x^2} \, dx} \\
&= z_1 e^{-\frac{x}{8} - \frac{11 \ln(x)}{24} - \frac{7 \ln(1+x)}{8}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{8}}}{x^{11/24} (1+x)^{7/8}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-\frac{x}{4}}}{x^{1/4} (1+x)^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+35x^2+11x}{12x^3+12x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x}{4} - \frac{11 \ln(x)}{12} - \frac{7 \ln(1+x)}{4}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \mathrm{e}^{-\frac{x}{4} - \frac{11 \ln(x)}{12} - \frac{7 \ln(1+x)}{4}} \sqrt{x} \, (1+x)^{3/2} \, \mathrm{e}^{\frac{x}{2}} dx \right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{e^{-\frac{x}{4}}}{x^{1/4}\left(1+x\right)^{3/4}}\right) + c_2 \left(\frac{e^{-\frac{x}{4}}}{x^{1/4}\left(1+x\right)^{3/4}}\left(\int e^{-\frac{x}{4}-\frac{11\ln(x)}{12}-\frac{7\ln(1+x)}{4}}\sqrt{x}\,(1+x)^{3/2}\,e^{\frac{x}{2}}dx\right)\right)$$

Will add steps showing solving for IC soon.

1.84.2 Maple step by step solution

Let's solve

$12x^2(1+x)\left(\frac{d}{dx}y'\right) + x(3x^2+35x+11)\,y' - (-5x^2-10x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(5x^2+10x-1)y}{12x^2(1+x)} - \frac{(3x^2+35x+11)y'}{12x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3x^2+35x+11)y'}{12x(1+x)} + \frac{(5x^2+10x-1)y}{12x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{3x^2+35x+11}{12x(1+x)}, P_3(x) = \frac{5x^2+10x-1}{12x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left.\left((1+x)\cdot P_2(x)\right)\right|_{x=-1} = \frac{7}{4}$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left.\left((1+x)^2\cdot P_3(x)\right)\right|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $12x^2(1+x)\left(\frac{d}{dx}y'\right) + x(3x^2+35x+11)\,y' + (5x^2+10x-1)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(12u^3-24u^2+12u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (3u^3+26u^2-50u+21)\left(\frac{d}{du}y(u)\right) + (5u^2-6)\,y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- >k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$3a_0r(3+4r)\, u^{-1+r} + (3a_1(1+r)\,(7+4r) - 2a_0(3+4r)\,(1+3r))\, u^r + (3a_2(2+r)\,(11+4r) - 2$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $3r(3+4r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{3}{4}\right\}$

- The coefficients of each power of u must be 0

  $[3a_1(1+r)\,(7+4r) - 2a_0(3+4r)\,(1+3r) = 0, 3a_2(2+r)\,(11+4r) - 2a_1(7+4r)\,(4+3r) + 2a$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = \frac{2a_0(12r^2+13r+3)}{3(4r^2+11r+7)}, a_2 = \frac{2a_0(54r^3+135r^2+101r+24)}{9(4r^3+23r^2+41r+22)}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $12(-2a_k + a_{k-1} + a_{k+1})\, k^2 + (24(-2a_k + a_{k-1} + a_{k+1})\, r - 26a_k + 3a_{k-2} - 10a_{k-1} + 33a_{k+1})\, k +$

- Shift index using $k- >k + 2$

$$12(-2a_{k+2}+a_{k+1}+a_{k+3})\,(k+2)^2+(24(-2a_{k+2}+a_{k+1}+a_{k+3})\,r-26a_{k+2}+3a_k-10a_{k+1}+3$$

- Recursion relation that defines series solution to ODE

$$a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+24kra_{k+1}-48kra_{k+2}+12r^2a_{k+1}-24r^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+3ra_k+38ra_{k+1}-122r}{3(4k^2+8kr+4r^2+27k+27r+45)}$$

- Recursion relation for $r=0$

$$a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a_k+26a_{k+1}-154a_{k+2}}{3(4k^2+27k+45)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a_k+26a_{k+1}-154a_{k+2}}{3(4k^2+27k+45)},a_1=\frac{2a_0}{7},a_2=\right.$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k,a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a_k+26a_{k+1}-154a_{k+2}}{3(4k^2+27k+45)},a_1=\frac{2a_0}{7},\right.$$

- Recursion relation for $r=-\frac{3}{4}$

$$a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+20ka_{k+1}-86ka_{k+2}+\frac{11}{4}a_k+\frac{17}{4}a_{k+1}-76a_{k+2}}{3(4k^2+21k+27)}$$

- Solution for $r=-\frac{3}{4}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-\frac{3}{4}},a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+20ka_{k+1}-86ka_{k+2}+\frac{11}{4}a_k+\frac{17}{4}a_{k+1}-76a_{k+2}}{3(4k^2+21k+27)},a_1=0,a_2\right.$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k-\frac{3}{4}},a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+20ka_{k+1}-86ka_{k+2}+\frac{11}{4}a_k+\frac{17}{4}a_{k+1}-76a_{k+2}}{3(4k^2+21k+27)},a_1=0,\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k(1+x)^k\right)+\left(\sum_{k=0}^{\infty}b_k(1+x)^{k-\frac{3}{4}}\right),a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a}{3(4k^2+27k+45)}\right.$$

1.84.3 Maple trace

```
Methods for second order ODEs:
```

1.84.4 Maple dsolve solution

Solving time : 0.053 (sec)
Leaf size : 43

```
dsolve(12*x^2*(1+x)*diff(diff(y(x),x),x)+x*(3*x^2+35*x+11)*diff(y(x),x)-(-5*x^2-10*x+1
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{x}{4}} \left(\mathrm{HeunC}\left(\frac{1}{4}, \frac{7}{12}, -\frac{3}{4}, -\frac{1}{12}, \frac{1}{2}, -x\right) x^{7/12} c_2 + \mathrm{HeunC}\left(\frac{1}{4}, -\frac{7}{12}, -\frac{3}{4}, -\frac{1}{12}, \frac{1}{2}, -x\right) c_1\right)}{x^{1/4} (1+x)^{3/4}}$$

1.84.5 Mathematica DSolve solution

Solving time : 21.072 (sec)
Leaf size : 61

```
DSolve[{12*x^2*(1+x)*D[y[x],{x,2}]+x*(11+35*x+3*x^2)*D[y[x],x]-(1-10*x-5*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x/4} \left(c_2 \int_1^x \frac{e^{\frac{K[1]}{4}}}{K[1]^{5/12} \sqrt[4]{K[1]+1}} dK[1] + c_1\right)}{\sqrt[4]{x}(x+1)^{3/4}}$$

1.85 problem 87

Internal problem ID [8223]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 87
Date solved : `Monday, October 21, 2024 at 05:03:43 PM`
CAS classification :
`[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]`

Solve

$$x^2(10x^2+x+5)\,y''+x(48x^2+3x+4)\,y'+(36x^2+x)\,y=0$$

1.85.1 Solved as second order ode using Kovacic algorithm

Time used: 1.176 (sec)

Writing the ode as

$$(10x^4+x^3+5x^2)\,y''+(48x^3+3x^2+4x)\,y'+(36x^2+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=10x^4+x^3+5x^2\\B&=48x^3+3x^2+4x\\C&=36x^2+x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-96x^4 - 16x^3 - 97x^2 - 12x - 24}{4\left(10x^3 + x^2 + 5x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -96x^4 - 16x^3 - 97x^2 - 12x - 24 \\
t &= 4\left(10x^3 + x^2 + 5x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-96x^4 - 16x^3 - 97x^2 - 12x - 24}{4\left(10x^3 + x^2 + 5x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 154: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(10x^3 + x^2 + 5x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{20} + \frac{i\sqrt{199}}{20}$ of order 2. There is a pole at $x = -\frac{1}{20} - \frac{i\sqrt{199}}{20}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned} r = &-\frac{6}{25x^2} - \frac{3}{125x} + \frac{-\frac{1}{19900} - \frac{i\sqrt{199}}{1990}}{\left(x + \frac{1}{20} - \frac{i\sqrt{199}}{20}\right)^2} \\ &+ \frac{-\frac{1}{19900} + \frac{i\sqrt{199}}{1990}}{\left(x + \frac{1}{20} + \frac{i\sqrt{199}}{20}\right)^2} + \frac{\frac{3}{250} - \frac{647i\sqrt{199}}{9900250}}{x + \frac{1}{20} - \frac{i\sqrt{199}}{20}} + \frac{\frac{3}{250} + \frac{647i\sqrt{199}}{9900250}}{x + \frac{1}{20} + \frac{i\sqrt{199}}{20}} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{6}{25}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{5} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{2}{5} \end{aligned}$$

For the pole at $x = -\frac{1}{20} + \frac{i\sqrt{199}}{20}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{20} - \frac{i\sqrt{199}}{20}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{19900} - \frac{i\sqrt{199}}{1990}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{989826 - 1990i\sqrt{199}}}{1990} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{989826 - 1990i\sqrt{199}}}{1990} \end{aligned}$$

For the pole at $x=-\frac{1}{20}-\frac{i\sqrt{199}}{20}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{20}+\frac{i\sqrt{199}}{20}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{1}{19900}+\frac{i\sqrt{199}}{1990}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}+\frac{\sqrt{989826+1990i\sqrt{199}}}{1990}\\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}-\frac{\sqrt{989826+1990i\sqrt{199}}}{1990}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty=0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{-96x^4-16x^3-97x^2-12x-24}{4\left(10x^3+x^2+5x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{6}{25}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{5}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{2}{5}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-96x^4-16x^3-97x^2-12x-24}{4\left(10x^3+x^2+5x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{5}$	$\frac{2}{5}$
$-\frac{1}{20}+\frac{i\sqrt{199}}{20}$	2	0	$\frac{1}{2}+\frac{\sqrt{989826-1990i\sqrt{199}}}{1990}$	$\frac{1}{2}-\frac{\sqrt{989826-1990i\sqrt{199}}}{1990}$
$-\frac{1}{20}-\frac{i\sqrt{199}}{20}$	2	0	$\frac{1}{2}+\frac{\sqrt{989826+1990i\sqrt{199}}}{1990}$	$\frac{1}{2}-\frac{\sqrt{989826+1990i\sqrt{199}}}{1990}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{5}$	$\frac{2}{5}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{5}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{3}{5} - \left(\frac{3}{5}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{5x} + \frac{\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}}{x + \frac{1}{20} - \frac{i\sqrt{199}}{20}} + \frac{\frac{1}{2} - \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}}{x + \frac{1}{20} + \frac{i\sqrt{199}}{20}} + (0) \\
&= \frac{3}{5x} + \frac{\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}}{x + \frac{1}{20} - \frac{i\sqrt{199}}{20}} + \frac{\frac{1}{2} - \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}}{x + \frac{1}{20} + \frac{i\sqrt{199}}{20}} \\
&= \frac{12x^2 + x + 6}{20x^3 + 2x^2 + 10x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{5x} + \frac{\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}}{x + \frac{1}{20} - \frac{i\sqrt{199}}{20}} + \frac{\frac{1}{2} - \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}}{x + \frac{1}{20} + \frac{i\sqrt{199}}{20}}\right)(0) + \left(\left(-\frac{3}{5x^2} - \frac{\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}}{\left(x + \frac{1}{20} - \frac{i\sqrt{199}}{20}\right)^2} - \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{5x} + \frac{\frac{1}{2} - \frac{\sqrt{989826 - 1990 i \sqrt{199}}}{1990}}{x + \frac{1}{20} - \frac{i\sqrt{199}}{20}} + \frac{\frac{1}{2} - \frac{\sqrt{989826 + 1990 i \sqrt{199}}}{1990}}{x + \frac{1}{20} + \frac{i\sqrt{199}}{20}}\right) dx} \\
&= x^{3/5} \mathrm{e}^{-\frac{\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{48x^3 + 3x^2 + 4x}{10x^4 + x^3 + 5x^2} \, dx} \\
&= z_1 e^{-\ln(10x^2 + x + 5) - \frac{\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995} - \frac{2\ln(x)}{5}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{(10x^2 + x + 5)\, x^{2/5}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{1/5} \mathrm{e}^{-\frac{2\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{10x^2 + x + 5}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{48x^3 + 3x^2 + 4x}{10x^4 + x^3 + 5x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-2\ln(10x^2 + x + 5) - \frac{2\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995} - \frac{4\ln(x)}{5}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-2\ln(10x^2 + x + 5) - \frac{2\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995} - \frac{4\ln(x)}{5}} (10x^2 + x + 5)^2 \, \mathrm{e}^{\frac{4\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{x^{2/5}} \, dx \right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x^{1/5} \mathrm{e}^{-\frac{2\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{10x^2 + x + 5} \right)$$
$$+ c_2 \left(\frac{x^{1/5} \mathrm{e}^{-\frac{2\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{10x^2 + x + 5} \left(\int \frac{\mathrm{e}^{-2\ln(10x^2+x+5) - \frac{2\sqrt{199}\, \arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995} - \frac{4\ln(x)}{5}} (10x^2 + x + 5)^2 \mathrm{e}^{\frac{4\sqrt{19}}{}}}{x^{2/5}} \right.\right.$$

Will add steps showing solving for IC soon.

1.85.2 Maple step by step solution

Let's solve

$x^2(10x^2 + x + 5) \left(\frac{d}{dx} y'\right) + x(48x^2 + 3x + 4) y' + (36x^2 + x) y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx} y'$

- Isolate 2nd derivative

$\frac{d}{dx} y' = -\frac{(36x+1)y}{x(10x^2+x+5)} - \frac{(48x^2+3x+4)y'}{x(10x^2+x+5)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx} y' + \frac{(48x^2+3x+4)y'}{x(10x^2+x+5)} + \frac{(36x+1)y}{x(10x^2+x+5)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x) = \frac{48x^2+3x+4}{x(10x^2+x+5)}, P_3(x) = \frac{36x+1}{x(10x^2+x+5)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left.(x \cdot P_2(x))\right|_{x=0} = \frac{4}{5}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $x(10x^2+x+5)\left(\frac{d}{dx}y'\right)+(48x^2+3x+4)\,y'+(36x+1)\,y=0$
- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$
  - Convert $x^m\cdot y'$ to series expansion for $m=0..2$

    $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=1..3$

    $x^m\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-1+5r)\,x^{-1+r}+\left(a_1(1+r)\,(4+5r)+a_0(1+r)^2\right)x^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)\,(5k+4+5r\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+5r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{1}{5}\right\}$
- Each term must be 0

  $a_1(1+r)\,(4+5r)+a_0(1+r)^2=0$
- Each term in the series must be 0, giving the recursion relation

  $((a_k+10a_{k-1}+5a_{k+1})\,k+(a_k+10a_{k-1}+5a_{k+1})\,r+a_k+8a_{k-1}+4a_{k+1})\,(k+r+1)=0$
- Shift index using $k->k+1$

$$((a_{k+1}+10a_k+5a_{k+2})(k+1)+(a_{k+1}+10a_k+5a_{k+2})r+a_{k+1}+8a_k+4a_{k+2})(k+r+2)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{10ka_k+ka_{k+1}+10ra_k+ra_{k+1}+18a_k+2a_{k+1}}{5k+5r+9}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{10ka_k+ka_{k+1}+18a_k+2a_{k+1}}{5k+9}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty}a_kx^k, a_{k+2}=-\frac{10ka_k+ka_{k+1}+18a_k+2a_{k+1}}{5k+9}, 4a_1+a_0=0\right]$$

- Recursion relation for $r=\frac{1}{5}$

$$a_{k+2}=-\frac{10ka_k+ka_{k+1}+20a_k+\frac{11}{5}a_{k+1}}{5k+10}$$

- Solution for $r=\frac{1}{5}$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{5}}, a_{k+2}=-\frac{10ka_k+ka_{k+1}+20a_k+\frac{11}{5}a_{k+1}}{5k+10}, 6a_1+\frac{36a_0}{25}=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{5}}\right), a_{k+2}=-\frac{10ka_k+ka_{k+1}+18a_k+2a_{k+1}}{5k+9}, 4a_1+a_0=0, b_{k+2}=-\frac{10kb_k+\cdots}{}\right.$$

1.85.3 Maple trace

```
Methods for second order ODEs:
```

1.85.4 Maple dsolve solution

Solving time : `0.108` (sec)
Leaf size : 162

```
dsolve(x^2*(10*x^2+x+5)*diff(diff(y(x),x),x)+x*(48*x^2+3*x+4)*diff(y(x),x)+(36*x^2+x)*
       y(x),singsol=all)
```

$$y=\frac{(20x+1+i\sqrt{199})^{-\frac{i\sqrt{199}}{1990}}(i\sqrt{199}-20x-1)^{\frac{i\sqrt{199}}{1990}}e^{-\frac{\sqrt{199}\arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}\left(\operatorname{HeunG}\left(\frac{\sqrt{199}+i}{i-\sqrt{199}},0,0,\frac{1}{5},\frac{6}{5},-\frac{i\sqrt{}}{99}\right.\right.}{10x^2+x+5}$$

1.85.5 Mathematica DSolve solution

Solving time : `3.597` (sec)
Leaf size : 88

```
DSolve[{x^2*(5+x+10*x^2)*D[y[x],{x,2}]+x*(4+3*x+48*x^2)*D[y[x],x]+(x+36*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[5]{x}e^{-\frac{2\arctan\left(\frac{20x+1}{\sqrt{199}}\right)}{5\sqrt{199}}}\left(c_2\int_1^x \frac{e^{\frac{2\arctan\left(\frac{20K[1]+1}{\sqrt{199}}\right)}{5\sqrt{199}}}}{K[1]^{6/5}}dK[1]+c_1\right)}{10x^2+x+5}$$

1.86 problem 88

Internal problem ID [8224]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 88
Date solved : `Monday, October 21, 2024 at 05:03:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$18x^2(1+x)\,y''+3x\left(x^2+11x+5\right)y'-\left(-5x^2-2x+1\right)y=0$$

1.86.1 Solved as second order ode using Kovacic algorithm

Time used: 0.386 (sec)

Writing the ode as

$$\left(18x^3+18x^2\right)y''+\left(3x^3+33x^2+15x\right)y'+\left(5x^2+2x-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=18x^3+18x^2\\B&=3x^3+33x^2+15x\\C&=5x^2+2x-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144\left(x^2 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^4 - 18x^3 - 45x^2 - 18x - 27 \\
t &= 144\left(x^2 + x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144\left(x^2 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 156: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 4 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{144} - \frac{3}{16x^2} - \frac{7}{18\,(1+x)} + \frac{1}{4x} - \frac{35}{144\,(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{12} - \frac{5}{6x} - \frac{53}{12x^2} - \frac{523}{12x^3} - \frac{6659}{12x^4} - \frac{94267}{12x^5} - \frac{1432421}{12x^6} - \frac{22802941}{12x^7} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{12}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{12}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{144}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2} \\
&= Q + \frac{R}{144x^4 + 288x^3 + 144x^2} \\
&= \left(\frac{1}{144}\right) + \left(\frac{-20x^3 - 46x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2}\right) \\
&= \frac{1}{144} + \frac{-20x^3 - 46x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -20. Dividing this by leading coefficient in t which is 144 gives $-\frac{5}{36}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{5}{36}\right) - (0) \\
&= -\frac{5}{36}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{12} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{36}}{\frac{1}{12}} - 0\right) &= -\frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{36}}{\frac{1}{12}} - 0\right) &= \frac{5}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{12}$	$\frac{5}{12}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{12}$	$-\frac{5}{6}$	$\frac{5}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\ &= \frac{5}{6} - \left(\frac{5}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{7}{12\,(1+x)} + \frac{1}{4x} + (-)\left(\frac{1}{12}\right) \\ &= \frac{7}{12\,(1+x)} + \frac{1}{4x} - \frac{1}{12} \\ &= \frac{7}{12+12x} + \frac{1}{4x} - \frac{1}{12} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{12\,(1+x)} + \frac{1}{4x} - \frac{1}{12}\right)(0) + \left(\left(-\frac{7}{12\,(1+x)^2} - \frac{1}{4x^2}\right) + \left(\frac{7}{12\,(1+x)} + \frac{1}{4x} - \frac{1}{12}\right)^2 - \left(\frac{x^4 - 1}{\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{7}{12(1+x)} + \frac{1}{4x} - \frac{1}{12}\right) dx} \\
&= x^{1/4} (1+x)^{7/12} \, \mathrm{e}^{-\frac{x}{12}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x^3+33x^2+15x}{18x^3+18x^2} \, dx} \\
&= z_1 e^{-\frac{x}{12} - \frac{5\ln(x)}{12} - \frac{5\ln(1+x)}{12}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{12}}}{x^{5/12} (1+x)^{5/12}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(1+x)^{1/6} \, \mathrm{e}^{-\frac{x}{6}}}{x^{1/6}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+33x^2+15x}{18x^3+18x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x}{6} - \frac{5\ln(x)}{6} - \frac{5\ln(1+x)}{6}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x}{6} - \frac{5\ln(x)}{6} - \frac{5\ln(1+x)}{6}} x^{1/3} \mathrm{e}^{\frac{x}{3}}}{(1+x)^{1/3}} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(1+x)^{1/6}\, \mathrm{e}^{-\frac{x}{6}}}{x^{1/6}}\right) + c_2 \left(\frac{(1+x)^{1/6}\, \mathrm{e}^{-\frac{x}{6}}}{x^{1/6}} \left(\int \frac{\mathrm{e}^{-\frac{x}{6}-\frac{5\ln(x)}{6}-\frac{5\ln(1+x)}{6}} x^{1/3} \mathrm{e}^{\frac{x}{3}}}{(1+x)^{1/3}} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.86.2 Maple step by step solution

Let's solve

$18x^2(1+x)\left(\frac{d}{dx}y'\right) + 3x(x^2+11x+5)\,y' - (-5x^2-2x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(5x^2+2x-1)y}{18x^2(1+x)} - \frac{(x^2+11x+5)y'}{6x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(x^2+11x+5)y'}{6x(1+x)} + \frac{(5x^2+2x-1)y}{18x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x^2+11x+5}{6x(1+x)}, P_3(x) = \frac{5x^2+2x-1}{18x^2(1+x)}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{5}{6}$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $18x^2(1+x)\left(\frac{d}{dx}y'\right) + 3x(x^2+11x+5)\,y' + (5x^2+2x-1)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(18u^3-36u^2+18u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (3u^3+24u^2-42u+15)\left(\frac{d}{du}y(u)\right) + (5u^2-8u+2)\,y(u) = 0$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m=0..2$

$$u^m \cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

$$u^m \cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(-1+6r)\,u^{-1+r}+\left(3a_1(1+r)\,(5+6r)-2a_0(1+3r)\,(-1+6r)\right)u^r+\left(3a_2(2+r)\,(11+6r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$3r(-1+6r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{1}{6}\right\}$$

- The coefficients of each power of u must be 0

$$[3a_1(1+r)\,(5+6r)-2a_0(1+3r)\,(-1+6r)=0, 3a_2(2+r)\,(11+6r)-2a_1(4+3r)\,(5+6r)+$$

- Solve for the dependent coefficient(s)

$$\left\{a_1=\frac{2a_0(18r^2+3r-1)}{3(6r^2+11r+5)}, a_2=\frac{2a_0(81r^3+126r^2+21r+4)}{9(6r^3+29r^2+45r+22)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$18(-2a_k+a_{k-1}+a_{k+1})\,k^2+3(12(-2a_k+a_{k-1}+a_{k+1})\,r-2a_k+a_{k-2}-10a_{k-1}+11a_{k+1})\,k+$$

- Shift index using $k->k+2$

$18(-2a_{k+2}+a_{k+1}+a_{k+3})(k+2)^2+3(12(-2a_{k+2}+a_{k+1}+a_{k+3})r-2a_{k+2}+a_k-10a_{k+1}+11$

- Recursion relation that defines series solution to ODE

$a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+36kra_{k+1}-72kra_{k+2}+18r^2a_{k+1}-36r^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+3ra_k+42ra_{k+1}-150r}{3(6k^2+12kr+6r^2+35k+35r+51)}$

- Recursion relation for $r=0$

$a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)}$

- Solution for $r=0$

$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k, a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)}, a_1=-\frac{2a_0}{15}, a\right.$

- Revert the change of variables $u=1+x$

$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k, a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)}, a_1=-\frac{2a_0}{15}\right.$

- Recursion relation for $r=\frac{1}{6}$

$a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+48ka_{k+1}-162ka_{k+2}+\frac{11}{2}a_k+\frac{47}{2}a_{k+1}-180a_{k+2}}{3(6k^2+37k+57)}$

- Solution for $r=\frac{1}{6}$

$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+\frac{1}{6}}, a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+48ka_{k+1}-162ka_{k+2}+\frac{11}{2}a_k+\frac{47}{2}a_{k+1}-180a_{k+2}}{3(6k^2+37k+57)}, a_1=0, a\right.$

- Revert the change of variables $u=1+x$

$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k+\frac{1}{6}}, a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+48ka_{k+1}-162ka_{k+2}+\frac{11}{2}a_k+\frac{47}{2}a_{k+1}-180a_{k+2}}{3(6k^2+37k+57)}, a_1=\right.$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty}a_k(1+x)^k\right)+\left(\sum_{k=0}^{\infty}b_k(1+x)^{k+\frac{1}{6}}\right), a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a}{3(6k^2+35k+51)}\right.$

1.86.3 Maple trace

```
Methods for second order ODEs:
```

1.86.4 Maple dsolve solution

Solving time : `0.049` (sec)
Leaf size : 38

```
dsolve(18*x^2*(1+x)*diff(diff(y(x),x),x)+3*x*(x^2+11*x+5)*diff(y(x),x)-(-5*x^2-2*x+1)*
       y(x),singsol=all)
```

$$
y=\frac{\mathrm{e}^{-\frac{x}{6}}\left(\operatorname{HeunC}\left(\frac{1}{6},\frac{1}{2},-\frac{1}{6},-\frac{5}{36},\frac{1}{4},-x\right)\sqrt{x}\,c_2+\operatorname{HeunC}\left(\frac{1}{6},-\frac{1}{2},-\frac{1}{6},-\frac{5}{36},\frac{1}{4},-x\right)c_1\right)}{x^{1/6}}
$$

1.86.5 Mathematica DSolve solution

Solving time : `6.388` (sec)
Leaf size : 73

```
DSolve[{18*x^2*(1+x)*D[y[x],{x,2}]+3*x*(5+11*x+x^2)*D[y[x],x]-(1-2*x-5*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x)\to\frac{e^{-x/6}\left(c_2\int_1^x\frac{e^{\frac{K[1]}{6}}\sqrt[3]{\frac{K[1]}{K[1]+1}}}{K[1]^{5/6}(K[1]+1)^{5/6}}dK[1]+c_1\right)}{\sqrt[6]{\frac{x}{x+1}}}
$$

1.87 problem 89

Internal problem ID [8225]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 89
Date solved : `Monday, October 21, 2024 at 05:03:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + x(3+2x)\,y' - (1-x)\,y = 0$$

1.87.1 Solved as second order ode using Kovacic algorithm

Time used: 0.302 (sec)

Writing the ode as

$$2x^2y'' + \left(2x^2+3x\right)y' + (x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 2x^2+3x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4x^2 + 4x + 5}{16x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4x^2 + 4x + 5 \\
t &= 16x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{4x^2 + 4x + 5}{16x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 158: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{5}{16x^2} + \frac{1}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4x} + \frac{1}{4x^2} - \frac{1}{8x^3} + \frac{1}{16x^5} - \frac{3}{64x^6} - \frac{1}{128x^7} + \frac{11}{256x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^2 + 4x + 5}{16x^2} \\
&= Q + \frac{R}{16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{4x + 5}{16x^2}\right) \\
&= \frac{1}{4} + \frac{4x + 5}{16x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{1}{4}\right) - (0) \\
&= \frac{1}{4}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &= \frac{1}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &= -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 4x + 5}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4x} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{4x} - \frac{1}{2} \\
&= -\frac{1}{4x} - \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{4x} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x} - \frac{1}{2}\right)^2 - \left(\frac{4x^2+4x+5}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{4x} - \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{2}}}{x^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2+3x}{2x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \frac{3\ln(x)}{4}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{x}{2}}}{x^{3/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+3x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\sqrt{x}\,e^{x} - \frac{\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{-x}}{x}\right) + c_2\left(\frac{e^{-x}}{x}\left(\sqrt{x}\,e^{x} - \frac{\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.87.2 Maple step by step solution

Let's solve

$2x^2\left(\frac{d}{dx}y'\right) + x(3+2x)\,y' - (1-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x-1)y}{2x^2} - \frac{(3+2x)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3+2x)y'}{2x} + \frac{(x-1)y}{2x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{3+2x}{2x}, P_3(x) = \frac{x-1}{2x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{3}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{2}$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $2x^2\left(\frac{d}{dx}y'\right) + x(3 + 2x)\, y' + (x - 1)\, y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$
    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$
    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$
    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$
    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$
    Rewrite ODE with series expansions
    $a_0(1 + r)\,(-1 + 2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k + r + 1)\,(2k + 2r - 1) + a_{k-1}(2k + 2r - 1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+r)(-1+2r)=0$
- Values of r that satisfy the indicial equation
  $r\in\left\{-1,\frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $2(a_k(k+r+1)+a_{k-1})\left(k+r-\frac{1}{2}\right)=0$
- Shift index using $k->k+1$
  $2(a_{k+1}(k+2+r)+a_k)\left(k+\frac{1}{2}+r\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k}{k+2+r}$
- Recursion relation for $r=-1$
  $a_{k+1}=-\frac{a_k}{k+1}$
- Solution for $r=-1$
  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-1}, a_{k+1}=-\frac{a_k}{k+1}\right]$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1}=-\frac{a_k}{k+\frac{5}{2}}$
- Solution for $r=\frac{1}{2}$
  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{a_k}{k+\frac{5}{2}}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{a_k}{k+1}, b_{k+1}=-\frac{b_k}{k+\frac{5}{2}}\right]$

1.87.3 Maple trace

```
Methods for second order ODEs:
```

1.87.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 52

```
dsolve(2*x^2*diff(diff(y(x),x),x)+x*(3+2*x)*diff(y(x),x)-(1-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{3\left(2c_1(-x)^{3/2} + \mathrm{e}^{-x}\left(xc_1\sqrt{\pi}\,\operatorname{erf}\left(\sqrt{-x}\right) - \frac{4c_2\sqrt{x}\,\sqrt{-x}}{3}\right)\right)}{4x^{3/2}\sqrt{-x}}$$

1.87.5 Mathematica DSolve solution

Solving time : `0.045` (sec)
Leaf size : 33

```
DSolve[{2*x^2*D[y[x],{x,2}]+x*(3+2*x)*D[y[x],x]-(1-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x}\left(c_2 x^{3/2} L_{-\frac{3}{2}}^{\frac{3}{2}}(x) + c_1\right)}{x}$$

1.88 problem 90

Internal problem ID [8226]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 90
Date solved : `Monday, October 21, 2024 at 05:03:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + x(5+x)\,y' - (2-3x)\,y = 0$$

1.88.1 Solved as second order ode using Kovacic algorithm

Time used: 0.335 (sec)

Writing the ode as

$$2x^2y'' + \left(x^2+5x\right)y' + (3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= x^2+5x \\ C &= 3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 14x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 14x + 21$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 14x + 21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 160: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} - \frac{7}{8x} + \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{7}{4x} - \frac{7}{2x^2} - \frac{49}{2x^3} - \frac{196}{x^4} - \frac{1715}{x^5} - \frac{31899}{2x^6} - \frac{309729}{2x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 14x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-14x + 21}{16x^2}\right) \\ &= \frac{1}{16} + \frac{-14x + 21}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -14. Dividing this by leading coefficient in t which is 16 gives $-\frac{7}{8}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{7}{8}\right) - (0)\\ &= -\frac{7}{8}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{4}\\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{7}{8}}{\frac{1}{4}} - 0\right) &= -\frac{7}{4}\\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{7}{8}}{\frac{1}{4}} - 0\right) &= \frac{7}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 14x + 21}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$-\frac{7}{4}$	$\frac{7}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right)\\ &= \frac{7}{4} - \left(\frac{7}{4}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{7}{4x} + (-)\left(\frac{1}{4}\right) \\
&= \frac{7}{4x} - \frac{1}{4} \\
&= -\frac{x-7}{4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{4x} - \frac{1}{4}\right)(0) + \left(\left(-\frac{7}{4x^2}\right) + \left(\frac{7}{4x} - \frac{1}{4}\right)^2 - \left(\frac{x^2 - 14x + 21}{16x^2}\right) \right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{7}{4x} - \frac{1}{4}\right) dx} \\
&= x^{7/4} \mathrm{e}^{-\frac{x}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2+5x}{2x^2} \, dx} \\
&= z_1 e^{-\frac{x}{4} - \frac{5\ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{4}}}{x^{5/4}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,e^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+5x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{2}-\frac{5\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{2\,e^{\frac{x}{2}}}{5x^{5/2}} - \frac{2\,e^{\frac{x}{2}}}{15x^{3/2}} - \frac{2\,e^{\frac{x}{2}}}{15\sqrt{x}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right)}{15}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,e^{-\frac{x}{2}}\right) + c_2\left(\sqrt{x}\,e^{-\frac{x}{2}}\left(-\frac{2\,e^{\frac{x}{2}}}{5x^{5/2}} - \frac{2\,e^{\frac{x}{2}}}{15x^{3/2}} - \frac{2\,e^{\frac{x}{2}}}{15\sqrt{x}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right)}{15}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.88.2 Maple step by step solution

Let's solve

$2x^2\left(\frac{d}{dx}y'\right) + x(5+x)\,y' - (2-3x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(3x-2)y}{2x^2} - \frac{(5+x)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(5+x)y'}{2x} + \frac{(3x-2)y}{2x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{5+x}{2x}, P_3(x) = \frac{3x-2}{2x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{5}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d}{dx}y'\right) + x(5+x)\,y' + (3x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(2+r)(-1+2r)x^r + \left(\sum_{k=1}^{\infty} (a_k(k+r+2)(2k+2r-1) + a_{k-1}(k+r+2))x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(2+r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-2, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $2(k+r+2)\left(\left(k+r-\frac{1}{2}\right)a_k + \frac{a_{k-1}}{2}\right) = 0$
- Shift index using $k->k+1$
  $2(k+r+3)\left(\left(k+\frac{1}{2}+r\right)a_{k+1} + \frac{a_k}{2}\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k}{2k+1+2r}$
- Recursion relation for $r = -2$
  $a_{k+1} = -\frac{a_k}{2k-3}$
- Solution for $r = -2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{a_k}{2k-3}\right]$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+1} = -\frac{a_k}{2k+2}$
- Solution for $r = \frac{1}{2}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{2k+2}\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{2k-3}, b_{k+1} = -\frac{b_k}{2k+2}\right]$

1.88.3 Maple trace

```
Methods for second order ODEs:
```

1.88.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 52

```
dsolve(2*x^2*diff(diff(y(x),x),x)+x*(5+x)*diff(y(x),x)-(2-3*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{i\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right)\sqrt{2}\,\mathrm{e}^{-\frac{x}{2}} x^{5/2} c_2 + c_1\, x^{5/2} \mathrm{e}^{-\frac{x}{2}} + 2c_2(x^2+x+3)}{x^2}$$

1.88.5 Mathematica DSolve solution

Solving time : 0.505 (sec)
Leaf size : 70

```
DSolve[{2*x^2*D[y[x],{x,2}]+x*(5+x)*D[y[x],x]-(2-3*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{15}\left(-\frac{2c_2(x^2+x+3)}{x^2} + 15c_1 e^{-x/2}\sqrt{x} + \sqrt{2}c_2 e^{-x/2}\sqrt{-x}\Gamma\left(\frac{1}{2}, -\frac{x}{2}\right)\right)$$

1.89 problem 91

Internal problem ID [8227]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 91
Date solved : `Monday, October 21, 2024 at 05:03:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2y'' + x(1+x)\,y' - y = 0$$

1.89.1 Solved as second order ode using Kovacic algorithm

Time used: 0.510 (sec)

Writing the ode as

$$3x^2y'' + \left(x^2+x\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= x^2+x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 + 2x + 7}{36x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 2x + 7 \\
t &= 36x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 + 2x + 7}{36x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 162: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{36} + \frac{7}{36x^2} + \frac{1}{18x}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{1}{6x} + \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{1}{4x^4} + \frac{7}{4x^5} - \frac{7}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 2x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{2x + 7}{36x^2}\right) \\ &= \frac{1}{36} + \frac{2x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 2. Dividing this by leading coefficient in t which is 36 gives $\frac{1}{18}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{18}\right) - (0) \\ &= \frac{1}{18} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{18}}{\frac{1}{6}} - 0\right) &= \frac{1}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{18}}{\frac{1}{6}} - 0\right) &= -\frac{1}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 2x + 7}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{6} - \left(-\frac{1}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{6x} + (-)\left(\frac{1}{6}\right) \\ &= -\frac{1}{6x} - \frac{1}{6} \\ &= -\frac{1+x}{6x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{6x} - \frac{1}{6}\right)(0) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} - \frac{1}{6}\right)^2 - \left(\frac{x^2+2x+7}{36x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{6x} - \frac{1}{6}\right) dx} \\ &= \frac{e^{-\frac{x}{6}}}{x^{1/6}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{3x^2} \, dx} \\ &= z_1 e^{-\frac{x}{6} - \frac{\ln(x)}{6}} \\ &= z_1 \left(\frac{e^{-\frac{x}{6}}}{x^{1/6}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{x}{3}}}{x^{1/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{3x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int e^{-\frac{x}{3}-\frac{\ln(x)}{3}} x^{2/3} e^{\frac{2x}{3}}\, dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-\frac{x}{3}}}{x^{1/3}}\right) + c_2 \left(\frac{e^{-\frac{x}{3}}}{x^{1/3}} \left(\int e^{-\frac{x}{3}-\frac{\ln(x)}{3}} x^{2/3} e^{\frac{2x}{3}}\, dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.89.2 Maple step by step solution

Let's solve

$3x^2 \left(\frac{d}{dx} y'\right) + x(1+x)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{y}{3x^2} - \frac{(1+x)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{(1+x)y'}{3x} - \frac{y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{1+x}{3x}, P_3(x) = -\frac{1}{3x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{1}{3}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{3}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2\left(\frac{d}{dx}y'\right) + x(1+x)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+3r)\,(-1+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r+1)\,(k+r-1) + a_{k-1}(k+r-1)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, -\frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

$$3(k+r-1)\left(\left(k+r+\frac{1}{3}\right)a_k+\frac{a_{k-1}}{3}\right)=0$$

- Shift index using $k->k+1$

$$3(k+r)\left(\left(k+\frac{4}{3}+r\right)a_{k+1}+\frac{a_k}{3}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{a_k}{3k+4+3r}$$

- Recursion relation for $r=1$

$$a_{k+1}=-\frac{a_k}{3k+7}$$

- Solution for $r=1$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+1},a_{k+1}=-\frac{a_k}{3k+7}\right]$$

- Recursion relation for $r=-\frac{1}{3}$

$$a_{k+1}=-\frac{a_k}{3k+3}$$

- Solution for $r=-\frac{1}{3}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{3}},a_{k+1}=-\frac{a_k}{3k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k-\frac{1}{3}}\right),a_{k+1}=-\frac{a_k}{3k+7},b_{k+1}=-\frac{b_k}{3k+3}\right]$$

1.89.3 Maple trace

```
Methods for second order ODEs:
```

1.89.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 30

```
dsolve(3*x^2*diff(diff(y(x),x),x)+x*(1+x)*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{\mathrm{e}^{-\frac{x}{6}}\left(x^{1/6}\,\mathrm{WhittakerM}\left(-\frac{1}{6},\frac{2}{3},\frac{x}{3}\right)c_1+\mathrm{e}^{-\frac{x}{6}}c_2\right)}{x^{1/3}}$$

1.89.5 Mathematica DSolve solution

Solving time : 0.034 (sec)
Leaf size : 50

```
DSolve[{3*x^2*D[y[x],{x,2}]+x*(1+x)*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x/3}\left(c_2x^{2/3} - 3\sqrt[3]{3}c_1(-x)^{2/3}\Gamma\left(\frac{4}{3}, -\frac{x}{3}\right)\right)}{x}$$

1.90 problem 92

Internal problem ID [8228]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 92
Date solved : `Monday, October 21, 2024 at 05:03:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' - xy' + (1-2x)\,y = 0$$

1.90.1 Solved as second order ode using Kovacic algorithm

Time used: 0.209 (sec)

Writing the ode as

$$2x^2y'' - xy' + (1-2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -x \\ C &= 1-2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3 + 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 + 16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3 + 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 164: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{1}{2}(1-(1)) \\
&= 0
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\
&= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\
&= \frac{1}{2x}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}
$$

Since $d=0$, then letting

$$
p = 1 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$
0 = 0
$$

And solving for p gives

$$
p = 1
$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta + \frac{p'}{p} \\
&= \frac{1}{2x}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0
$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1-16x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+4\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+4\sqrt{x}}{4x} dx} \\ &= x^{1/4} \mathrm{e}^{2\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{2x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{4}} \\ &= z_1 \left(x^{1/4}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\, \mathrm{e}^{2\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{2x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right) \end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\sqrt{x}\, e^{2\sqrt{x}}\right) + c_2 \left(\sqrt{x}\, e^{2\sqrt{x}} \left(-\frac{e^{-4\sqrt{x}}}{2}\right)\right)$$

Will add steps showing solving for IC soon.

1.90.2 Maple step by step solution

Let's solve

$2x^2 \left(\frac{d}{dx} y'\right) - x y' + (1 - 2x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{(2x-1)y}{2x^2} + \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' - \frac{y'}{2x} - \frac{(2x-1)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = -\frac{2x-1}{2x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -\frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{2}$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2 \left(\frac{d}{dx} y'\right) - x y' + (1 - 2x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) - 2a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k+r-1)\left(k+r-\frac{1}{2}\right) a_k - 2a_{k-1} = 0$
- Shift index using $k->k+1$

  $2(k+r)\left(k+\frac{1}{2}+r\right) a_{k+1} - 2a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k}{(k+r)(2k+1+2r)}$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{2a_k}{(k+1)(2k+3)}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{2a_k}{(k+1)(2k+3)}\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{2a_k}{\left(k+\frac{1}{2}\right)(2k+2)}$
- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = \frac{2a_k}{(k+1)(2k+3)}, b_{k+1} = \frac{2b_k}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

1.90.3 Maple trace

```
Methods for second order ODEs:
```

1.90.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve(2*x^2*diff(diff(y(x),x),x)-x*diff(y(x),x)+(1-2*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\left(c_1 \sinh\left(2\sqrt{x}\right) + c_2 \cosh\left(2\sqrt{x}\right)\right)$$

1.90.5 Mathematica DSolve solution

Solving time : 0.076 (sec)
Leaf size : 41

```
DSolve[{2*x^2*D[y[x],{x,2}]-x*D[y[x],x]+(1-2*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-2\sqrt{x}} \sqrt{x}\left(2c_1 e^{4\sqrt{x}} - c_2\right)$$

1.91 problem 93

Internal problem ID [8229]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 93
Date solved : `Monday, October 21, 2024 at 05:03:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2y'' + x(1 + x)\, y' - (1 + 3x)\, y = 0$$

1.91.1 Solved as second order ode using Kovacic algorithm

Time used: 0.648 (sec)

Writing the ode as

$$3x^2y'' + \left(x^2 + x\right) y' + (-3x - 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= x^2 + x \\ C &= -3x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 38x + 7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 38x + 7$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 38x + 7}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 166: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{19}{18x} + \frac{7}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{19}{6x} - \frac{59}{2x^2} + \frac{1121}{2x^3} - \frac{53041}{4x^4} + \frac{1404613}{4x^5} - \frac{39845827}{4x^6} + \frac{1184064097}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 38x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{38x + 7}{36x^2}\right) \\ &= \frac{1}{36} + \frac{38x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 38. Dividing this by leading coefficient in t which is 36 gives $\frac{19}{18}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{19}{18}\right) - (0) \\ &= \frac{19}{18} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{19}{18}}{\frac{1}{6}} - 0\right) &&= \frac{19}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{19}{18}}{\frac{1}{6}} - 0\right) &&= -\frac{19}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 38x + 7}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{19}{6}$	$-\frac{19}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{19}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= \frac{19}{6} - \left(\frac{7}{6}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{6x} + \left(\frac{1}{6}\right) \\
&= \frac{7}{6x} + \frac{1}{6} \\
&= \frac{7+x}{6x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{7}{6x} + \frac{1}{6}\right)(2x + a_1) + \left(\left(-\frac{7}{6x^2}\right) + \left(\frac{7}{6x} + \frac{1}{6}\right)^2 - \left(\frac{x^2 + 38x + 7}{36x^2}\right) \right) = 0$$

$$\frac{(-a_1 + 20)\, x - 2a_0 + 7a_1}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 70, a_1 = 20\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 20x + 70$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^2 + 20x + 70)\, e^{\int \left(\frac{7}{6x} + \frac{1}{6}\right) dx} \\
&= (x^2 + 20x + 70)\, e^{\frac{x}{6} + \frac{7 \ln(x)}{6}} \\
&= (x^2 + 20x + 70)\, x^{7/6} e^{\frac{x}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{3x^2}\,dx} \\
&= z_1 e^{-\frac{x}{6}-\frac{\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{6}}}{x^{1/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x^2 + 20x + 70)\, x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{3x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(x^2+20x+70)^2\, x^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x^2+20x+70)\, x\right) + c_2\left((x^2+20x+70)\, x \left(\int \frac{\mathrm{e}^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(x^2+20x+70)^2\, x^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.91.2 Maple step by step solution

Let's solve

$3x^2\left(\frac{d}{dx}y'\right)+x(1+x)\,y'-(1+3x)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{(1+3x)y}{3x^2}-\frac{(1+x)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(1+x)y'}{3x}-\frac{(1+3x)y}{3x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1+x}{3x}, P_3(x)=-\frac{1+3x}{3x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=\frac{1}{3}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=-\frac{1}{3}$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

$3x^2\left(\frac{d}{dx}y'\right)+x(1+x)\,y'+(-3x-1)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (1+3r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (3k+3r+1)\,(k+r-1) + a_{k-1}(k-4+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+3r)\,(-1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{1, -\frac{1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $3\left(k+r+\frac{1}{3}\right)(k+r-1)\, a_k + a_{k-1}(k-4+r) = 0$
- Shift index using $k- >k+1$
  $3\left(k+\frac{4}{3}+r\right)(k+r)\, a_{k+1} + a_k(k+r-3) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k (k+r-3)}{(3k+4+3r)(k+r)}$
- Recursion relation for $r = 1$; series terminates at $k = 2$
  $a_{k+1} = -\frac{a_k (k-2)}{(3k+7)(k+1)}$
- Apply recursion relation for $k = 0$
  $a_1 = \frac{2a_0}{7}$
- Apply recursion relation for $k = 1$
  $a_2 = \frac{a_1}{20}$
- Express in terms of a_0
  $a_2 = \frac{a_0}{70}$
- Terminating series solution of the ODE for $r = 1$. Use reduction of order to find the second li
  $y = a_0 \cdot \left(1 + \frac{2}{7}x + \frac{1}{70}x^2\right)$
- Recursion relation for $r = -\frac{1}{3}$
  $a_{k+1} = -\frac{a_k \left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}$

- Solution for $r=-\frac{1}{3}$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+1}=-\frac{a_k\left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}\right]$$

- Combine solutions and rename parameters

  $$\left[y=a_0\cdot\left(1+\frac{2}{7}x+\frac{1}{70}x^2\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{3}}\right), b_{k+1}=-\frac{b_k\left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}\right]$$

1.91.3 Maple trace

```
Methods for second order ODEs:
```

1.91.4 Maple dsolve solution

Solving time : `0.015` (sec)
Leaf size : 41

```
dsolve(3*x^2*diff(diff(y(x),x),x)+x*(1+x)*diff(y(x),x)-(1+3*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_2\,\mathrm{e}^{-\frac{x}{3}}\,\mathrm{hypergeom}\left([3],\left[-\frac{1}{3}\right],\frac{x}{3}\right)+70c_1\left(x^{4/3}+\frac{2x^{7/3}}{7}+\frac{x^{10/3}}{70}\right)}{x^{1/3}}$$

1.91.5 Mathematica DSolve solution

Solving time : `2.468` (sec)
Leaf size : 78

```
DSolve[{3*x^2*D[y[x],{x,2}]+x*(1+x)*D[y[x],x]-(1+3*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_1x\left(x^2+20x+70\right)-\frac{c_2x(x^2+20x+70)\,\Gamma\left(\frac{2}{3},\frac{x}{3}\right)}{1680\sqrt[3]{3}}+\frac{c_2e^{-x/3}(x^3+19x^2+54x-18)}{1680\sqrt[3]{x}}$$

1.92 problem 94

Internal problem ID [8230]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 94
Date solved : `Monday, October 21, 2024 at 05:03:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(3+x)\,y''+x(1+5x)\,y'+(1+x)\,y=0$$

1.92.1 Solved as second order ode using Kovacic algorithm

Time used: 0.345 (sec)

Writing the ode as

$$\left(2x^3+6x^2\right)y''+\left(5x^2+x\right)y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+6x^2\\B&=5x^2+x\\C&=1+x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 30x - 35}{16\left(x^2 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 - 30x - 35$$
$$t = 16\left(x^2 + 3x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 30x - 35}{16\left(x^2 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 168: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{5}{108\,(3+x)} + \frac{7}{36\,(3+x)^2} - \frac{35}{144x^2} - \frac{5}{108x}
$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(3+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-3x^2 - 30x - 35}{16\,(x^2+3x)^2}
$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-3x^2-30x-35}{16\left(x^2+3x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-3	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{1}{4}-\left(\frac{1}{4}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{6\left(3+x\right)} + \frac{5}{12x} + (-)\left(0\right) \\
&= -\frac{1}{6\left(3+x\right)} + \frac{5}{12x} \\
&= \frac{x+5}{4x\left(3+x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{6\left(3+x\right)} + \frac{5}{12x}\right)(0) + \left(\left(\frac{1}{6\left(3+x\right)^2} - \frac{5}{12x^2}\right) + \left(-\frac{1}{6\left(3+x\right)} + \frac{5}{12x}\right)^2 - \left(\frac{-3x^2 - 30x - 3}{16\left(x^2+3x\right)^2}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{6(3+x)} + \frac{5}{12x}\right) dx} \\
&= \frac{x^{5/12}}{\left(3+x\right)^{1/6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x^2+x}{2x^3+6x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{12} - \frac{7\ln(3+x)}{6}} \\
&= z_1\left(\frac{1}{x^{1/12}\left(3+x\right)^{7/6}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{(3+x)^{4/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2+x}{2x^3+6x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{6}-\frac{7\ln(3+x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{6}-\frac{7\ln(3+x)}{3}} (3+x)^{8/3}}{x^{2/3}} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/3}}{(3+x)^{4/3}} \right) + c_2 \left(\frac{x^{1/3}}{(3+x)^{4/3}} \left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{6}-\frac{7\ln(3+x)}{3}} (3+x)^{8/3}}{x^{2/3}} dx \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.92.2 Maple step by step solution

Let's solve

$2x^2(3+x)\left(\frac{d}{dx}y'\right) + x(1+5x)\,y' + (1+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(1+x)y}{2x^2(3+x)} - \frac{(1+5x)y'}{2x(3+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(1+5x)y'}{2x(3+x)} + \frac{(1+x)y}{2x^2(3+x)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{1+5x}{2x(3+x)}, P_3(x) = \frac{1+x}{2x^2(3+x)}\right]$
- ○ $(3+x) \cdot P_2(x)$ is analytic at $x = -3$

  $\left.((3+x) \cdot P_2(x))\right|_{x=-3} = \frac{7}{3}$
- ○ $(3+x)^2 \cdot P_3(x)$ is analytic at $x = -3$

  $\left.((3+x)^2 \cdot P_3(x))\right|_{x=-3} = 0$
- ○ $x = -3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -3$

- Multiply by denominators

  $2x^2(3+x)\left(\frac{d}{dx}y'\right) + x(1+5x)\,y' + (1+x)\,y = 0$
- Change variables using $x = u - 3$ so that the regular singular point is at $u = 0$

  $(2u^3 - 12u^2 + 18u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (5u^2 - 29u + 42)\left(\frac{d}{du}y(u)\right) + (-2+u)\,y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
- ○ Shift index using $k- > k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
- ○ Shift index using $k- > k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
- ○ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^{m} \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m} \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$6a_0 r(4+3r)\, u^{-1+r} + \left(6a_1(1+r)(7+3r) - a_0(12r^2+17r+2)\right) u^{r} + \left(\sum_{k=1}^{\infty} (6a_{k+1}(k+r+1)(3k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$6r(4+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{4}{3}\right\}$$

- Each term must be 0

$$6a_1(1+r)(7+3r) - a_0(12r^2+17r+2) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(-6a_k + a_{k-1} + 9a_{k+1})\, k^2 + (4(-6a_k + a_{k-1} + 9a_{k+1})\, r - 17a_k - a_{k-1} + 60a_{k+1})\, k + 2(-6a_k +$$

- Shift index using $k->k+1$

$$2(-6a_{k+1} + a_k + 9a_{k+2})\,(k+1)^2 + (4(-6a_{k+1} + a_k + 9a_{k+2})\, r - 17a_{k+1} - a_k + 60a_{k+2})\,(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 4kra_k - 24kra_{k+1} + 2r^2a_k - 12r^2a_{k+1} + 3ka_k - 41ka_{k+1} + 3ra_k - 41ra_{k+1} + a_k - 31a_{k+1}}{6(3k^2+6kr+3r^2+16k+16r+20)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}, 42a_1 - 2a_0 = 0\right]$$

- Revert the change of variables $u = 3 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (3+x)^k, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}, 42a_1 - 2a_0 = 0\right]$$

- Recursion relation for $r = -\frac{4}{3}$

$$a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} - \frac{7}{3}ka_k - 9ka_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}$$

- Solution for $r = -\frac{4}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} - \frac{7}{3}ka_k - 9ka_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}, -6a_1 - \frac{2a_0}{3} = 0\right]$$

- Revert the change of variables $u = 3 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (3+x)^{k-\frac{4}{3}}, a_{k+2} = -\frac{2k^2 a_k - 12k^2 a_{k+1} - \frac{7}{3} k a_k - 9k a_{k+1} + \frac{5}{9} a_k + \frac{7}{3} a_{k+1}}{6(3k^2+8k+4)}, -6a_1 - \frac{2a_0}{3} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (3+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (3+x)^{k-\frac{4}{3}}\right), a_{k+2} = -\frac{2k^2 a_k - 12k^2 a_{k+1} + 3k a_k - 41k a_{k+1} + a_k - 31 a_{k+1}}{6(3k^2+16k+20)}, 42a$$

1.92.3 Maple trace

```
Methods for second order ODEs:
```

1.92.4 Maple dsolve solution

Solving time : `0.015` (sec)
Leaf size : 36

```
dsolve(2*x^2*(3+x)*diff(diff(y(x),x),x)+x*(1+5*x)*diff(y(x),x)+(1+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \sqrt{x}\, \text{hypergeom}\left(\left[1, \frac{3}{2}\right], \left[\frac{7}{6}\right], -\frac{x}{3}\right) + \frac{c_2\, x^{1/3}}{(3+x)\left(1+\frac{x}{3}\right)^{1/3}}$$

1.92.5 Mathematica DSolve solution

Solving time : `1.195` (sec)
Leaf size : 50

```
DSolve[{2*x^2*(3+x)*D[y[x],{x,2}]+x*(1+5*x)*D[y[x],x]+(1+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[3]{x}\left(6\sqrt[3]{3}c_2\sqrt[6]{x}\,\text{Hypergeometric2F1}\left(-\frac{1}{3}, \frac{1}{6}, \frac{7}{6}, -\frac{x}{3}\right) + c_1\right)}{(x+3)^{4/3}}$$

1.93 problem 95

Internal problem ID [8231]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 95
Date solved : `Monday, October 21, 2024 at 05:03:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(4+x)\,y''-x(1-3x)\,y'+y=0$$

1.93.1 Solved as second order ode using Kovacic algorithm

Time used: 0.585 (sec)

Writing the ode as

$$x^2(4+x)\,y''+\left(3x^2-x\right)y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(4+x) \\ B &= 3x^2-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 6x - 7}{4\left(x^2 + 4x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2 - 6x - 7 \\ t &= 4\left(x^2 + 4x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 6x - 7}{4\left(x^2 + 4x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 170: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 4x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -4$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{128\,(4+x)} - \frac{7}{64x^2} + \frac{65}{64\,(4+x)^2} - \frac{5}{128x}$$

For the pole at $x = -4$ let b be the coefficient of $\frac{1}{(4+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{8} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 6x - 7}{4\,(x^2 + 4x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{3x^2-6x-7}{4\left(x^2+4x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-4	2	0	$\frac{13}{8}$	$-\frac{5}{8}$
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=-\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= -\frac{1}{2}-\left(-\frac{1}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{8\,(4+x)} + \frac{1}{8x} + (-)\,(0) \\
&= -\frac{5}{8\,(4+x)} + \frac{1}{8x} \\
&= -\frac{x-1}{2x\,(4+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{5}{8\,(4+x)} + \frac{1}{8x}\right)(0) + \left(\left(\frac{5}{8\,(4+x)^2} - \frac{1}{8x^2}\right) + \left(-\frac{5}{8\,(4+x)} + \frac{1}{8x}\right)^2 - \left(\frac{3x^2-6x-7}{4\,(x^2+4x)^2}\right)\right) = 0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{8(4+x)} + \frac{1}{8x}\right) dx} \\
&= \frac{x^{1/8}}{(4+x)^{5/8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^2-x}{x^2(4+x)}\, dx} \\
&= z_1 e^{\frac{\ln(x)}{8} - \frac{13\ln(4+x)}{8}} \\
&= z_1 \left(\frac{x^{1/8}}{(4+x)^{13/8}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x^{1/4}}{(4+x)^{9/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2-x}{x^2(4+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{4}-\frac{13\ln(4+x)}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{\ln(x)}{4}-\frac{13\ln(4+x)}{4}}(4+x)^{9/2}}{\sqrt{x}}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/4}}{(4+x)^{9/4}}\right) + c_2\left(\frac{x^{1/4}}{(4+x)^{9/4}}\left(\int \frac{\mathrm{e}^{\frac{\ln(x)}{4}-\frac{13\ln(4+x)}{4}}(4+x)^{9/2}}{\sqrt{x}}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.93.2 Maple step by step solution

Let's solve

$x^2(4+x)\left(\frac{d}{dx}y'\right) - x(1-3x)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x^2(4+x)} - \frac{(3x-1)y'}{x(4+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(3x-1)y'}{x(4+x)} + \frac{y}{x^2(4+x)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3x-1}{x(4+x)}, P_3(x) = \frac{1}{x^2(4+x)}\right]$

- $(4+x) \cdot P_2(x)$ is analytic at $x = -4$

  $\left((4+x) \cdot P_2(x)\right)\bigg|_{x=-4} = \frac{13}{4}$

- $(4+x)^2 \cdot P_3(x)$ is analytic at $x = -4$

  $\left((4+x)^2 \cdot P_3(x)\right)\bigg|_{x=-4} = 0$

- $x = -4$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -4$

- Multiply by denominators

  $x^2(4+x)\left(\frac{d}{dx}y'\right) + x(3x-1)y' + y = 0$

- Change variables using $x = u - 4$ so that the regular singular point is at $u = 0$

  $(u^3 - 8u^2 + 16u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (3u^2 - 25u + 52)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$4a_0r(9+4r)\,u^{-1+r}+\left(4a_1(1+r)\,(13+4r)-a_0(8r^2+17r-1)\right)u^r+\left(\sum_{k=1}^{\infty}\left(4a_{k+1}(k+1+r)\,(4k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r(9+4r)=0$
- Values of r that satisfy the indicial equation
  $r\in\left\{0,-\frac{9}{4}\right\}$
- Each term must be 0
  $4a_1(1+r)\,(13+4r)-a_0(8r^2+17r-1)=0$
- Each term in the series must be 0, giving the recursion relation
  $(-8a_k+a_{k-1}+16a_{k+1})\,k^2+(2(-8a_k+a_{k-1}+16a_{k+1})\,r-17a_k+68a_{k+1})\,k+(-8a_k+a_{k-1}+$
- Shift index using $k->k+1$
  $(-8a_{k+1}+a_k+16a_{k+2})\,(k+1)^2+(2(-8a_{k+1}+a_k+16a_{k+2})\,r-17a_{k+1}+68a_{k+2})\,(k+1)+(-$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}+2kra_k-16kra_{k+1}+r^2a_k-8r^2a_{k+1}+2ka_k-33ka_{k+1}+2ra_k-33ra_{k+1}-24a_{k+1}}{4(4k^2+8kr+4r^2+25k+25r+34)}$
- Recursion relation for $r=0$
  $a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}+2ka_k-33ka_{k+1}-24a_{k+1}}{4(4k^2+25k+34)}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^k, a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}+2ka_k-33ka_{k+1}-24a_{k+1}}{4(4k^2+25k+34)}, 52a_1+a_0=0\right]$
- Revert the change of variables $u=4+x$
  $\left[y=\sum_{k=0}^{\infty}a_k(4+x)^k, a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}+2ka_k-33ka_{k+1}-24a_{k+1}}{4(4k^2+25k+34)}, 52a_1+a_0=0\right]$
- Recursion relation for $r=-\frac{9}{4}$
  $a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}-\frac{5}{2}ka_k+3ka_{k+1}+\frac{9}{16}a_k+\frac{39}{4}a_{k+1}}{4(4k^2+7k-2)}$
- Solution for $r=-\frac{9}{4}$
  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-\frac{9}{4}}, a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}-\frac{5}{2}ka_k+3ka_{k+1}+\frac{9}{16}a_k+\frac{39}{4}a_{k+1}}{4(4k^2+7k-2)}, -20a_1-\frac{5a_0}{4}=0\right]$
- Revert the change of variables $u=4+x$
  $\left[y=\sum_{k=0}^{\infty}a_k(4+x)^{k-\frac{9}{4}}, a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}-\frac{5}{2}ka_k+3ka_{k+1}+\frac{9}{16}a_k+\frac{39}{4}a_{k+1}}{4(4k^2+7k-2)}, -20a_1-\frac{5a_0}{4}=0\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty}a_k(4+x)^k\right)+\left(\sum_{k=0}^{\infty}b_k(4+x)^{k-\frac{9}{4}}\right), a_{k+2}=-\frac{k^2a_k-8k^2a_{k+1}+2ka_k-33ka_{k+1}-24a_{k+1}}{4(4k^2+25k+34)}, 52a_1+\right.$

1.93.3 Maple trace

```
Methods for second order ODEs:
```

1.93.4 Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 27

```
dsolve(x^2*(4+x)*diff(diff(y(x),x),x)-x*(1-3*x)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 x^{1/4}}{(4+x)^{9/4}} + c_2 \operatorname{hypergeom}\left([1,3], \left[\frac{7}{4}\right], -\frac{x}{4}\right) x$$

1.93.5 Mathematica DSolve solution

Solving time : 0.36 (sec)
Leaf size : 89

```
DSolve[{x^2*(4+x)*D[y[x],{x,2}]-x*(1-3*x)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[4]{x}\left(-10c_2 \arctan\left(\sqrt[4]{\frac{x}{x+4}}\right) + 10c_2 \operatorname{arctanh}\left(\sqrt[4]{\frac{x}{x+4}}\right) + c_2\sqrt[4]{x+4}x^{7/4} + 9c_2\sqrt[4]{x+4}x^{3/4} + 2c_1\right)}{2(x+4)^{9/4}}$$

1.94 problem 96

Internal problem ID [8232]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 96
Date solved : `Monday, October 21, 2024 at 05:03:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + 5xy' + (1+x)\,y = 0$$

1.94.1 Solved as second order ode using Kovacic algorithm

Time used: 0.279 (sec)

Writing the ode as

$$2x^2y'' + 5xy' + (1+x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 5x \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-3 - 8x}{16x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -3 - 8x \\
t &= 16x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-3 - 8x}{16x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 172: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} - \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{1}{2}(1-(1)) \\
&= 0
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\
&= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\
&= \frac{1}{2x}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}
$$

Since $d=0$, then letting

$$
p = 1 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$
0 = 0
$$

And solving for p gives

$$
p = 1
$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta + \frac{p'}{p} \\
&= \frac{1}{2x}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0
$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1+8x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\,\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{2}\,\sqrt{-x}}{4x} dx} \\ &= x^{1/4} \mathrm{e}^{\sqrt{2}\,\sqrt{-x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x}{2x^2}\, dx} \\ &= z_1 e^{-\frac{5\ln(x)}{4}} \\ &= z_1 \left(\frac{1}{x^{5/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\sqrt{2}\,\sqrt{-x}}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{2x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\sqrt{2}\,\sqrt{-x}\left(1 - \mathrm{e}^{-2\sqrt{2}\,\sqrt{-x}}\right)}{2\sqrt{x}} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\sqrt{2}\,\sqrt{-x}}}{x}\right) + c_2 \left(\frac{e^{\sqrt{2}\,\sqrt{-x}}}{x}\left(-\frac{\sqrt{2}\,\sqrt{-x}\left(1-e^{-2\sqrt{2}\,\sqrt{-x}}\right)}{2\sqrt{x}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.94.2 Maple step by step solution

Let's solve

$2x^2\left(\frac{d}{dx}y'\right) + 5xy' + (1+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(1+x)y}{2x^2} - \frac{5y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{5y'}{2x} + \frac{(1+x)y}{2x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{5}{2x}, P_3(x) = \frac{1+x}{2x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{2}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d}{dx}y'\right) + 5xy' + (1+x)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r+1) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-1, -\frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k+r+1)\left(k+r+\frac{1}{2}\right) a_k + a_{k-1} = 0$
- Shift index using $k- >k + 1$

  $2(k+2+r)\left(k+\frac{3}{2}+r\right) a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+2+r)(2k+3+2r)}$
- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}\right]$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}$

- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}, b_{k+1} = -\frac{b_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

1.94.3 Maple trace

```
Methods for second order ODEs:
```

1.94.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 29

```
dsolve(2*x^2*diff(diff(y(x),x),x)+5*x*diff(y(x),x)+(1+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(\sqrt{x}\,\sqrt{2}\right) + c_2 \cos\left(\sqrt{x}\,\sqrt{2}\right)}{x}$$

1.94.5 Mathematica DSolve solution

Solving time : `0.133` (sec)
Leaf size : 60

```
DSolve[{2*x^2*D[y[x],{x,2}]+5*x*D[y[x],x]+(1+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{i\sqrt{2}\sqrt{x}} + i\sqrt{2}c_2 e^{-i\sqrt{2}\sqrt{x}}}{2x}$$

1.95 problem 97

Internal problem ID [8233]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 97
Date solved : `Monday, October 21, 2024 at 05:03:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2y'' + x(10 - x)\,y' - (2 + x)\,y = 0$$

1.95.1 Solved as second order ode using Kovacic algorithm

Time used: 0.389 (sec)

Writing the ode as

$$6x^2y'' + \left(-x^2 + 10x\right)y' + (-x - 2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= -x^2 + 10x \\ C &= -x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 28}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x + 28$$
$$t = 144x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 28}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 174: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{144} + \frac{7}{36x^2} + \frac{1}{36x}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{12} + \frac{1}{6x} + \frac{1}{x^2} - \frac{2}{x^3} - \frac{2}{x^4} + \frac{28}{x^5} - \frac{56}{x^6} - \frac{272}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{12}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{12}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{144}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 28}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{1}{144}\right) + \left(\frac{4x + 28}{144x^2}\right) \\ &= \frac{1}{144} + \frac{4x + 28}{144x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 144 gives $\frac{1}{36}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{36}\right) - (0) \\ &= \frac{1}{36} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{12} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{36}}{\frac{1}{12}} - 0\right) &= \frac{1}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{36}}{\frac{1}{12}} - 0\right) &= -\frac{1}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 28}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{12}$	$\frac{1}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{6} - \left(-\frac{1}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{6x} + (-)\left(\frac{1}{12}\right) \\
&= -\frac{1}{6x} - \frac{1}{12} \\
&= -\frac{2 + x}{12x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{6x} - \frac{1}{12}\right)(0) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} - \frac{1}{12}\right)^2 - \left(\frac{x^2 + 4x + 28}{144x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{6x} - \frac{1}{12}\right) dx} \\
&= \frac{e^{-\frac{x}{12}}}{x^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2 + 10x}{6x^2} \, dx} \\
&= z_1 e^{\frac{x}{12} - \frac{5 \ln(x)}{6}} \\
&= z_1 \left(\frac{e^{\frac{x}{12}}}{x^{5/6}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+10x}{6x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x}{6}-\frac{5\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \mathrm{e}^{\frac{x}{6}-\frac{5\ln(x)}{3}} x^2 dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\int \mathrm{e}^{\frac{x}{6}-\frac{5\ln(x)}{3}} x^2 dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.95.2 Maple step by step solution

Let's solve
$6x^2\left(\frac{d}{dx}y'\right) + x(10-x)\, y' - (2+x)\, y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = \frac{(2+x)y}{6x^2} + \frac{(x-10)y'}{6x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{(x-10)y'}{6x} - \frac{(2+x)y}{6x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x-10}{6x}, P_3(x) = -\frac{2+x}{6x^2}\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{3}$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{3}$
- $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $6x^2\left(\frac{d}{dx}y'\right) - x(x-10)\,y' + (-x-2)\,y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions
- Convert $x^m \cdot y$ to series expansion for $m = 0..1$
  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$
  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$
  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k- >k + 1 - m$
  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$
  Rewrite ODE with series expansions
  $2a_0(1+r)\,(-1+3r)\,x^r + \left(\sum_{k=1}^{\infty} (2a_k(k+r+1)\,(3k+3r-1) - a_{k-1}(k+r))\,x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2(1+r)(-1+3r)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-1, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$6(k+r+1)\left(k+r-\frac{1}{3}\right)a_k - a_{k-1}(k+r)=0$$

- Shift index using $k->k+1$

$$6(k+2+r)\left(k+\frac{2}{3}+r\right)a_{k+1} - a_k(k+r+1)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(k+r+1)}{2(k+2+r)(3k+2+3r)}$$

- Recursion relation for $r=-1$

$$a_{k+1}=\frac{a_k k}{2(k+1)(3k-1)}$$

- Solution for $r=-1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1}=\frac{a_k k}{2(k+1)(3k-1)}\right]$$

- Recursion relation for $r=\frac{1}{3}$

$$a_{k+1}=\frac{a_k\left(k+\frac{4}{3}\right)}{2\left(k+\frac{7}{3}\right)(3k+3)}$$

- Solution for $r=\frac{1}{3}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1}=\frac{a_k\left(k+\frac{4}{3}\right)}{2\left(k+\frac{7}{3}\right)(3k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+1}=\frac{a_k k}{2(k+1)(3k-1)}, b_{k+1}=\frac{b_k\left(k+\frac{4}{3}\right)}{2\left(k+\frac{7}{3}\right)(3k+3)}\right]$$

1.95.3 Maple trace

```
Methods for second order ODEs:
```

1.95.4 Maple dsolve solution

Solving time : `0.014` (sec)
Leaf size : 27

```
dsolve(6*x^2*diff(diff(y(x),x),x)+x*(10-x)*diff(y(x),x)-(2+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\, x^{5/6} + c_1 \,\mathrm{WhittakerM}\left(-\frac{1}{6}, \frac{2}{3}, \frac{x}{6}\right) \mathrm{e}^{\frac{x}{12}} x}{x^{11/6}}$$

1.95.5 Mathematica DSolve solution

Solving time : `0.049` (sec)
Leaf size : 38

```
DSolve[{6*x^2*D[y[x],{x,2}]+x*(10-x)*D[y[x],x]-(2+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 \sqrt[3]{x} L_{-\frac{4}{3}}^{\frac{4}{3}}\left(\frac{x}{6}\right) + \frac{6\sqrt[3]{6} c_1}{x}$$

1.96 problem 98

Internal problem ID [8234]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 98
Date solved : `Monday, October 21, 2024 at 05:03:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(3+4x)\,y''+x(11+4x)\,y'-(3+4x)\,y=0$$

1.96.1 Solved as second order ode using Kovacic algorithm

Time used: 0.379 (sec)

Writing the ode as

$$\left(4x^3+3x^2\right)y''+\left(4x^2+11x\right)y'+(-3-4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+3x^2 \\ B &= 4x^2+11x \\ C &= -3-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{48x^2 + 8x + 91}{4(4x^2 + 3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 48x^2 + 8x + 91$$
$$t = 4(4x^2 + 3x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{48x^2 + 8x + 91}{4(4x^2 + 3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 176: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(4x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{3}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{28}{9\left(x + \frac{3}{4}\right)^2} + \frac{176}{27\left(x + \frac{3}{4}\right)} - \frac{176}{27x} + \frac{91}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{91}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{13}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{7}{6} \end{aligned}$$

For the pole at $x = -\frac{3}{4}$ let b be the coefficient of $\frac{1}{\left(x+\frac{3}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{28}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{4}{3} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{48x^2 + 8x + 91}{4\left(4x^2 + 3x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{48x^2 + 8x + 91}{4\left(4x^2 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{13}{6}$	$-\frac{7}{6}$
$-\frac{3}{4}$	2	0	$\frac{7}{3}$	$-\frac{4}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{5}{2}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{7}{6x} - \frac{4}{3\left(x+\frac{3}{4}\right)} + (-)(0) \\
&= -\frac{7}{6x} - \frac{4}{3\left(x+\frac{3}{4}\right)} \\
&= \frac{-7-20x}{8x^2+6x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{7}{6x} - \frac{4}{3\left(x+\frac{3}{4}\right)}\right)(2x+a_1) + \left(\left(\frac{7}{6x^2} + \frac{4}{3\left(x+\frac{3}{4}\right)^2}\right) + \left(-\frac{7}{6x} - \frac{4}{3\left(x+\frac{3}{4}\right)}\right)^2 - \left(\frac{48x^2+8x}{4(4x^2+3
$$

$$
\frac{12a_1x - 8x + 32a_0 -}{x(3+4x)}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{7}{48}, a_1 = \frac{2}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + \frac{2}{3}x + \frac{7}{48}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + \frac{2}{3}x + \frac{7}{48}\right) e^{\int \left(-\frac{7}{6x} - \frac{4}{3\left(x+\frac{3}{4}\right)}\right) dx} \\
&= \left(x^2 + \frac{2}{3}x + \frac{7}{48}\right) e^{-\frac{4\ln(3+4x)}{3} - \frac{7\ln(x)}{6}} \\
&= \frac{x^2 + \frac{2}{3}x + \frac{7}{48}}{(3+4x)^{4/3} x^{7/6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^2+11x}{4x^3+3x^2}\,dx} \\
&= z_1 e^{\frac{4\ln(3+4x)}{3}-\frac{11\ln(x)}{6}} \\
&= z_1\left(\frac{(3+4x)^{4/3}}{x^{11/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{4x^2+11x}{4x^3+3x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1\int \frac{e^{\frac{8\ln(3+4x)}{3}-\frac{11\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{\frac{8\ln(3+4x)}{3}-\frac{11\ln(x)}{3}}x^6}{\left(x^2+\frac{2}{3}x+\frac{7}{48}\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left(\frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3}\right)+c_2\left(\frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3}\left(\int \frac{\mathrm{e}^{\frac{8\ln(3+4x)}{3}-\frac{11\ln(x)}{3}}x^6}{\left(x^2+\frac{2}{3}x+\frac{7}{48}\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.96.2 Maple step by step solution

Let's solve

$x^2(3+4x)\left(\frac{d}{dx}y'\right)+x(11+4x)\,y'-(3+4x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{y}{x^2}-\frac{(11+4x)y'}{x(3+4x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(11+4x)y'}{x(3+4x)}-\frac{y}{x^2}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{11+4x}{x(3+4x)},P_3(x)=-\frac{1}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{11}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-1$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(3+4x)\left(\frac{d}{dx}y'\right)+x(11+4x)\,y'+(-3-4x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(3+r)\,(-1+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+3)\,(3k+3r-1) + 4a_{k-1}(k+r)\,(k-2+r)\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(3+r)\,(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-3, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3(k+r+3)\left(k+r-\frac{1}{3}\right) a_k + 4a_{k-1}(k+r)\,(k-2+r) = 0$$

- Shift index using $k->k+1$

$$3(k+4+r)\left(k+\frac{2}{3}+r\right) a_{k+1} + 4a_k(k+r+1)\,(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{4a_k(k+r+1)(k+r-1)}{(k+4+r)(3k+2+3r)}$$

- Recursion relation for $r = -3$; series terminates at $k = 2$

$$a_{k+1} = -\frac{4a_k(k-2)(k-4)}{(k+1)(3k-7)}$$

- Apply recursion relation for $k = 0$

$$a_1 = \frac{32a_0}{7}$$

- Apply recursion relation for $k = 1$

$$a_2 = \frac{3a_1}{2}$$

- Express in terms of a_0

$$a_2 = \frac{48a_0}{7}$$

- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

$$y = a_0 \cdot \left(\frac{48}{7}x^2 + \frac{32}{7}x + 1\right)$$

- Recursion relation for $r=\frac{1}{3}$

  $a_{k+1}=-\frac{4a_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{\left(k+\frac{13}{3}\right)(3k+3)}$
- Solution for $r=\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1}=-\frac{4a_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{\left(k+\frac{13}{3}\right)(3k+3)}\right]$
- Combine solutions and rename parameters

  $\left[y=a_0\cdot\left(\frac{48}{7}x^2+\frac{32}{7}x+1\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), b_{k+1}=-\frac{4b_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{\left(k+\frac{13}{3}\right)(3k+3)}\right]$

1.96.3 Maple trace

```
Methods for second order ODEs:
```

1.96.4 Maple dsolve solution

Solving time : 0.028 (sec)
Leaf size : 41

```
dsolve(x^2*(3+4*x)*diff(diff(y(x),x),x)+x*(11+4*x)*diff(y(x),x)-(3+4*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1(48x^2+32x+7)}{x^3}+c_2\,\text{hypergeom}\left([3,5],\left[\frac{13}{3}\right],-\frac{4x}{3}\right)(3+4x)^{11/3}x^{1/3}$$

1.96.5 Mathematica DSolve solution

Solving time : 1.955 (sec)
Leaf size : 367

```
DSolve[{x^2*(3+4*x)*D[y[x],{x,2}]+x*(11+4*x)*D[y[x],x]-(3+4*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$y(x)$

$$\rightarrow \frac{12\sqrt[3]{2}\sqrt{3}c_2(48x^2+32x+7)\arctan\left(\frac{\sqrt{3}\sqrt[3]{4x+3}}{2\,2^{2/3}\sqrt[3]{x}+\sqrt[3]{4x+3}}\right)+384c_2(4x+3)^{2/3}x^{10/3}+576c_2(4x+3)^{2/3}x}{}$$

1.97 problem 99

Internal problem ID [8235]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 99
Date solved : `Monday, October 21, 2024 at 05:04:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+3x)\,y''+x(4+11x)\,y'-(1-x)\,y=0$$

1.97.1 Solved as second order ode using Kovacic algorithm

Time used: 0.225 (sec)

Writing the ode as

$$\left(6x^3+4x^2\right)y''+\left(11x^2+4x\right)y'+(x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=6x^3+4x^2\\B&=11x^2+4x\\C&=x-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-35}{16\left(2+3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -35$$
$$t = 16(2+3x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{35}{16\left(2+3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 178: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2+3x)^2$. There is a pole at $x = -\frac{2}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144\left(x + \frac{2}{3}\right)^2}$$

For the pole at $x = -\frac{2}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{35}{16\,(2+3x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{35}{16\left(2+3x\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{2}{3}$	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{12}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{5}{12} - \left(\frac{5}{12}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{12\left(x+\frac{2}{3}\right)} + (-)\left(0\right) \\ &= \frac{5}{12\left(x+\frac{2}{3}\right)} \\ &= \frac{5}{8+12x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{12\left(x + \frac{2}{3}\right)}\right)(0) + \left(\left(-\frac{5}{12\left(x + \frac{2}{3}\right)^2}\right) + \left(\frac{5}{12\left(x + \frac{2}{3}\right)}\right)^2 - \left(-\frac{35}{16\,(2+3x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{5}{12\left(x+\frac{2}{3}\right)} dx} \\
&= (2+3x)^{5/12}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{11x^2+4x}{6x^3+4x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} - \frac{5\ln(2+3x)}{12}} \\
&= z_1 \left(\frac{1}{\sqrt{x}\,(2+3x)^{5/12}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^2+4x}{6x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)-\frac{5\ln(2+3x)}{6}}}{(y_1)^2}\, dx \\
&= y_1 \left(2\,\mathrm{e}^{-\ln(x)-\frac{5\ln(2+3x)}{6}} x(2+3x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\sqrt{x}}\right) + c_2 \left(\frac{1}{\sqrt{x}} \left(2\,\mathrm{e}^{-\ln(x)-\frac{5\ln(2+3x)}{6}} x(2+3x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.97.2 Maple step by step solution

Let's solve

$2x^2(2+3x)\left(\frac{d}{dx}y'\right) + x(4+11x)\,y' - (1-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x-1)y}{2x^2(2+3x)} - \frac{(4+11x)y'}{2x(2+3x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(4+11x)y'}{2x(2+3x)} + \frac{(x-1)y}{2x^2(2+3x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{4+11x}{2x(2+3x)}, P_3(x) = \frac{x-1}{2x^2(2+3x)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $2x^2(2+3x)\left(\frac{d}{dx}y'\right) + x(4+11x)\,y' + (x-1)\,y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$
  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k-m$
  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$
  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k- >k+1-m$
  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$
  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
- Shift index using $k- >k+2-m$
  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions
  $a_0(1+2r)\,(-1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + a_{k-1}(2k+2r-1)\,(3k-2+3r\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $4\left(k+r-\frac{1}{2}\right)\left(\left(\frac{3k}{2}+\frac{3r}{2}-1\right)a_{k-1}+a_k\left(k+r+\frac{1}{2}\right)\right)=0$
- Shift index using $k->k+1$
  $4\left(k+r+\frac{1}{2}\right)\left(\left(\frac{3k}{2}+\frac{1}{2}+\frac{3r}{2}\right)a_k+a_{k+1}\left(k+\frac{3}{2}+r\right)\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{(3k+3r+1)a_k}{2k+3+2r}$
- Recursion relation for $r=-\frac{1}{2}$
  $a_{k+1}=-\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}$
- Solution for $r=-\frac{1}{2}$
  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}, a_{k+1}=-\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}\right]$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1}=-\frac{\left(3k+\frac{5}{2}\right)a_k}{2k+4}$
- Solution for $r=\frac{1}{2}$
  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{\left(3k+\frac{5}{2}\right)a_k}{2k+4}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}, b_{k+1}=-\frac{\left(3k+\frac{5}{2}\right)b_k}{2k+4}\right]$

1.97.3 Maple trace

```
Methods for second order ODEs:
```

1.97.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 19

```
dsolve(2*x^2*(2+3*x)*diff(diff(y(x),x),x)+x*(4+11*x)*diff(y(x),x)-(1-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_2(2+3x)^{1/6}+c_1}{\sqrt{x}}$$

1.97.5 Mathematica DSolve solution

Solving time : 0.092 (sec)
Leaf size : 32

```
DSolve[{2*x^2*(2+3*x)*D[y[x],{x,2}]+x*(4+11*x)*D[y[x],x]-(1-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 \sqrt[6]{6x+4} + 2^{5/6} c_1}{\sqrt{x}}$$

1.98 problem 100

Internal problem ID [8236]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 100
Date solved : `Monday, October 21, 2024 at 05:04:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(2+x)\,y''+5x(1-x)\,y'-(2-8x)\,y=0$$

1.98.1 Solved as second order ode using Kovacic algorithm

Time used: 0.957 (sec)

Writing the ode as

$$x^2(2+x)\,y''+\left(-5x^2+5x\right)y'+(8x-2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(2+x)\\B&=-5x^2+5x\\C&=8x-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int\frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 126x + 21}{4(x^2 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^2 - 126x + 21$$
$$t = 4(x^2 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 126x + 21}{4(x^2 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 180: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{147}{16x} + \frac{147}{16\,(2+x)} + \frac{21}{16x^2} + \frac{285}{16\,(2+x)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{285}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{15}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 126x + 21}{4\,(x^2 + 2x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{3x^2-126x+21}{4\left(x^2+2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{19}{4}$	$-\frac{15}{4}$
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=-\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= -\frac{1}{2}-\left(-\frac{9}{2}\right)\\
&= 4
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{15}{4\left(2+x\right)} - \frac{3}{4x} + (-)\left(0\right) \\ &= -\frac{15}{4\left(2+x\right)} - \frac{3}{4x} \\ &= -\frac{3(3x+1)}{2x\left(2+x\right)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(-\frac{15}{4\left(2+x\right)} - \frac{3}{4x}\right)\left(4x^3 + 3x^2a_3 + 2a_2x + a_1\right) + \left(\left(\frac{15}{4\left(2+x\right)^2} + \frac{3}{4x^2}\right) + \left(-\frac{}{4}\right.\right.$$

$$\frac{3(4+a_3)\,x^3 + (8a_2 + }{}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{40}, a_1 = \frac{1}{5}, a_2 = \frac{3}{2}, a_3 = -4\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right) e^{\int \left(-\frac{15}{4(2+x)} - \frac{3}{4x}\right)dx} \\ &= \left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right) e^{-\frac{15\ln(2+x)}{4} - \frac{3\ln(x)}{4}} \\ &= \frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40\left(2+x\right)^{15/4} x^{3/4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x^2+5x}{x^2(2+x)}\, dx} \\
&= z_1 e^{\frac{15\ln(2+x)}{4}-\frac{5\ln(x)}{4}} \\
&= z_1 \left(\frac{(2+x)^{15/4}}{x^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5x^2+5x}{x^2(2+x)}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{15\ln(2+x)}{2}-\frac{5\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{10x^{5/2}\sqrt{2+x}\left(8x^5\sqrt{x(2+x)} + 4200\ln\left(\frac{x+\sqrt{x(2+x)}}{x}\right)x^4 - 4200\ln\left(\frac{\sqrt{x(2+x)}-x}{x}\right)x^4 + 328x^4\sqrt{x(2
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2}\right) \\
&\quad + c_2\left(\frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2}\left(\frac{10x^{5/2}\sqrt{2+x}\left(8x^5\sqrt{x(2+x)} + 4200\ln\left(\frac{x+\sqrt{x(2+x)}}{x}\right)x^4 - 420
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.98.2 Maple step by step solution

Let's solve

$x^2(2+x)\left(\frac{d}{dx}y'\right)+5x(1-x)\,y'-(2-8x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{2(4x-1)y}{x^2(2+x)}+\frac{5(x-1)y'}{x(2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{5(x-1)y'}{x(2+x)}+\frac{2(4x-1)y}{x^2(2+x)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{5(x-1)}{x(2+x)},P_3(x)=\frac{2(4x-1)}{x^2(2+x)}\right]$

- ◦ $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((2+x)\cdot P_2(x)\right)\Big|_{x=-2}=-\frac{15}{2}$

- ◦ $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((2+x)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

- ◦ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $x^2(2+x)\left(\frac{d}{dx}y'\right)-5x(x-1)\,y'+(8x-2)\,y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(u^3-4u^2+4u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-5u^2+25u-30)\left(\frac{d}{du}y(u)\right)+(8u-18)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ◦ Shift index using $k->k-m$

$$u^{m} \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^{m} \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^{m} \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^{m} \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m} \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$2a_{0}r(-17+2r)\, u^{-1+r} + \left(2a_{1}(1+r)\,(-15+2r) - a_{0}(4r^{2}-29r+18)\right) u^{r} + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$2r(-17+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{17}{2}\right\}$$

- Each term must be 0

$$2a_{1}(1+r)\,(-15+2r) - a_{0}(4r^{2}-29r+18) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-4a_{k}+a_{k-1}+4a_{k+1})\, k^{2} + \left((-8a_{k}+2a_{k-1}+8a_{k+1})\, r + 29a_{k} - 8a_{k-1} - 26a_{k+1}\right) k + (-4a_{k}+a$$

- Shift index using $k->k+1$

$$(-4a_{k+1}+a_{k}+4a_{k+2})\,(k+1)^{2} + \left((-8a_{k+1}+2a_{k}+8a_{k+2})\, r + 29a_{k+1} - 8a_{k} - 26a_{k+2}\right)(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^{2}a_{k}-4k^{2}a_{k+1}+2kra_{k}-8kra_{k+1}+r^{2}a_{k}-4r^{2}a_{k+1}-6ka_{k}+21ka_{k+1}-6ra_{k}+21ra_{k+1}+8a_{k}+7a_{k+1}}{2(2k^{2}+4kr+2r^{2}-9k-9r-26)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^{2}a_{k}-4k^{2}a_{k+1}-6ka_{k}+21ka_{k+1}+8a_{k}+7a_{k+1}}{2(2k^{2}-9k-26)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_{k}u^{k}, a_{k+2} = -\frac{k^{2}a_{k}-4k^{2}a_{k+1}-6ka_{k}+21ka_{k+1}+8a_{k}+7a_{k+1}}{2(2k^{2}-9k-26)}, -30a_{1}-18a_{0} = 0\right]$$

- Revert the change of variables $u = 2+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^k, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 6k a_k + 21k a_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2 - 9k - 26)}, -30a_1 - 18a_0 = 0\right]$$

- Recursion relation for $r = \frac{17}{2}$

$$a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 11k a_k - 47k a_{k+1} + \frac{117}{4} a_k - \frac{207}{2} a_{k+1}}{2(2k^2 + 25k + 42)}$$

- Solution for $r = \frac{17}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{17}{2}}, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 11k a_k - 47k a_{k+1} + \frac{117}{4} a_k - \frac{207}{2} a_{k+1}}{2(2k^2 + 25k + 42)}, 38a_1 - \frac{121 a_0}{2} = 0\right]$$

- Revert the change of variables $u = 2 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k+\frac{17}{2}}, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 11k a_k - 47k a_{k+1} + \frac{117}{4} a_k - \frac{207}{2} a_{k+1}}{2(2k^2 + 25k + 42)}, 38a_1 - \frac{121 a_0}{2} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (2+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (2+x)^{k+\frac{17}{2}}\right), a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 6k a_k + 21k a_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2 - 9k - 26)}, -30\right.$$

1.98.3 Maple trace

```
Methods for second order ODEs:
```

1.98.4 Maple dsolve solution

Solving time : `0.028` (sec)
Leaf size : 113

```
dsolve(x^2*(2+x)*diff(diff(y(x),x),x)+5*x*(1-x)*diff(y(x),x)-(2-8*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1(40x^4 - 160x^3 + 60x^2 + 8x + 1)}{x^2} + \frac{4(-2-x)^{3/4}\left(1050x^{3/2}\left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right)\operatorname{arcsinh}\left(\frac{\sqrt{2}\sqrt{x}}{2}\right) + \sqrt{2+x}\,x^2\left(x^5 + 41x^4 - \frac{6987}{4}x^3\right.\right.}{x^{7/2}(2+x)^{3/4}}$$

1.98.5 Mathematica DSolve solution

Solving time : `3.612` (sec)
Leaf size : 114

```
DSolve[{x^2*(2+x)*D[y[x],{x,2}]+5*x*(1-x)*D[y[x],x]-(2-8*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1050c_2(40x^4-160x^3+60x^2+8x+1)\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{x}{x+2}}}\right)+2c_1(40x^4-160x^3+60x^2+8x+1)+5c_2\sqrt{}}{80x^2}$$

1.99 problem 101

Internal problem ID [8237]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 101
Date solved : `Monday, October 21, 2024 at 05:04:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$8x^2\left(-x^2+1\right)y''+2x\left(-13x^2+1\right)y'+\left(-9x^2+1\right)y=0$$

1.99.1 Solved as second order ode using Kovacic algorithm

Time used: 0.404 (sec)

Writing the ode as

$$\left(-8x^4+8x^2\right)y''+\left(-26x^3+2x\right)y'+\left(-9x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-8x^4+8x^2\\B&=-26x^3+2x\\C&=-9x^2+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7x^4 - 26x^2 - 15}{64 (x^3 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -7x^4 - 26x^2 - 15$$
$$t = 64(x^3 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-7x^4 - 26x^2 - 15}{64 (x^3 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 182: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{15}{64x^2} - \frac{1}{4(x+1)} - \frac{3}{16(x-1)^2} - \frac{3}{16(x+1)^2} + \frac{1}{4x-4}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x=-1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty=0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r=\frac{s}{t}=\frac{-7x^4-26x^2-15}{64\left(x^3-x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{8} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-7x^4-26x^2-15}{64\left(x^3-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{8}$	$\frac{3}{8}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{7}{8} - \left(\frac{7}{8}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4} + (0) \\
&= \frac{3}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4} \\
&= \frac{7x^2 - 3}{8x^3 - 8x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4}\right)(0) + \left(\left(-\frac{3}{8x^2} - \frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2}\right) + \left(\frac{3}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right) dx} \\
&= x^{3/8} (x + 1)^{1/4} (x - 1)^{1/4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-26x^3+2x}{-8x^4+8x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{8}-\frac{3\ln(x+1)}{4}-\frac{3\ln(x-1)}{4}} \\
&= z_1 \left(\frac{1}{x^{1/8}\,(x+1)^{3/4}\,(x-1)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{1/4}(x^2-1)^{1/4}}{(x+1)^{3/4}\,(x-1)^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-26x^3+2x}{-8x^4+8x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{4}-\frac{3\ln(x+1)}{2}-\frac{3\ln(x-1)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{4}-\frac{3\ln(x+1)}{2}-\frac{3\ln(x-1)}{2}}\,(x+1)^{3/2}\,(x-1)^{3/2}}{\sqrt{x}\,\sqrt{x^2-1}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/4}(x^2-1)^{1/4}}{(x+1)^{3/4}\,(x-1)^{3/4}}\right) + c_2\left(\frac{x^{1/4}(x^2-1)^{1/4}}{(x+1)^{3/4}\,(x-1)^{3/4}}\left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{4}-\frac{3\ln(x+1)}{2}-\frac{3\ln(x-1)}{2}}\,(x+1)^{3/2}\,(x-1)^{3/2}}{\sqrt{x}\,\sqrt{x^2-1}}d
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.99.2 Maple step by step solution

Let's solve

$8x^2(-x^2+1)\left(\frac{d}{dx}y'\right)+2x(-13x^2+1)\,y'+(-9x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(9x^2-1)y}{8x^2(x^2-1)}-\frac{(13x^2-1)y'}{4x(x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(13x^2-1)y'}{4x(x^2-1)}+\frac{(9x^2-1)y}{8x^2(x^2-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{13x^2-1}{4x(x^2-1)},P_3(x)=\frac{9x^2-1}{8x^2(x^2-1)}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=\frac{3}{2}$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$8x^2(x^2-1)\left(\frac{d}{dx}y'\right)+2x(13x^2-1)\,y'+(9x^2-1)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(8u^4-32u^3+40u^2-16u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(26u^3-78u^2+76u-24)\left(\frac{d}{du}y(u)\right)+(9u^2-18u+8)\,y$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..4$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-8a_0r(1+2r)\,u^{-1+r} + (-8a_1(1+r)\,(3+2r) + 4a_0(1+2r)\,(2+5r))\,u^r + (-8a_2(2+r)\,(5+2r)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$-8r(1+2r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{0, -\frac{1}{2}\right\}$$
- The coefficients of each power of u must be 0
$$[-8a_1(1+r)\,(3+2r) + 4a_0(1+2r)\,(2+5r) = 0, -8a_2(2+r)\,(5+2r) + 4a_1(3+2r)\,(7+5r) -$$
- Solve for the dependent coefficient(s)
$$\left\{a_1 = \frac{a_0\,(10r^2+9r+2)}{2(2r^2+5r+3)}, a_2 = \frac{a_0\,(34r^3+76r^2+41r+5)}{4(2r^3+11r^2+19r+10)}\right\}$$
- Each term in the series must be 0, giving the recursion relation
$$8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\,k^2 + 2(8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\,r + 18a_k - 7a_{k-2} + 9a_{k-1}$$
- Shift index using $k- >k+2$
$$8(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\,(k+2)^2 + 2(8(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\,r + 18a_{k+2} - 7a_k + 9$$
- Recursion relation that defines series solution to ODE
$$a_{k+3} = \frac{8k^2a_k - 32k^2a_{k+1} + 40k^2a_{k+2} + 16kra_k - 64kra_{k+1} + 80kra_{k+2} + 8r^2a_k - 32r^2a_{k+1} + 40r^2a_{k+2} + 18ka_k - 110ka_{k+1} + 196ka_{k+}}{8(2k^2+4kr+2r^2+13k+13r+21)}$$
- Recursion relation for $r = 0$
$$a_{k+3} = \frac{8k^2a_k - 32k^2a_{k+1} + 40k^2a_{k+2} + 18ka_k - 110ka_{k+1} + 196ka_{k+2} + 9a_k - 96a_{k+1} + 240a_{k+2}}{8(2k^2+13k+21)}$$
- Solution for $r = 0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)}, a_1=\frac{a_0}{3}\right.$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)}, a_1=\right.$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}, a_1=\right.$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{1}{2}}, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}, a_1\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k(x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{1}{2}}\right), a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka}{8(2k^2+13k+21)}\right.$$

1.99.3 Maple trace

```
Methods for second order ODEs:
```

1.99.4 Maple dsolve solution

Solving time : `0.026` (sec)
Leaf size : 34

```
dsolve(8*x^2*(-x^2+1)*diff(diff(y(x),x),x)+2*x*(-13*x^2+1)*diff(y(x),x)+(-9*x^2+1)*y(x
       y(x),singsol=all)
```

$$y=\frac{x^{1/4}\left(\text{LegendreQ}\left(-\frac{1}{8},\frac{1}{8},\sqrt{-x^2+1}\right)c_2\,x^{1/8}+c_1\right)}{\sqrt{x^2-1}}$$

1.99.5 Mathematica DSolve solution

Solving time : `0.144` (sec)
Leaf size : 47

```
DSolve[{8*x^2*(1-x^2)*D[y[x],{x,2}]+2*x*(1-13*x^2)*D[y[x],x]+(1-9*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[4]{x}\left(4c_2\sqrt[4]{x}\,\text{Hypergeometric2F1}\left(\frac{1}{8},\frac{1}{2},\frac{9}{8},x^2\right)+c_1\right)}{\sqrt{1-x^2}}$$

1.100 problem 102

Internal problem ID [8238]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 102
Date solved : `Monday, October 21, 2024 at 05:04:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+1)\,y''-2x(-x^2+2)\,y'+4y=0$$

1.100.1 Solved as second order ode using Kovacic algorithm

Time used: 0.400 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(2x^3-4x)\,y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2\\ B &= 2x^3-4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-x^2 + 2}{(x^3 + x)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -x^2 + 2 \\
t &= (x^3 + x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-x^2 + 2}{(x^3 + x)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 184: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 6 - 2\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{7i}{4(x-i)} - \frac{7i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^2+2}{(x^3+x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}+\alpha_{c_3}^{-}\right) \\
&= 1-(1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{2}{x} - \frac{1}{2\,(x-i)} - \frac{1}{2\,(x+i)} + (-)\,(0) \\
&= \frac{2}{x} - \frac{1}{2\,(x-i)} - \frac{1}{2\,(x+i)} \\
&= \frac{x^2+2}{x^3+x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{x} - \frac{1}{2\,(x-i)} - \frac{1}{2\,(x+i)}\right)(0) + \left(\left(-\frac{2}{x^2} + \frac{1}{2\,(x-i)^2} + \frac{1}{2\,(x+i)^2}\right) + \left(\frac{2}{x} - \frac{1}{2\,(x-i)} - \frac{1}{2\,(x+\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right) dx} \\
&= \frac{x^2}{\sqrt{x^2+1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^3-4x}{x^4+x^2}\, dx} \\ &= z_1 e^{-\frac{3\ln\left(x^2+1\right)}{2}+2\ln(x)} \\ &= z_1\left(\frac{x^2}{\left(x^2+1\right)^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^4}{\left(x^2+1\right)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-4x}{x^4+x^2}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1 \int \frac{e^{-3\ln\left(x^2+1\right)+4\ln(x)}}{\left(y_1\right)^2}\, dx \\ &= y_1\left(-\frac{\left(3x^2+1\right)\left(x^2+1\right)^3 \mathrm{e}^{-3\ln\left(x^2+1\right)+4\ln(x)}}{3x^7}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{x^4}{\left(x^2+1\right)^2}\right) + c_2\left(\frac{x^4}{\left(x^2+1\right)^2}\left(-\frac{\left(3x^2+1\right)\left(x^2+1\right)^3 \mathrm{e}^{-3\ln\left(x^2+1\right)+4\ln(x)}}{3x^7}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.100.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)-2x(-x^2+2)\,y'+4y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{4y}{x^2(x^2+1)}-\frac{2(x^2-2)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{2(x^2-2)y'}{x(x^2+1)}+\frac{4y}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

  $\left[P_2(x)=\frac{2(x^2-2)}{x(x^2+1)}, P_3(x)=\frac{4}{x^2(x^2+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-4$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=4$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)+2x(x^2-2)\,y'+4y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y'$ to series expansion for $m=1..3$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=2..4$

  $$x^m\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)\,(-4+r)\,x^r+a_1r(-3+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r-1)\,(k+r-4)+a_{k-2}(k-2+r)\,(\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+r)\,(-4+r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\{1,4\}$$

- Each term must be 0

  $$a_1r(-3+r)=0$$

- Solve for the dependent coefficient(s)

  $$a_1=0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-1)\,(a_k(k+r-4)+a_{k-2}(k-2+r))=0$$

- Shift index using $k->k+2$

  $$(k+r+1)\,(a_{k+2}(k-2+r)+a_k(k+r))=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{a_k(k+r)}{k-2+r}$$

- Recursion relation for $r=1$

  $$a_{k+2}=-\frac{a_k(k+1)}{k-1}$$

- Solution for $r=1$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^{k+1},a_{k+2}=-\frac{a_k(k+1)}{k-1},a_1=0\right]$$

- Recursion relation for $r=4$

  $$a_{k+2}=-\frac{a_k(k+4)}{k+2}$$

- Solution for $r=4$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^{k+4},a_{k+2}=-\frac{a_k(k+4)}{k+2},a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+4}\right), a_{k+2} = -\frac{a_k(k+1)}{k-1}, a_1 = 0, b_{k+2} = -\frac{b_k(k+4)}{k+2}, b_1 = 0\right]$$

1.100.3 Maple trace

```
Methods for second order ODEs:
```

1.100.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 26

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-2*x*(-x^2+2)*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x(c_1 x^3 + 3c_2 x^2 + c_2)}{(x^2+1)^2}$$

1.100.5 Mathematica DSolve solution

Solving time : 0.084 (sec)
Leaf size : 35

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-2*x*(2-x^2)*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{-3c_1 x^4 + 3c_2 x^3 + c_2 x}{3(x^2+1)^2}$$

1.101 problem 103

Internal problem ID [8239]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 103
Date solved : `Monday, October 21, 2024 at 05:04:06 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x(x^2+3)\,y''+(-x^2+2)\,y'-8xy=0$$

1.101.1 Solved as second order ode using Kovacic algorithm

Time used: 0.348 (sec)

Writing the ode as

$$(x^3+3x)\,y''+(-x^2+2)\,y'-8xy=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3+3x \\ B &= -x^2+2 \\ C &= -8x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35x^4 + 74x^2 - 8$$
$$t = 4\left(x^3 + 3x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 186: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9x^2} + \frac{85}{144\left(x - i\sqrt{3}\right)^2} + \frac{85}{144\left(x + i\sqrt{3}\right)^2} - \frac{187i\sqrt{3}}{144\left(x - i\sqrt{3}\right)} + \frac{187i\sqrt{3}}{144\left(x + i\sqrt{3}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{85}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{12} \end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{3}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{85}{144}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{12} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$
$i\sqrt{3}$	2	0	$\frac{17}{12}$	$-\frac{5}{12}$
$-i\sqrt{3}$	2	0	$\frac{17}{12}$	$-\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{7}{2} - \left(\frac{7}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)} + (0) \\
&= \frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)} \\
&= \frac{2}{3x} + \frac{17x}{6x^2+18}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)}\right)(0) + \left(\left(-\frac{2}{3x^2} - \frac{17}{12\left(x-i\sqrt{3}\right)^2} - \frac{17}{12\left(x+i\sqrt{3}\right)^2}\right) + \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{3x} + \frac{17}{12\left(x - i\sqrt{3}\right)} + \frac{17}{12\left(x + i\sqrt{3}\right)}\right) dx} \\
&= x^{2/3} \left(x^2 + 3\right)^{17/12}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2+2}{x^3+3x} \, dx} \\
&= z_1 e^{\frac{5 \ln\left(x^2+3\right)}{12} - \frac{\ln(x)}{3}} \\
&= z_1 \left(\frac{\left(x^2+3\right)^{5/12}}{x^{1/3}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \left(x^2 + 3\right)^{11/6} x^{1/3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^3+3x} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{5 \ln\left(x^2+3\right)}{6} - \frac{2 \ln(x)}{3}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(-\frac{x^{1/3} (8x^4 + 44x^2 + 55) \, \mathrm{e}^{\frac{5 \ln\left(x^2+3\right)}{6} - \frac{2 \ln(x)}{3}}}{55 \left(x^2+3\right)^{8/3}}\right)
\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\left(x^2+3\right)^{11/6} x^{1/3}\right)$$
$$+ c_2\left(\left(x^2+3\right)^{11/6} x^{1/3}\left(-\frac{x^{1/3}(8x^4+44x^2+55)\,\mathrm{e}^{\frac{5\ln\left(x^2+3\right)}{6}-\frac{2\ln(x)}{3}}}{55\left(x^2+3\right)^{8/3}}\right)\right)$$

Will add steps showing solving for IC soon.

1.101.2 Maple step by step solution

Let's solve

$x(x^2+3)\left(\frac{d}{dx}y'\right) + (-x^2+2)\,y' - 8xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{8y}{x^2+3} + \frac{(x^2-2)y'}{x(x^2+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x^2-2)y'}{x(x^2+3)} - \frac{8y}{x^2+3} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x^2-2}{x(x^2+3)}, P_3(x) = -\frac{8}{x^2+3}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{2}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x^2+3)\left(\frac{d}{dx}y'\right) + (-x^2+2)\,y' - 8xy = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

◦ Convert $x \cdot y$ to series expansion

$$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

◦ Shift index using $k->k-1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

◦ Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

◦ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

◦ Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+3r)\, x^{-1+r} + a_1(1+r)\,(2+3r)\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)\,(3k+2+3r) + a_{k-1}(k+r-$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-1+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{1}{3}\right\}$
- Each term must be 0
  $a_1(1+r)\,(2+3r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r+1)\left(a_{k-1}(k-5+r) + 3a_{k+1}\left(k+\frac{2}{3}+r\right)\right) = 0$
- Shift index using $k->k+1$
  $(k+r+2)\left(a_k(k+r-4) + 3a_{k+2}\left(k+\frac{5}{3}+r\right)\right) = 0$
- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k(k+r-4)}{3k+5+3r}$

- Recursion relation for $r = 0$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k(k-4)}{3k+5}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k-4)}{3k+5}, 2a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{a_k\left(k-\frac{11}{3}\right)}{3k+6}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k\left(k-\frac{11}{3}\right)}{3k+6}, 4a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{a_k(k-4)}{3k+5}, 2a_1 = 0, b_{k+2} = -\frac{b_k\left(k-\frac{11}{3}\right)}{3k+6}, 4b_1 = 0\right]$

1.101.3 Maple trace

```
Methods for second order ODEs:
```

1.101.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 32

```
dsolve(x*(x^2+3)*diff(diff(y(x),x),x)+(-x^2+2)*diff(y(x),x)-8*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \left(x^2 + 3\right)^{11/6} x^{1/3} + \frac{c_2(8x^4 + 44x^2 + 55)}{8}$$

1.101.5 Mathematica DSolve solution

Solving time : `0.196` (sec)
Leaf size : 41

```
DSolve[{x*(3+x^2)*D[y[x],{x,2}]+(2-x^2)*D[y[x],x]-8*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(x^2+3\right)^{11/6} - \frac{1}{55} c_2 \left(8x^4+44x^2+55\right)$$

1.102 problem 104

Internal problem ID [8240]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 104
Date solved : `Monday, October 21, 2024 at 05:04:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(-x^2+1\right)y''+x\left(-19x^2+7\right)y'-\left(14x^2+1\right)y=0$$

1.102.1 Solved as second order ode using Kovacic algorithm

Time used: 0.392 (sec)

Writing the ode as

$$\left(-4x^4+4x^2\right)y''+\left(-19x^3+7x\right)y'+\left(-14x^2-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-4x^4+4x^2\\B&=-19x^3+7x\\C&=-14x^2-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-15x^4 - 42x^2 + 9}{64\left(x^3 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -15x^4 - 42x^2 + 9$$
$$t = 64\left(x^3 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-15x^4 - 42x^2 + 9}{64\left(x^3 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 188: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(x+1)^2} - \frac{3}{16(x-1)^2} + \frac{9}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-15x^4 - 42x^2 + 9}{64\left(x^3 - x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{15}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-15x^4 - 42x^2 + 9}{64\left(x^3 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{8}$	$-\frac{1}{8}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{8}$	$\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{3}{8} - \left(\frac{3}{8}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} + (-)\left(0\right) \\ &= -\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} \\ &= \frac{3x^2+1}{8x^3-8x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right)(0) + \left(\left(\frac{1}{8x^2} - \frac{1}{4\left(x-1\right)^2} - \frac{1}{4\left(x+1\right)^2}\right) + \left(-\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right) dx} \\ &= \frac{\left(x+1\right)^{1/4} \left(x-1\right)^{1/4}}{x^{1/8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-19x^3+7x}{-4x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{7\ln(x)}{8}-\frac{3\ln(x+1)}{4}-\frac{3\ln(x-1)}{4}} \\
&= z_1\left(\frac{1}{x^{7/8}(x+1)^{3/4}(x-1)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x^2-1)^{1/4}}{x(x+1)^{3/4}(x-1)^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-19x^3+7x}{-4x^4+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(x)}{4}-\frac{3\ln(x+1)}{2}-\frac{3\ln(x-1)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{7\ln(x)}{4}-\frac{3\ln(x+1)}{2}-\frac{3\ln(x-1)}{2}}x^2(x+1)^{3/2}(x-1)^{3/2}}{\sqrt{x^2-1}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{(x^2-1)^{1/4}}{x(x+1)^{3/4}(x-1)^{3/4}}\right) + c_2\left(\frac{(x^2-1)^{1/4}}{x(x+1)^{3/4}(x-1)^{3/4}}\left(\int \frac{\mathrm{e}^{-\frac{7\ln(x)}{4}-\frac{3\ln(x+1)}{2}-\frac{3\ln(x-1)}{2}}x^2(x+1)^{3/2}(x-}{\sqrt{x^2-1}}\right.\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.102.2 Maple step by step solution

Let's solve

$4x^2(-x^2+1)\left(\frac{d}{dx}y'\right)+x(-19x^2+7)\,y'-(14x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(14x^2+1)y}{4x^2(x^2-1)}-\frac{(19x^2-7)y'}{4x(x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(19x^2-7)y'}{4x(x^2-1)}+\frac{(14x^2+1)y}{4x^2(x^2-1)}=0$

☐ Check to see if x_0 is a regular singular point

- ∘ Define functions

$\left[P_2(x)=\frac{19x^2-7}{4x(x^2-1)},P_3(x)=\frac{14x^2+1}{4x^2(x^2-1)}\right]$

- ∘ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=\frac{3}{2}$

- ∘ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

- ∘ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$4x^2(x^2-1)\left(\frac{d}{dx}y'\right)+x(19x^2-7)\,y'+(14x^2+1)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(4u^4-16u^3+20u^2-8u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(19u^3-57u^2+50u-12)\left(\frac{d}{du}y(u)\right)+(14u^2-28u+15$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ∘ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ∘ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..4$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-4a_0 r(1+2r)\, u^{-1+r} + \left(-4a_1(1+r)\,(3+2r) + 5a_0(4r^2+6r+3)\right) u^r + \left(-4a_2(2+r)\,(5+2r) +\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$-4r(1+2r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{0, -\frac{1}{2}\right\}$$
- The coefficients of each power of u must be 0
$$[-4a_1(1+r)\,(3+2r) + 5a_0(4r^2+6r+3) = 0, -4a_2(2+r)\,(5+2r) + 5a_1(4r^2+14r+13) - a_0$$
- Solve for the dependent coefficient(s)
$$\left\{a_1 = \frac{5a_0(4r^2+6r+3)}{4(2r^2+5r+3)}, a_2 = \frac{a_0(272r^4+1352r^3+2464r^2+1948r+639)}{16(4r^4+28r^3+71r^2+77r+30)}\right\}$$
- Each term in the series must be 0, giving the recursion relation
$$4(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, k^2 + (8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, r + 30a_k - a_{k-2} - 9a_{k-1} -$$
- Shift index using $k- >k+2$
$$4(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\,(k+2)^2 + (8(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\, r + 30a_{k+2} - a_k - 9a_k$$
- Recursion relation that defines series solution to ODE
$$a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 8kra_k - 32kra_{k+1} + 40kra_{k+2} + 4r^2a_k - 16r^2a_{k+1} + 20r^2a_{k+2} + 15ka_k - 73ka_{k+1} + 110ka_{k+2}}{4(2k^2+4kr+2r^2+13k+13r+21)}$$
- Recursion relation for $r = 0$
$$a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 15ka_k - 73ka_{k+1} + 110ka_{k+2} + 14a_k - 85a_{k+1} + 155a_{k+2}}{4(2k^2+13k+21)}$$
- Solution for $r = 0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+3}=\frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}+14a_k-85a_{k+1}+155a_{k+2}}{4(2k^2+13k+21)}, a_1=\frac{5a}{4}\right.$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3}=\frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}+14a_k-85a_{k+1}+155a_{k+2}}{4(2k^2+13k+21)}, a_1=\right.$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+3}=\frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+2}}{4(2k^2+11k+15)}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+3}=\frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+2}}{4(2k^2+11k+15)}, a_1=\right.$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{1}{2}}, a_{k+3}=\frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+2}}{4(2k^2+11k+15)}, a\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k(x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{1}{2}}\right), a_{k+3}=\frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka}{4(2k^2+13k+21)}\right.$$

1.102.3 Maple trace

```
Methods for second order ODEs:
```

1.102.4 Maple dsolve solution

Solving time : `0.027` (sec)
Leaf size : 44

```
dsolve(4*x^2*(-x^2+1)*diff(diff(y(x),x),x)+x*(-19*x^2+7)*diff(y(x),x)-(14*x^2+1)*y(x)
       y(x),singsol=all)
```

$$y=\frac{c_1 \operatorname{LegendreP}\left(-\frac{3}{8}, \frac{5}{8}, \sqrt{-x^2+1}\right)+c_2 \operatorname{LegendreQ}\left(-\frac{3}{8}, \frac{5}{8}, \sqrt{-x^2+1}\right)}{x^{3/8}\sqrt{x^2-1}}$$

1.102.5 Mathematica DSolve solution

Solving time : 0.136 (sec)
Leaf size : 50

```
DSolve[{4*x^2*(1-x^2)*D[y[x],{x,2}]+x*(7-19*x^2)*D[y[x],x]-(1+14*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_2x^{5/4}\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{5}{8},\frac{13}{8},x^2\right)+5c_1}{5x\sqrt{1-x^2}}$$

1.103 problem 105

Internal problem ID [8241]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 105
Date solved : `Monday, October 21, 2024 at 05:04:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2\left(-x^2+2\right)y''+x\left(-11x^2+1\right)y'+\left(-5x^2+1\right)y=0$$

1.103.1 Solved as second order ode using Kovacic algorithm

Time used: 0.431 (sec)

Writing the ode as

$$\left(-3x^4+6x^2\right)y''+\left(-11x^3+x\right)y'+\left(-5x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -3x^4+6x^2 \\ B &= -11x^3+x \\ C &= -5x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5x^4 - 4x^2 - 35}{36\left(x^3 - 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5x^4 - 4x^2 - 35$$
$$t = 36\left(x^3 - 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-5x^4 - 4x^2 - 35}{36\left(x^3 - 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 190: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144x^2} - \frac{7}{64\left(x - \sqrt{2}\right)^2} - \frac{7}{64\left(x + \sqrt{2}\right)^2} + \frac{31\sqrt{2}}{384\left(x - \sqrt{2}\right)} - \frac{31\sqrt{2}}{384\left(x + \sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-5x^4 - 4x^2 - 35}{36\left(x^3 - 2x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-5x^4 - 4x^2 - 35}{36\left(x^3 - 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$
$\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{5}{6} - \left(\frac{5}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{7}{12x} + \frac{1}{8x - 8\sqrt{2}} + \frac{1}{8x + 8\sqrt{2}} + (0) \\ &= \frac{7}{12x} + \frac{1}{8x - 8\sqrt{2}} + \frac{1}{8x + 8\sqrt{2}} \\ &= \frac{5x^2 - 7}{6x^3 - 12x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{12x} + \frac{1}{8x - 8\sqrt{2}} + \frac{1}{8x + 8\sqrt{2}}\right)(0) + \left(\left(-\frac{7}{12x^2} - \frac{1}{8\left(x - \sqrt{2}\right)^2} - \frac{1}{8\left(x + \sqrt{2}\right)^2}\right) + \left(\frac{7}{12x} + \frac{}{8x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{7}{12x} + \frac{1}{8x - 8\sqrt{2}} + \frac{1}{8x + 8\sqrt{2}}\right) dx} \\ &= x^{7/12} \left(x + \sqrt{2}\right)^{1/8} \left(x - \sqrt{2}\right)^{1/8} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-11x^3+x}{-3x^4+6x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{12}-\frac{7\ln\left(x^2-2\right)}{8}} \\
&= z_1 \left(\frac{1}{x^{1/12}\left(x^2-2\right)^{7/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x}}{\left(x^2-2\right)^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-11x^3+x}{-3x^4+6x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{6}-\frac{7\ln\left(x^2-2\right)}{4}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{6}-\frac{7\ln\left(x^2-2\right)}{4}}\left(x^2-2\right)^{3/2}}{x} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{\left(x^2-2\right)^{3/4}}\right) + c_2 \left(\frac{\sqrt{x}}{\left(x^2-2\right)^{3/4}}\left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{6}-\frac{7\ln\left(x^2-2\right)}{4}}\left(x^2-2\right)^{3/2}}{x} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.103.2 Maple step by step solution

Let's solve

$3x^2(-x^2+2)\left(\frac{d}{dx}y'\right)+x(-11x^2+1)\,y'+(-5x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(5x^2-1)y}{3x^2(x^2-2)}-\frac{(11x^2-1)y'}{3x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(11x^2-1)y'}{3x(x^2-2)}+\frac{(5x^2-1)y}{3x^2(x^2-2)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{11x^2-1}{3x(x^2-2)},P_3(x)=\frac{5x^2-1}{3x^2(x^2-2)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{6}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{6}$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$3x^2(x^2-2)\left(\frac{d}{dx}y'\right)+x(11x^2-1)\,y'+(5x^2-1)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+3r)\,(-1+2r)\,x^r - a_1(2+3r)\,(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (-a_k(3k+3r-1)\,(2k+2r-1)+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$-(-1+3r)\,(-1+2r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$$
- Each term must be 0
$$-a_1(2+3r)\,(1+2r) = 0$$
- Solve for the dependent coefficient(s)
$$a_1 = 0$$
- Each term in the series must be 0, giving the recursion relation
$$-6\left(k+r-\frac{1}{3}\right)\left(\frac{(-k-r+1)a_{k-2}}{2} + a_k\left(k+r-\frac{1}{2}\right)\right) = 0$$
- Shift index using $k- > k+2$
$$-6\left(k+\frac{5}{3}+r\right)\left(\frac{(-k-1-r)a_k}{2} + a_{k+2}\left(k+\frac{3}{2}+r\right)\right) = 0$$
- Recursion relation that defines series solution to ODE
$$a_{k+2} = \frac{(k+r+1)a_k}{2k+3+2r}$$
- Recursion relation for $r = \frac{1}{2}$
$$a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}$$
- Solution for $r = \frac{1}{2}$
$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = \frac{\left(k+\frac{4}{3}\right)a_k}{2k+\frac{11}{3}}$
- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{\left(k+\frac{4}{3}\right)a_k}{2k+\frac{11}{3}}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}, a_1 = 0, b_{k+2} = \frac{\left(k+\frac{4}{3}\right)b_k}{2k+\frac{11}{3}}, b_1 = 0\right]$

1.103.3 Maple trace

```
Methods for second order ODEs:
```

1.103.4 Maple dsolve solution

Solving time : `0.021` (sec)
Leaf size : 35

```
dsolve(3*x^2*(-x^2+2)*diff(diff(y(x),x),x)+x*(-11*x^2+1)*diff(y(x),x)+(-5*x^2+1)*y(x)
       y(x),singsol=all)
```

$$y = \frac{c_1 \sqrt{x}}{\left(-2x^2+4\right)^{3/4}} + c_2 \, x^{1/3} \operatorname{hypergeom}\left(\left[\frac{2}{3}, 1\right], \left[\frac{11}{12}\right], \frac{x^2}{2}\right)$$

1.103.5 Mathematica DSolve solution

Solving time : `0.168` (sec)
Leaf size : 57

```
DSolve[{3*x^2*(2-x^2)*D[y[x],{x,2}]+x*(1-11*x^2)*D[y[x],x]+(1-5*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1 \sqrt{x} - 3\ 2^{3/4} c_2 \sqrt[3]{x} \operatorname{Hypergeometric2F1}\left(-\frac{1}{12}, \frac{1}{4}, \frac{11}{12}, \frac{x^2}{2}\right)}{\left(2-x^2\right)^{3/4}}$$

1.104 problem 106

Internal problem ID [8242]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 106
Date solved : `Monday, October 21, 2024 at 05:04:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+2\right)y''-x\left(-7x^2+12\right)y'+\left(3x^2+7\right)y=0$$

1.104.1 Solved as second order ode using Kovacic algorithm

Time used: 0.448 (sec)

Writing the ode as

$$\left(2x^4+4x^2\right)y''+\left(7x^3-12x\right)y'+\left(3x^2+7\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+4x^2 \\ B &= 7x^3-12x \\ C &= 3x^2+7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^4 - 72x^2 + 128$$
$$t = 16\left(x^3 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 192: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + \frac{65}{64\left(x - i\sqrt{2}\right)^2} + \frac{65}{64\left(x + i\sqrt{2}\right)^2} + \frac{135i\sqrt{2}}{128\left(x - i\sqrt{2}\right)} - \frac{135i\sqrt{2}}{128\left(x + i\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{8} \end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{8} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1
$i\sqrt{2}$	2	0	$\frac{13}{8}$	$-\frac{5}{8}$
$-i\sqrt{2}$	2	0	$\frac{13}{8}$	$-\frac{5}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{2}{x} - \frac{5}{8\left(x-i\sqrt{2}\right)} - \frac{5}{8\left(x+i\sqrt{2}\right)} + (0) \\
&= \frac{2}{x} - \frac{5}{8\left(x-i\sqrt{2}\right)} - \frac{5}{8\left(x+i\sqrt{2}\right)} \\
&= \frac{2}{x} - \frac{5x}{4x^2+8}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{2}{x} - \frac{5}{8\left(x-i\sqrt{2}\right)} - \frac{5}{8\left(x+i\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{2}{x^2} + \frac{5}{8\left(x-i\sqrt{2}\right)^2} + \frac{5}{8\left(x+i\sqrt{2}\right)^2}\right) + \left(\frac{2}{x} - \frac{}{8(}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{x} - \frac{5}{8\left(x - i\sqrt{2}\right)} - \frac{5}{8\left(x + i\sqrt{2}\right)}\right) dx} \\
&= \frac{x^2}{\left(x^2+2\right)^{5/8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^3-12x}{2x^4+4x^2} \, dx} \\
&= z_1 e^{\frac{3\ln(x)}{2} - \frac{13\ln\left(x^2+2\right)}{8}} \\
&= z_1 \left(\frac{x^{3/2}}{\left(x^2+2\right)^{13/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{7/2}}{\left(x^2+2\right)^{9/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3-12x}{2x^4+4x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{3\ln(x) - \frac{13\ln\left(x^2+2\right)}{4}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{3\ln(x) - \frac{13\ln\left(x^2+2\right)}{4}} \left(x^2+2\right)^{9/2}}{x^7} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{7/2}}{(x^2+2)^{9/4}} \right) + c_2 \left(\frac{x^{7/2}}{(x^2+2)^{9/4}} \left(\int \frac{e^{3\ln(x) - \frac{13\ln(x^2+2)}{4}} (x^2+2)^{9/2}}{x^7} dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.104.2 Maple step by step solution

Let's solve
$2x^2(x^2+2)\left(\frac{d}{dx}y'\right) - x(-7x^2+12)\,y' + (3x^2+7)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{(3x^2+7)y}{2x^2(x^2+2)} - \frac{(7x^2-12)y'}{2x(x^2+2)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + \frac{(7x^2-12)y'}{2x(x^2+2)} + \frac{(3x^2+7)y}{2x^2(x^2+2)} = 0$
- □ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{7x^2-12}{2x(x^2+2)}, P_3(x) = \frac{3x^2+7}{2x^2(x^2+2)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $(x \cdot P_2(x))\bigg|_{x=0} = -3$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{7}{4}$
  - $x = 0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $2x^2(x^2+2)\left(\frac{d}{dx}y'\right) + x(7x^2-12)\,y' + (3x^2+7)\,y = 0$
- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1 + 2r)\,(-7 + 2r)\, x^r + a_1(1 + 2r)\,(-5 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r - 1)\,(2k + 2r - 7) + a$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1 + 2r)\,(-7 + 2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{\frac{1}{2}, \frac{7}{2}\right\}$
- Each term must be 0
  $a_1(1 + 2r)\,(-5 + 2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $4\left(\frac{a_{k-2}(k+r-1)}{2} + a_k\left(k + r - \frac{7}{2}\right)\right)\left(k + r - \frac{1}{2}\right) = 0$
- Shift index using $k- >k + 2$

$$4\left(\frac{a_k(k+r+1)}{2}+a_{k+2}\left(k-\frac{3}{2}+r\right)\right)\left(k+\frac{3}{2}+r\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a_k(k+r+1)}{2k-3+2r}$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}},a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2},a_1=0\right]$$

- Recursion relation for $r=\frac{7}{2}$

$$a_{k+2}=-\frac{a_k\left(k+\frac{9}{2}\right)}{2k+4}$$

- Solution for $r=\frac{7}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{7}{2}},a_{k+2}=-\frac{a_k\left(k+\frac{9}{2}\right)}{2k+4},a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{7}{2}}\right),a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2},a_1=0,b_{k+2}=-\frac{b_k\left(k+\frac{9}{2}\right)}{2k+4},b_1=0\right]$$

1.104.3 Maple trace

```
Methods for second order ODEs:
```

1.104.4 Maple dsolve solution

Solving time : `0.026` (sec)
Leaf size : 35

```
dsolve(2*x^2*(x^2+2)*diff(diff(y(x),x),x)-x*(-7*x^2+12)*diff(y(x),x)+(3*x^2+7)*y(x) =
       y(x),singsol=all)
```

$$y=\frac{c_1\,x^{7/2}}{\left(2x^2+4\right)^{9/4}}+c_2\sqrt{x}\ \text{hypergeom}\left(\left[\frac{3}{4},1\right],\left[-\frac{1}{2}\right],-\frac{x^2}{2}\right)$$

1.104.5 Mathematica DSolve solution

Solving time : 0.166 (sec)
Leaf size : 57

```
DSolve[{2*x^2*(2+x^2)*D[y[x],{x,2}]-x*(12-7*x^2)*D[y[x],x]+(7+3*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}\left(3c_1x^3 - 2\sqrt[4]{2}c_2 \,\text{Hypergeometric2F1}\left(-\frac{3}{2}, -\frac{5}{4}, -\frac{1}{2}, -\frac{x^2}{2}\right)\right)}{3\left(x^2+2\right)^{9/4}}$$

1.105 problem 107

Internal problem ID [8243]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 107
Date solved : `Monday, October 21, 2024 at 05:04:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(x^2+2)\,y''+x(7x^2+4)\,y'-(-3x^2+1)\,y=0$$

1.105.1 Solved as second order ode using Kovacic algorithm

Time used: 0.378 (sec)

Writing the ode as

$$(2x^4+4x^2)\,y''+(7x^3+4x)\,y'+(3x^2-1)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+4x^2 \\ B &= 7x^3+4x \\ C &= 3x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 + 24}{16\left(x^2 + 2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 + 24$$
$$t = 16\left(x^2 + 2\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 + 24}{16\left(x^2 + 2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 194: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+2)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{15}{64\left(x - i\sqrt{2}\right)^2} - \frac{15}{64\left(x + i\sqrt{2}\right)^2} - \frac{9i\sqrt{2}}{128\left(x - i\sqrt{2}\right)} + \frac{9i\sqrt{2}}{128\left(x + i\sqrt{2}\right)}
$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}
$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 24}{16\left(x^2+2\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 + 24}{16\left(x^2+2\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$i\sqrt{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$
$-i\sqrt{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)} + (0) \\
&= \frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)} \\
&= \frac{3x}{4x^2 + 8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{3}{8\left(x - i\sqrt{2}\right)^2} - \frac{3}{8\left(x + i\sqrt{2}\right)^2}\right) + \left(\frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{}{8(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)} \right) dx} \\
&= \left(-x^2 - 2\right)^{3/8}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^3+4x}{2x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln\left(x^2+2\right)}{8}-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\left(x^2+2\right)^{5/8}\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(-1)^{3/8}}{\left(x^2+2\right)^{1/4}\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3+4x}{2x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(x^2+2\right)}{4}-\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int -\mathrm{e}^{-\frac{5\ln\left(x^2+2\right)}{4}-\ln(x)}\sqrt{x^2+2}\,x(-1)^{1/4}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(-1)^{3/8}}{\left(x^2+2\right)^{1/4}\sqrt{x}}\right) + c_2\left(\frac{(-1)^{3/8}}{\left(x^2+2\right)^{1/4}\sqrt{x}}\left(\int -\mathrm{e}^{-\frac{5\ln\left(x^2+2\right)}{4}-\ln(x)}\sqrt{x^2+2}\,x(-1)^{1/4}\,dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.105.2 Maple step by step solution

Let's solve

$2x^2(x^2+2)\left(\frac{d}{dx}y'\right)+x(7x^2+4)\,y'-(-3x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(3x^2-1)y}{2x^2(x^2+2)}-\frac{(7x^2+4)y'}{2x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(7x^2+4)y'}{2x(x^2+2)}+\frac{(3x^2-1)y}{2x^2(x^2+2)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{7x^2+4}{2x(x^2+2)},P_3(x)=\frac{3x^2-1}{2x^2(x^2+2)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=-\frac{1}{4}$

- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(x^2+2)\left(\frac{d}{dx}y'\right)+x(7x^2+4)\,y'+(3x^2-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ○ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\frac{a_{k-2}(k+r-1)}{2} + a_k\left(k+r+\frac{1}{2}\right)\right)\left(k+r-\frac{1}{2}\right) = 0$

- Shift index using $k->k+2$

  $4\left(\frac{a_k(k+r+1)}{2} + a_{k+2}\left(k+\frac{5}{2}+r\right)\right)\left(k+\frac{3}{2}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+1)}{2k+5+2r}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+6}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+6}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0, b_{k+2} = -\frac{b_k\left(k+\frac{3}{2}\right)}{2k+6}, b_1 = 0\right]$$

1.105.3 Maple trace

```
Methods for second order ODEs:
```

1.105.4 Maple dsolve solution

Solving time : `0.024` (sec)
Leaf size : 35

```
dsolve(2*x^2*(x^2+2)*diff(diff(y(x),x),x)+x*(7*x^2+4)*diff(y(x),x)-(-3*x^2+1)*y(x) = 0
       y(x),singsol=all)
```

$$y = \frac{c_2 \operatorname{LegendreQ}\left(-\frac{1}{4}, \frac{1}{4}, \frac{i\sqrt{2}\,x}{2}\right)\left(x^2+2\right)^{1/8} + c_1}{\left(x^2+2\right)^{1/4}\sqrt{x}}$$

1.105.5 Mathematica DSolve solution

Solving time : `0.108` (sec)
Leaf size : 68

```
DSolve[{2*x^2*(2+x^2)*D[y[x],{x,2}]+x*(4+7*x^2)*D[y[x],x]-(1-3*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 \sqrt[8]{x^2+2}\,\text{Gamma}\left(\frac{3}{4}\right) Q_{-\frac{1}{4}}^{\frac{1}{4}}\left(\frac{ix}{\sqrt{2}}\right) + 2^{3/8} c_1}{\sqrt{x}\sqrt[4]{x^2+2}\,\text{Gamma}\left(\frac{3}{4}\right)}$$

1.106 problem 108

Internal problem ID [8244]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 108
Date solved : `Monday, October 21, 2024 at 05:04:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(2x^2+1\right)y''+5x\left(6x^2+1\right)y'-\left(-40x^2+2\right)y=0$$

1.106.1 Solved as second order ode using Kovacic algorithm

Time used: 0.621 (sec)

Writing the ode as

$$\left(4x^4+2x^2\right)y''+\left(30x^3+5x\right)y'+\left(40x^2-2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+2x^2 \\ B &= 30x^3+5x \\ C &= 40x^2-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{20x^4 + 12x^2 + 21}{16\,(2x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 20x^4 + 12x^2 + 21$$
$$t = 16(2x^3 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{20x^4 + 12x^2 + 21}{16\,(2x^3 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 196: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16x^2} + \frac{5}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{5}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} + \frac{13i\sqrt{2}}{16\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{13i\sqrt{2}}{16\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4}\end{aligned}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}\end{aligned}$$

For the pole at $x=-\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{20x^4+12x^2+21}{16\left(2x^3+x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{20x^4+12x^2+21}{16\left(2x^3+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{5}{4} - \left(\frac{5}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{7}{4x} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)} + (0) \\ &= \frac{7}{4x} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)} \\ &= \frac{10x^2+7}{8x^3+4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{4x} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)}\right)(0) + \left(\left(-\frac{7}{4x^2} + \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)^2} + \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{7}{4x} - \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{7}{4x} - \frac{1}{4\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x + \frac{i\sqrt{2}}{2}\right)}\right) dx} \\
&= \frac{2^{3/4} x^{7/4}}{2\left(2x^2+1\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{30x^3+5x}{4x^4+2x^2} \, dx} \\
&= z_1 e^{-\frac{5\ln\left(x\left(2x^2+1\right)\right)}{4}} \\
&= z_1 \left(\frac{1}{\left(2x^3+x\right)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{3/4} x^{3/4}}{2\left(2x^2+1\right)^{5/4}\left(2x^3+x\right)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{30x^3+5x}{4x^4+2x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(2x^3+x\right)}{2}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{\left(2x^2+1\right)^{5/2}\sqrt{2}}{\left(2x^3+x\right)^2 x^{3/2}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{2^{3/4} x^{3/4}}{2 \left(2x^2+1\right)^{5/4} \left(2x^3+x\right)^{1/4}}\right) + c_2 \left(\frac{2^{3/4} x^{3/4}}{2 \left(2x^2+1\right)^{5/4} \left(2x^3+x\right)^{1/4}} \left(\int \frac{\left(2x^2+1\right)^{5/2} \sqrt{2}}{\left(2x^3+x\right)^2 x^{3/2}} dx\right)\right)$$

Will add steps showing solving for IC soon.

1.106.2 Maple step by step solution

Let's solve

$2x^2(2x^2+1)\left(\frac{d}{dx}y'\right) + 5x(6x^2+1)\,y' - (-40x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(20x^2-1)y}{x^2(2x^2+1)} - \frac{5(6x^2+1)y'}{2x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{5(6x^2+1)y'}{2x(2x^2+1)} + \frac{(20x^2-1)y}{x^2(2x^2+1)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{5(6x^2+1)}{2x(2x^2+1)}, P_3(x) = \frac{20x^2-1}{x^2(2x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x=0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -1$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2(2x^2+1)\left(\frac{d}{dx}y'\right) + 5x(6x^2+1)\,y' + (40x^2-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-1+2r)\, x^r + a_1(3+r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(2k+2r-1) + 2a_{k-2}(k -\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-2, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r+2)\left(a_{k-2}(2k+1+2r) + a_k\left(k+r-\frac{1}{2}\right)\right) = 0$

- Shift index using $k- >k + 2$

  $2(k+r+4)\left(a_k(2k+2r+5) + a_{k+2}\left(k+\frac{3}{2}+r\right)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k(2k+2r+5)}{2k+3+2r}$

- Recursion relation for $r = -2$

$a_{k+2} = -\frac{2a_k(2k+1)}{2k-1}$

- Solution for $r = -2$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2a_k(2k+1)}{2k-1}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+2} = -\frac{2a_k(2k+6)}{2k+4}$

- Solution for $r = \frac{1}{2}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{2a_k(2k+6)}{2k+4}, a_1 = 0\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{2a_k(2k+1)}{2k-1}, a_1 = 0, b_{k+2} = -\frac{2b_k(2k+6)}{2k+4}, b_1 = 0\right]$

1.106.3 Maple trace

```
Methods for second order ODEs:
```

1.106.4 Maple dsolve solution

Solving time : `0.024` (sec)
Leaf size : 35

```
dsolve(2*x^2*(2*x^2+1)*diff(diff(y(x),x),x)+5*x*(6*x^2+1)*diff(y(x),x)-(-40*x^2+2)*y(x
       y(x),singsol=all)
```

$$y = \frac{c_1\sqrt{x}}{(2x^2+1)^{3/2}} + \frac{c_2 \operatorname{hypergeom}\left(\left[\frac{1}{4}, 1\right], \left[-\frac{1}{4}\right], -2x^2\right)}{x^2}$$

1.106.5 Mathematica DSolve solution

Solving time : 0.175 (sec)
Leaf size : 52

```
DSolve[{2*x^2*(1+2*x^2)*D[y[x],{x,2}]+5*x*(1+6*x^2)*D[y[x],x]-(2-40*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{5c_1x^{5/2} - 2c_2 \,\text{Hypergeometric2F1}\left(-\frac{5}{4}, -\frac{1}{2}, -\frac{1}{4}, -2x^2\right)}{5x^2\left(2x^2+1\right)^{3/2}}$$

1.107 problem 109

Internal problem ID [8245]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 109
Date solved : `Monday, October 21, 2024 at 05:04:15 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x\left(x^2+1\right)y''+\left(7x^2+4\right)y'+8xy=0$$

1.107.1 Solved as second order ode using Kovacic algorithm

Time used: 0.336 (sec)

Writing the ode as

$$\left(x^3+x\right)y''+\left(7x^2+4\right)y'+8xy=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3+x \\ B &= 7x^2+4 \\ C &= 8x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^4 + 14x^2 + 8$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 198: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 4 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{3}{16\,(x-i)^2} - \frac{3}{16\,(x+i)^2} + \frac{7i}{16\,(x-i)} - \frac{7i}{16\,(x+i)} + \frac{2}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{4}\\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^4+14x^2+8}{4\left(x^3+x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^4+14x^2+8}{4\left(x^3+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1
i	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-i$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} + (-)\,(0) \\
&= -\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} \\
&= -\frac{1}{x} + \frac{x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)(0) + \left(\left(\frac{1}{x^2} - \frac{1}{4\,(x-i)^2} - \frac{1}{4\,(x+i)^2}\right) + \left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right) dx} \\
&= \frac{\left(x^2+1\right)^{1/4}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^2+4}{x^3+x}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+1\right)}{4}-2\ln(x)} \\
&= z_1\left(\frac{1}{\left(x^2+1\right)^{3/4} x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{\sqrt{x^2+1}\,x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2+4}{x^3+x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+1\right)}{2}-4\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{x^5}{\sqrt{x^2+1}} - \frac{x^3}{3\sqrt{x^2+1}} + \frac{4x^7}{\sqrt{x^2+1}} + \frac{8x^9}{3\sqrt{x^2+1}} + \frac{x\sqrt{x^2+1}}{2} - \frac{\operatorname{arcsinh}(x)}{2}\right. \\
&\quad \left. + \frac{\sqrt{x^2+1}\,x^3}{3} - \frac{4x^5\sqrt{x^2+1}}{3} - \frac{8x^7\sqrt{x^2+1}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{\sqrt{x^2+1}\,x^3}\right) + c_2\left(\frac{1}{\sqrt{x^2+1}\,x^3}\left(\frac{x^5}{\sqrt{x^2+1}} - \frac{x^3}{3\sqrt{x^2+1}} + \frac{4x^7}{\sqrt{x^2+1}} + \frac{8x^9}{3\sqrt{x^2+1}}\right.\right. \\
&\quad \left.\left. + \frac{x\sqrt{x^2+1}}{2} - \frac{\operatorname{arcsinh}(x)}{2} + \frac{\sqrt{x^2+1}\,x^3}{3} - \frac{4x^5\sqrt{x^2+1}}{3} - \frac{8x^7\sqrt{x^2+1}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.107.2 Maple step by step solution

Let's solve

$x(x^2+1)\left(\frac{d}{dx}y'\right)+(7x^2+4)\,y'+8xy=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{8y}{x^2+1}-\frac{(7x^2+4)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(7x^2+4)y'}{x(x^2+1)}+\frac{8y}{x^2+1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{7x^2+4}{x(x^2+1)},P_3(x)=\frac{8}{x^2+1}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=4$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x(x^2+1)\left(\frac{d}{dx}y'\right)+(7x^2+4)\,y'+8xy=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x\cdot y$ to series expansion

  $x\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+1}$

  ◦ Shift index using $k->k-1$

$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx} y'\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(3+r)\, x^{-1+r} + a_1(1+r)\,(4+r)\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)\,(k+r+4) + a_{k-1}(k+r+3)\,(k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$
- Each term must be 0

  $a_1(1+r)\,(4+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r+1)\,(a_{k+1}(k+r+4) + a_{k-1}(k+r+3)) = 0$
- Shift index using $k->k+1$

  $(k+r+2)\,(a_{k+2}(k+5+r) + a_k(k+r+4)) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+4)}{k+5+r}$
- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{a_k(k+1)}{k+2}$
- Solution for $r = -3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a_k(k+1)}{k+2}, -2a_1 = 0\right]$
- Recursion relation for $r = 0$

$a_{k+2} = -\frac{a_k(k+4)}{k+5}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{k+5}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k(k+1)}{k+2}, -2a_1 = 0, b_{k+2} = -\frac{b_k(k+4)}{k+5}, 4b_1 = 0\right]$$

1.107.3 Maple trace

```
Methods for second order ODEs:
```

1.107.4 Maple dsolve solution

Solving time : `0.015` (sec)
Leaf size : 32

```
dsolve(x*(x^2+1)*diff(diff(y(x),x),x)+(7*x^2+4)*diff(y(x),x)+8*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-\sqrt{x^2+1}\, c_2 x + \operatorname{arcsinh}(x)\, c_2 + c_1}{\sqrt{x^2+1}\, x^3}$$

1.107.5 Mathematica DSolve solution

Solving time : `0.199` (sec)
Leaf size : 55

```
DSolve[{x*(1+x^2)*D[y[x],{x,2}]+(4+7*x^2)*D[y[x],x]+8*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-c_2 \operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right) + c_2 x\sqrt{x^2+1} + 2c_1}{2x^3\sqrt{x^2+1}}$$

1.108 problem 110

Internal problem ID [8246]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 110
Date solved : `Monday, October 21, 2024 at 05:04:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+1\right)y''+x\left(8x^2+3\right)y'-\left(-4x^2+3\right)y=0$$

1.108.1 Solved as second order ode using Kovacic algorithm

Time used: 0.384 (sec)

Writing the ode as

$$\left(2x^4+2x^2\right)y''+\left(8x^3+3x\right)y'+\left(4x^2-3\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+2x^2 \\ B &= 8x^3+3x \\ C &= 4x^2-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{36x^2 + 21}{16\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 36x^2 + 21$$
$$t = 16\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{36x^2 + 21}{16\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 200: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 6 - 2\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16x^2} - \frac{15}{64(x-i)^2} - \frac{15}{64(x+i)^2} + \frac{27i}{64(x-i)} - \frac{27i}{64(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{36x^2 + 21}{16\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
i	2	0	$\frac{5}{8}$	$\frac{3}{8}$
$-i$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)} + (0) \\
&= -\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)} \\
&= -\frac{3}{4x(x^2+1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)}\right)(0) + \left(\left(\frac{3}{4x^2} - \frac{3}{8(x-i)^2} - \frac{3}{8(x+i)^2}\right) + \left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{3/8}}{x^{3/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x^3+3x}{2x^4+2x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4}-\frac{5\ln\left(x^2+1\right)}{8}} \\
&= z_1\left(\frac{1}{x^{3/4}\left(x^2+1\right)^{5/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^{3/2}\left(x^2+1\right)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x^3+3x}{2x^4+2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}-\frac{5\ln\left(x^2+1\right)}{4}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \mathrm{e}^{-\frac{3\ln(x)}{2}-\frac{5\ln\left(x^2+1\right)}{4}} x^3\sqrt{x^2+1}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^{3/2}\left(x^2+1\right)^{1/4}}\right) + c_2\left(\frac{1}{x^{3/2}\left(x^2+1\right)^{1/4}}\left(\int \mathrm{e}^{-\frac{3\ln(x)}{2}-\frac{5\ln\left(x^2+1\right)}{4}} x^3\sqrt{x^2+1}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.108.2 Maple step by step solution

Let's solve

$2x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(8x^2+3)\,y'-(-4x^2+3)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(4x^2-3)y}{2x^2(x^2+1)}-\frac{(8x^2+3)y'}{2x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(8x^2+3)y'}{2x(x^2+1)}+\frac{(4x^2-3)y}{2x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{8x^2+3}{2x(x^2+1)},P_3(x)=\frac{4x^2-3}{2x^2(x^2+1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{3}{2}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-\frac{3}{2}$

- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(8x^2+3)\,y'+(4x^2-3)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2r+3)\,(-1+r)\, x^r + a_1(5+2r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+3)\,(k+r-1) + 2a_{k-2}(k+r)\,(k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2r+3)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, -\frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(5+2r)\, r = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(\left(k+r+\frac{3}{2}\right) a_k + a_{k-2}(k+r)\right)(k+r-1) = 0$$

- Shift index using $k- >k+2$

$$2\left(\left(k+\frac{7}{2}+r\right) a_{k+2} + a_k(k+r+2)\right)(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2a_k(k+r+2)}{2k+7+2r}$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{2a_k(k+3)}{2k+9}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_k(k+3)}{2k+9}, a_1 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$a_{k+2} = -\frac{2a_k\left(k+\frac{1}{2}\right)}{2k+4}$

- Solution for $r = -\frac{3}{2}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{2a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2a_k(k+3)}{2k+9}, a_1 = 0, b_{k+2} = -\frac{2b_k\left(k+\frac{1}{2}\right)}{2k+4}, b_1 = 0\right]$

1.108.3 Maple trace

```
Methods for second order ODEs:
```

1.108.4 Maple dsolve solution

Solving time : `0.106` (sec)
Leaf size : 31

```
dsolve(2*x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(8*x^2+3)*diff(y(x),x)-(-4*x^2+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x \,\mathrm{hypergeom}\left(\left[1, \frac{3}{2}\right], \left[\frac{9}{4}\right], -x^2\right) + \frac{c_2}{x^{3/2}\left(x^2+1\right)^{1/4}}$$

1.108.5 Mathematica DSolve solution

Solving time : `0.144` (sec)
Leaf size : 60

```
DSolve[{2*x^2*(1+x^2)*D[y[x],{x,2}]+x*(3+8*x^2)*D[y[x],x]-(3-4*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{c_2 \,\mathrm{Hypergeometric2F1}\left(\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, -x^2\right)}{x\sqrt[4]{x^2+1}} + \frac{c_1}{x^{3/2}\sqrt[4]{x^2+1}} + \frac{c_2}{x}$$

1.109 problem 111

Internal problem ID [8247]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 111
Date solved : `Monday, October 21, 2024 at 05:04:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' + 3x\left(x^2+3\right)y' - \left(-5x^2+1\right)y = 0$$

1.109.1 Solved as second order ode using Kovacic algorithm

Time used: 0.351 (sec)

Writing the ode as

$$9x^2y'' + \left(3x^3+9x\right)y' + \left(5x^2-1\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= 3x^3+9x \\ C &= 5x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^2 - 5}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 8x^2 - 5 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^2 - 5}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 202: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 4\\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{36} - \frac{2}{9} - \frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} - \frac{2}{3x} - \frac{7}{4x^3} - \frac{7}{x^5} - \frac{595}{16x^7} - \frac{889}{4x^9} - \frac{45647}{32x^{11}} - \frac{76811}{8x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{6}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^4 - 8x^2 - 5}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{x^2}{36} - \frac{2}{9}\right) + \left(-\frac{5}{36x^2}\right) \\ &= \frac{x^2}{36} - \frac{2}{9} - \frac{5}{36x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{2}{9}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{2}{9}\right) - (0) \\ &= -\frac{2}{9}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{6} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{2}{9}}{\frac{1}{6}} - 1\right) &= -\frac{7}{6} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{2}{9}}{\frac{1}{6}} - 1\right) &= \frac{1}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 8x^2 - 5}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{6}$	$-\frac{7}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{6} - \left(\frac{1}{6}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{6x} + (-)\left(\frac{x}{6}\right) \\
&= \frac{1}{6x} - \frac{x}{6} \\
&= \frac{1}{6x} - \frac{x}{6}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{6x} - \frac{x}{6}\right)(0) + \left(\left(-\frac{1}{6x^2} - \frac{1}{6}\right) + \left(\frac{1}{6x} - \frac{x}{6}\right)^2 - \left(\frac{x^4 - 8x^2 - 5}{36x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{6x} - \frac{x}{6}\right) dx} \\
&= x^{1/6} \mathrm{e}^{-\frac{x^2}{12}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x^3 + 9x}{9x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{12} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x^2}{12}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{x^2}{6}}}{x^{1/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+9x}{9x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{6}-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\int e^{-\frac{x^2}{6}-\ln(x)} x^{2/3} e^{\frac{x^2}{3}}\, dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-\frac{x^2}{6}}}{x^{1/3}}\right) + c_2 \left(\frac{e^{-\frac{x^2}{6}}}{x^{1/3}} \left(\int e^{-\frac{x^2}{6}-\ln(x)} x^{2/3} e^{\frac{x^2}{3}}\, dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.109.2 Maple step by step solution

Let's solve

$9x^2\left(\frac{d}{dx}y'\right) + 3x(x^2+3)\, y' - (-5x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(5x^2-1)y}{9x^2} - \frac{(x^2+3)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(x^2+3)y'}{3x} + \frac{(5x^2-1)y}{9x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $$\left[P_2(x) = \frac{x^2+3}{3x}, P_3(x) = \frac{5x^2-1}{9x^2}\right]$$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $$(x \cdot P_2(x))\bigg|_{x=0} = 1$$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{9}$$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $$9x^2\left(\frac{d}{dx}y'\right) + 3x(x^2+3)\,y' + (5x^2-1)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- > k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k- > k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

    Rewrite ODE with series expansions

$$a_0(1+3r)(-1+3r)x^r + a_1(4+3r)(2+3r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r+1)(3k+3r-1) + a_{k-2}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+3r)(-1+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$
- Each term must be 0
  $a_1(4+3r)(2+3r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(3k+3r-1)(3a_k k + 3a_k r + a_k + a_{k-2}) = 0$
- Shift index using $k- > k+2$
  $(3k+3r+5)(3a_{k+2}(k+2) + 3a_{k+2}r + a_{k+2} + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{3k+7+3r}$
- Recursion relation for $r = -\frac{1}{3}$
  $a_{k+2} = -\frac{a_k}{3k+6}$
- Solution for $r = -\frac{1}{3}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = -\frac{a_k}{3k+6}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{3}$
  $a_{k+2} = -\frac{a_k}{3k+8}$
- Solution for $r = \frac{1}{3}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k}{3k+8}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{a_k}{3k+6}, a_1 = 0, b_{k+2} = -\frac{b_k}{3k+8}, b_1 = 0\right]$

1.109.3 Maple trace

```
Methods for second order ODEs:
```

1.109.4 Maple dsolve solution

Solving time : 0.011 (sec)
Leaf size : 37

```
dsolve(9*x^2*diff(diff(y(x),x),x)+3*x*(x^2+3)*diff(y(x),x)-(-5*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{x^2}{12}} \left(\mathrm{WhittakerM}\left(\frac{1}{3}, \frac{1}{6}, \frac{x^2}{6}\right) x^{1/3} c_1 + e^{-\frac{x^2}{12}} c_2 x\right)}{x^{4/3}}$$

1.109.5 Mathematica DSolve solution

Solving time : 0.394 (sec)
Leaf size : 61

```
DSolve[{9*x^2*D[y[x],{x,2}]+3*x*(3+x^2)*D[y[x],x]-(1-5*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{x^2}{6}} \left(2c_1 x^{4/3} + \sqrt[3]{6} c_2 (-x^2)^{2/3} \Gamma\left(\frac{1}{3}, -\frac{x^2}{6}\right)\right)}{2x^{5/3}}$$

1.110 problem 112

Internal problem ID [8248]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 112
Date solved : `Monday, October 21, 2024 at 05:04:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2y'' + x(6x^2+1)\,y' + (9x^2+1)\,y = 0$$

1.110.1 Solved as second order ode using Kovacic algorithm

Time used: 0.439 (sec)

Writing the ode as

$$6x^2y'' + (6x^3+x)\,y' + (9x^2+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= 6x^3 + x \\ C &= 9x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{36x^4 - 132x^2 - 35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 36x^4 - 132x^2 - 35$$
$$t = 144x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{36x^4 - 132x^2 - 35}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 204: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - \frac{11}{12} - \frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{11}{12x} - \frac{13}{12x^3} - \frac{143}{72x^5} - \frac{130}{27x^7} - \frac{17017}{1296x^9} - \frac{597961}{15552x^{11}} - \frac{11016863}{93312x^{13}} + \dots \quad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \quad (10)$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{36x^4 - 132x^2 - 35}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{x^2}{4} - \frac{11}{12}\right) + \left(-\frac{35}{144x^2}\right) \\ &= \frac{x^2}{4} - \frac{11}{12} - \frac{35}{144x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{11}{12}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{11}{12}\right) - (0) \\ &= -\frac{11}{12} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{11}{12}}{\frac{1}{2}} - 1\right) &&= -\frac{17}{12} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{11}{12}}{\frac{1}{2}} - 1\right) &&= \frac{5}{12}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{36x^4 - 132x^2 - 35}{144x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	$-\frac{17}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{12}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{5}{12} - \left(\frac{5}{12}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{5}{12x} + (-)\left(\frac{x}{2}\right) \\
&= \frac{5}{12x} - \frac{x}{2} \\
&= \frac{5}{12x} - \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{5}{12x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{5}{12x^2} - \frac{1}{2}\right) + \left(\frac{5}{12x} - \frac{x}{2}\right)^2 - \left(\frac{36x^4 - 132x^2 - 35}{144x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{5}{12x} - \frac{x}{2}\right) dx} \\
&= x^{5/12} \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{6x^3 + x}{6x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{4} - \frac{\ln(x)}{12}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x^2}{4}}}{x^{1/12}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^{1/3}e^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x^3+x}{6x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}-\frac{\ln(x)}{6}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{e^{-\frac{x^2}{2}-\frac{\ln(x)}{6}}e^{x^2}}{x^{2/3}}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{1/3}e^{-\frac{x^2}{2}}\right) + c_2\left(x^{1/3}e^{-\frac{x^2}{2}}\left(\int \frac{e^{-\frac{x^2}{2}-\frac{\ln(x)}{6}}e^{x^2}}{x^{2/3}}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.110.2 Maple step by step solution

Let's solve

$6x^2\left(\frac{d}{dx}y'\right) + x(6x^2+1)\,y' + (9x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(9x^2+1)y}{6x^2} - \frac{(6x^2+1)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(6x^2+1)y'}{6x} + \frac{(9x^2+1)y}{6x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{6x^2+1}{6x}, P_3(x) = \frac{9x^2+1}{6x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{1}{6}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{6}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2\left(\frac{d}{dx}y'\right) + x(6x^2+1)\, y' + (9x^2+1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+3r)\,(-1+2r)\,x^r + a_1(2+3r)\,(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)\,(2k+2r-1) + 3a\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$(-1+3r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation
$r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$
- Each term must be 0
$a_1(2+3r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)
$a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
$6\left(\left(k+r-\frac{1}{3}\right)a_k + a_{k-2}\right)\left(k+r-\frac{1}{2}\right) = 0$
- Shift index using $k- >k+2$
$6\left(\left(k+\frac{5}{3}+r\right)a_{k+2} + a_k\right)\left(k+\frac{3}{2}+r\right) = 0$
- Recursion relation that defines series solution to ODE
$a_{k+2} = -\frac{3a_k}{3k+5+3r}$
- Recursion relation for $r = \frac{1}{2}$
$a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}$
- Solution for $r = \frac{1}{2}$
$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{3}$
$a_{k+2} = -\frac{3a_k}{3k+6}$
- Solution for $r = \frac{1}{3}$
$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{3a_k}{3k+6}, a_1 = 0\right]$
- Combine solutions and rename parameters
$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}, a_1 = 0, b_{k+2} = -\frac{3b_k}{3k+6}, b_1 = 0\right]$

1.110.3 Maple trace

```
Methods for second order ODEs:
```

1.110.4 Maple dsolve solution

Solving time : 0.016 (sec)
Leaf size : 36

```
dsolve(6*x^2*diff(diff(y(x),x),x)+x*(6*x^2+1)*diff(y(x),x)+(9*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{x^2}{4}}\left(x^{11/12}e^{-\frac{x^2}{4}}c_2 + \text{WhittakerM}\left(\frac{11}{24}, \frac{1}{24}, \frac{x^2}{2}\right)c_1\right)}{x^{7/12}}$$

1.110.5 Mathematica DSolve solution

Solving time : 0.658 (sec)
Leaf size : 61

```
DSolve[{6*x^2*D[y[x],{x,2}]+x*(1+6*x^2)*D[y[x],x]+(1+9*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{x^2}{2}}\left(2c_1x^{11/6} + \sqrt[12]{2}c_2(-x^2)^{11/12}\,\Gamma\left(\frac{1}{12}, -\frac{x^2}{2}\right)\right)}{2x^{3/2}}$$

1.111 problem 113

Internal problem ID [8249]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 113
Date solved : `Monday, October 21, 2024 at 05:04:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2\left(x^2+1\right)y''+3x\left(13x^2+3\right)y'-\left(-25x^2+1\right)y=0$$

1.111.1 Solved as second order ode using Kovacic algorithm

Time used: 0.381 (sec)

Writing the ode as

$$\left(9x^4+9x^2\right)y''+\left(39x^3+9x\right)y'+\left(25x^2-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^4+9x^2 \\ B &= 39x^3+9x \\ C &= 25x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9x^4 + 6x^2 - 5}{36\,(x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -9x^4 + 6x^2 - 5$$
$$t = 36\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-9x^4 + 6x^2 - 5}{36\,(x^3 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 206: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36(x-i)^2} - \frac{5}{36(x+i)^2} - \frac{i}{12(x-i)} + \frac{i}{12x+12i} - \frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{6}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-9x^4+6x^2-5}{36\left(x^3+x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-9x^4+6x^2-5}{36\left(x^3+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$
i	2	0	$\frac{5}{6}$	$\frac{1}{6}$
$-i$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i} + (-)\,(0) \\
&= \frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i} \\
&= \frac{1}{6x} + \frac{x}{3x^2+3}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i}\right)(0) + \left(\left(-\frac{1}{6x^2} - \frac{1}{6\left(x-i\right)^2} - \frac{1}{6\left(x+i\right)^2}\right) + \left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i}\right) dx} \\
&= \left(x^2+1\right)^{1/6} (-x)^{1/6}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{39x^3+9x}{9x^4+9x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln\left(x^2+1\right)}{6}-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\left(x^2+1\right)^{5/6}\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(-x)^{1/6}}{\left(x^2+1\right)^{2/3}\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{39x^3+9x}{9x^4+9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(x^2+1\right)}{3}-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{5\ln\left(x^2+1\right)}{3}-\ln(x)}\left(x^2+1\right)^{4/3}x}{(-x)^{1/3}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{(-x)^{1/6}}{\left(x^2+1\right)^{2/3}\sqrt{x}}\right) + c_2\left(\frac{(-x)^{1/6}}{\left(x^2+1\right)^{2/3}\sqrt{x}}\left(\int \frac{\mathrm{e}^{-\frac{5\ln\left(x^2+1\right)}{3}-\ln(x)}\left(x^2+1\right)^{4/3}x}{(-x)^{1/3}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.111.2 Maple step by step solution

Let's solve

$9x^2(x^2+1)\left(\frac{d}{dx}y'\right)+3x(13x^2+3)\,y'-(-25x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(25x^2-1)y}{9x^2(x^2+1)}-\frac{(13x^2+3)y'}{3x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(13x^2+3)y'}{3x(x^2+1)}+\frac{(25x^2-1)y}{9x^2(x^2+1)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{13x^2+3}{3x(x^2+1)},P_3(x)=\frac{25x^2-1}{9x^2(x^2+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$(x\cdot P_2(x))\bigg|_{x=0}=1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2\cdot P_3(x))\bigg|_{x=0}=-\frac{1}{9}$

- ○ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$9x^2(x^2+1)\left(\frac{d}{dx}y'\right)+3x(13x^2+3)\,y'+(25x^2-1)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ○ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+3r)\,(-1+3r)\, x^r + a_1(4+3r)\,(2+3r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+1)\,(3k+3r-1) + a_{k-2}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+3r)\,(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$$

- Each term must be 0

$$a_1(4+3r)\,(2+3r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$9\left(\left(k+r-\frac{1}{3}\right) a_{k-2} + a_k\left(k+r+\frac{1}{3}\right)\right)\left(k+r-\frac{1}{3}\right) = 0$$

- Shift index using $k- >k+2$

$$9\left(\left(k+\frac{5}{3}+r\right) a_k + a_{k+2}\left(k+\frac{7}{3}+r\right)\right)\left(k+\frac{5}{3}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{(3k+3r+5)a_k}{3k+7+3r}$$

- Recursion relation for $r = -\frac{1}{3}$

$$a_{k+2} = -\frac{(3k+4)a_k}{3k+6}$$

- Solution for $r = -\frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = -\frac{(3k+4)a_k}{3k+6}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{3}$

$a_{k+2} = -\frac{(3k+6)a_k}{3k+8}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{(3k+6)a_k}{3k+8}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{(3k+4)a_k}{3k+6}, a_1 = 0, b_{k+2} = -\frac{(3k+6)b_k}{3k+8}, b_1 = 0\right]$

1.111.3 Maple trace

```
Methods for second order ODEs:
```

1.111.4 Maple dsolve solution

Solving time : `0.032` (sec)
Leaf size : 33

```
dsolve(9*x^2*(x^2+1)*diff(diff(y(x),x),x)+3*x*(13*x^2+3)*diff(y(x),x)-(-25*x^2+1)*y(x
       y(x),singsol=all)
```

$$y = \frac{c_1}{(x^2+1)^{2/3}\, x^{1/3}} + c_2\, x^{1/3} \operatorname{hypergeom}\left([1,1], \left[\frac{4}{3}\right], -x^2\right)$$

1.111.5 Mathematica DSolve solution

Solving time : `0.342` (sec)
Leaf size : 124

```
DSolve[{9*x^2*(1+x^2)*D[y[x],{x,2}]+3*x*(3+13*x^2)*D[y[x],x]-(1-25*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{2\sqrt{3}c_2 \arctan\left(\frac{\sqrt{3}x^{2/3}}{x^{2/3}+2\sqrt[3]{x^2+1}}\right) - 2c_2 \log\left(\sqrt[3]{x^2+1} - x^{2/3}\right) + c_2 \log\left(x^{4/3} + (x^2+1)^{2/3} + \sqrt[3]{x^2+1}x^{2/3}\right.}{4\sqrt[3]{x}\,(x^2+1)^{2/3}}$$

1.112 problem 114

Internal problem ID [8250]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 114
Date solved : `Monday, October 21, 2024 at 05:04:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+1\right)y''+4x\left(6x^2+1\right)y'-\left(-25x^2+1\right)y=0$$

1.112.1 Solved as second order ode using Kovacic algorithm

Time used: 0.337 (sec)

Writing the ode as

$$\left(4x^4+4x^2\right)y''+\left(24x^3+4x\right)y'+\left(25x^2-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 24x^3+4x \\ C &= 25x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 6}{4(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 6$$
$$t = 4(x^2+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 6}{4(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 208: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\,(x-i)^2} + \frac{5}{16\,(x+i)^2} + \frac{7i}{16\,(x-i)} - \frac{7i}{16\,(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 6}{4\,(x^2+1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2-6}{4\left(x^2+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{1}{2} - \left(-\frac{1}{2}\right)\\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} + \left(-\right)\left(0\right) \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} \\
&= -\frac{x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)}\right)(1) + \left(\left(\frac{1}{4\left(x-i\right)^2} + \frac{1}{4\left(x+i\right)^2}\right) + \left(-\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)}\right)^2 - \left(\frac{-}{4(\cdots}\right.\right.
$$

$$
\frac{(x^2+1\cdots}{(-x+i)^2(\cdots}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\
&= (x)\, \frac{1}{\left(\left(-x+i\right)\left(x+i\right)\right)^{1/4}} \\
&= \frac{x}{\left(-x^2-1\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{24x^3+4x}{4x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}-\frac{5\ln\left(x^2+1\right)}{4}} \\
&= z_1\left(\frac{1}{\sqrt{x}\,\left(x^2+1\right)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{x}\,\sqrt{2}}{\left(x^2+1\right)^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{24x^3+4x}{4x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)-\frac{5\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(i\left(-\frac{\left(x^2+1\right)^{3/2}}{x}+x\sqrt{x^2+1}+\operatorname{arcsinh}(x)\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{x}\,\sqrt{2}}{\left(x^2+1\right)^{3/2}}\right) \\
&\quad + c_2\left(\frac{\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{x}\,\sqrt{2}}{\left(x^2+1\right)^{3/2}}\left(i\left(-\frac{\left(x^2+1\right)^{3/2}}{x}+x\sqrt{x^2+1}+\operatorname{arcsinh}(x)\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.112.2 Maple step by step solution

Let's solve

$4x^2(x^2+1)\left(\frac{d}{dx}y'\right)+4x(6x^2+1)\,y'-(-25x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(25x^2-1)y}{4x^2(x^2+1)}-\frac{(6x^2+1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(6x^2+1)y'}{x(x^2+1)}+\frac{(25x^2-1)y}{4x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ∘ Define functions

  $\left[P_2(x)=\frac{6x^2+1}{x(x^2+1)},P_3(x)=\frac{25x^2-1}{4x^2(x^2+1)}\right]$

- ∘ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=1$

- ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=-\frac{1}{4}$

- ∘ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+1)\left(\frac{d}{dx}y'\right)+4x(6x^2+1)\,y'+(25x^2-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ∘ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ∘ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ∘ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\,x^r + a_1(3+2r)\,(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + a_{k-2}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0
  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $4\left(k+r+\frac{1}{2}\right)\left(\left(k+r+\frac{1}{2}\right)a_{k-2} + a_k\left(k+r-\frac{1}{2}\right)\right) = 0$
- Shift index using $k->k+2$
  $4\left(k+\frac{5}{2}+r\right)\left(\left(k+\frac{5}{2}+r\right)a_k + a_{k+2}\left(k+\frac{3}{2}+r\right)\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{(2k+2r+5)a_k}{2k+3+2r}$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+2} = -\frac{(2k+4)a_k}{2k+2}$
- Solution for $r = -\frac{1}{2}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{(2k+4)a_k}{2k+2}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

$a_{k+2} = -\frac{(2k+6)a_k}{2k+4}$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{(2k+6)a_k}{2k+4}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{(2k+4)a_k}{2k+2}, a_1 = 0, b_{k+2} = -\frac{(2k+6)b_k}{2k+4}, b_1 = 0\right]$$

1.112.3 Maple trace

```
Methods for second order ODEs:
```

1.112.4 Maple dsolve solution

Solving time : `0.021` (sec)
Leaf size : 34

```
dsolve(4*x^2*(x^2+1)*diff(diff(y(x),x),x)+4*x*(6*x^2+1)*diff(y(x),x)-(-25*x^2+1)*y(x)
       y(x),singsol=all)
```

$$y = \frac{-\sqrt{x^2+1}\, c_2 + x(c_2 \operatorname{arcsinh}(x) + c_1)}{(x^2+1)^{3/2}\sqrt{x}}$$

1.112.5 Mathematica DSolve solution

Solving time : `0.19` (sec)
Leaf size : 54

```
DSolve[{4*x^2*(1+x^2)*D[y[x],{x,2}]+4*x*(1+6*x^2)*D[y[x],x]-(1-25*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x \operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right) - c_2\sqrt{x^2+1} + c_1 x}{\sqrt{x}\,(x^2+1)^{3/2}}$$

1.113 problem 115

Internal problem ID [8251]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 115
Date solved : `Monday, October 21, 2024 at 05:04:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$8x^2(2x^2+1)\,y''+2x(34x^2+5)\,y'-(-30x^2+1)\,y=0$$

1.113.1 Solved as second order ode using Kovacic algorithm

Time used: 0.558 (sec)

Writing the ode as

$$(16x^4+8x^2)\,y''+(68x^3+10x)\,y'+(30x^2-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^4+8x^2 \\ B &= 68x^3+10x \\ C &= 30x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 132x^4 + 148x^2 - 7 \\ t &= 64\left(2x^3 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 210: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(2x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2} - \frac{3}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{3}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{i\sqrt{2}}{2\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{i\sqrt{2}}{2x + i\sqrt{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{11}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= \frac{11}{8} - \left(\frac{11}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}} + (0) \\
&= \frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}} \\
&= \frac{22x^2+7}{16x^3+8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}}\right)(0) + \left(\left(-\frac{7}{8x^2} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{7}{8x} + \frac{}{4x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{7}{8x} + \frac{1}{4x - 2i\sqrt{2}} + \frac{1}{4x + 2i\sqrt{2}}\right) dx} \\
&= 2^{1/4} \left(2x^2 + 1\right)^{1/4} x^{7/8}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{68x^3 + 10x}{16x^4 + 8x^2} \, dx} \\
&= z_1 e^{-\frac{3 \ln\left(2x^2+1\right)}{4} - \frac{5 \ln(x)}{8}} \\
&= z_1 \left(\frac{1}{\left(2x^2 + 1\right)^{3/4} x^{5/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{1/4} 2^{1/4}}{\sqrt{2x^2 + 1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{68x^3 + 10x}{16x^4 + 8x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{3 \ln\left(2x^2+1\right)}{2} - \frac{5 \ln(x)}{4}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{3 \ln\left(2x^2+1\right)}{2} - \frac{5 \ln(x)}{4}} \left(2x^2 + 1\right) \sqrt{2}}{2\sqrt{x}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/4}2^{1/4}}{\sqrt{2x^2+1}}\right) + c_2\left(\frac{x^{1/4}2^{1/4}}{\sqrt{2x^2+1}}\left(\int \frac{\mathrm{e}^{-\frac{3\ln\left(2x^2+1\right)}{2}-\frac{5\ln(x)}{4}}\left(2x^2+1\right)\sqrt{2}}{2\sqrt{x}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.113.2 Maple step by step solution

Let's solve

$8x^2(2x^2+1)\left(\frac{d}{dx}y'\right)+2x(34x^2+5)\,y'-(-30x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(30x^2-1)y}{8x^2(2x^2+1)}-\frac{(34x^2+5)y'}{4x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(34x^2+5)y'}{4x(2x^2+1)}+\frac{(30x^2-1)y}{8x^2(2x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{34x^2+5}{4x(2x^2+1)},P_3(x)=\frac{30x^2-1}{8x^2(2x^2+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=\frac{5}{4}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=-\frac{1}{8}$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $8x^2(2x^2+1)\left(\frac{d}{dx}y'\right)+2x(34x^2+5)\,y'+(30x^2-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx} y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+4r)\, x^r + a_1(3+2r)\,(3+4r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(4k+4r-1) + 2a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+2r)\,(-1+4r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$

- Each term must be 0

$a_1(3+2r)\,(3+4r) = 0$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$8\left(k+r+\frac{1}{2}\right)\left(\left(2k+2r-\frac{5}{2}\right) a_{k-2} + a_k\left(k+r-\frac{1}{4}\right)\right) = 0$

- Shift index using $k- >k + 2$

$8\left(k+\frac{5}{2}+r\right)\left(\left(2k+\frac{3}{2}+2r\right) a_k + a_{k+2}\left(k+\frac{7}{4}+r\right)\right) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{2(4k+4r+3)a_k}{4k+7+4r}$

- Recursion relation for $r = -\frac{1}{2}$

$a_{k+2} = -\frac{2(4k+1)a_k}{4k+5}$

- Solution for $r = -\frac{1}{2}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{2(4k+1)a_k}{4k+5}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{4}$

$a_{k+2} = -\frac{2(4k+4)a_k}{4k+8}$

- Solution for $r = \frac{1}{4}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{2(4k+4)a_k}{4k+8}, a_1 = 0\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2} = -\frac{2(4k+1)a_k}{4k+5}, a_1 = 0, b_{k+2} = -\frac{2(4k+4)b_k}{4k+8}, b_1 = 0\right]$

1.113.3 Maple trace

```
Methods for second order ODEs:
```

1.113.4 Maple dsolve solution

Solving time : `0.023` (sec)
Leaf size : 46

```
dsolve(8*x^2*(2*x^2+1)*diff(diff(y(x),x),x)+2*x*(34*x^2+5)*diff(y(x),x)-(-30*x^2+1)*y(
       y(x),singsol=all)
```

$$y = \frac{c_1 \operatorname{LegendreP}\left(\frac{3}{8}, \frac{3}{8}, \sqrt{2x^2+1}\right) + c_2 \operatorname{LegendreQ}\left(\frac{3}{8}, \frac{3}{8}, \sqrt{2x^2+1}\right)}{\sqrt{2x^2+1}\, x^{1/8}}$$

1.113.5 Mathematica DSolve solution

Solving time : 0.164 (sec)
Leaf size : 54

```
DSolve[{8*x^2*(1+2*x^2)*D[y[x],{x,2}]+2*x*(5+34*x^2)*D[y[x],x]-(1-30*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3c_1x^{3/4} - 4c_2 \,\text{Hypergeometric2F1}\left(-\frac{3}{8}, \frac{1}{2}, \frac{5}{8}, -2x^2\right)}{3\sqrt{x}\sqrt{2x^2+1}}$$

1.114 problem 116

Internal problem ID [8252]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 116
Date solved : `Monday, October 21, 2024 at 05:04:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(1+x)\,y''-x(1-3x)\,y'+y=0$$

1.114.1 Solved as second order ode using Kovacic algorithm

Time used: 0.206 (sec)

Writing the ode as

$$\left(2x^3+2x^2\right)y''+\left(3x^2-x\right)y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+2x^2\\ B &= 3x^2-x\\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 212: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} + (-)(0) \\ &= \frac{1}{4x} \\ &= \frac{1}{4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x}\right)(0) + \left(\left(-\frac{1}{4x^2}\right) + \left(\frac{1}{4x}\right)^2 - \left(-\frac{3}{16x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \frac{1}{4x} dx} \\
&= x^{1/4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^2 - x}{2x^3 + 2x^2}\, dx} \\
&= z_1 e^{-\ln(1+x) + \frac{\ln(x)}{4}} \\
&= z_1 \left(\frac{x^{1/4}}{1+x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{1+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{3x^2-x}{2x^3+2x^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-2\ln(1+x)+\frac{\ln(x)}{2}}}{(y_1)^2}\,dx$$
$$= y_1\left(2\,\mathrm{e}^{-2\ln(1+x)+\frac{\ln(x)}{2}}(1+x)^2\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{\sqrt{x}}{1+x}\right) + c_2\left(\frac{\sqrt{x}}{1+x}\left(2\,\mathrm{e}^{-2\ln(1+x)+\frac{\ln(x)}{2}}(1+x)^2\right)\right)$$

Will add steps showing solving for IC soon.

1.114.2 Maple step by step solution

Let's solve

$2x^2(1+x)\left(\frac{d}{dx}y'\right) - x(1-3x)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{2x^2(1+x)} - \frac{(3x-1)y'}{2x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3x-1)y'}{2x(1+x)} + \frac{y}{2x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

    $\left[P_2(x) = \frac{3x-1}{2x(1+x)}, P_3(x) = \frac{1}{2x^2(1+x)}\right]$

  ○ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.\left((1+x)\cdot P_2(x)\right)\right|_{x=-1} = 2$

  ○ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- ○ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- • Multiply by denominators

  $2x^2(1+x)\left(\frac{d}{dx}y'\right) + x(3x-1)\,y' + y = 0$

- • Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(2u^3 - 4u^2 + 2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (3u^2 - 7u + 4)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- • Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - ○ Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - ○ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

  - ○ Shift index using $k- >k+2-m$

    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(1+r)\,u^{-1+r} + (2a_1(1+r)(2+r) - a_0(1+r)(-1+4r))\,u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+r+1)(k+\right.$

- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(1+r) = 0$

- • Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- • Each term must be 0

  $2a_1(1+r)(2+r) - a_0(1+r)(-1+4r) = 0$

- • Each term in the series must be 0, giving the recursion relation

$(-4a_k+2a_{k-1}+2a_{k+1})\,k^2+((-8a_k+4a_{k-1}+4a_{k+1})\,r-3a_k-3a_{k-1}+6a_{k+1})\,k+(-4a_k+2a$

- Shift index using $k->k+1$

$(-4a_{k+1}+2a_k+2a_{k+2})\,(k+1)^2+((-8a_{k+1}+4a_k+4a_{k+2})\,r-3a_{k+1}-3a_k+6a_{k+2})\,(k+1)+$

- Recursion relation that defines series solution to ODE

$a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}+4kra_k-8kra_{k+1}+2r^2a_k-4r^2a_{k+1}+ka_k-11ka_{k+1}+ra_k-11ra_{k+1}-6a_{k+1}}{2(k^2+2kr+r^2+5k+5r+6)}$

- Recursion relation for $r=-1$

$a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}-3ka_k-3ka_{k+1}+a_k+a_{k+1}}{2(k^2+3k+2)}$

- Solution for $r=-1$

$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-1},a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}-3ka_k-3ka_{k+1}+a_k+a_{k+1}}{2(k^2+3k+2)},0=0\right]$

- Revert the change of variables $u=1+x$

$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k-1},a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}-3ka_k-3ka_{k+1}+a_k+a_{k+1}}{2(k^2+3k+2)},0=0\right]$

- Recursion relation for $r=0$

$a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}+ka_k-11ka_{k+1}-6a_{k+1}}{2(k^2+5k+6)}$

- Solution for $r=0$

$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}+ka_k-11ka_{k+1}-6a_{k+1}}{2(k^2+5k+6)},4a_1+a_0=0\right]$

- Revert the change of variables $u=1+x$

$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k,a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}+ka_k-11ka_{k+1}-6a_{k+1}}{2(k^2+5k+6)},4a_1+a_0=0\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty}a_k(1+x)^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k(1+x)^k\right),a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}-3ka_k-3ka_{k+1}+a_k+a_{k+1}}{2(k^2+3k+2)},0=0,\right.$

1.114.3 Maple trace

```
Methods for second order ODEs:
```

1.114.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 19

```
dsolve(2*x^2*(1+x)*diff(diff(y(x),x),x)-x*(1-3*x)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\sqrt{x} + c_1 x}{1 + x}$$

1.114.5 Mathematica DSolve solution

Solving time : 0.055 (sec)
Leaf size : 25

```
DSolve[{2*x^2*(1+x)*D[y[x],{x,2}]-x*(1-3*x)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1\sqrt{x} + 2c_2 x}{x + 1}$$

1.115 problem 117

Internal problem ID [8253]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 117
Date solved : `Monday, October 21, 2024 at 05:04:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2\left(2x^2+1\right)y''+x\left(50x^2+1\right)y'+\left(30x^2+1\right)y=0$$

1.115.1 Solved as second order ode using Kovacic algorithm

Time used: 0.214 (sec)

Writing the ode as

$$\left(12x^4+6x^2\right)y''+\left(50x^3+x\right)y'+\left(30x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 12x^4+6x^2 \\ B &= 50x^3+x \\ C &= 30x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -35$$
$$t = 144x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{35}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 214: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{35}{144x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{35}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{12}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{5}{12} - \left(\frac{5}{12}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{12x} + (-)\,(0) \\ &= \frac{5}{12x} \\ &= \frac{5}{12x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{12x}\right)(0) + \left(\left(-\frac{5}{12x^2}\right) + \left(\frac{5}{12x}\right)^2 - \left(-\frac{35}{144x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{5}{12x} dx} \\
&= x^{5/12}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{50x^3 + x}{12x^4 + 6x^2} \, dx} \\
&= z_1 e^{-\ln(2x^2+1) - \frac{\ln(x)}{12}} \\
&= z_1 \left(\frac{1}{(2x^2+1) \, x^{1/12}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{2x^2 + 1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{50x^3+x}{12x^4+6x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(2x^2+1)-\frac{\ln(x)}{6}}}{(y_1)^2}\,dx \\
&= y_1 \left(6x^{1/3}\mathrm{e}^{-2\ln(2x^2+1)-\frac{\ln(x)}{6}}\left(2x^2+1\right)^2\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/3}}{2x^2+1}\right) + c_2 \left(\frac{x^{1/3}}{2x^2+1}\left(6x^{1/3}\mathrm{e}^{-2\ln(2x^2+1)-\frac{\ln(x)}{6}}\left(2x^2+1\right)^2\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.115.2 Maple step by step solution

Let's solve

$6x^2(2x^2+1)\left(\frac{d}{dx}y'\right) + x(50x^2+1)\,y' + (30x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(30x^2+1)y}{6x^2(2x^2+1)} - \frac{(50x^2+1)y'}{6x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(50x^2+1)y'}{6x(2x^2+1)} + \frac{(30x^2+1)y}{6x^2(2x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{50x^2+1}{6x(2x^2+1)}, P_3(x) = \frac{30x^2+1}{6x^2(2x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{6}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{6}$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$6x^2(2x^2+1)\left(\frac{d}{dx}y'\right) + x(50x^2+1)\,y' + (30x^2+1)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+3r)\,(-1+2r)\,x^r + a_1(2+3r)\,(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)\,(2k+2r-1) + 2a\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+3r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(2+3r)(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(3k+3r-1)(2k+2r-1)(a_k+2a_{k-2}) = 0$
- Shift index using $k->k+2$

  $(3k+3r+5)(2k+2r+3)(a_{k+2}+2a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -2a_k$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -2a_k$
- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -2a_k, a_1 = 0\right]$$
- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -2a_k$
- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -2a_k, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -2a_k, a_1 = 0, b_{k+2} = -2b_k, b_1 = 0\right]$$

1.115.3 Maple trace

```
Methods for second order ODEs:
```

1.115.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 24

```
dsolve(6*x^2*(2*x^2+1)*diff(diff(y(x),x),x)+x*(50*x^2+1)*diff(y(x),x)+(30*x^2+1)*y(x)
       y(x),singsol=all)
```

$$y = \frac{x^{1/3}\left(c_1\, x^{1/6} + c_2\right)}{2x^2 + 1}$$

1.115.5 Mathematica DSolve solution

Solving time : `0.074` (sec)
Leaf size : 32

```
DSolve[{6*x^2*(1+2*x^2)*D[y[x],{x,2}]+x*(1+50*x^2)*D[y[x],x]+(1+30*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[3]{x}\left(6c_2\sqrt[6]{x} + c_1\right)}{2x^2 + 1}$$

1.116 problem 118

Internal problem ID [8254]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 118
Date solved : `Monday, October 21, 2024 at 05:04:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$28x^2(1-3x)\,y''-7x(5+9x)\,y'+7(2+9x)\,y=0$$

1.116.1 Solved as second order ode using Kovacic algorithm

Time used: 0.213 (sec)

Writing the ode as

$$\left(-84x^3+28x^2\right)y''+\left(-63x^2-35x\right)y'+(63x+14)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-84x^3+28x^2\\B&=-63x^2-35x\\C&=63x+14\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{33}{64x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 33 \\
t &= 64x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{33}{64x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 216: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{33}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{33}{64x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{33}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{3}{8} - \left(-\frac{3}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{8x} + (-)(0) \\
&= -\frac{3}{8x} \\
&= -\frac{3}{8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}(0) + 2\left(-\frac{3}{8x}\right)(0) + \left(\left(\frac{3}{8x^2}\right) + \left(-\frac{3}{8x}\right)^2 - \left(\frac{33}{64x^2}\right)\right) = 0 \\ 0 = 0\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int -\frac{3}{8x} dx} \\ &= \frac{1}{x^{3/8}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-63x^2 - 35x}{-84x^3 + 28x^2}\, dx} \\ &= z_1 e^{\frac{5\ln(x)}{8} - \ln(-1+3x)} \\ &= z_1 \left(\frac{x^{5/8}}{-1+3x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}}{-1+3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-63x^2-35x}{-84x^3+28x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{5\ln(x)}{4}-2\ln(-1+3x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{4\sqrt{x}\,\mathrm{e}^{\frac{5\ln(x)}{4}-2\ln(-1+3x)}(-1+3x)^2}{7}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/4}}{-1+3x}\right) + c_2\left(\frac{x^{1/4}}{-1+3x}\left(\frac{4\sqrt{x}\,\mathrm{e}^{\frac{5\ln(x)}{4}-2\ln(-1+3x)}(-1+3x)^2}{7}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.116.2 Maple step by step solution

Let's solve

$28x^2(1-3x)\left(\frac{d}{dx}y'\right) - 7x(5+9x)\,y' + 7(2+9x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(2+9x)y}{4x^2(-1+3x)} - \frac{(5+9x)y'}{4x(-1+3x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(5+9x)y'}{4x(-1+3x)} - \frac{(2+9x)y}{4x^2(-1+3x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{5+9x}{4x(-1+3x)}, P_3(x) = -\frac{2+9x}{4x^2(-1+3x)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = -\frac{5}{4}$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{2}$$

- $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$4x^2(-1+3x)\left(\frac{d}{dx}y'\right) + x(5+9x)\,y' + (-9x-2)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+4r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(4k+4r-1)\,(k+r-2) + 3a_{k-1}(4k+4r-1)\,(k+r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-(-1+4r)(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{2, \frac{1}{4}\}$

- Each term in the series must be 0, giving the recursion relation

  $-4(k+r-2)\left(k+r-\frac{1}{4}\right)(a_k - 3a_{k-1}) = 0$

- Shift index using $k->k+1$

  $-4(k+r-1)\left(k+\frac{3}{4}+r\right)(a_{k+1} - 3a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = 3a_k$

- Recursion relation for $r = 2$

  $a_{k+1} = 3a_k$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = 3a_k\right]$

- Recursion relation for $r = \frac{1}{4}$

  $a_{k+1} = 3a_k$

- Solution for $r = \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = 3a_k\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+1} = 3a_k, b_{k+1} = 3b_k\right]$

1.116.3 Maple trace

```
Methods for second order ODEs:
```

1.116.4 Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 23

```
dsolve(28*x^2*(1-3*x)*diff(diff(y(x),x),x)-7*x*(5+9*x)*diff(y(x),x)+7*(2+9*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1\, x^2 + c_2\, x^{1/4}}{-1+3x}$$

1.116.5 Mathematica DSolve solution

Solving time : 0.072 (sec)
Leaf size : 30

```
DSolve[{28*x^2*(1-3*x)*D[y[x],{x,2}]-7*x*(5+9*x)*D[y[x],x]+7*(2+9*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_2x^2 + 7c_1\sqrt[4]{x}}{7 - 21x}$$

1.117 problem 119

Internal problem ID [8255]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 119
Date solved : `Monday, October 21, 2024 at 05:04:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$8x^2\left(-x^2+2\right)y''+2x\left(-21x^2+10\right)y'-\left(35x^2+2\right)y=0$$

1.117.1 Solved as second order ode using Kovacic algorithm

Time used: 0.217 (sec)

Writing the ode as

$$\left(-8x^4+16x^2\right)y''+\left(-42x^3+20x\right)y'+\left(-35x^2-2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -8x^4+16x^2 \\ B &= -42x^3+20x \\ C &= -35x^2-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -7$$
$$t = 64x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{7}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 218: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{7}{64x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{7}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{8} - \left(\frac{1}{8}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{8x} + (-)(0) \\ &= \frac{1}{8x} \\ &= \frac{1}{8x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{8x}\right)(0)+\left(\left(-\frac{1}{8x^2}\right)+\left(\frac{1}{8x}\right)^2-\left(-\frac{7}{64x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int\omega\,dx}\\
&=\mathrm{e}^{\int\frac{1}{8x}dx}\\
&=x^{1/8}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int-\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int\frac{1}{2}\frac{-42x^3+20x}{-8x^4+16x^2}\,dx}\\
&=z_1e^{-\ln(x^2-2)-\frac{5\ln(x)}{8}}\\
&=z_1\left(\frac{1}{(x^2-2)\,x^{5/8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{1}{(x^2-2)\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int-\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-42x^3+20x}{-8x^4+16x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x^2-2)-\frac{5\ln(x)}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{4x^2 \mathrm{e}^{-2\ln(x^2-2)-\frac{5\ln(x)}{4}} (x^2-2)^2}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{(x^2-2)\sqrt{x}}\right) + c_2 \left(\frac{1}{(x^2-2)\sqrt{x}} \left(\frac{4x^2 \mathrm{e}^{-2\ln(x^2-2)-\frac{5\ln(x)}{4}} (x^2-2)^2}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.117.2 Maple step by step solution

Let's solve

$8x^2(-x^2+2)\left(\frac{d}{dx}y'\right) + 2x(-21x^2+10)\, y' - (35x^2+2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(35x^2+2)y}{8x^2(x^2-2)} - \frac{(21x^2-10)y'}{4x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(21x^2-10)y'}{4x(x^2-2)} + \frac{(35x^2+2)y}{8x^2(x^2-2)} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{21x^2-10}{4x(x^2-2)}, P_3(x) = \frac{35x^2+2}{8x^2(x^2-2)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{4}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{8}$
- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $8x^2(x^2-2)\left(\frac{d}{dx}y'\right) + 2x(21x^2-10)\,y' + (35x^2+2)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0(1+2r)\,(-1+4r)\,x^r - 2a_1(3+2r)\,(3+4r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (-2a_k(2k+2r+1)\,(4k+4r-1)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2(1+2r)\,(-1+4r) = 0$
- Values of r that satisfy the indicial equation

$r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$

- Each term must be 0

  $-2a_1(3+2r)(3+4r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $-(2k+2r+1)(4k+4r-1)(2a_k - a_{k-2}) = 0$

- Shift index using $k- > k+2$

  $-(2k+2r+5)(4k+4r+7)(2a_{k+2} - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{2}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{a_k}{2}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{4}$

  $a_{k+2} = \frac{a_k}{2}$

- Solution for $r = \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2} = \frac{a_k}{2}, a_1 = 0, b_{k+2} = \frac{b_k}{2}, b_1 = 0\right]$

1.117.3 Maple trace

```
Methods for second order ODEs:
```

1.117.4 Maple dsolve solution

Solving time : 0.012 (sec)
Leaf size : 22

```
dsolve(8*x^2*(-x^2+2)*diff(diff(y(x),x),x)+2*x*(-21*x^2+10)*diff(y(x),x)-(35*x^2+2)*y(
       y(x),singsol=all)
```

$$y = \frac{c_2\, x^{3/4} + c_1}{(x^2-2)\sqrt{x}}$$

1.117.5 Mathematica DSolve solution

Solving time : 0.083 (sec)
Leaf size : 34

```
DSolve[{8*x^2*(2-x^2)*D[y[x],{x,2}]+2*x*(10-21*x^2)*D[y[x],x]-(2+35*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\frac{3c_1}{\sqrt{x}} + 4c_2\sqrt[4]{x}}{6-3x^2}$$

1.118 problem 120

Internal problem ID [8256]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 120
Date solved : `Monday, October 21, 2024 at 05:04:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+3x+1\right)y''-4x\left(-3x^2-3x+1\right)y'+3\left(x^2-x+1\right)y=0$$

1.118.1 Solved as second order ode using Kovacic algorithm

Time used: 0.146 (sec)

Writing the ode as

$$\left(4x^4+12x^3+4x^2\right)y''+\left(12x^3+12x^2-4x\right)y'+\left(3x^2-3x+3\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+12x^3+4x^2 \\ B &= 12x^3+12x^2-4x \\ C &= 3x^2-3x+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 220: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{12x^3+12x^2-4x}{4x^4+12x^3+4x^2}\,dx}\\ &= z_1 e^{\frac{\ln(x)}{2}-\ln(x^2+3x+1)}\\ &= z_1\left(\frac{\sqrt{x}}{x^2+3x+1}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{x^2+3x+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{12x^3+12x^2-4x}{4x^4+12x^3+4x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{\ln(x)-2\ln(x^2+3x+1)}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{x^2+3x+1}\right) + c_2\left(\frac{\sqrt{x}}{x^2+3x+1}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.118.2 Maple step by step solution

Let's solve

$4x^2(x^2+3x+1)\left(\frac{d}{dx}y'\right) - 4x(-3x^2-3x+1)\,y' + 3(x^2-x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{3(x^2-x+1)y}{4x^2(x^2+3x+1)} - \frac{(3x^2+3x-1)y'}{x(x^2+3x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3x^2+3x-1)y'}{x(x^2+3x+1)} + \frac{3(x^2-x+1)y}{4x^2(x^2+3x+1)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{3x^2+3x-1}{x(x^2+3x+1)}, P_3(x) = \frac{3(x^2-x+1)}{4x^2(x^2+3x+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$

  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+3x+1)\left(\frac{d}{dx}y'\right) + 4x(3x^2+3x-1)\,y' + (3x^2-3x+3)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)\,(-3+2r)\, x^r + (a_1(1+2r)\,(-1+2r) + 3a_0(1+2r)\,(-1+2r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(1+2r)\,(-1+2r) + 3a_0(1+2r)\,(-1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -3a_0$$

- Each term in the series must be 0, giving the recursion relation

$$(2k+2r-1)\,(2k+2r-3)\,(a_k + 3a_{k-1} + a_{k-2}) = 0$$

- Shift index using $k->k+2$

$$(2k+2r+3)\,(2k+2r+1)\,(a_{k+2} + 3a_{k+1} + a_k) = 0$$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -3a_{k+1} - a_k$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -3a_{k+1} - a_k$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = -3a_{k+1} - a_k$
- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0, b_{k+2} = -3b_{k+1} - b_k, b_1 = \right.$

1.118.3 Maple trace

```
Methods for second order ODEs:
```

1.118.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 23

```
dsolve(4*x^2*(x^2+3*x+1)*diff(diff(y(x),x),x)-4*x*(-3*x^2-3*x+1)*diff(y(x),x)+3*(x^2-x
       y(x),singsol=all)
```

$$y = \frac{\sqrt{x}\,(c_2 x + c_1)}{x^2 + 3x + 1}$$

1.118.5 Mathematica DSolve solution

Solving time : 0.086 (sec)
Leaf size : 28

```
DSolve[{4*x^2*(1+3*x+x^2)*D[y[x],{x,2}]-4*x*(1-3*x-3*x^2)*D[y[x],x]+3*(1-x+x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}(c_2x + c_1)}{x^2 + 3x + 1}$$

1.119 problem 121

Internal problem ID [8257]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 121
Date solved : `Monday, October 21, 2024 at 05:04:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2(1+x)^2 y'' - x\left(-11x^2 - 10x + 1\right) y' + \left(5x^2+1\right) y = 0$$

1.119.1 Solved as second order ode using Kovacic algorithm

Time used: 0.208 (sec)

Writing the ode as

$$3x^2(1+x)^2 y'' + \left(11x^3 + 10x^2 - x\right) y' + \left(5x^2+1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2(1+x)^2 \\ B &= 11x^3 + 10x^2 - x \\ C &= 5x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{5}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 222: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{36x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{6} - \left(\frac{1}{6}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{6x} + (-)(0) \\
&= \frac{1}{6x} \\
&= \frac{1}{6x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{6x}\right)(0) + \left(\left(-\frac{1}{6x^2}\right) + \left(\frac{1}{6x}\right)^2 - \left(-\frac{5}{36x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{6x} dx} \\ &= x^{1/6} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{11x^3+10x^2-x}{3x^2(1+x)^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{6} - 2\ln(1+x)} \\ &= z_1 \left(\frac{x^{1/6}}{(1+x)^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{(1+x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+10x^2-x}{3x^2(1+x)^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{3}-4\ln(1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{3x^{1/3}\mathrm{e}^{\frac{\ln(x)}{3}-4\ln(1+x)}(1+x)^4}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/3}}{(1+x)^2}\right) + c_2 \left(\frac{x^{1/3}}{(1+x)^2}\left(\frac{3x^{1/3}\mathrm{e}^{\frac{\ln(x)}{3}-4\ln(1+x)}(1+x)^4}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.119.2 Maple step by step solution

Let's solve

$3x^2(1+x)^2 \left(\frac{d}{dx}y'\right) - x(-11x^2-10x+1)\, y' + (5x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(5x^2+1)y}{3x^2(1+x)^2} - \frac{y'(11x-1)}{3x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'(11x-1)}{3x(1+x)} + \frac{(5x^2+1)y}{3x^2(1+x)^2} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{11x-1}{3x(1+x)}, P_3(x) = \frac{5x^2+1}{3x^2(1+x)^2}\right]$

- ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = 4$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $$\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 2$$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $$3x^2(1+x)^2\left(\frac{d}{dx}y'\right) + x(1+x)(11x-1)\,y' + (5x^2+1)\,y = 0$$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $$(3u^4-6u^3+3u^2)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (11u^3-23u^2+12u)\left(\frac{d}{du}y(u)\right) + (5u^2-10u+6)\,y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 2..4$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$3a_0(2+r)\,(1+r)\,u^r + \left(3a_1(3+r)\,(2+r) - a_0(2+r)\,(5+6r)\right)u^{1+r} + \left(\sum_{k=2}^{\infty} \left(3a_k(k+r+2)\,(k+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $3(2+r)(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$
- Each term must be 0

  $3a_1(3+r)(2+r) - a_0(2+r)(5+6r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0(5+6r)}{3(3+r)}$
- Each term in the series must be 0, giving the recursion relation

  $3(a_k + a_{k-2} - 2a_{k-1})k^2 + (6(a_k + a_{k-2} - 2a_{k-1})r + 9a_k - 4a_{k-2} - 5a_{k-1})k + 3(a_k + a_{k-2} - 2a_{}$
- Shift index using $k- >k+2$

  $3(a_{k+2} + a_k - 2a_{k+1})(k+2)^2 + (6(a_{k+2} + a_k - 2a_{k+1})r + 9a_{k+2} - 4a_k - 5a_{k+1})(k+2) + 3(a_{k+}$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 6kra_k - 12kra_{k+1} + 3r^2a_k - 6r^2a_{k+1} + 8ka_k - 29ka_{k+1} + 8ra_k - 29ra_{k+1} + 5a_k - 33a_{k+1}}{3(k^2 + 2kr + r^2 + 7k + 7r + 12)}$
- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2 + 3k + 2)}$
- Solution for $r = -2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-2}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2 + 3k + 2)}, a_1 = -\frac{7a_0}{3}\right]$
- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-2}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2 + 3k + 2)}, a_1 = -\frac{7a_0}{3}\right]$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 2ka_k - 17ka_{k+1} - 10a_{k+1}}{3(k^2 + 5k + 6)}$
- Solution for $r = -1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 2ka_k - 17ka_{k+1} - 10a_{k+1}}{3(k^2 + 5k + 6)}, a_1 = -\frac{a_0}{6}\right]$
- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-1}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 2ka_k - 17ka_{k+1} - 10a_{k+1}}{3(k^2 + 5k + 6)}, a_1 = -\frac{a_0}{6}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k-1}\right), a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2 + 3k + 2)}, a_1 =\right.$

1.119.3 Maple trace

```
Methods for second order ODEs:
```

1.119.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 19

```
dsolve(3*x^2*(1+x)^2*diff(diff(y(x),x),x)-x*(-11*x^2-10*x+1)*diff(y(x),x)+(5*x^2+1)*y(
       y(x),singsol=all)
```

$$y = \frac{c_2 x^{1/3} + c_1 x}{(1+x)^2}$$

1.119.5 Mathematica DSolve solution

Solving time : 0.062 (sec)
Leaf size : 29

```
DSolve[{3*x^2*(1+x)^2*D[y[x],{x,2}]-x*(1-10*x-11*x^2)*D[y[x],x]+(1+5*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 \sqrt[3]{x} + 3c_2 x}{2(x+1)^2}$$

1.120 problem 122

Internal problem ID [8258]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 122
Date solved : `Monday, October 21, 2024 at 05:04:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+2x+3\right)y''-x\left(-15x^2-14x+3\right)y'+\left(7x^2+3\right)y=0$$

1.120.1 Solved as second order ode using Kovacic algorithm

Time used: 0.222 (sec)

Writing the ode as

$$\left(4x^4+8x^3+12x^2\right)y''+\left(15x^3+14x^2-3x\right)y'+\left(7x^2+3\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+8x^3+12x^2 \\ B &= 15x^3+14x^2-3x \\ C &= 7x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -7$$
$$t = 64x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{7}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 224: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{7}{64x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{7}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{8} - \left(\frac{1}{8}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{8x} + (-)(0) \\ &= \frac{1}{8x} \\ &= \frac{1}{8x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}(0) + 2\left(\frac{1}{8x}\right)(0) + \left(\left(-\frac{1}{8x^2}\right) + \left(\frac{1}{8x}\right)^2 - \left(-\frac{7}{64x^2}\right)\right) = 0 \\ 0 = 0\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \frac{1}{8x} dx} \\ &= x^{1/8}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{15x^3+14x^2-3x}{4x^4+8x^3+12x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{8} - \ln(x^2+2x+3)} \\ &= z_1 \left(\frac{x^{1/8}}{x^2+2x+3}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}}{x^2+2x+3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{15x^3+14x^2-3x}{4x^4+8x^3+12x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{4}-2\ln(x^2+2x+3)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{4\sqrt{x}\,\mathrm{e}^{\frac{\ln(x)}{4}-2\ln(x^2+2x+3)}\,(x^2+2x+3)^2}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/4}}{x^2+2x+3}\right) + c_2\left(\frac{x^{1/4}}{x^2+2x+3}\left(\frac{4\sqrt{x}\,\mathrm{e}^{\frac{\ln(x)}{4}-2\ln(x^2+2x+3)}\,(x^2+2x+3)^2}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.120.2 Maple step by step solution

Let's solve

$4x^2(x^2+2x+3)\left(\frac{d}{dx}y'\right) - x(-15x^2-14x+3)\,y' + (7x^2+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(7x^2+3)y}{4x^2(x^2+2x+3)} - \frac{(15x^2+14x-3)y'}{4x(x^2+2x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(15x^2+14x-3)y'}{4x(x^2+2x+3)} + \frac{(7x^2+3)y}{4x^2(x^2+2x+3)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{15x^2+14x-3}{4x(x^2+2x+3)}, P_3(x) = \frac{7x^2+3}{4x^2(x^2+2x+3)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\Big|_{x=0} = -\frac{1}{4}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+2x+3)\left(\frac{d}{dx}y'\right) + x(15x^2+14x-3)\,y' + (7x^2+3)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $3a_0(-1+4r)\,(-1+r)\,x^r + (3a_1(3+4r)\,r + 2a_0r(3+4r))\,x^{1+r} + \left(\sum_{k=2}^{\infty} (3a_k(4k+4r-1)\,(k+r\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$3(-1+4r)(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{1,\frac{1}{4}\right\}$

- Each term must be 0

  $3a_1(3+4r)r+2a_0r(3+4r)=0$

- Solve for the dependent coefficient(s)

  $a_1=-\frac{2a_0}{3}$

- Each term in the series must be 0, giving the recursion relation

  $(4k+4r-1)(k+r-1)(3a_k+2a_{k-1}+a_{k-2})=0$

- Shift index using $k->k+2$

  $(4k+4r+7)(k+r+1)(3a_{k+2}+2a_{k+1}+a_k)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{2a_{k+1}}{3}-\frac{a_k}{3}$

- Recursion relation for $r=1$

  $a_{k+2}=-\frac{2a_{k+1}}{3}-\frac{a_k}{3}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+1},a_{k+2}=-\frac{2a_{k+1}}{3}-\frac{a_k}{3},a_1=-\frac{2a_0}{3}\right]$

- Recursion relation for $r=\frac{1}{4}$

  $a_{k+2}=-\frac{2a_{k+1}}{3}-\frac{a_k}{3}$

- Solution for $r=\frac{1}{4}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{4}},a_{k+2}=-\frac{2a_{k+1}}{3}-\frac{a_k}{3},a_1=-\frac{2a_0}{3}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k+1}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{4}}\right),a_{k+2}=-\frac{2a_{k+1}}{3}-\frac{a_k}{3},a_1=-\frac{2a_0}{3},b_{k+2}=-\frac{2b_{k+1}}{3}-\frac{b_k}{3},b_1=-\right.$

1.120.3 Maple trace

```
Methods for second order ODEs:
```

1.120.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 24

```
dsolve(4*x^2*(x^2+2*x+3)*diff(diff(y(x),x),x)-x*(-15*x^2-14*x+3)*diff(y(x),x)+(7*x^2+3
       y(x),singsol=all)
```

$$y = \frac{c_2 x^{1/4} + c_1 x}{x^2 + 2x + 3}$$

1.120.5 Mathematica DSolve solution

Solving time : `0.09` (sec)
Leaf size : 33

```
DSolve[{4*x^2*(3+2*x+x^2)*D[y[x],{x,2}]-x*(3-14*x-15*x^2)*D[y[x],x]+(3+7*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3c_1 \sqrt[4]{x} + 4c_2 x}{3x^2 + 6x + 9}$$

1.121 problem 123

Internal problem ID [8259]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 123
Date solved : `Monday, October 21, 2024 at 05:04:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2-2x+1)\,y''-x(3+x)\,y'+(4+x)\,y=0$$

1.121.1 Solved as second order ode using Kovacic algorithm

Time used: 0.370 (sec)

Writing the ode as

$$x^2(x-1)^2\,y''+\left(-x^2-3x\right)y'+(4+x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(x-1)^2 \\ B &= -x^2-3x \\ C &= 4+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{7x^2 + 10x - 1}{4x^2 (x-1)^4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 7x^2 + 10x - 1 \\
t &= 4x^2(x-1)^4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{7x^2 + 10x - 1}{4x^2 (x-1)^4}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 226: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2(x-1)^4$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{2(x-1)} + \frac{3}{2x} + \frac{4}{(x-1)^4} - \frac{2}{(x-1)^3} + \frac{7}{4(x-1)^2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r=-\frac{3}{2(x-1)}+\frac{3}{2x}+\frac{4}{(x-1)^4}-\frac{2}{(x-1)^3}+\frac{7}{4(x-1)^2}-\frac{1}{4x^2}$$

There is pole in r at $x=1$ of order 4, hence $v=2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c=1$ gives

$$[\sqrt{r}]_c \approx \frac{2}{(x-1)^2}-\frac{1}{2(x-1)}+\frac{21}{32}-\frac{9x}{32}+\frac{53(x-1)^2}{256}-\frac{149(x-1)^3}{1024}+\dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v=2$ the above becomes

$$[\sqrt{r}]_c=\frac{2}{(x-1)^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-1)^2}$ is

$$a=2$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c=1$. This term becomes $\frac{1}{(x-1)^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -2. Therefore

$$\begin{aligned}b&=(-2)-(0)\\&=-2\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_c&=\frac{2}{(x-1)^2}\\ \alpha_c^{+}&=\frac{1}{2}\left(\frac{b}{a}+v\right)&&=\frac{1}{2}\left(\frac{-2}{2}+2\right)&&=\frac{1}{2}\\ \alpha_c^{-}&=\frac{1}{2}\left(-\frac{b}{a}+v\right)&&=\frac{1}{2}\left(-\frac{-2}{2}+2\right)&&=\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}[\sqrt{r}]_\infty&=0\\ \alpha_\infty^{+}&=0\\ \alpha_\infty^{-}&=1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2 + 10x - 1}{4x^2 (x-1)^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	4	$\frac{2}{(x-1)^2}$	$\frac{1}{2}$	$\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} + (-)(0) \\
&= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} \\
&= \frac{2x^2 + x + 1}{2x (x-1)^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{4}{(x-1)^3} - \frac{1}{2\,(x-1)^2}\right) + \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right) dx} \\ &= \sqrt{x-1}\,\sqrt{x}\,\mathrm{e}^{-\frac{2}{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2(x-1)^2}\, dx} \\ &= z_1 e^{-\frac{2}{x-1} - \frac{3\ln(x-1)}{2} + \frac{3\ln(x)}{2}} \\ &= z_1 \left(\frac{x^{3/2}\mathrm{e}^{-\frac{2}{x-1}}}{(x-1)^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}}{(x-1)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2(x-1)^2} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-\frac{4}{x-1}-3\ln(x-1)+3\ln(x)}}{(y_1)^2} dx$$
$$= y_1 \left(\mathrm{e}^{-4} \operatorname{Ei}_1\left(-\frac{4}{x-1}-4\right)\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x^{3/2} \mathrm{e}^{-\frac{4}{x-1}} \sqrt{x(x-1)}}{(x-1)^{3/2}}\right) + c_2 \left(\frac{x^{3/2} \mathrm{e}^{-\frac{4}{x-1}} \sqrt{x(x-1)}}{(x-1)^{3/2}} \left(\mathrm{e}^{-4} \operatorname{Ei}_1\left(-\frac{4}{x-1}-4\right)\right)\right)$$

Will add steps showing solving for IC soon.

1.121.2 Maple step by step solution

Let's solve

$x^2(x^2-2x+1)\left(\frac{d}{dx}y'\right) - x(3+x)\,y' + (4+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4+x)y}{x^2(x^2-2x+1)} + \frac{(3+x)y'}{x(x^2-2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(3+x)y'}{x(x^2-2x+1)} + \frac{(4+x)y}{x^2(x^2-2x+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{3+x}{x(x^2-2x+1)}, P_3(x) = \frac{4+x}{x^2(x^2-2x+1)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$(x^2 \cdot P_3(x))\bigg|_{x=0} = 4$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$
- Multiply by denominators

  $x^2(x^2 - 2x + 1)\left(\frac{d}{dx}y'\right) - x(3 + x)\, y' + (4 + x)\, y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ○ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$
  - ○ Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$
  - ○ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r-2+m}$
  - ○ Shift index using $k- >k + 2 - m$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2 + r)^2 x^r + \left(a_1(-1 + r)^2 - a_0(1 + 2r)\,(-1 + r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k + r - 2)^2 - a_{k-1}(2k -\right.\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2 + r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 2$

- Each term must be 0

  $$a_1(-1+r)^2 - a_0(1+2r)(-1+r) = 0$$
- Solve for the dependent coefficient(s)

  $$a_1 = \frac{a_0(1+2r)}{-1+r}$$
- Each term in the series must be 0, giving the recursion relation

  $$((a_k + a_{k-2} - 2a_{k-1})k + (a_k + a_{k-2} - 2a_{k-1})r - 2a_k - 3a_{k-2} + a_{k-1})(k+r-2) = 0$$
- Shift index using $k->k+2$

  $$((a_{k+2} + a_k - 2a_{k+1})(k+2) + (a_{k+2} + a_k - 2a_{k+1})r - 2a_{k+2} - 3a_k + a_{k+1})(k+r) = 0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{ka_k - 2ka_{k+1} + ra_k - 2ra_{k+1} - a_k - 3a_{k+1}}{k+r}$$
- Recursion relation for $r = 2$

  $$a_{k+2} = -\frac{ka_k - 2ka_{k+1} + a_k - 7a_{k+1}}{k+2}$$
- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{ka_k - 2ka_{k+1} + a_k - 7a_{k+1}}{k+2}, a_1 = 5a_0\right]$$

1.121.3 Maple trace

```
Methods for second order ODEs:
```

1.121.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 45

```
dsolve(x^2*(x^2-2*x+1)*diff(diff(y(x),x),x)-x*(3+x)*diff(y(x),x)+(4+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^2\left(c_2\,\mathrm{e}^{-\frac{4x}{x-1}}\,\mathrm{Ei}_1\left(-\frac{4x}{x-1}\right) + \mathrm{e}^{-\frac{4}{x-1}}c_1\right)}{x-1}$$

1.121.5 Mathematica DSolve solution

Solving time : 0.368 (sec)
Leaf size : 54

```
DSolve[{x^2*(1-2*x+x^2)*D[y[x],{x,2}]-x*(3+x)*D[y[x],x]+(4+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{4x}{x-1}}\sqrt{1-x}x^2\left(c_2 \operatorname{ExpIntegralEi}\left(\frac{4x}{x-1}\right) + e^4 c_1\right)}{(x-1)^{3/2}}$$

1.122 problem 124

Internal problem ID [8260]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 124
Date solved : `Monday, October 21, 2024 at 05:04:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''+5x^2y'+(1+x)\,y=0$$

1.122.1 Solved as second order ode using Kovacic algorithm

Time used: 0.257 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y''+5x^2y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+4x^2\\B&=5x^2\\C&=1+x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2 - 24x - 16 \\ t &= 16\left(x^2 + 2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 228: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{5}{16\,(2+x)^2} - \frac{1}{8x} + \frac{1}{16+8x}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 24x - 16}{16\,(x^2+2x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-3x^2-24x-16}{16\left(x^2+2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= \frac{1}{4}-\left(\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(2+x\right)} + \frac{1}{2x} + (-)\left(0\right) \\
&= -\frac{1}{4\left(2+x\right)} + \frac{1}{2x} \\
&= \frac{x+4}{4x\left(2+x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(2+x\right)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{4\left(2+x\right)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{4\left(2+x\right)} + \frac{1}{2x}\right)^2 - \left(\frac{-3x^2-24x-16}{16\left(x^2+2x\right)^2}\right)
$$

$$
0 =
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{4(2+x)} + \frac{1}{2x}\right) dx} \\
&= \frac{\sqrt{x}}{\left(2+x\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x^2}{2x^3+4x^2} \, dx} \\
&= z_1 e^{-\frac{5\ln(2+x)}{4}} \\
&= z_1 \left(\frac{1}{\left(2+x\right)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(2+x)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2}{2x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(2+x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(2\sqrt{2+x} - 2\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{(2+x)^{3/2}}\right) + c_2\left(\frac{\sqrt{x}}{(2+x)^{3/2}}\left(2\sqrt{2+x} - 2\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.122.2 Maple step by step solution

Let's solve

$2x^2(2+x)\left(\frac{d}{dx}y'\right) + 5x^2y' + (1+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(1+x)y}{2x^2(2+x)} - \frac{5y'}{2(2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + \frac{5y'}{2(2+x)} + \frac{(1+x)y}{2x^2(2+x)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions
    $\left[P_2(x) = \frac{5}{2(2+x)}, P_3(x) = \frac{1+x}{2x^2(2+x)}\right]$
  - $(2+x) \cdot P_2(x)$ is analytic at $x = -2$
    $\left((2+x) \cdot P_2(x)\right)\Big|_{x=-2} = \frac{5}{2}$
  - $(2+x)^2 \cdot P_3(x)$ is analytic at $x = -2$
    $\left((2+x)^2 \cdot P_3(x)\right)\Big|_{x=-2} = 0$
  - $x = -2$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = -2$

- Multiply by denominators
  $2x^2(2+x)\left(\frac{d}{dx}y'\right) + 5x^2y' + (1+x)\,y = 0$
- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$
  $(2u^3 - 8u^2 + 8u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (5u^2 - 20u + 20)\left(\frac{d}{du}y(u)\right) + (-1+u)\,y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$
    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k- >k - m$
    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)(k+1-m+r) u^{k+r}$$

  Rewrite ODE with series expansions

  $$4a_0 r(3+2r) u^{-1+r} + (4a_1(1+r)(5+2r) - a_0(8r^2+12r+1)) u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)(2k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4r(3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{3}{2}\right\}$$

- Each term must be 0

  $$4a_1(1+r)(5+2r) - a_0(8r^2+12r+1) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$2(-4a_k + a_{k-1} + 4a_{k+1}) k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1}) r - 12a_k - a_{k-1} + 28a_{k+1}) k + 2(-4a_k +$$

- Shift index using $k->k+1$

  $$2(-4a_{k+1} + a_k + 4a_{k+2}) (k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2}) r - 12a_{k+1} - a_k + 28a_{k+2}) (k+1) -$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 4kr a_k - 16kr a_{k+1} + 2r^2 a_k - 8r^2 a_{k+1} + 3k a_k - 28k a_{k+1} + 3r a_k - 28r a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2+4kr+2r^2+11k+11r+14)}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2+11k+14)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = 2 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

  $$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3 a_{k+1}}{4(2k^2+5k+2)}$$

- Solution for $r = -\frac{3}{2}$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3 a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = 2 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (2+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (2+x)^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2+11k+14)}, 20a\right.$$

1.122.3 Maple trace

```
Methods for second order ODEs:
```

1.122.4 Maple dsolve solution

Solving time : `0.011` (sec)
Leaf size : 39

```
dsolve(2*x^2*(2+x)*diff(diff(y(x),x),x)+5*x^2*diff(y(x),x)+(1+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\left(\sqrt{2}\,\sqrt{2+x}\,c_2 - 2\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right) c_2 + c_1\right)\sqrt{x}}{(2+x)^{3/2}}$$

1.122.5 Mathematica DSolve solution

Solving time : `0.162` (sec)
Leaf size : 55

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}]+5*x^2*D[y[x],x]+(1+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}\left(-2\sqrt{2}c_2\text{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right) + 2c_2\sqrt{x+2} + c_1\right)}{(x+2)^{3/2}}$$

1.123 problem 125

Internal problem ID [8261]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 125
Date solved : `Monday, October 21, 2024 at 05:04:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-x^2+2\right) y''-2x\left(2x^2+1\right) y'+\left(-2x^2+2\right) y=0$$

1.123.1 Solved as second order ode using Kovacic algorithm

Time used: 0.442 (sec)

Writing the ode as

$$\left(-x^4+2x^2\right) y''+\left(-4x^3-2x\right) y'+\left(-2x^2+2\right) y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+2x^2 \\ B &= -4x^3-2x \\ C &= -2x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3x^2 - 1}{(x^3 - 2x)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3x^2 - 1 \\
t &= (x^3 - 2x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3x^2 - 1}{(x^3 - 2x)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 230: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{5}{16\left(x - \sqrt{2}\right)^2} + \frac{5}{16\left(x + \sqrt{2}\right)^2} - \frac{3\sqrt{2}}{32\left(x - \sqrt{2}\right)} + \frac{3\sqrt{2}}{32\left(x + \sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^2 - 1}{\left(x^3 - 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\sqrt{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\sqrt{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{1}{4\left(x - \sqrt{2}\right)} - \frac{1}{4\left(x + \sqrt{2}\right)} + (0) \\
&= \frac{1}{2x} - \frac{1}{4\left(x - \sqrt{2}\right)} - \frac{1}{4\left(x + \sqrt{2}\right)} \\
&= -\frac{1}{x^3 - 2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{4\left(x - \sqrt{2}\right)} - \frac{1}{4\left(x + \sqrt{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{4\left(x - \sqrt{2}\right)^2} + \frac{1}{4\left(x + \sqrt{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{4\left(x - \sqrt{2}\right)} - \frac{1}{4\left(x + \sqrt{2}\right)} \right) dx} \\
&= \frac{\sqrt{x}}{\left(x - \sqrt{2}\right)^{1/4} \left(x + \sqrt{2}\right)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^3-2x}{-x^4+2x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}-\frac{5\ln\left(x^2-2\right)}{4}} \\
&= z_1 \left(\frac{\sqrt{x}}{\left(x^2-2\right)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{\left(x^2-2\right)^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3-2x}{-x^4+2x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{5\ln\left(x^2-2\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\sqrt{x^2-2}+\sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{\left(x^2-2\right)^{3/2}}\right) + c_2 \left(\frac{x}{\left(x^2-2\right)^{3/2}}\left(\sqrt{x^2-2}+\sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.123.2 Maple step by step solution

Let's solve

$x^2(-x^2+2)\left(\frac{d}{dx}y'\right)-2x(2x^2+1)\,y'+(-2x^2+2)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{2(x^2-1)y}{x^2(x^2-2)}-\frac{2(2x^2+1)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{2(2x^2+1)y'}{x(x^2-2)}+\frac{2(x^2-1)y}{x^2(x^2-2)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{2(2x^2+1)}{x(x^2-2)},P_3(x)=\frac{2(x^2-1)}{x^2(x^2-2)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=-1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=1$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(x^2-2)\left(\frac{d}{dx}y'\right)+2x(2x^2+1)\,y'+(2x^2-2)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0(-1+r)^2\, x^r - 2a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(-2a_k(k+r-1)^2 + a_{k-2}(k+r)\,(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 1$$

- Each term must be 0

$$-2a_1 r^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-2a_k(k+r-1)^2 + a_{k-2}(k+r)\,(k+r-1) = 0$$

- Shift index using $k->k+2$

$$-2a_{k+2}(k+r+1)^2 + a_k(k+r+2)\,(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a_k(k+r+2)}{2(k+r+1)}$$

- Recursion relation for $r = 1$

$$a_{k+2} = \frac{a_k(k+3)}{2(k+2)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k(k+3)}{2(k+2)}, a_1 = 0\right]$$

1.123.3 Maple trace

```
Methods for second order ODEs:
```

1.123.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 42

```
dsolve(x^2*(-x^2+2)*diff(diff(y(x),x),x)-2*x*(2*x^2+1)*diff(y(x),x)+(-2*x^2+2)*y(x) =
       y(x),singsol=all)
```

$$y = \frac{x\left(\sqrt{2}\, c_2 \sqrt{x^2-2} + 2 \arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right) c_2 + c_1\right)}{\left(x^2-2\right)^{3/2}}$$

1.123.5 Mathematica DSolve solution

Solving time : `0.208` (sec)
Leaf size : 58

```
DSolve[{x^2*(2-x^2)*D[y[x],{x,2}]-2*x*(1+2*x^2)*D[y[x],x]+(2-2*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x\left(-\sqrt{2} c_2 \text{arctanh}\left(\sqrt{1-\frac{x^2}{2}}\right) + c_2\sqrt{2-x^2} + c_1\right)}{\left(2-x^2\right)^{3/2}}$$

1.124 problem 126

Internal problem ID [8262]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 126
Date solved : `Monday, October 21, 2024 at 05:04:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(5-x)\,y' + (9-4x)\,y = 0$$

1.124.1 Solved as second order ode using Kovacic algorithm

Time used: 0.545 (sec)

Writing the ode as

$$x^2y'' + \left(x^2-5x\right)y' + (9-4x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2-5x \\ C &= 9-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 + 6x - 1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 6x - 1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 + 6x - 1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 232: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{2x} - \frac{5}{2x^2} + \frac{15}{2x^3} - \frac{115}{4x^4} + \frac{495}{4x^5} - \frac{2285}{4x^6} + \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{6x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 6. Dividing this by leading coefficient in t which is 4 gives $\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{3}{2}\right) - (0) \\ &= \frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2x} + \frac{1}{2} \\
&= \frac{1+x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2+6x-1}{4x^2}\right) \right) = 0$$

$$\frac{1-a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (1+x)\, e^{\int \left(\frac{1}{2x} + \frac{1}{2}\right) dx} \\
&= (1+x)\, e^{\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= (1+x)\sqrt{x}\, e^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2-5x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\frac{5\ln(x)}{2}} \\
&= z_1\left(x^{5/2}\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^3(1+x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-5x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\operatorname{Ei}_1(x) - \frac{\mathrm{e}^{-x}}{-1-x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^3(1+x)\right) + c_2\left(x^3(1+x)\left(-\operatorname{Ei}_1(x) - \frac{\mathrm{e}^{-x}}{-1-x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.124.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x(5-x)\,y' + (9-4x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(-9+4x)y}{x^2} - \frac{(-5+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(-5+x)y'}{x} - \frac{(-9+4x)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{-5+x}{x}, P_3(x) = -\frac{-9+4x}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -5$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 9$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x(-5+x)\,y' + (9-4x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-3)^2 + a_{k-1}(k-5+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 3$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-3)^2 + a_{k-1}(k-5+r) = 0$$

- Shift index using $k- >k+1$

$$a_{k+1}(k-2+r)^2 + a_k(k+r-4) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r-4)}{(k-2+r)^2}$$

- Recursion relation for $r=3$; series terminates at $k=1$

$$a_{k+1} = -\frac{a_k(k-1)}{(k+1)^2}$$

- Apply recursion relation for $k=0$

$$a_1 = a_0$$

- Terminating series solution of the ODE for $r=3$. Use reduction of order to find the second li

$$y = a_0 \cdot (1+x)$$

1.124.3 Maple trace

```
Methods for second order ODEs:
```

1.124.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 27

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(5-x)*diff(y(x),x)+(9-4*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^3\left(-c_2\, e^{-x} + \left(\mathrm{Ei}_1\left(x\right) c_2 + c_1\right)\left(1 + x\right)\right)$$

1.124.5 Mathematica DSolve solution

Solving time : 0.415 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]-x*(5-x)*D[y[x],x]+(9-4*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x}x^3(c_2e^x(x+1)\,\mathrm{ExpIntegralEi}(-x) + c_1e^x(x+1) + c_2)$$

1.125 problem 127

Internal problem ID [8263]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 127
Date solved : `Monday, October 21, 2024 at 05:04:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+x+1\right)y''+12x^2(1+x)\,y'+\left(3x^2+3x+1\right)y=0$$

1.125.1 Solved as second order ode using Kovacic algorithm

Time used: 0.865 (sec)

Writing the ode as

$$\left(4x^4+4x^3+4x^2\right)y''+\left(12x^3+12x^2\right)y'+\left(3x^2+3x+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 4x^4+4x^3+4x^2\\ B &= 12x^3+12x^2\\ C &= 3x^2+3x+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 4x - 1}{4\left(x^3 + x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 - 4x - 1$$
$$t = 4\left(x^3 + x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 4x - 1}{4\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 234: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{3}{8} - \frac{i\sqrt{3}}{8}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{3}{8} + \frac{i\sqrt{3}}{8}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{\frac{1}{4} - \frac{5i\sqrt{3}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{4} + \frac{5i\sqrt{3}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} - \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{8} - \frac{i\sqrt{3}}{8}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{-2 - 2i\sqrt{3}}}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{-2 - 2i\sqrt{3}}}{4} \end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{8} + \frac{i\sqrt{3}}{8}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{-2+2i\sqrt{3}}}{4} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^2 - 4x - 1}{4\left(x^3 + x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{-2-2i\sqrt{3}}}{4}$	$\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}$
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{-2+2i\sqrt{3}}}{4}$	$\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} + (-)\,(0) \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \\
&= \frac{2x^2+1}{2x\,(x^2+x+1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} \right) -$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right) dx} \\
&= \sqrt{x}\,(x^2+x+1)^{1/4}\,\sqrt{2}\, e^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{12x^3+12x^2}{4x^4+4x^3+4x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+x+1\right)}{4}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}}{\left(x^2+x+1\right)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\sqrt{x}\,\sqrt{2}}{\sqrt{x^2+x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{12x^3+12x^2}{4x^4+4x^3+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+x+1\right)}{2}-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{3\ln\left(x^2+x+1\right)}{2}-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\left(x^2+x+1\right)\mathrm{e}^{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{2x}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\sqrt{x}\,\sqrt{2}}{\sqrt{x^2+x+1}}\right) \\
&\quad + c_2 \left(\frac{e^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\sqrt{x}\,\sqrt{2}}{\sqrt{x^2+x+1}}\left(\int \frac{e^{-\frac{3\ln\left(x^2+x+1\right)}{2}-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\left(x^2+x+1\right)e^{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{2x}dx
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.125.2 Maple step by step solution

Let's solve

$4x^2(x^2+x+1)\left(\frac{d}{dx}y'\right)+12x^2(1+x)\,y'+(3x^2+3x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(3x^2+3x+1)y}{4x^2(x^2+x+1)}-\frac{3(1+x)y'}{x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{3(1+x)y'}{x^2+x+1}+\frac{(3x^2+3x+1)y}{4x^2(x^2+x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{3(1+x)}{x^2+x+1}, P_3(x)=\frac{3x^2+3x+1}{4x^2(x^2+x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+x+1)\left(\frac{d}{dx}y'\right)+12x^2(1+x)\,y'+(3x^2+3x+1)\,y=0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 2..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx} y'\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r-2+m}$

  - Shift index using $k- >k + 2 - m$

    $x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + 2r)^2 x^r + \left(a_1(1 + 2r)^2 + a_0(3 + 2r)\,(1 + 2r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k + 2r - 1)^2 + a_{k-1}(2k\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + 2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$

- Each term must be 0

  $a_1(1 + 2r)^2 + a_0(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -\frac{(3+2r)a_0}{1+2r}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k + r - \frac{1}{2}\right)\left((a_k + a_{k-2} + a_{k-1})\, k + (a_k + a_{k-2} + a_{k-1})\, r - \frac{a_k}{2} - \frac{3a_{k-2}}{2} + \frac{a_{k-1}}{2}\right) = 0$

- Shift index using $k->k+2$

$$4\left(k+\frac{3}{2}+r\right)\left(\left(a_{k+2}+a_k+a_{k+1}\right)(k+2)+\left(a_{k+2}+a_k+a_{k+1}\right)r-\frac{a_{k+2}}{2}-\frac{3a_k}{2}+\frac{a_{k+1}}{2}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2ka_k+2ka_{k+1}+2ra_k+2ra_{k+1}+a_k+5a_{k+1}}{2k+2r+3}$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{2ka_k+2ka_{k+1}+2a_k+6a_{k+1}}{2k+4}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{2ka_k+2ka_{k+1}+2a_k+6a_{k+1}}{2k+4}, a_1=-2a_0\right]$$

1.125.3 Maple trace

```
Methods for second order ODEs:
```

1.125.4 Maple dsolve solution

Solving time : `0.313` (sec)
Leaf size : 143

```
dsolve(4*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+12*x^2*(1+x)*diff(y(x),x)+(3*x^2+3*x+1)*y(
       y(x),singsol=all)
```

$$y=\frac{\sqrt{i\sqrt{3}-2x-1}\,\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}\sqrt{x}\left(c_1\left(\frac{-2ix-i+\sqrt{3}}{\sqrt{3}+2ix+i}\right)^{\frac{1}{4}-\frac{i\sqrt{3}}{4}}+c_2\left(\frac{-2ix-i+\sqrt{3}}{\sqrt{3}+2ix+i}\right)^{\frac{3}{4}+\frac{i\sqrt{3}}{4}}\text{hypergeom}\left(\left[1,\frac{1}{2}+\right.\right.\right.}{\left(x^2+x+1\right)^{3/4}}$$

1.125.5 Mathematica DSolve solution

Solving time : `1.496` (sec)
Leaf size : 93

```
DSolve[{4*x^2*(1+x+x^2)*D[y[x],{x,2}]+12*x^2*(1+x)*D[y[x],x]+(1+3*x+3*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}e^{-\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}\left(c_2\int_1^x \frac{e^{\sqrt{3}\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}}{K[1]\sqrt{K[1]^2+K[1]+1}}dK[1]+c_1\right)}{\sqrt{x^2+x+1}}$$

1.126 problem 128

Internal problem ID [8264]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 128
Date solved : `Monday, October 21, 2024 at 05:04:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+x+1)\,y''-x(-2x^2-4x+1)\,y'+y=0$$

1.126.1 Solved as second order ode using Kovacic algorithm

Time used: 0.914 (sec)

Writing the ode as

$$x^2(x^2+x+1)\,y''+(2x^3+4x^2-x)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2(x^2+x+1)\\ B&=2x^3+4x^2-x\\ C&=1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{10x^2 - 8x - 1}{4\left(x^3 + x^2 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 10x^2 - 8x - 1 \\
t &= 4\left(x^3 + x^2 + x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{10x^2 - 8x - 1}{4\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 236: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{2x} - \frac{1}{4x^2} + \frac{-\frac{29}{24} - \frac{7i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{29}{24} + \frac{7i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{\frac{3}{4} - \frac{41i\sqrt{3}}{36}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{3}{4} + \frac{41i\sqrt{3}}{36}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{24} - \frac{7i\sqrt{3}}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{-138 - 42i\sqrt{3}}}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{-138 - 42i\sqrt{3}}}{12} \end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{24} + \frac{7i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{-138+42i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{10x^2 - 8x - 1}{4\left(x^3 + x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{-138-42i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}$
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{-138+42i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} + (-)(0) \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \\
&= \frac{2x^2 - 2x + 1}{2x(x^2 + x + 1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right.}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right) dx} \\
&= (x^2 + x + 1)^{1/4} \sqrt{2}\, \sqrt{x}\, e^{-\frac{7\sqrt{3} \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+4x^2-x}{x^2(x^2+x+1)}\,dx} \\
&= z_1 e^{-\frac{3\ln(x^2+x+1)}{4}-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}+\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{\sqrt{x}\,\mathrm{e}^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{(x^2+x+1)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x\,\mathrm{e}^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{\sqrt{x^2+x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+4x^2-x}{x^2(x^2+x+1)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x^2+x+1)}{2}-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{3\ln(x^2+x+1)}{2}-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}+\ln(x)}(x^2+x+1)\,\mathrm{e}^{\frac{14\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{2x^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$

$$= c_1\left(\frac{x\,\mathrm{e}^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{\sqrt{x^2+x+1}}\right)$$

$$+c_2\left(\frac{x\,\mathrm{e}^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{\sqrt{x^2+x+1}}\left(\int \frac{\mathrm{e}^{-\frac{3\ln\left(x^2+x+1\right)}{2}-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}+\ln(x)}(x^2+x+1)\,\mathrm{e}^{\frac{14\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{2x^2}\right.\right.$$

Will add steps showing solving for IC soon.

1.126.2 Maple step by step solution

Let's solve

$x^2(x^2+x+1)\left(\frac{d}{dx}y'\right) - x(-2x^2-4x+1)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x^2(x^2+x+1)} - \frac{(2x^2+4x-1)\,y'}{x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(2x^2+4x-1)\,y'}{x(x^2+x+1)} + \frac{y}{x^2(x^2+x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x^2+4x-1}{x(x^2+x+1)}, P_3(x) = \frac{1}{x^2(x^2+x+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+x+1)\left(\frac{d}{dx}y'\right) + x(2x^2+4x-1)\,y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  ○ Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  ○ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  ○ Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2 x^r + \left(a_1 r^2 + a_0 r(3+r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-1}(k+r-1)\,(k+2+r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $a_1 r^2 + a_0 r(3+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -\frac{(3+r)a_0}{r}$

- Each term in the series must be 0, giving the recursion relation

  $\left((a_k + a_{k-2} + a_{k-1})\, k + (a_k + a_{k-2} + a_{k-1})\, r - a_k - 2a_{k-2} + 2a_{k-1}\right)(k+r-1) = 0$

- Shift index using $k->k+2$

  $\left((a_{k+2} + a_k + a_{k+1})\,(k+2) + (a_{k+2} + a_k + a_{k+1})\, r - a_{k+2} - 2a_k + 2a_{k+1}\right)(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k a_k + k a_{k+1} + r a_k + r a_{k+1} + 4 a_{k+1}}{k+r+1}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{k a_k + k a_{k+1} + a_k + 5 a_{k+1}}{k+2}$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{k a_k + k a_{k+1} + a_k + 5 a_{k+1}}{k+2}, a_1 = -4 a_0\right]$$

1.126.3 Maple trace

```
Methods for second order ODEs:
```

1.126.4 Maple dsolve solution

Solving time : `0.156` (sec)
Leaf size : 147

```
dsolve(x^2*(x^2+x+1)*diff(diff(y(x),x),x)-x*(-2*x^2-4*x+1)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} x \left(c_2 \left(2x+1+i\sqrt{3}\right)^{\frac{3}{4}+\frac{7i\sqrt{3}}{12}} \left(i\sqrt{3}-2x-1\right)^{-\frac{1}{4}-\frac{7i\sqrt{3}}{12}} \operatorname{hypergeom}\left(\left[1, \frac{1}{2}+\frac{7i\sqrt{3}}{6}\right], \left[\frac{3}{2}+\right.\right.\right.}{\left(x^2+x+1\right)^{3/4}}$$

1.126.5 Mathematica DSolve solution

Solving time : `1.535` (sec)
Leaf size : 90

```
DSolve[{x^2*(1+x+x^2)*D[y[x],{x,2}]-x*(1-4*x-2*x^2)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x e^{-\frac{7\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}} \left(c_2 \int_1^x \frac{e^{\frac{7\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}{\sqrt{3}}}}{K[1]\sqrt{K[1]^2+K[1]+1}} dK[1] + c_1\right)}{\sqrt{x^2+x+1}}$$

1.127 problem 129

Internal problem ID [8265]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 129
Date solved : `Monday, October 21, 2024 at 05:04:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' + 3x\left(-2x^2+3x+5\right)y' + \left(-14x^2+12x+1\right)y = 0$$

1.127.1 Solved as second order ode using Kovacic algorithm

Time used: 0.576 (sec)

Writing the ode as

$$9x^2y'' + \left(-6x^3+9x^2+15x\right)y' + \left(-14x^2+12x+1\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= -6x^3+9x^2+15x \\ C &= -14x^2+12x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^4 - 12x^3 + 33x^2 - 18x - 9$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 238: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{9} - \frac{x}{3} + \frac{11}{12} - \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{3} - \frac{1}{2} + \frac{1}{x} + \frac{3}{4x^2} - \frac{3}{4x^3} - \frac{27}{8x^4} - \frac{117}{32x^5} + \frac{405}{64x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{3}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= -\frac{1}{2} + \frac{x}{3}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4} - \frac{1}{3}x + \frac{1}{9}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{9}x^2 - \frac{1}{3}x + \frac{11}{12}\right) + \left(\frac{-18x - 9}{36x^2}\right) \\ &= \frac{x^2}{9} - \frac{x}{3} + \frac{11}{12} + \frac{-18x - 9}{36x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is $\frac{11}{12}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{11}{12}\right) - \left(\frac{1}{4}\right) \\ &= \frac{2}{3}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= -\frac{1}{2} + \frac{x}{3} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-\frac{1}{2} + \frac{x}{3}$	$\frac{1}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \left(-\frac{1}{2} + \frac{x}{3}\right) \\
&= \frac{1}{2x} - \frac{1}{2} + \frac{x}{3} \\
&= \frac{1}{2x} - \frac{1}{2} + \frac{x}{3}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{3}\right) + \left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right)^2 - \left(\frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}\right)\right) = 0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{\frac{x(x-3)}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-6x^3 + 9x^2 + 15x}{9x^2} \, dx} \\
&= z_1 e^{\frac{x^2}{6} - \frac{x}{2} - \frac{5 \ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x(x-3)}{6}}}{x^{5/6}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^{\frac{x(x-3)}{3}}}{x^{1/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x^3+9x^2+15x}{9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{3}-x-\frac{5\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int e^{\frac{x^2}{3}-x-\frac{5\ln(x)}{3}} x^{2/3} e^{-\frac{2x(x-3)}{3}}\,dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\frac{x(x-3)}{3}}}{x^{1/3}}\right) + c_2 \left(\frac{e^{\frac{x(x-3)}{3}}}{x^{1/3}} \left(\int e^{\frac{x^2}{3}-x-\frac{5\ln(x)}{3}} x^{2/3} e^{-\frac{2x(x-3)}{3}}\,dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.127.2 Maple step by step solution

Let's solve

$9x^2\left(\frac{d}{dx}y'\right) + 3x(-2x^2+3x+5)\,y' + (-14x^2+12x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(14x^2-12x-1)y}{9x^2} + \frac{(2x^2-3x-5)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(2x^2-3x-5)y'}{3x} - \frac{(14x^2-12x-1)y}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{2x^2-3x-5}{3x}, P_3(x) = -\frac{14x^2-12x-1}{9x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Bigg|_{x=0} = \frac{5}{3}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = \frac{1}{9}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- • Multiply by denominators

  $9x^2\left(\frac{d}{dx}y'\right) - 3x(2x^2 - 3x - 5)\, y' + (-14x^2 + 12x + 1)\, y = 0$

- • Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$

- ○ Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

- ○ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(1+3r)^2 x^r + \left(a_1(4+3r)^2 + 3a_0(4+3r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+1)^2 + 3a_{k-1}(3k+3r+1\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+3r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = -\frac{1}{3}$
- Each term must be 0
  $a_1(4+3r)^2 + 3a_0(4+3r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = -\frac{3a_0}{4+3r}$
- Each term in the series must be 0, giving the recursion relation
  $a_k(3k+3r+1)^2 + (3k+3r+1)(-2a_{k-2} + 3a_{k-1}) = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(3k+3r+7)^2 + (3k+3r+7)(-2a_k + 3a_{k+1}) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+3r+7}$
- Recursion relation for $r = -\frac{1}{3}$
  $a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+6}$
- Solution for $r = -\frac{1}{3}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+6}, a_1 = -a_0\right]$

1.127.3 Maple trace

```
Methods for second order ODEs:
```

1.127.4 Maple dsolve solution

Solving time : 0.049 (sec)
Leaf size : 32

```
dsolve(9*x^2*diff(diff(y(x),x),x)+3*x*(-2*x^2+3*x+5)*diff(y(x),x)+(-14*x^2+12*x+1)*y(x
       y(x),singsol=all)
```

$$y = \frac{e^{\frac{x(x-3)}{3}} \left(c_2 \left(\int \frac{e^{-\frac{x(x-3)}{3}}}{x} dx\right) + c_1\right)}{x^{1/3}}$$

1.127.5 Mathematica DSolve solution

Solving time : 0.816 (sec)
Leaf size : 52

```
DSolve[{9*x^2*D[y[x],{x,2}]+3*x*(5+3*x-2*x^2)*D[y[x],x]+(1+12*x-14*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{\frac{1}{3}(x-3)x} \left(c_2 \int_1^x \frac{e^{K[1]-\frac{K[1]^2}{3}}}{K[1]} dK[1] + c_1\right)}{\sqrt[3]{x}}$$

1.128 problem 130

Internal problem ID [8266]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 130
Date solved : `Monday, October 21, 2024 at 05:04:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+2x)\,y''+x\left(3x^2+14x+5\right)y'+\left(12x^2+18x+4\right)y=0$$

1.128.1 Solved as second order ode using Kovacic algorithm

Time used: 0.645 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(3x^3+14x^2+5x\right)y'+\left(12x^2+18x+4\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+x^2 \\ B &= 3x^3+14x^2+5x \\ C &= 12x^2+18x+4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{4\left(2x^2 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9x^4 - 12x^3 - 16x^2 - 4x - 1 \\
t &= 4\left(2x^2 + x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{4\left(2x^2 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 240: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 4 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{16} - \frac{1}{4x^2} - \frac{15}{64\left(x + \frac{1}{2}\right)^2} - \frac{21}{16\left(x + \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{3}{8}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{3}{4} - \frac{7}{8x} - \frac{19}{48x^2} - \frac{151}{288x^3} - \frac{139}{192x^4} - \frac{11383}{10368x^5} - \frac{38729}{20736x^6} - \frac{1212655}{373248x^7} + \dots \qquad (9)
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{3}{4}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{3}{4} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{16}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2} \\
&= Q + \frac{R}{16x^4 + 16x^3 + 4x^2} \\
&= \left(\frac{9}{16}\right) + \left(\frac{-21x^3 - \frac{73}{4}x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2}\right) \\
&= \frac{9}{16} + \frac{-21x^3 - \frac{73}{4}x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2}
\end{aligned}
$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -21. Dividing this by leading coefficient in t which is 16 gives $-\frac{21}{16}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{21}{16}\right) - (0) \\
&= -\frac{21}{16}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{4} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{21}{16}}{\frac{3}{4}} - 0\right) &= -\frac{7}{8} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{21}{16}}{\frac{3}{4}} - 0\right) &= \frac{7}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{4\left(2x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{3}{4}$	$-\frac{7}{8}$	$\frac{7}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{7}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-}\right) \\ &= \frac{7}{8} - \left(\frac{7}{8}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} + (-)\left(\frac{3}{4}\right) \\ &= \frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4} \\ &= \frac{-3x^2+2x+1}{4x^2+2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{3}{8\left(x+\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4}\right)^2 - \left(\frac{9x^4 - 12x}{4}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4}\right) dx} \\
&= (1+2x)^{3/8} \sqrt{x}\, \mathrm{e}^{-\frac{3x}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x^3+14x^2+5x}{2x^3+x^2} \, dx} \\
&= z_1 e^{-\frac{3x}{4} - \frac{5\ln(1+2x)}{8} - \frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{3x}{4}}}{(1+2x)^{5/8} x^{5/2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{3x}{2}}}{(1+2x)^{1/4} x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+14x^2+5x}{2x^3+x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{3x}{2} - \frac{5\ln(1+2x)}{4} - 5\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \mathrm{e}^{-\frac{3x}{2} - \frac{5\ln(1+2x)}{4} - 5\ln(x)} \sqrt{1+2x}\, x^4 \mathrm{e}^{3x} dx \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-\frac{3x}{2}}}{(1+2x)^{1/4}\, x^2}\right) + c_2 \left(\frac{e^{-\frac{3x}{2}}}{(1+2x)^{1/4}\, x^2} \left(\int e^{-\frac{3x}{2}-\frac{5\ln(1+2x)}{4}-5\ln(x)} \sqrt{1+2x}\, x^4 e^{3x} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.128.2 Maple step by step solution

Let's solve

$x^2(1+2x)\left(\frac{d}{dx}y'\right)+x(3x^2+14x+5)\,y'+(12x^2+18x+4)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2(6x^2+9x+2)y}{x^2(1+2x)} - \frac{(3x^2+14x+5)y'}{x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3x^2+14x+5)y'}{x(1+2x)} + \frac{2(6x^2+9x+2)y}{x^2(1+2x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3x^2+14x+5}{x(1+2x)}, P_3(x) = \frac{2(6x^2+9x+2)}{x^2(1+2x)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

    $(x \cdot P_2(x))\Big|_{x=0} = 5$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(1+2x)\left(\frac{d}{dx}y'\right)+x(3x^2+14x+5)\,y'+(12x^2+18x+4)\,y=0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)^2 x^r + \left(a_1(3+r)^2 + 2a_0(3+r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)^2 + 2a_{k-1}(k+r+2)^2 + 3a\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = -2$
- Each term must be 0

  $a_1(3+r)^2 + 2a_0(3+r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = -2a_0$
- Each term in the series must be 0, giving the recursion relation

  $((2k+2r+4)\, a_{k-1} + a_k(k+r+2) + 3a_{k-2})\,(k+r+2) = 0$
- Shift index using $k- >k + 2$

  $((2k+8+2r)\, a_{k+1} + a_{k+2}(k+r+4) + 3a_k)\,(k+r+4) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2ka_{k+1}+2ra_{k+1}+3a_k+8a_{k+1}}{k+r+4}$
- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{2ka_{k+1} + 3a_k + 4a_{k+1}}{k+2}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2ka_{k+1} + 3a_k + 4a_{k+1}}{k+2}, a_1 = -2a_0\right]$$

1.128.3 Maple trace

```
Methods for second order ODEs:
```

1.128.4 Maple dsolve solution

Solving time : `0.050` (sec)
Leaf size : 53

```
dsolve(x^2*(1+2*x)*diff(diff(y(x),x),x)+x*(3*x^2+14*x+5)*diff(y(x),x)+(12*x^2+18*x+4)*
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{3x}{2}} \left((1+2x)^{1/4} \operatorname{HeunC}\left(-\frac{3}{4}, \frac{1}{4}, 0, \frac{21}{32}, -\frac{5}{32}, 1+2x\right) c_2 + \operatorname{HeunC}\left(-\frac{3}{4}, -\frac{1}{4}, 0, \frac{21}{32}, -\frac{5}{32}, 1+2x\right) c_1\right)}{(1+2x)^{1/4} x^2}$$

1.128.5 Mathematica DSolve solution

Solving time : `12.879` (sec)
Leaf size : 61

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]+x*(5+14*x+3*x^2)*D[y[x],x]+(4+18*x+12*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-3x/2} \left(c_2 \int_1^x \frac{e^{\frac{3K[1]}{2}}}{K[1](2K[1]+1)^{3/4}} dK[1] + c_1\right)}{x^2 \sqrt[4]{2x+1}}$$

1.129 problem 131

Internal problem ID [8267]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 131
Date solved : `Monday, October 21, 2024 at 05:04:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$16x^2y'' + 4x\left(2x^2 + x + 6\right) y' + \left(18x^2 + 5x + 1\right) y = 0$$

1.129.1 Solved as second order ode using Kovacic algorithm

Time used: 0.542 (sec)

Writing the ode as

$$16x^2y'' + \left(8x^3 + 4x^2 + 24x\right) y' + \left(18x^2 + 5x + 1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^2 \\ B &= 8x^3 + 4x^2 + 24x \\ C &= 18x^2 + 5x + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 + 4x^3 - 31x^2 - 8x - 16 \\ t &= 64x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 242: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} + \frac{x}{16} - \frac{31}{64} - \frac{1}{8x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} + \frac{1}{8} - \frac{1}{x} + \frac{1}{4x^2} - \frac{21}{8x^3} + \frac{37}{16x^4} - \frac{377}{32x^5} + \frac{1137}{64x^6} + \cdots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{1}{8} + \frac{x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64} + \frac{1}{16}x + \frac{1}{16}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{64}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2} \\ &= Q + \frac{R}{64x^2} \\ &= \left(\frac{1}{16}x^2 + \frac{1}{16}x - \frac{31}{64}\right) + \left(\frac{-8x - 16}{64x^2}\right) \\ &= \frac{x^2}{16} + \frac{x}{16} - \frac{31}{64} + \frac{-8x - 16}{64x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{31}{64}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{31}{64}\right) - \left(\frac{1}{64}\right) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{8}+\frac{x}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{4}}-1\right) &&= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{4}}-1\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4+4x^3-31x^2-8x-16}{64x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{1}{8}+\frac{x}{4}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{1}{8} + \frac{x}{4}\right) \\
&= \frac{1}{2x} - \frac{1}{8} - \frac{x}{4} \\
&= \frac{1}{2x} - \frac{1}{8} - \frac{x}{4}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right)^2 - \left(\frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-\frac{x(x+1)}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{8x^3 + 4x^2 + 24x}{16x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{8} - \frac{x}{8} - \frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x(x+1)}{8}}}{x^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{x(x+1)}{4}}}{x^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8x^3+4x^2+24x}{16x^2} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{4}-\frac{x}{4}-\frac{3\ln(x)}{2}}}{(y_1)^2} dx \\ &= y_1 \left(\int e^{-\frac{x^2}{4}-\frac{x}{4}-\frac{3\ln(x)}{2}} \sqrt{x}\, e^{\frac{x(x+1)}{2}} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-\frac{x(x+1)}{4}}}{x^{1/4}} \right) + c_2 \left(\frac{e^{-\frac{x(x+1)}{4}}}{x^{1/4}} \left(\int e^{-\frac{x^2}{4}-\frac{x}{4}-\frac{3\ln(x)}{2}} \sqrt{x}\, e^{\frac{x(x+1)}{2}} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.129.2 Maple step by step solution

Let's solve

$16x^2 \left(\frac{d}{dx} y' \right) + 4x(2x^2 + x + 6) y' + (18x^2 + 5x + 1) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(18x^2+5x+1)y}{16x^2} - \frac{(2x^2+x+6)y'}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{(2x^2+x+6)y'}{4x} + \frac{(18x^2+5x+1)y}{16x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2x^2+x+6}{4x}, P_3(x) = \frac{18x^2+5x+1}{16x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{3}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{16}$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $16x^2\left(\frac{d}{dx}y'\right) + 4x(2x^2 + x + 6)\, y' + (18x^2 + 5x + 1)\, y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

    Rewrite ODE with series expansions

$$a_0(1+4r)^2 x^r + \left(a_1(5+4r)^2 + a_0(5+4r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r+1)^2 + a_{k-1}(4k+4r+1) \right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+4r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = -\frac{1}{4}$
- Each term must be 0
  $a_1(5+4r)^2 + a_0(5+4r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = -\frac{a_0}{5+4r}$
- Each term in the series must be 0, giving the recursion relation
  $a_k(4k+4r+1)^2 + (4k+4r+1)(2a_{k-2} + a_{k-1}) = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(4k+4r+9)^2 + (4k+4r+9)(2a_k + a_{k+1}) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+4r+9}$
- Recursion relation for $r = -\frac{1}{4}$
  $a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+8}$
- Solution for $r = -\frac{1}{4}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}, a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+8}, a_1 = -\frac{a_0}{4}\right]$

1.129.3 Maple trace

```
Methods for second order ODEs:
```

1.129.4 Maple dsolve solution

Solving time : `0.051` (sec)
Leaf size : 32

```
dsolve(16*x^2*diff(diff(y(x),x),x)+4*x*(2*x^2+x+6)*diff(y(x),x)+(18*x^2+5*x+1)*y(x) =
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{x(x+1)}{4}} \left(c_2 \left(\int \frac{e^{\frac{x(x+1)}{4}}}{x} dx\right) + c_1\right)}{x^{1/4}}$$

1.129.5 Mathematica DSolve solution

Solving time : `0.645` (sec)
Leaf size : 51

```
DSolve[{16*x^2*D[y[x],{x,2}]+4*x*(6+x+2*x^2)*D[y[x],x]+(1+5*x+18*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{1}{4}x(x+1)} \left(c_2 \int_1^x \frac{e^{\frac{1}{4}K[1](K[1]+1)}}{K[1]} dK[1] + c_1\right)}{\sqrt[4]{x}}$$

1.130 problem 132

Internal problem ID [8268]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 132
Date solved : `Monday, October 21, 2024 at 05:04:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2(1+x)\,y''+3x\left(-x^2+11x+5\right)y'+\left(-7x^2+16x+1\right)y=0$$

1.130.1 Solved as second order ode using Kovacic algorithm

Time used: 0.416 (sec)

Writing the ode as

$$\left(9x^3+9x^2\right)y''+\left(-3x^3+33x^2+15x\right)y'+\left(-7x^2+16x+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=9x^3+9x^2\\B&=-3x^3+33x^2+15x\\C&=-7x^2+16x+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36\,(x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 + 6x^3 + 3x^2 - 18x - 9 \\ t &= 36(x^2 + x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36\,(x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 244: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{7}{36\,(1+x)^2} + \frac{1}{9+9x} - \frac{1}{4x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{6} + \frac{1}{3x} - \frac{5}{6x^2} + \frac{5}{6x^3} - \frac{7}{3x^4} + \frac{41}{6x^5} - \frac{149}{6x^6} + \frac{277}{3x^7} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{6}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{6}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t}\\
&= \frac{x^4+6x^3+3x^2-18x-9}{36x^4+72x^3+36x^2}\\
&= Q + \frac{R}{36x^4+72x^3+36x^2}\\
&= \left(\frac{1}{36}\right) + \left(\frac{4x^3+2x^2-18x-9}{36x^4+72x^3+36x^2}\right)\\
&= \frac{1}{36} + \frac{4x^3+2x^2-18x-9}{36x^4+72x^3+36x^2}
\end{aligned}
$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is 4. Dividing this by leading coefficient in t which is 36 gives $\frac{1}{9}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{1}{9}\right) - (0)\\
&= \frac{1}{9}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{6}\\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{9}}{\frac{1}{6}} - 0\right) &&= \frac{1}{3}\\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{9}}{\frac{1}{6}} - 0\right) &&= -\frac{1}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4+6x^3+3x^2-18x-9}{36\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to

determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{+} = \frac{1}{3}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\ &= \frac{1}{3} - \left(\frac{1}{3}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (+)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{6\,(1+x)} + \frac{1}{2x} + \left(\frac{1}{6}\right) \\ &= -\frac{1}{6\,(1+x)} + \frac{1}{2x} + \frac{1}{6} \\ &= -\frac{1}{6+6x} + \frac{1}{2x} + \frac{1}{6} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6\,(1+x)} + \frac{1}{2x} + \frac{1}{6}\right)(0) + \left(\left(\frac{1}{6\,(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{6\,(1+x)} + \frac{1}{2x} + \frac{1}{6}\right)^2 - \left(\frac{x^4 + 6x^3 +}{36\,(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{6(1+x)}+\frac{1}{2x}+\frac{1}{6}\right)dx} \\
&= \frac{\sqrt{x}\,\mathrm{e}^{\frac{x}{6}}}{(1+x)^{1/6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x^3+33x^2+15x}{9x^3+9x^2}\, dx} \\
&= z_1 e^{\frac{x}{6}-\frac{7\ln(1+x)}{6}-\frac{5\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{6}}}{(1+x)^{7/6}\, x^{5/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{\frac{x}{3}}}{(1+x)^{4/3}\, x^{1/3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+33x^2+15x}{9x^3+9x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x}{3}-\frac{7\ln(1+x)}{3}-\frac{5\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \mathrm{e}^{\frac{x}{3}-\frac{7\ln(1+x)}{3}-\frac{5\ln(x)}{3}} (1+x)^{8/3}\, x^{2/3} \mathrm{e}^{-\frac{2x}{3}}\, dx\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{e^{\frac{x}{3}}}{(1+x)^{4/3} x^{1/3}}\right) + c_2 \left(\frac{e^{\frac{x}{3}}}{(1+x)^{4/3} x^{1/3}} \left(\int e^{\frac{x}{3} - \frac{7\ln(1+x)}{3} - \frac{5\ln(x)}{3}} (1+x)^{8/3} x^{2/3} e^{-\frac{2x}{3}} dx\right)\right)$$

Will add steps showing solving for IC soon.

1.130.2 Maple step by step solution

Let's solve
$9x^2(1+x)\left(\frac{d}{dx}y'\right) + 3x(-x^2+11x+5)\,y' + (-7x^2+16x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = \frac{(7x^2-16x-1)y}{9x^2(1+x)} + \frac{(x^2-11x-5)y'}{3x(1+x)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{(x^2-11x-5)y'}{3x(1+x)} - \frac{(7x^2-16x-1)y}{9x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{x^2-11x-5}{3x(1+x)}, P_3(x) = -\frac{7x^2-16x-1}{9x^2(1+x)}\right]$
  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$
    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{7}{3}$
  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$
    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$
  - $x=-1$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = -1$

- Multiply by denominators
  $9x^2(1+x)\left(\frac{d}{dx}y'\right) - 3x(x^2-11x-5)\,y' + (-7x^2+16x+1)\,y = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$
  $(9u^3-18u^2+9u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-3u^3+42u^2-60u+21)\left(\frac{d}{du}y(u)\right) + (-7u^2+30u-22)\,y(u) =$
- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0 r(4+3r)\, u^{-1+r} + \left(3a_1(1+r)(7+3r) - 2a_0\left(9r^2+21r+11\right)\right) u^r + \left(3a_2(2+r)(10+3r) - 2a\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $3r(4+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{4}{3}\right\}$
- The coefficients of each power of u must be 0
  $[3a_1(1+r)(7+3r) - 2a_0(9r^2+21r+11) = 0, 3a_2(2+r)(10+3r) - 2a_1(9r^2+39r+41) + 3a$
- Solve for the dependent coefficient(s)
  $\left\{a_1 = \frac{2a_0(9r^2+21r+11)}{3(3r^2+10r+7)}, a_2 = \frac{a_0(243r^4+1593r^3+3699r^2+3567r+1174)}{9(9r^4+78r^3+241r^2+312r+140)}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $9(-2a_k + a_{k-1} + a_{k+1})\, k^2 + 3(6(-2a_k + a_{k-1} + a_{k+1})\, r - 14a_k - a_{k-2} + 5a_{k-1} + 10a_{k+1})\, k + 9($
- Shift index using $k- >k + 2$

$$9(-2a_{k+2}+a_{k+1}+a_{k+3})(k+2)^2+3(6(-2a_{k+2}+a_{k+1}+a_{k+3})r-14a_{k+2}-a_k+5a_{k+1}+10a_{k}$$

- Recursion relation that defines series solution to ODE

$$a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}+18kra_{k+1}-36kra_{k+2}+9r^2a_{k+1}-18r^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-3ra_k+51ra_{k+1}-114ra}{3(3k^2+6kr+3r^2+22k+22r+39)}$$

- Recursion relation for $r=0$

$$a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k, a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}, a_1=\frac{22a_0}{21}, a_2=\right.$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k, a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}, a_1=\frac{22a_0}{21},\right.$$

- Recursion relation for $r=-\frac{4}{3}$

$$a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}$$

- Solution for $r=-\frac{4}{3}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-\frac{4}{3}}, a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}, a_1=\frac{2a_0}{3}, a_2=\right.$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k-\frac{4}{3}}, a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}, a_1=\frac{2a_0}{3},\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k(1+x)^k\right)+\left(\sum_{k=0}^{\infty}b_k(1+x)^{k-\frac{4}{3}}\right), a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a}{3(3k^2+22k+39)}\right.$$

1.130.3 Maple trace

```
Methods for second order ODEs:
```

1.130.4 Maple dsolve solution

Solving time : `0.049` (sec)
Leaf size : 36

```
dsolve(9*x^2*(1+x)*diff(diff(y(x),x),x)+3*x*(-x^2+11*x+5)*diff(y(x),x)+(-7*x^2+16*x+1)
       y(x),singsol=all)
```

$$y = \frac{\frac{c_1 \operatorname{HeunC}\left(-\frac{1}{3},-\frac{4}{3},0,-\frac{1}{9},\frac{11}{18},1+x\right)}{(1+x)^{4/3}} + c_2 \operatorname{HeunC}\left(-\frac{1}{3},\frac{4}{3},0,-\frac{1}{9},\frac{11}{18},1+x\right)}{x^{1/3}}$$

1.130.5 Mathematica DSolve solution

Solving time : `4.456` (sec)
Leaf size : 50

```
DSolve[{9*x^2*(1+x)*D[y[x],{x,2}]+3*x*(5+11*x-x^2)*D[y[x],x]+(1+16*x-7*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{x/3}\left(c_1 - \sqrt[3]{3e}\, c_2 \Gamma\left(\frac{1}{3}, \frac{x+1}{3}\right)\right)}{\sqrt[3]{x}(x+1)^{4/3}}$$

1.131 problem 133

Internal problem ID [8269]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 133
Date solved : `Monday, October 21, 2024 at 05:04:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$36x^2(1-2x)\,y''+24x(1-9x)\,y'+(1-70x)\,y=0$$

1.131.1 Solved as second order ode using Kovacic algorithm

Time used: 0.407 (sec)

Writing the ode as

$$\left(-72x^3+36x^2\right)y''+\left(-216x^2+24x\right)y'+(1-70x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-72x^3+36x^2\\ B&=-216x^2+24x\\ C&=1-70x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-32x^2 + 48x - 9}{36 \left(2x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -32x^2 + 48x - 9 \\ t &= 36\left(2x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-32x^2 + 48x - 9}{36 \left(2x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 246: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{1}{3x} + \frac{7}{36\left(x - \frac{1}{2}\right)^2} - \frac{1}{3\left(x - \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-32x^2 + 48x - 9}{36 (2x^2 - x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{2}{3} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{3}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-32x^2 + 48x - 9}{36 (2x^2 - x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^+ + \alpha_{c_2}^-) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)} + (-)\,(0) \\
&= \frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)} \\
&= \frac{-3+4x}{12x^2-6x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{6\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{-32x^2+48x-9}{36\left(2x^2-x\right)^2}\right.\right.
$$

$$
0 =
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)}\right) dx} \\
&= \frac{\sqrt{x}}{(-1+2x)^{1/6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-216x^2+24x}{-72x^3+36x^2} \, dx} \\
&= z_1 e^{-\frac{7\ln(-1+2x)}{6} - \frac{\ln(x)}{3}} \\
&= z_1 \left(\frac{1}{(-1+2x)^{7/6}\, x^{1/3}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x^{1/6}}{(-1+2x)^{4/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-216x^2+24x}{-72x^3+36x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(-1+2x)}{3}-\frac{2\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(3(-1+2x)^{1/3} + \frac{\ln\left((-1+2x)^{2/3}-(-1+2x)^{1/3}+1\right)}{2} \right. \\
&\quad \left. - \sqrt{3}\arctan\left(\frac{\left(-1+2(-1+2x)^{1/3}\right)\sqrt{3}}{3}\right) - \ln\left((-1+2x)^{1/3}+1\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/6}}{(-1+2x)^{4/3}}\right) \\
&\quad + c_2 \left(\frac{x^{1/6}}{(-1+2x)^{4/3}} \left(3(-1+2x)^{1/3} + \frac{\ln\left((-1+2x)^{2/3}-(-1+2x)^{1/3}+1\right)}{2} - \sqrt{3}\arctan\left(\frac{\left(-1+2(-\right.}{}\right.\right.\right.
\end{aligned}$$

Will add steps showing solving for IC soon.

1.131.2 Maple step by step solution

Let's solve

$36x^2(1-2x)\left(\frac{d}{dx}y'\right)+24x(1-9x)\,y'+(1-70x)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(70x-1)y}{36x^2(-1+2x)}-\frac{2(-1+9x)y'}{3x(-1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{2(-1+9x)y'}{3x(-1+2x)}+\frac{(70x-1)y}{36x^2(-1+2x)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{2(-1+9x)}{3x(-1+2x)},P_3(x)=\frac{70x-1}{36x^2(-1+2x)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$(x\cdot P_2(x))\bigg|_{x=0}=\frac{2}{3}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2\cdot P_3(x))\bigg|_{x=0}=\frac{1}{36}$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$36x^2(-1+2x)\left(\frac{d}{dx}y'\right)+24x(-1+9x)\,y'+(70x-1)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

$x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+6r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(6k+6r-1)^2 + 2a_{k-1}(6k+1+6r)\,(6k+6r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-1+6r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{6}$$

- Each term in the series must be 0, giving the recursion relation

$$-36\left(k+r-\frac{1}{6}\right)\left(\left(-2k-2r-\frac{1}{3}\right) a_{k-1} + a_k\left(k+r-\frac{1}{6}\right)\right) = 0$$

- Shift index using $k->k+1$

$$-36\left(k+\frac{5}{6}+r\right)\left(\left(-2k-\frac{7}{3}-2r\right) a_k + a_{k+1}\left(k+\frac{5}{6}+r\right)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2(6k+6r+7)a_k}{6k+6r+5}$$

- Recursion relation for $r = \frac{1}{6}$

$$a_{k+1} = \frac{2(6k+8)a_k}{6k+6}$$

- Solution for $r = \frac{1}{6}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{6}}, a_{k+1} = \frac{2(6k+8)a_k}{6k+6}\right]$$

1.131.3 Maple trace

```
Methods for second order ODEs:
```

1.131.4 Maple dsolve solution

Solving time : 0.037 (sec)
Leaf size : 93

```
dsolve(36*x^2*(1-2*x)*diff(diff(y(x),x),x)+24*x*(1-9*x)*diff(y(x),x)+(1-70*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^{1/6}\left(2\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,(-1+2x)^{1/3}}{-2+(-1+2x)^{1/3}}\right)c_2 - 2\ln\left((-1+2x)^{1/3}+1\right)c_2 + \ln\left((-1+2x)^{2/3}-(-1+2x)^{1/3}+\right.\right.}{3\,(-1+2x)^{4/3}}$$

1.131.5 Mathematica DSolve solution

Solving time : 0.263 (sec)
Leaf size : 111

```
DSolve[{36*x^2*(1-2*x)*D[y[x],{x,2}]+24*x*(1-9*x)*D[y[x],x]+(1-70*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[6]{x}\left(-2\sqrt{3}c_2\arctan\left(\frac{2\sqrt[3]{1-2x}+1}{\sqrt{3}}\right)+6c_2\sqrt[3]{1-2x}+2c_2\log\left(\sqrt[3]{1-2x}-1\right)-c_2\log\left((1-2x)^{2/3}+\sqrt[3]{1}\right.\right.}{2(1-2x)^{4/3}}$$

1.132 problem 134

Internal problem ID [8270]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 134
Date solved : `Monday, October 21, 2024 at 05:04:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''-x(3-x)\,y'+4y=0$$

1.132.1 Solved as second order ode using Kovacic algorithm

Time used: 0.283 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(x^2-3x\right)y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\B&=x^2-3x\\C&=4\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -x^2 - 10x - 1 \\
t &= 4\left(x^2 + x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 248: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x} + \frac{2}{(1+x)^2} - \frac{1}{4x^2} + \frac{2}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4(x^2 + x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^2-10x-1}{4\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{1+x} + \frac{1}{2x} + (-)(0) \\
&= -\frac{1}{1+x} + \frac{1}{2x} \\
&= -\frac{x-1}{2x(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{1+x} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{1+x} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2-10x-1}{4(x^2+x)^2}\right)\right) = 0
$$

$$
\frac{1+a_0}{x(1+x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x-1)\, e^{\int \left(-\frac{1}{1+x} + \frac{1}{2x}\right) dx} \\
&= (x-1)\, e^{\frac{\ln(x)}{2} - \ln(1+x)} \\
&= \frac{(x-1)\sqrt{x}}{1+x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2-3x}{x^2(1+x)}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}-2\ln(1+x)} \\
&= z_1\left(\frac{x^{3/2}}{(1+x)^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2(x-1)}{(1+x)^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-3x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)-4\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{4}{x-1}+\ln(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2(x-1)}{(1+x)^3}\right) + c_2\left(\frac{x^2(x-1)}{(1+x)^3}\left(-\frac{4}{x-1}+\ln(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.132.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right)-x(3-x)\,y'+4y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{4y}{x^2(1+x)}-\frac{(x-3)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(x-3)y'}{x(1+x)}+\frac{4y}{x^2(1+x)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{x-3}{x(1+x)},P_3(x)=\frac{4}{x^2(1+x)}\right]$

- ○ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=4$

- ○ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\left(\frac{d}{dx}y'\right)+x(x-3)\,y'+4y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(u^2-5u+4)\left(\frac{d}{du}y(u)\right)+4y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty} a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)$ to series expansion for $m =1..3$

$$u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =0}^{\infty} a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- Shift index using $k - >k +2-m$

$$u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty} a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$a_{0} r \left(3+r \right) u^{-1+r}+\left(a_{1} \left(1+r \right) \left(4+r \right)-a_{0} \left(2 r^{2}+3 r -4\right)\right) u^{r}+\left(\sum_{k =1}^{\infty} \left(a_{k +1} \left(k +1+r \right) \left(k +4+r \right)\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$r \left(3+r \right)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-3, 0\right\}$$

- Each term must be 0

$$a_{1} \left(1+r \right) \left(4+r \right)-a_{0} \left(2 r^{2}+3 r -4\right)=0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k +1} \left(k +1+r \right) \left(k +4+r \right)-a_{k} \left(2 k^{2}+4 k r +2 r^{2}+3 k +3 r -4\right)+a_{k -1} \left(k +r -1\right)^{2}=0$$

- Shift index using $k - >k +1$

$$a_{k +2} \left(k +2+r \right) \left(k +5+r \right)-a_{k +1} \left(2 \left(k +1\right)^{2}+4 \left(k +1\right) r +2 r^{2}+3 k -1+3 r \right)+a_{k} \left(k +r \right)^{2}=$$

- Recursion relation that defines series solution to ODE

$$a_{k +2}=-\frac{k^{2} a_{k}-2 k^{2} a_{k +1}+2 k r a_{k}-4 k r a_{k +1}+r^{2} a_{k}-2 r^{2} a_{k +1}-7 k a_{k +1}-7 r a_{k +1}-a_{k +1}}{\left(k +2+r \right) \left(k +5+r \right)}$$

- Recursion relation for $r =-3$

$$a_{k +2}=-\frac{k^{2} a_{k}-2 k^{2} a_{k +1}-6 k a_{k}+5 k a_{k +1}+9 a_{k}+2 a_{k +1}}{\left(k -1\right) \left(k +2\right)}$$

- Series not valid for $r =-3$, division by 0 in the recursion relation at $k =1$

$$a_{k +2}=-\frac{k^{2} a_{k}-2 k^{2} a_{k +1}-6 k a_{k}+5 k a_{k +1}+9 a_{k}+2 a_{k +1}}{\left(k -1\right) \left(k +2\right)}$$

- Recursion relation for $r =0$

$$a_{k +2}=-\frac{k^{2} a_{k}-2 k^{2} a_{k +1}-7 k a_{k +1}-a_{k +1}}{\left(k +2\right) \left(k +5\right)}$$

- Solution for $r =0$

$$\left[y \left(u \right)=\sum_{k =0}^{\infty} a_{k} u^{k}, a_{k +2}=-\frac{k^{2} a_{k}-2 k^{2} a_{k +1}-7 k a_{k +1}-a_{k +1}}{\left(k +2\right) \left(k +5\right)}, 4 a_{1}+4 a_{0}=0\right]$$

- Revert the change of variables $u =1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 7k a_{k+1} - a_{k+1}}{(k+2)(k+5)}, 4a_1 + 4a_0 = 0\right]$$

1.132.3 Maple trace

```
Methods for second order ODEs:
```

1.132.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 30

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)-x*(3-x)*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^2 (c_2 (x-1) \ln (x) + c_1 x - c_1 - 4c_2)}{(1+x)^3}$$

1.132.5 Mathematica DSolve solution

Solving time : `0.093` (sec)
Leaf size : 33

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-x*(3-x)*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^2 (c_1 (x-1) + c_2 (x-1) \log(x) - 4c_2)}{(x+1)^3}$$

1.133 problem 135

Internal problem ID [8271]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 135
Date solved : `Monday, October 21, 2024 at 05:04:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-2x)\,y''-x(5-4x)\,y'+(9-4x)\,y=0$$

1.133.1 Solved as second order ode using Kovacic algorithm

Time used: 0.267 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(4x^2-5x\right)y'+(9-4x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= 4x^2-5x \\ C &= 9-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x - 1}{4(2x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8x - 1 \\ t &= 4(2x^2 - x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x - 1}{4(2x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 250: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{1}{4x^2} + \frac{3}{4\left(x - \frac{1}{2}\right)^2} - \frac{1}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8x - 1}{4 \left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\ &= 0 - (0) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} - \frac{1}{2 \left(x - \frac{1}{2}\right)} + (0) \\ &= \frac{1}{2x} - \frac{1}{2 \left(x - \frac{1}{2}\right)} \\ &= -\frac{1}{2x \left(-1 + 2x\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2\left(x - \frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{2\left(x - \frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{1}{2\left(x - \frac{1}{2}\right)}\right)^2 - \left(\frac{8x - 1}{4\left(2x^2 - x\right)^2}\right)\right) =$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2\left(x - \frac{1}{2}\right)}\right) dx} \\
&= \frac{\sqrt{x}}{\sqrt{-1 + 2x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4x^2 - 5x}{-2x^3 + x^2} \, dx} \\
&= z_1 e^{-\frac{3 \ln(-1+2x)}{2} + \frac{5 \ln(x)}{2}} \\
&= z_1 \left(\frac{x^{5/2}}{(-1 + 2x)^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^3}{(-1 + 2x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^2-5x}{-2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(-1+2x)+5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(2x - \ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^3}{(-1+2x)^2}\right) + c_2\left(\frac{x^3}{(-1+2x)^2}(2x - \ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.133.2 Maple step by step solution

Let's solve

$x^2(1-2x)\left(\frac{d}{dx}y'\right) - x(5-4x)\,y' + (9-4x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(-9+4x)y}{x^2(-1+2x)} + \frac{(4x-5)y'}{x(-1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(4x-5)y'}{x(-1+2x)} + \frac{(-9+4x)y}{x^2(-1+2x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{4x-5}{x(-1+2x)}, P_3(x) = \frac{-9+4x}{x^2(-1+2x)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -5$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 9$$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(-1+2x)\left(\frac{d}{dx}y'\right) - x(4x-5)\,y' + (-9+4x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ◦ Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - ◦ Shift index using $k- >k+2-m$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(-3+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-3)^2 + 2a_{k-1}(k+r-2)\,(k+r-3)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-3+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 3$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r-3)^2 + 2a_{k-1}(k+r-2)(k+r-3) = 0$
- Shift index using $k->k+1$

  $-a_{k+1}(k+r-2)^2 + 2a_k(k+r-1)(k+r-2) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r-1)}{k+r-2}$
- Recursion relation for $r = 3$

  $a_{k+1} = \frac{2a_k(k+2)}{k+1}$
- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{2a_k(k+2)}{k+1}\right]$

1.133.3 Maple trace

```
Methods for second order ODEs:
```

1.133.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 26

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)-x*(5-4*x)*diff(y(x),x)+(9-4*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^3(2c_2x - c_2 \ln(x) + c_1)}{(-1+2x)^2}$$

1.133.5 Mathematica DSolve solution

Solving time : `0.081` (sec)
Leaf size : 29

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]-x*(5-4*x)*D[y[x],x]+(9-4*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^3(-2c_2x + c_2 \log(x) + c_1)}{(1-2x)^2}$$

1.134 problem 136

Internal problem ID [8272]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 136
Date solved : `Monday, October 21, 2024 at 05:04:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''+x^2y'+(1-x)\,y=0$$

1.134.1 Solved as second order ode using Kovacic algorithm

Time used: 0.262 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y''+x^2y'+(1-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2 \\ B &= x^2 \\ C &= 1-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{5x^2 + 8x - 16}{16\left(x^2 + 2x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 5x^2 + 8x - 16 \\
t &= 16\left(x^2 + 2x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{5x^2 + 8x - 16}{16\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 252: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{8x} - \frac{1}{4x^2} - \frac{3}{16(2+x)^2} - \frac{3}{8(2+x)}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 + 8x - 16}{16(x^2 + 2x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 + 8x - 16}{16\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{4} - \left(\frac{5}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4\,(2+x)} + \frac{1}{2x} + (0) \\
&= \frac{3}{4\,(2+x)} + \frac{1}{2x} \\
&= \frac{5x+4}{4x\,(2+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{3}{4\,(2+x)} + \frac{1}{2x}\right)(0) + \left(\left(-\frac{3}{4\,(2+x)^2} - \frac{1}{2x^2}\right) + \left(\frac{3}{4\,(2+x)} + \frac{1}{2x}\right)^2 - \left(\frac{5x^2+8x-16}{16\,(x^2+2x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{4(2+x)} + \frac{1}{2x}\right) dx} \\
&= (2+x)^{3/4}\sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2}{2x^3+4x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(2+x)}{4}} \\
&= z_1\left(\frac{1}{(2+x)^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \sqrt{2+x}\,\sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{2x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(2+x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{2} + \frac{1}{\sqrt{2+x}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{2+x}\,\sqrt{x}\right) + c_2\left(\sqrt{2+x}\,\sqrt{x}\left(-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{2} + \frac{1}{\sqrt{2+x}}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.134.2 Maple step by step solution

Let's solve

$2x^2(2+x)\left(\frac{d}{dx}y'\right) + x^2y' + (1-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(x-1)y}{2x^2(2+x)} - \frac{y'}{2(2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{2(2+x)} - \frac{(x-1)y}{2x^2(2+x)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{2(2+x)}, P_3(x) = -\frac{x-1}{2x^2(2+x)}\right]$
  - $(2+x) \cdot P_2(x)$ is analytic at $x = -2$

    $\left((2+x) \cdot P_2(x)\right)\Big|_{x=-2} = \frac{1}{2}$
  - $(2+x)^2 \cdot P_3(x)$ is analytic at $x = -2$

    $\left((2+x)^2 \cdot P_3(x)\right)\Big|_{x=-2} = 0$
  - $x = -2$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -2$

- Multiply by denominators

  $2x^2(2+x)\left(\frac{d}{dx}y'\right) + x^2y' + (1-x)\,y = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u = 0$

  $(2u^3 - 8u^2 + 8u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (u^2 - 4u + 4)\left(\frac{d}{du}y(u)\right) + (3-u)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k- >k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $4a_0 r(-1+2r)\, u^{-1+r} + (4a_1(1+r)(1+2r) - a_0(8r^2-4r-3))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)(2k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term must be 0

  $4a_1(1+r)(1+2r) - a_0(8r^2-4r-3) = 0$

- Each term in the series must be 0, giving the recursion relation

  $2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\, r + 4a_k - 5a_{k-1} + 12a_{k+1})\, k + 2(-4a_k +$

- Shift index using $k- >k+1$

  $2(-4a_{k+1} + a_k + 4a_{k+2})\, (k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\, r + 4a_{k+1} - 5a_k + 12a_{k+2})\, (k+1) -$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} - ka_k - 12ka_{k+1} - ra_k - 12ra_{k+1} - a_k - a_{k+1}}{4(2k^2+4kr+2r^2+7k+7r+6)}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 + 3a_0 = 0\right]$

- Revert the change of variables $u = 2 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (2+x)^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 + 3a_0 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + ka_k - 20ka_{k+1} - a_k - 9a_{k+1}}{4(2k^2+9k+10)}$

- Solution for $r = \frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + ka_k - 20ka_{k+1} - a_k - 9a_{k+1}}{4(2k^2+9k+10)}, 12a_1 + 3a_0 = 0\right]$

- Revert the change of variables $u = 2+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k+\frac{1}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + k a_k - 20k a_{k+1} - a_k - 9a_{k+1}}{4(2k^2+9k+10)}, 12a_1 + 3a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (2+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (2+x)^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - k a_k - 12k a_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 +$$

1.134.3 Maple trace

```
Methods for second order ODEs:
```

1.134.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 50

```
dsolve(2*x^2*(2+x)*diff(diff(y(x),x),x)+x^2*diff(y(x),x)+(1-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \sqrt{x(2+x)} + \frac{c_2 \left((2+x) \operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right) - \sqrt{2+x}\,\sqrt{2}\right)\sqrt{x}}{\sqrt{2+x}}$$

1.134.5 Mathematica DSolve solution

Solving time : `0.206` (sec)
Leaf size : 65

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}]+x^2*D[y[x],x]+(1-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}\left(2\left(c_1\sqrt{x+2} + c_2\right) - \sqrt{2}c_2\sqrt{x+2}\operatorname{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right)\right)}{2\sqrt[4]{2}}$$

1.135 problem 137

Internal problem ID [8273]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 137
Date solved : `Monday, October 21, 2024 at 05:05:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(1+x)\,y''-x(6-x)\,y'+(8-x)\,y=0$$

1.135.1 Solved as second order ode using Kovacic algorithm

Time used: 0.265 (sec)

Writing the ode as

$$\left(2x^3+2x^2\right)y''+\left(x^2-6x\right)y'+(8-x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+2x^2 \\ B &= x^2-6x \\ C &= 8-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 - 20x - 4}{16\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 - 20x - 4$$
$$t = 16\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 - 20x - 4}{16\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 254: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4x} + \frac{3}{4(1+x)} - \frac{1}{4x^2} + \frac{21}{16(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 - 20x - 4}{16(x^2 + x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4}\\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{5x^2-20x-4}{16\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-\frac{1}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^- -\left(\alpha_{c_1}^- +\alpha_{c_2}^+\right)\\ &= -\frac{1}{4}-\left(-\frac{1}{4}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{3}{4\,(1+x)} + \frac{1}{2x} + (-)\,(0) \\ &= -\frac{3}{4\,(1+x)} + \frac{1}{2x} \\ &= -\frac{x-2}{4x\,(1+x)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4\,(1+x)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{3}{4\,(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{3}{4\,(1+x)} + \frac{1}{2x}\right)^2 - \left(\frac{5x^2-20x-4}{16\,(x^2+x)^2}\right)\right) =$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{3}{4(1+x)} + \frac{1}{2x}\right) dx} \\ &= \frac{\sqrt{x}}{(1+x)^{3/4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2-6x}{2x^3+2x^2}\, dx} \\ &= z_1 e^{-\frac{7\ln(1+x)}{4} + \frac{3\ln(x)}{2}} \\ &= z_1 \left(\frac{x^{3/2}}{(1+x)^{7/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2}{(1+x)^{5/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-6x}{2x^3+2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(1+x)}{2}+3\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2(1+x)^{3/2}}{3} + 2\sqrt{1+x} + \ln\left(\sqrt{1+x}-1\right) - \ln\left(1+\sqrt{1+x}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2}{(1+x)^{5/2}}\right) \\
&\quad + c_2 \left(\frac{x^2}{(1+x)^{5/2}} \left(\frac{2(1+x)^{3/2}}{3} + 2\sqrt{1+x} + \ln\left(\sqrt{1+x}-1\right) - \ln\left(1+\sqrt{1+x}\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.135.2 Maple step by step solution

Let's solve

$2x^2(1+x)\left(\frac{d}{dx}y'\right)-x(6-x)\,y'+(8-x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{(x-8)y}{2x^2(1+x)}-\frac{(x-6)y'}{2x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(x-6)y'}{2x(1+x)}-\frac{(x-8)y}{2x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{x-6}{2x(1+x)},P_3(x)=-\frac{x-8}{2x^2(1+x)}\right]$

- ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=\frac{7}{2}$

- ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $2x^2(1+x)\left(\frac{d}{dx}y'\right)+x(x-6)\,y'+(8-x)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(2u^3-4u^2+2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(u^2-8u+7)\left(\frac{d}{du}y(u)\right)+(9-u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ◦ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(5+2r)\, u^{-1+r} + \left(a_1(1+r)\,(7+2r) - a_0(4r^2+4r-9)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+7+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(5+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{5}{2}\right\}$$

- Each term must be 0

$$a_1(1+r)\,(7+2r) - a_0(4r^2+4r-9) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-4a_k + 2a_{k-1} + 2a_{k+1})\, k^2 + \left((-8a_k + 4a_{k-1} + 4a_{k+1})\, r - 4a_k - 5a_{k-1} + 9a_{k+1}\right) k + (-4a_k + 2a$$

- Shift index using $k->k+1$

$$(-4a_{k+1} + 2a_k + 2a_{k+2})\,(k+1)^2 + \left((-8a_{k+1} + 4a_k + 4a_{k+2})\, r - 4a_{k+1} - 5a_k + 9a_{k+2}\right)(k+1) +$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + 4kra_k - 8kra_{k+1} + 2r^2a_k - 4r^2a_{k+1} - ka_k - 12ka_{k+1} - ra_k - 12ra_{k+1} - a_k + a_{k+1}}{2k^2 + 4kr + 2r^2 + 13k + 13r + 18}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}, 7a_1 + 9a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y=\sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2}=-\frac{2k^2 a_k-4k^2 a_{k+1}-k a_k-12k a_{k+1}-a_k+a_{k+1}}{2k^2+13k+18}, 7a_1+9a_0=0\right]$$

- Recursion relation for $r=-\frac{5}{2}$

$$a_{k+2}=-\frac{2k^2 a_k-4k^2 a_{k+1}-11k a_k+8k a_{k+1}+14a_k+6a_{k+1}}{2k^2+3k-2}$$

- Solution for $r=-\frac{5}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{5}{2}}, a_{k+2}=-\frac{2k^2 a_k-4k^2 a_{k+1}-11k a_k+8k a_{k+1}+14a_k+6a_{k+1}}{2k^2+3k-2}, -3a_1-6a_0=0\right]$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty} a_k (1+x)^{k-\frac{5}{2}}, a_{k+2}=-\frac{2k^2 a_k-4k^2 a_{k+1}-11k a_k+8k a_{k+1}+14a_k+6a_{k+1}}{2k^2+3k-2}, -3a_1-6a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (1+x)^k\right)+\left(\sum_{k=0}^{\infty} b_k (1+x)^{k-\frac{5}{2}}\right), a_{k+2}=-\frac{2k^2 a_k-4k^2 a_{k+1}-k a_k-12k a_{k+1}-a_k+a_{k+1}}{2k^2+13k+18}, 7a_1+\right.$$

1.135.3 Maple trace

```
Methods for second order ODEs:
```

1.135.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 50

```
dsolve(2*x^2*(1+x)*diff(diff(y(x),x),x)-x*(6-x)*diff(y(x),x)+(8-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{2\left(-\frac{3\ln\left(1+\sqrt{1+x}\right)c_2}{2}+\frac{3\ln\left(\sqrt{1+x}-1\right)c_2}{2}+(x+4)\,c_2\sqrt{1+x}+\frac{3c_1}{2}\right)x^2}{3\,(1+x)^{5/2}}$$

1.135.5 Mathematica DSolve solution

Solving time : 0.141 (sec)
Leaf size : 50

```
DSolve[{2*x^2*(1+x)*D[y[x],{x,2}]-x*(6-x)*D[y[x],x]+(8-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^2\left(-6c_2\operatorname{arctanh}\left(\sqrt{x+1}\right)+2c_2\sqrt{x+1}(x+4)+3c_1\right)}{3(x+1)^{5/2}}$$

1.136 problem 138

Internal problem ID [8274]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 138
Date solved : `Monday, October 21, 2024 at 05:05:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+2x)\,y''+x(5+9x)\,y'+(4+3x)\,y=0$$

1.136.1 Solved as second order ode using Kovacic algorithm

Time used: 0.286 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(9x^2+5x\right)y'+(4+3x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+x^2 \\ B &= 9x^2+5x \\ C &= 4+3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 21x^2 + 6x - 1$$
$$t = 4\left(2x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 256: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{5}{2x} + \frac{5}{16\left(x + \frac{1}{2}\right)^2} - \frac{5}{2\left(x + \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{7}{4} - \left(\frac{7}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)} + (0) \\
&= \frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)} \\
&= \frac{1+7x}{4x^2+2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{5}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{21x^2+6x-1}{4\left(2x^2+x\right)^2}\right)\right)$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)}\right) dx} \\
&= \sqrt{x}\,(1+2x)^{5/4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{9x^2+5x}{2x^3+x^2}\, dx} \\
&= z_1 e^{-\frac{5\ln(x)}{2} + \frac{\ln(1+2x)}{4}} \\
&= z_1 \left(\frac{(1+2x)^{1/4}}{x^{5/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(1+2x)^{3/2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{9x^2+5x}{2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-5\ln(x)+\frac{\ln(1+2x)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\ln\left(\sqrt{1+2x}-1\right)+\frac{2}{3\,(1+2x)^{3/2}}+\frac{2}{\sqrt{1+2x}}-\ln\left(\sqrt{1+2x}+1\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(1+2x)^{3/2}}{x^2}\right) \\
&\quad + c_2\left(\frac{(1+2x)^{3/2}}{x^2}\left(\ln\left(\sqrt{1+2x}-1\right)+\frac{2}{3\,(1+2x)^{3/2}}+\frac{2}{\sqrt{1+2x}}-\ln\left(\sqrt{1+2x}+1\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.136.2 Maple step by step solution

Let's solve

$x^2(1+2x)\left(\frac{d}{dx}y'\right)+x(5+9x)\,y'+(4+3x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(4+3x)y}{x^2(1+2x)}-\frac{(5+9x)y'}{x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(5+9x)y'}{x(1+2x)}+\frac{(4+3x)y}{x^2(1+2x)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{5+9x}{x(1+2x)},P_3(x)=\frac{4+3x}{x^2(1+2x)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=5$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=4$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(1+2x)\left(\frac{d}{dx}y'\right)+x(5+9x)\,y'+(4+3x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)^2 + a_{k-1}(k+r+2)\,(2k-1+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -2$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r+2)\left(a_k(k+r+2) + a_{k-1}(2k-1+2r)\right) = 0$$

- Shift index using $k->k+1$

$$(k+r+3)\left(a_{k+1}(k+r+3) + a_k(2k+2r+1)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(2k+2r+1)}{k+r+3}$$

- Recursion relation for $r = -2$

$$a_{k+1} = -\frac{a_k(2k-3)}{k+1}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{a_k(2k-3)}{k+1}\right]$$

1.136.3 Maple trace

```
Methods for second order ODEs:
```

1.136.4 Maple dsolve solution

Solving time : 0.011 (sec)
Leaf size : 73

```
dsolve(x^2*(1+2*x)*diff(diff(y(x),x),x)+x*(5+9*x)*diff(y(x),x)+(4+3*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\left(x+\frac{1}{2}\right)^2 \ln\left(\sqrt{1+2x}-1\right) - c_2\left(x+\frac{1}{2}\right)^2 \ln\left(\sqrt{1+2x}+1\right) + c_2\left(x+\frac{2}{3}\right)\sqrt{1+2x} + 4c_1\left(x+\frac{1}{2}\right)^2}{x^2\sqrt{1+2x}}$$

1.136.5 Mathematica DSolve solution

Solving time : 0.181 (sec)
Leaf size : 56

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]+x*(5+9*x)*D[y[x],x]+(4+3*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_2\left(-3(2x+1)^{3/2}\text{arctanh}\left(\sqrt{2x+1}\right)+6x+4\right)+3c_1(2x+1)^{3/2}}{3x^2}$$

1.137 problem 139

Internal problem ID [8275]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 139
Date solved : `Monday, October 21, 2024 at 05:05:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-2x)\,y''-x(5+4x)\,y'+(9+4x)\,y=0$$

1.137.1 Solved as second order ode using Kovacic algorithm

Time used: 0.316 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(-4x^2-5x\right)y'+(9+4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= -4x^2-5x \\ C &= 9+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{32x^2 + 56x - 1}{4(2x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 32x^2 + 56x - 1$$
$$t = 4(2x^2 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{32x^2 + 56x - 1}{4(2x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 258: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4\left(x - \frac{1}{2}\right)^2} - \frac{13}{x - \frac{1}{2}} + \frac{13}{x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{32x^2 + 56x - 1}{4\left(2x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{32x^2 + 56x - 1}{4\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right)\\ &= -1 - (-2)\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)} + (-)(0) \\
&= \frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)} \\
&= \frac{-1-8x}{4x^2-2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right)(1) + \left(\left(-\frac{1}{2x^2} + \frac{5}{2\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{32x^2+56x-1}{4\left(2x^2-x\right)^2}\right)\right)
$$

$$
\frac{-1+8a_0}{x(-1+2x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{1}{8}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{1}{8}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{8}\right) e^{\int \left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right) dx} \\
&= \left(x + \frac{1}{8}\right) e^{\frac{\ln(x)}{2} - \frac{5\ln(-1+2x)}{2}} \\
&= \frac{\left(x+\frac{1}{8}\right)\sqrt{x}}{(-1+2x)^{5/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^2-5x}{-2x^3+x^2}\,dx} \\
&= z_1 e^{\frac{5\ln(x)}{2}-\frac{7\ln(-1+2x)}{2}} \\
&= z_1 \left(\frac{x^{5/2}}{(-1+2x)^{7/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^3\left(x+\frac{1}{8}\right)}{(-1+2x)^6}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-5x}{-2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(x)-7\ln(-1+2x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{32x^3}{3} - 44x^2 + \frac{203x}{2} - 64\ln(x) - \frac{3125}{16(1+8x)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^3\left(x+\frac{1}{8}\right)}{(-1+2x)^6}\right) + c_2\left(\frac{x^3\left(x+\frac{1}{8}\right)}{(-1+2x)^6}\left(\frac{32x^3}{3} - 44x^2 + \frac{203x}{2} - 64\ln(x) - \frac{3125}{16(1+8x)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.137.2 Maple step by step solution

Let's solve

$x^2(1-2x)\left(\frac{d}{dx}y'\right)-x(5+4x)\,y'+(9+4x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{(9+4x)y}{x^2(-1+2x)}-\frac{(5+4x)y'}{x(-1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(5+4x)y'}{x(-1+2x)}-\frac{(9+4x)y}{x^2(-1+2x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{5+4x}{x(-1+2x)},P_3(x)=-\frac{9+4x}{x^2(-1+2x)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-5$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=9$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(-1+2x)\left(\frac{d}{dx}y'\right)+x(5+4x)\,y'+(-4x-9)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-3+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-3)^2 + 2a_{k-1}(k+1+r)\,(k-2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 3$$

- Each term in the series must be 0, giving the recursion relation

$$-a_k(k+r-3)^2 + 2a_{k-1}(k+1+r)\,(k-2+r) = 0$$

- Shift index using $k->k+1$

$$-a_{k+1}(k-2+r)^2 + 2a_k(k+r+2)\,(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k(k+r+2)(k+r-1)}{(k-2+r)^2}$$

- Recursion relation for $r = 3$

$$a_{k+1} = \frac{2a_k(k+5)(k+2)}{(k+1)^2}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{2a_k(k+5)(k+2)}{(k+1)^2}\right]$$

1.137.3 Maple trace

```
Methods for second order ODEs:
```

1.137.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 54

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)-x*(5+4*x)*diff(y(x),x)+(9+4*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\left(-6c_2\left(x+\frac{1}{8}\right)\ln\left(x\right)+c_2\,x^4-4c_2\,x^3+9c_2\,x^2+\left(8c_1+\frac{609c_2}{512}\right)x+c_1-\frac{9375c_2}{4096}\right)x^3}{\left(-1+2x\right)^6}$$

1.137.5 Mathematica DSolve solution

Solving time : 0.144 (sec)
Leaf size : 63

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]-x*(5+4*x)*D[y[x],x]+(9+4*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{x^3(c_2(4096x^4-16384x^3+36864x^2+4872x-9375)-48c_1(8x+1)-3072c_2(8x+1)\log(x))}{384(1-2x)^6}$$

1.138 problem 140

Internal problem ID [8276]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 140
Date solved : `Monday, October 21, 2024 at 05:05:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-x)\,y'' + x(7+x)\,y' + (9-x)\,y = 0$$

1.138.1 Solved as second order ode using Kovacic algorithm

Time used: 0.320 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y'' + \left(x^2+7x\right)y' + (9-x)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+x^2 \\ B &= x^2+7x \\ C &= 9-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-x^2 + 82x - 1}{4\left(x^2 - x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -x^2 + 82x - 1 \\
t &= 4\left(x^2 - x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-x^2 + 82x - 1}{4\left(x^2 - x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 260: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{20}{(-1+x)^2} - \frac{20}{-1+x} + \frac{20}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 5 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -4 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 82x - 1}{4(x^2 - x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-x^2+82x-1}{4\left(x^2-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}\right)\\
&= \frac{1}{2}-\left(-\frac{7}{2}\right)\\
&= 4
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} - \frac{4}{-1+x} + (-)\,(0) \\
&= \frac{1}{2x} - \frac{4}{-1+x} \\
&= -\frac{1+7x}{2x\,(-1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\left(12x^2 + 6x a_3 + 2a_2\right) + 2\left(\frac{1}{2x} - \frac{4}{-1+x}\right)\left(4x^3 + 3x^2 a_3 + 2a_2 x + a_1\right) + \left(\left(-\frac{1}{2x^2} + \frac{4}{(-1+x)^2}\right) + \left(\frac{1}{2x}\right.\right.
$$

$$
\frac{(a_3 - 16)\,x^3 + (4a_2 - 9a_3)}{x\,(}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1, a_1 = 16, a_2 = 36, a_3 = 16\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^4 + 16x^3 + 36x^2 + 16x + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 + 16x^3 + 36x^2 + 16x + 1\right) e^{\int \left(\frac{1}{2x} - \frac{4}{-1+x}\right) dx} \\
&= \left(x^4 + 16x^3 + 36x^2 + 16x + 1\right) e^{\frac{\ln(x)}{2} - 4\ln(-1+x)} \\
&= \frac{\left(x^4 + 16x^3 + 36x^2 + 16x + 1\right)\sqrt{x}}{(-1+x)^4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+7x}{-x^3+x^2}\,dx} \\
&= z_1 e^{-\frac{7\ln(x)}{2}+4\ln(-1+x)} \\
&= z_1\left(\frac{(-1+x)^4}{x^{7/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^4+16x^3+36x^2+16x+1}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+7x}{-x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-7\ln(x)+8\ln(-1+x)}}{(y_1)^2}\,dx \\
&= y_1\left(\ln(x) - \frac{20\left(-2x^3-\frac{15}{2}x^2-\frac{14}{3}x-\frac{5}{12}\right)}{x^4+16x^3+36x^2+16x+1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^4+16x^3+36x^2+16x+1}{x^3}\right) \\
&\quad + c_2\left(\frac{x^4+16x^3+36x^2+16x+1}{x^3}\left(\ln(x) - \frac{20\left(-2x^3-\frac{15}{2}x^2-\frac{14}{3}x-\frac{5}{12}\right)}{x^4+16x^3+36x^2+16x+1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.138.2 Maple step by step solution

Let's solve

$x^2(1-x)\left(\frac{d}{dx}y'\right)+x(7+x)\,y'+(9-x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(-9+x)y}{x^2(-1+x)}+\frac{(7+x)y'}{x(-1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{(7+x)y'}{x(-1+x)}+\frac{(-9+x)y}{x^2(-1+x)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{7+x}{x(-1+x)},P_3(x)=\frac{-9+x}{x^2(-1+x)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=7$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=9$

- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(-1+x)\left(\frac{d}{dx}y'\right)-x(7+x)\,y'+(-9+x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx} y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 (3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k (k+r+3)^2 + a_{k-1} (k-2+r)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -3$$

- Each term in the series must be 0, giving the recursion relation

$$-a_k (k+r+3)^2 + a_{k-1} (k-2+r)^2 = 0$$

- Shift index using $k->k+1$

$$-a_{k+1} (k+4+r)^2 + a_k (k+r-1)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k (k+r-1)^2}{(k+4+r)^2}$$

- Recursion relation for $r = -3$; series terminates at $k = 4$

$$a_{k+1} = \frac{a_k (k-4)^2}{(k+1)^2}$$

- Apply recursion relation for $k = 0$

$$a_1 = 16 a_0$$

- Apply recursion relation for $k = 1$

$$a_2 = \frac{9 a_1}{4}$$

- Express in terms of a_0

$$a_2 = 36 a_0$$

- Apply recursion relation for $k = 2$

$$a_3 = \frac{4 a_2}{9}$$

- Express in terms of a_0

$a_3 = 16a_0$

- Apply recursion relation for $k = 3$

  $a_4 = \frac{a_3}{16}$
- Express in terms of a_0

  $a_4 = a_0$
- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y = a_0 \cdot (x^4 + 16x^3 + 36x^2 + 16x + 1)$

1.138.3 Maple trace

```
Methods for second order ODEs:
```

1.138.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 72

```
dsolve(x^2*(1-x)*diff(diff(y(x),x),x)+x*(7+x)*diff(y(x),x)+(9-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{3c_2(x^4 + 16x^3 + 36x^2 + 16x + 1)\ln(x) + c_1 x^4 + (16c_1 + 120c_2)x^3 + (36c_1 + 450c_2)x^2 + (16c_1 + 280c_2}{x^3}$$

1.138.5 Mathematica DSolve solution

Solving time : `0.155` (sec)
Leaf size : 78

```
DSolve[{x^2*(1-x)*D[y[x],{x,2}]+x*(7+x)*D[y[x],x]+(9-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{5c_2(24x^3 + 90x^2 + 56x + 5) + 3c_1(x^4 + 16x^3 + 36x^2 + 16x + 1) + 3c_2(x^4 + 16x^3 + 36x^2 + 16x + 1)\log}{3x^3}$$

1.139 problem 141

Internal problem ID [8277]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 141
Date solved : `Monday, October 21, 2024 at 05:05:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x\left(-x^2+1\right)y' + \left(x^2+1\right)y = 0$$

1.139.1 Solved as second order ode using Kovacic algorithm

Time used: 0.301 (sec)

Writing the ode as

$$x^2y'' + \left(x^3-x\right)y' + \left(x^2+1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 - x \\ C &= x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 4x^2 - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 4x^2 - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 262: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{1}{x} - \frac{5}{4x^3} - \frac{5}{2x^5} - \frac{105}{16x^7} - \frac{155}{8x^9} - \frac{1965}{32x^{11}} - \frac{3265}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 4x^2 - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{x^2}{4} - 1\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{4} - 1 - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) &&= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) &&= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 4x^2 - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{x}{2}\right) \\
&= \frac{1}{2x} - \frac{x}{2} \\
&= \frac{1}{2x} - \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{2}\right) + \left(\frac{1}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{x}{2}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^3 - x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{4} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.139.2 Maple step by step solution

Let's solve
$x^2\left(\frac{d}{dx}y'\right) - x(-x^2+1)\,y' + (x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d}{dx}y'$
- Isolate 2nd derivative
$\frac{d}{dx}y' = -\frac{(x^2+1)y}{x^2} - \frac{(x^2-1)y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(x^2-1)y'}{x} + \frac{(x^2+1)y}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- • Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x(x^2-1)\,y' + (x^2+1)\,y = 0$
- • Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - ◦ Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

    Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 1$
- Each term must be 0
  $a_1 r^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\,(a_k(k+r-1) + a_{k-2}) = 0$
- Shift index using $k- >k+2$
  $(k+r+1)\,(a_{k+2}(k+r+1) + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{k+r+1}$
- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{a_k}{k+2}$
- Solution for $r = 1$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$

1.139.3 Maple trace

```
Methods for second order ODEs:
```

1.139.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-x^2+1)*diff(y(x),x)+(x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x\,\mathrm{e}^{-\frac{x^2}{2}}\left(c_1 + \mathrm{Ei}_1\left(-\frac{x^2}{2}\right) c_2\right)$$

1.139.5 Mathematica DSolve solution

Solving time : 0.056 (sec)
Leaf size : 35

```
DSolve[{x^2*D[y[x],{x,2}]-x*(1-x^2)*D[y[x],x]+(1+x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-\frac{x^2}{2}}x\left(c_1 \operatorname{ExpIntegralEi}\left(\frac{x^2}{2}\right) + 2c_2\right)$$

1.140 problem 142

Internal problem ID [8278]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 142
Date solved : `Monday, October 21, 2024 at 05:05:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''-3x\left(-x^2+1\right)y'+4y=0$$

1.140.1 Solved as second order ode using Kovacic algorithm

Time used: 0.412 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(3x^3-3x\right)y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^4+x^2\\ B&=3x^3-3x\\ C&=4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^4 - 10x^2 - 1$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 264: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 4 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{i}{4x-4i} - \frac{i}{4(x+i)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} + (-)(0) \\
&= \frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} \\
&= \frac{1}{2x} - \frac{x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{2(x-i)^2} + \frac{1}{2(x+i)^2}\right) + \left(\frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right)dx} \\
&= \frac{\sqrt{x}}{\sqrt{x^2+1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3-3x}{x^4+x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}-\frac{3\ln\left(x^2+1\right)}{2}} \\
&= z_1\left(\frac{x^{3/2}}{\left(x^2+1\right)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2}{\left(x^2+1\right)^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3-3x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)-3\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{x^2}{2}+\ln\left(x\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2}{\left(x^2+1\right)^2}\right) + c_2\left(\frac{x^2}{\left(x^2+1\right)^2}\left(\frac{x^2}{2}+\ln\left(x\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.140.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)-3x(-x^2+1)\,y'+4y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{4y}{x^2(x^2+1)}-\frac{3(x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{3(x^2-1)y'}{x(x^2+1)}+\frac{4y}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3(x^2-1)}{x(x^2+1)},P_3(x)=\frac{4}{x^2(x^2+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-3$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=4$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d}{dx}y'\right)+3x(x^2-1)\,y'+4y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^{m} \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^{m} \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_{k}(k+r)(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^{m} \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-2+r)^{2} x^{r} + a_{1}(-1+r)^{2} x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_{k}(k+r-2)^{2} + a_{k-2}(k+r-2)(k+r)\right) x^{k+r}\right) = 0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^{2} = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term must be 0

  $a_{1}(-1+r)^{2} = 0$
- Solve for the dependent coefficient(s)

  $a_{1} = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left(a_{k}(k+r-2) + a_{k-2}(k+r)\right) = 0$
- Shift index using $k->k+2$

  $(k+r)\left(a_{k+2}(k+r) + a_{k}(k+r+2)\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_{k}(k+r+2)}{k+r}$
- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_{k}(k+4)}{k+2}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_{k} x^{k+2}, a_{k+2} = -\frac{a_{k}(k+4)}{k+2}, a_{1} = 0\right]$

1.140.3 Maple trace

```
Methods for second order ODEs:
```

1.140.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 27

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-3*x*(-x^2+1)*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^2 \left(c_1 + c_2 \left(\frac{x^2}{2} + \ln\left(x\right)\right)\right)}{\left(x^2+1\right)^2}$$

1.140.5 Mathematica DSolve solution

Solving time : 0.086 (sec)
Leaf size : 36

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-3*x*(1-x^2)*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^2 (c_2 x^2 + 2c_2 \log(x) + 2c_1)}{2\left(x^2+1\right)^2}$$

1.141 problem 143

Internal problem ID [8279]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 143
Date solved : `Monday, October 21, 2024 at 05:05:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 2x^3y' + \left(3x^2+1\right)y = 0$$

1.141.1 Solved as second order ode using Kovacic algorithm

Time used: 0.300 (sec)

Writing the ode as

$$4x^2y'' + 2x^3y' + \left(3x^2+1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 2x^3 \\ C &= 3x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^2 - 4}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 8x^2 - 4$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^2 - 4}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 266: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} - \frac{1}{2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{1}{x} - \frac{5}{2x^3} - \frac{10}{x^5} - \frac{105}{2x^7} - \frac{310}{x^9} - \frac{1965}{x^{11}} - \frac{13060}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{4}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 8x^2 - 4}{16x^2} \\
&= Q + \frac{R}{16x^2} \\
&= \left(\frac{x^2}{16} - \frac{1}{2}\right) + \left(-\frac{1}{4x^2}\right) \\
&= \frac{x^2}{16} - \frac{1}{2} - \frac{1}{4x^2}
\end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 8x^2 - 4}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{4}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} + (-)\left(\frac{x}{4}\right) \\ &= \frac{1}{2x} - \frac{x}{4} \\ &= \frac{1}{2x} - \frac{x}{4}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x} - \frac{x}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{x}{4}\right)^2 - \left(\frac{x^4 - 8x^2 - 4}{16x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{x}{4}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x^2}{8}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^3}{4x^2} \, dx} \\ &= z_1 e^{-\frac{x^2}{8}} \\ &= z_1 \left(e^{-\frac{x^2}{8}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{4}} \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3}{4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{4}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{4}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x^2}{4}} \sqrt{x}\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{4}} \sqrt{x} \left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{4}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.141.2 Maple step by step solution

Let's solve

$4x^2 \left(\frac{d}{dx} y'\right) + 2x^3 y' + (3x^2 + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(3x^2+1)y}{4x^2} - \frac{y'x}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{y'x}{2} + \frac{(3x^2+1)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{x}{2}, P_3(x) = \frac{3x^2+1}{4x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{4}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 2x^3y' + (3x^2+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^3 \cdot y'$ to series expansion

  $x^3 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+2}$

- ◦ Shift index using $k- >k-2$

  $x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2}(k-2+r)\,x^{k+r}$

- ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+2r)^2 x^r + a_1(1+2r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+2r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = \frac{1}{2}$
- Each term must be 0
  $a_1(1+2r)^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1) = 0$
- Shift index using $k->k+2$
  $a_{k+2}(2k+2r+3)^2 + a_k(2k+2r+3) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{2k+2r+3}$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = -\frac{a_k}{2k+4}$
- Solution for $r = \frac{1}{2}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{2k+4}, a_1 = 0\right]$

1.141.3 Maple trace

```
Methods for second order ODEs:
```

1.141.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 25

```
dsolve(4*x^2*diff(diff(y(x),x),x)+2*x^3*diff(y(x),x)+(3*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-\frac{x^2}{4}} \sqrt{x} \left(c_1 + \mathrm{Ei}_1\left(-\frac{x^2}{4}\right) c_2\right)$$

1.141.5 Mathematica DSolve solution

Solving time : 0.191 (sec)
Leaf size : 39

```
DSolve[{4*x^2*D[y[x],{x,2}]+2*x^3*D[y[x],x]+(1+3*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-\frac{x^2}{4}}\sqrt{x}\left(c_2 \operatorname{ExpIntegralEi}\left(\frac{x^2}{4}\right) + 2c_1\right)$$

1.142 problem 144

Internal problem ID [8280]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 144
Date solved : `Monday, October 21, 2024 at 05:05:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+1)\,y''-x(-2x^2+1)\,y'+y=0$$

1.142.1 Solved as second order ode using Kovacic algorithm

Time used: 0.300 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(2x^3-x)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 2x^3-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2 - 1 \\ t &= 4\left(x^3 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 268: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{3}{16(x-i)^2} - \frac{3}{16(x+i)^2} - \frac{5i}{16(x-i)} + \frac{5i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^2 - 1}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-i$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{1}{4x - 4i} + \frac{1}{4x + 4i} + (-)(0) \\
&= \frac{1}{2x} + \frac{1}{4x - 4i} + \frac{1}{4x + 4i} \\
&= \frac{1}{2x} + \frac{x}{2x^2 + 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{4x - 4i} + \frac{1}{4x + 4i}\right)(0) + \Bigg(\left(-\frac{1}{2x^2} - \frac{1}{4(x - i)^2} - \frac{1}{4(x + i)^2}\right) + \Bigg(\frac{1}{2x} + \frac{1}{4x - 4i} + \frac{1}{4x +$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{1}{4x - 4i} + \frac{1}{4x + 4i}\right) dx} \\
&= \left(x^2 + 1\right)^{1/4} \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3-x}{x^4+x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+1\right)}{4}+\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\sqrt{x}}{\left(x^2+1\right)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{\sqrt{x^2+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+1\right)}{2}+\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{\sqrt{x^2+1}}\right) + c_2 \left(\frac{x}{\sqrt{x^2+1}}\left(-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.142.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)-x(-2x^2+1)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{y}{x^2(x^2+1)}-\frac{(2x^2-1)\,y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(2x^2-1)\,y'}{x(x^2+1)}+\frac{y}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{2x^2-1}{x(x^2+1)},P_3(x)=\frac{1}{x^2(x^2+1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=1$

- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(2x^2-1)\,y'+y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)^2\,x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k-2+r)\,(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 1$$

- Each term must be 0

  $$a_1 r^2 = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-1)\,(a_k(k+r-1) + a_{k-2}(k-2+r)) = 0$$

- Shift index using $k- > k+2$

  $$(k+r+1)\,(a_{k+2}(k+r+1) + a_k(k+r)) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k(k+r)}{k+r+1}$$

- Recursion relation for $r = 1$

  $$a_{k+2} = -\frac{a_k(k+1)}{k+2}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+1)}{k+2}, a_1 = 0\right]$$

1.142.3 Maple trace

```
Methods for second order ODEs:
```

1.142.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 25

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-2*x^2+1)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x\left(c_2 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) + c_1\right)}{\sqrt{x^2+1}}$$

1.142.5 Mathematica DSolve solution

Solving time : 0.102 (sec)
Leaf size : 33

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(1-2*x^2)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x\left(c_1 - c_2 \operatorname{arctanh}\left(\sqrt{x^2+1}\right)\right)}{\sqrt{x^2+1}}$$

1.143 problem 145

Internal problem ID [8281]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 145
Date solved : `Monday, October 21, 2024 at 05:05:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+2\right)y''+7x^3y'+\left(3x^2+1\right)y=0$$

1.143.1 Solved as second order ode using Kovacic algorithm

Time used: 0.380 (sec)

Writing the ode as

$$\left(2x^4+4x^2\right)y''+7x^3y'+\left(3x^2+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+4x^2 \\ B &= 7x^3 \\ C &= 3x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^4 - 16}{16(x^3 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^4 - 16$$
$$t = 16(x^3 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^4 - 16}{16(x^3 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 270: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 4 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2} - \frac{7}{64\left(x - i\sqrt{2}\right)^2} - \frac{7}{64\left(x + i\sqrt{2}\right)^2} - \frac{9i\sqrt{2}}{128\left(x - i\sqrt{2}\right)} + \frac{9i\sqrt{2}}{128\left(x + i\sqrt{2}\right)}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}
$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8}
\end{aligned}
$$

For the pole at $x=-i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-3x^4-16}{16\left(x^3+2x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-3x^4-16}{16\left(x^3+2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$i\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-i\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{3}{4} - \left(\frac{3}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}} + (0) \\ &= \frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}} \\ &= \frac{1}{2x} + \frac{x}{4x^2 + 8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{8\left(x - i\sqrt{2}\right)^2} - \frac{1}{8\left(x + i\sqrt{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{}{8x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}}\right) dx} \\ &= \left(x^2 + 2\right)^{1/8} \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^3}{2x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(x^2+2\right)}{8}} \\
&= z_1\left(\frac{1}{\left(x^2+2\right)^{7/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x}}{\left(x^2+2\right)^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3}{2x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(x^2+2\right)}{4}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int \frac{1}{\left(x^2+2\right)^{1/4} x}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{\left(x^2+2\right)^{3/4}}\right) + c_2\left(\frac{\sqrt{x}}{\left(x^2+2\right)^{3/4}}\left(\int \frac{1}{\left(x^2+2\right)^{1/4} x}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.143.2 Maple step by step solution

Let's solve

$2x^2(x^2+2)\left(\frac{d}{dx}y'\right)+7x^3y'+(3x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(3x^2+1)y}{2x^2(x^2+2)}-\frac{7xy'}{2(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{7xy'}{2(x^2+2)}+\frac{(3x^2+1)y}{2x^2(x^2+2)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{7x}{2(x^2+2)},P_3(x)=\frac{3x^2+1}{2x^2(x^2+2)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\Big|_{x=0}=0$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=\frac{1}{4}$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$2x^2(x^2+2)\left(\frac{d}{dx}y'\right)+7x^3y'+(3x^2+1)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^3\cdot y'$ to series expansion

$$x^3 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+2}$$

- Shift index using $k- >k-2$

$$x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2}(k-2+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)^2\, x^r + a_1(1+2r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1)\,(k+r-1)\right) x\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{2}$$

- Each term must be 0

$$a_1(1+2r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(\frac{a_{k-2}(k+r-1)}{2} + a_k\left(k+r-\frac{1}{2}\right)\right)\left(k+r-\frac{1}{2}\right) = 0$$

- Shift index using $k- >k+2$

$$4\left(\frac{a_k(k+r+1)}{2} + a_{k+2}\left(k+\frac{3}{2}+r\right)\right)\left(k+\frac{3}{2}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+1)}{2k+2r+3}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+4}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+4}, a_1 = 0\right]$$

1.143.3 Maple trace

```
Methods for second order ODEs:
```

1.143.4 Maple dsolve solution

Solving time : 0.055 (sec)
Leaf size : 81

```
dsolve(2*x^2*(x^2+2)*diff(diff(y(x),x),x)+7*x^3*diff(y(x),x)+(3*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\sqrt{x}\left(2^{3/4}c_1 + \ln\left(-\sqrt{2}\left(2x^2+4\right)^{1/4}+2\right)c_2 - \ln\left(\sqrt{2}\left(2x^2+4\right)^{1/4}+2\right)c_2 + 2\arctan\left(\frac{\sqrt{2}\left(2x^2+4\right)^{1/4}}{2}\right)c_2\right.}{2\left(x^2+2\right)^{3/4}}$$

1.143.5 Mathematica DSolve solution

Solving time : 0.289 (sec)
Leaf size : 77

```
DSolve[{2*x^2*(2+x^2)*D[y[x],{x,2}]+7*x^3*D[y[x],x]+(1+3*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}\left(2^{3/4}c_2\arctan\left(\frac{\sqrt[4]{x^2+2}}{\sqrt[4]{2}}\right) - 2^{3/4}c_2\text{arctanh}\left(\frac{\sqrt[4]{x^2+2}}{\sqrt[4]{2}}\right) + 2c_1\right)}{2\left(x^2+2\right)^{3/4}}$$

1.144 problem 146

Internal problem ID [8282]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 146
Date solved : `Monday, October 21, 2024 at 05:05:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+1)\,y''-x(-4x^2+1)\,y'+(2x^2+1)\,y=0$$

1.144.1 Solved as second order ode using Kovacic algorithm

Time used: 0.333 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(4x^3-x)\,y'+(2x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^4+x^2\\B&=4x^3-x\\C&=2x^2+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-6x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -6x^2 - 1$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-6x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 272: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 2 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2} + \frac{5}{16\,(x-i)^2} + \frac{5}{16\,(x+i)^2} + \frac{3i}{16\,(x-i)} - \frac{3i}{16\,(x+i)}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-6x^2 - 1}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{1}{4(x-i)} - \frac{1}{4(x+i)} + (0) \\
&= \frac{1}{2x} - \frac{1}{4(x-i)} - \frac{1}{4(x+i)} \\
&= \frac{1}{2x^3 + 2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{4(x-i)} - \frac{1}{4(x+i)} \right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{4(x-i)^2} + \frac{1}{4(x+i)^2} \right) + \left(\frac{1}{2x} - \frac{1}{4(x-i)} - \frac{1}{4(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{4(x-i)} - \frac{1}{4(x+i)} \right) dx} \\
&= \frac{\sqrt{x}}{(x^2+1)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^3-x}{x^4+x^2}\, dx} \\
&= z_1 e^{\frac{\ln(x)}{2}-\frac{5\ln(x^2+1)}{4}} \\
&= z_1 \left(\frac{\sqrt{x}}{(x^2+1)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{(x^2+1)^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^3-x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{5\ln(x^2+1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{(x^2+1)^{3/2}}\right) + c_2 \left(\frac{x}{(x^2+1)^{3/2}} \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.144.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)-x(-4x^2+1)\,y'+(2x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(2x^2+1)y}{x^2(x^2+1)}-\frac{(4x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(4x^2-1)y'}{x(x^2+1)}+\frac{(2x^2+1)y}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{4x^2-1}{x(x^2+1)},P_3(x)=\frac{2x^2+1}{x^2(x^2+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=1$

  ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(4x^2-1)\,y'+(2x^2+1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2\, x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r)\,(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 1$
- Each term must be 0

  $a_1 r^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r-1) + a_{k-2}(k+r)) = 0$
- Shift index using $k->k+2$

  $(k+r+1)\,(a_{k+2}(k+r+1) + a_k(k+r+2)) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+2)}{k+r+1}$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k(k+3)}{k+2}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+3)}{k+2}, a_1 = 0\right]$

1.144.3 Maple trace

```
Methods for second order ODEs:
```

1.144.4 Maple dsolve solution

Solving time : 0.013 (sec)
Leaf size : 35

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-4*x^2+1)*diff(y(x),x)+(2*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\left(-c_2 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)+c_2\sqrt{x^2+1}+c_1\right)x}{\left(x^2+1\right)^{3/2}}$$

1.144.5 Mathematica DSolve solution

Solving time : 0.156 (sec)
Leaf size : 45

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(1-4*x^2)*D[y[x],x]+(1+2*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x\left(-c_2\operatorname{arctanh}\left(\sqrt{x^2+1}\right)+c_2\sqrt{x^2+1}+c_1\right)}{\left(x^2+1\right)^{3/2}}$$

1.145 problem 147

Internal problem ID [8283]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 147
Date solved : `Monday, October 21, 2024 at 05:05:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+4\right)y''+3x\left(3x^2+8\right)y'+\left(-9x^2+1\right)y=0$$

1.145.1 Solved as second order ode using Kovacic algorithm

Time used: 0.460 (sec)

Writing the ode as

$$\left(4x^4+16x^2\right)y''+\left(9x^3+24x\right)y'+\left(-9x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+16x^2 \\ B &= 9x^3+24x \\ C &= -9x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{153x^4 + 704x^2 - 256}{64\,(x^3 + 4x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 153x^4 + 704x^2 - 256$$
$$t = 64(x^3 + 4x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{153x^4 + 704x^2 - 256}{64\,(x^3 + 4x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 274: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 + 4x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2i$ of order 2. There is a pole at $x = -2i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{39}{256(x-2i)^2} - \frac{39}{256(x+2i)^2} - \frac{377i}{512(x-2i)} + \frac{377i}{512(x+2i)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 2i$ let b be the coefficient of $\frac{1}{(x-2i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{39}{256}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{16} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{16} \end{aligned}$$

For the pole at $x=-2i$ let b be the coefficient of $\frac{1}{(x+2i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{39}{256}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{13}{16} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{3}{16}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{153x^4+704x^2-256}{64\left(x^3+4x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{153}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{17}{8} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{9}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{153x^4+704x^2-256}{64\left(x^3+4x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$2i$	2	0	$\frac{13}{16}$	$\frac{3}{16}$
$-2i$	2	0	$\frac{13}{16}$	$\frac{3}{16}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{17}{8}$	$-\frac{9}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{17}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+ + \alpha_{c_3}^+\right)\\
&= \frac{17}{8} - \left(\frac{17}{8}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty\\
&= \frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)} + (0)\\
&= \frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)}\\
&= \frac{1}{2x} + \frac{13x}{8x^2+32}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{13}{16\,(x-2i)^2} - \frac{13}{16\,(x+2i)^2}\right) + \left(\frac{1}{2x} + \frac{1}{16\,(x
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx}\\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{13}{16(x-2i)} + \frac{13}{16(x+2i)}\right)dx}\\
&= \left(x^2+4\right)^{13/16}\sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{9x^3+24x}{4x^4+16x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+4\right)}{16}-\frac{3\ln(x)}{4}} \\
&= z_1\left(\frac{1}{\left(x^2+4\right)^{3/16} x^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+4\right)^{5/8}}{x^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{9x^3+24x}{4x^4+16x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+4\right)}{8}-\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{3\ln\left(x^2+4\right)}{8}-\frac{3\ln(x)}{2}}\sqrt{x}}{\left(x^2+4\right)^{5/4}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x^2+4\right)^{5/8}}{x^{1/4}}\right) + c_2\left(\frac{\left(x^2+4\right)^{5/8}}{x^{1/4}}\left(\int \frac{\mathrm{e}^{-\frac{3\ln\left(x^2+4\right)}{8}-\frac{3\ln(x)}{2}}\sqrt{x}}{\left(x^2+4\right)^{5/4}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.145.2 Maple step by step solution

Let's solve

$4x^2(x^2+4)\left(\frac{d}{dx}y'\right)+3x(3x^2+8)\,y'+(-9x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{(9x^2-1)y}{4x^2(x^2+4)}-\frac{3(3x^2+8)y'}{4x(x^2+4)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{3(3x^2+8)y'}{4x(x^2+4)}-\frac{(9x^2-1)y}{4x^2(x^2+4)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{3(3x^2+8)}{4x(x^2+4)},P_3(x)=-\frac{9x^2-1}{4x^2(x^2+4)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{3}{2}$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{16}$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+4)\left(\frac{d}{dx}y'\right)+3x(3x^2+8)\,y'+(-9x^2+1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ○ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+4r)^2 x^r + a_1(5+4r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r+1)^2 + a_{k-2}(4k+4r+1)\,(k-3+r)\right) x^{k+}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+4r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -\frac{1}{4}$$

- Each term must be 0

$$a_1(5+4r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$16\left(k+r+\frac{1}{4}\right)\left(\frac{a_{k-2}(k-3+r)}{4} + a_k\left(k+r+\frac{1}{4}\right)\right) = 0$$

- Shift index using $k- >k+2$

$$16\left(k+\frac{9}{4}+r\right)\left(\frac{a_k(k+r-1)}{4} + a_{k+2}\left(k+\frac{9}{4}+r\right)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r-1)}{4k+4r+9}$$

- Recursion relation for $r = -\frac{1}{4}$

$$a_{k+2} = -\frac{a_k\left(k-\frac{5}{4}\right)}{4k+8}$$

- Solution for $r = -\frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}, a_{k+2} = -\frac{a_k\left(k-\frac{5}{4}\right)}{4k+8}, a_1 = 0\right]$$

1.145.3 Maple trace

```
Methods for second order ODEs:
```

1.145.4 Maple dsolve solution

Solving time : `0.037` (sec)
Leaf size : 66

```
dsolve(4*x^2*(x^2+4)*diff(diff(y(x),x),x)+3*x*(3*x^2+8)*diff(y(x),x)+(-9*x^2+1)*y(x) =
       y(x),singsol=all)
```

$$y = \frac{\left(x^2 \operatorname{hypergeom}\left(\left[1, 1, \frac{13}{8}\right], [2, 2], -\frac{x^2}{4}\right) - \frac{32\gamma}{5} + \frac{64\ln(2)}{5} - \frac{64\ln(x)}{5} - \frac{32\Psi\left(\frac{5}{8}\right)}{5}\right)(x^2+4)^{5/8} c_2 2^{3/4} + c_1(x^2+4}{x^{1/4}}$$

1.145.5 Mathematica DSolve solution

Solving time : `0.86` (sec)
Leaf size : 198

```
DSolve[{4*x^2*(4+x^2)*D[y[x],{x,2}]+3*x*(8+3*x^2)*D[y[x],x]+(1-9*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2\left(5\, 2^{3/4}(x^2+4)^{5/8} \arctan\left(\frac{\sqrt[8]{x^2+4}}{\sqrt[4]{2}}\right) + 5\sqrt[4]{2}(x^2+4)^{5/8} \arctan\left(\frac{\sqrt{2} - \sqrt[4]{x^2+4}}{2^{3/4}\sqrt[8]{x^2+4}}\right) - 5\, 2^{3/4}(x^2+4)^{5/8} a}{80\sqrt[4]{x}}$$

1.146 problem 148

Internal problem ID [8284]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 148
Date solved : `Monday, October 21, 2024 at 05:05:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2\left(x^2+3\right)y''+x\left(11x^2+3\right)y'+\left(5x^2+1\right)y=0$$

1.146.1 Solved as second order ode using Kovacic algorithm

Time used: 0.382 (sec)

Writing the ode as

$$\left(3x^4+9x^2\right)y''+\left(11x^3+3x\right)y'+\left(5x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^4+9x^2 \\ B &= 11x^3+3x \\ C &= 5x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5x^4 + 18x^2 - 81$$
$$t = 36\left(x^3 + 3x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 276: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{5}{36\left(x - i\sqrt{3}\right)^2} - \frac{5}{36\left(x + i\sqrt{3}\right)^2} - \frac{7i\sqrt{3}}{108\left(x - i\sqrt{3}\right)} + \frac{7i\sqrt{3}}{108\left(x + i\sqrt{3}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}\end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$i\sqrt{3}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$
$-i\sqrt{3}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{5}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right)\\
&= \frac{5}{6} - \left(\frac{5}{6}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_\infty\\
&= \frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}} + (0)\\
&= \frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}}\\
&= \frac{1}{2x} + \frac{x}{3x^2+9}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{6\left(x-i\sqrt{3}\right)^2} - \frac{1}{6\left(x+i\sqrt{3}\right)^2}\right) + \left(\frac{1}{2x} + \frac{}{6x}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx}\\
&= e^{\int \left(\frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}}\right)dx}\\
&= \left(x^2+3\right)^{1/6}\sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^3+3x}{3x^4+9x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{6}-\frac{5\ln\left(x^2+3\right)}{6}} \\
&= z_1\left(\frac{1}{x^{1/6}\left(x^2+3\right)^{5/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{\left(x^2+3\right)^{2/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+3x}{3x^4+9x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{3}-\frac{5\ln\left(x^2+3\right)}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{3}-\frac{5\ln\left(x^2+3\right)}{3}}\left(x^2+3\right)^{4/3}}{x^{2/3}}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/3}}{\left(x^2+3\right)^{2/3}}\right) + c_2\left(\frac{x^{1/3}}{\left(x^2+3\right)^{2/3}}\left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{3}-\frac{5\ln\left(x^2+3\right)}{3}}\left(x^2+3\right)^{4/3}}{x^{2/3}}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.146.2 Maple step by step solution

Let's solve

$3x^2(x^2+3)\left(\frac{d}{dx}y'\right)+x(11x^2+3)\,y'+(5x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(5x^2+1)y}{3x^2(x^2+3)}-\frac{(11x^2+3)y'}{3x(x^2+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(11x^2+3)y'}{3x(x^2+3)}+\frac{(5x^2+1)y}{3x^2(x^2+3)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{11x^2+3}{3x(x^2+3)},P_3(x)=\frac{5x^2+1}{3x^2(x^2+3)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{9}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $3x^2(x^2+3)\left(\frac{d}{dx}y'\right)+x(11x^2+3)\,y'+(5x^2+1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+3r)^2\, x^r + a_1(2+3r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 + a_{k-2}(3k+3r-1)\,(k+r-1)\right) x\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+3r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{3}$$

- Each term must be 0

$$a_1(2+3r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$9\left(\frac{a_{k-2}(k+r-1)}{3} + a_k\left(k+r-\frac{1}{3}\right)\right)\left(k+r-\frac{1}{3}\right) = 0$$

- Shift index using $k->k+2$

$$9\left(\frac{a_k(k+r+1)}{3} + a_{k+2}\left(k+\frac{5}{3}+r\right)\right)\left(k+\frac{5}{3}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+1)}{3k+3r+5}$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+2} = -\frac{a_k\left(k+\frac{4}{3}\right)}{3k+6}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k\left(k+\frac{4}{3}\right)}{3k+6}, a_1 = 0\right]$$

1.146.3 Maple trace

```
Methods for second order ODEs:
```

1.146.4 Maple dsolve solution

Solving time : 0.057 (sec)
Leaf size : 102

```
dsolve(3*x^2*(x^2+3)*diff(diff(y(x),x),x)+x*(11*x^2+3)*diff(y(x),x)+(5*x^2+1)*y(x) = (
       y(x),singsol=all)
```

$$y = \frac{x^{1/3}\left(2\sqrt{3}\arctan\left(\frac{(9x^2+27)^{1/3}\sqrt{3}}{6+(9x^2+27)^{1/3}}\right)c_2+3\,3^{1/3}c_1-\ln\left((9x^2+27)^{2/3}+3(9x^2+27)^{1/3}+9\right)c_2+2\ln\left(3-\right.\right.}{9\left(x^2+3\right)^{2/3}}$$

1.146.5 Mathematica DSolve solution

Solving time : 0.091 (sec)
Leaf size : 94

```
DSolve[{3*x^2*(3+x^2)*D[y[x],x]+x*(3+11*x^2)*D[y[x],x]+(1+5*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1\exp\left(\frac{1}{3}\text{RootSum}\left[3\#1^3+11\#1^2+9\#1+3\&,\frac{3\#1^2\log(x-\#1)-4\#1\log(x-\#1)+9\log(x-\#1)}{9\#1^2+22\#1+9}\&\right]\right)}{\sqrt[3]{x}}$$

$$y(x) \to 0$$

1.147 problem 149

Internal problem ID [8285]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 149
Date solved : `Monday, October 21, 2024 at 05:05:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' - 3x\left(-2x^2+7\right)y' + \left(2x^2+25\right)y = 0$$

1.147.1 Solved as second order ode using Kovacic algorithm

Time used: 0.339 (sec)

Writing the ode as

$$9x^2y'' + \left(6x^3 - 21x\right)y' + \left(2x^2+25\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= 6x^3 - 21x \\ C &= 2x^2 + 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 24x^2 - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^4 - 24x^2 - 9$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 24x^2 - 9}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 278: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{9} - \frac{2}{3} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{3} - \frac{1}{x} - \frac{15}{8x^3} - \frac{45}{8x^5} - \frac{2835}{128x^7} - \frac{12555}{128x^9} - \frac{477495}{1024x^{11}} - \frac{2380185}{1024x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{3}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{3} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{9}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 24x^2 - 9}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{x^2}{9} - \frac{2}{3}\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{9} - \frac{2}{3} - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{2}{3}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{2}{3}\right) - (0) \\ &= -\frac{2}{3} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{3} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4 - 24x^2 - 9}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{3}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{x}{3}\right) \\
&= \frac{1}{2x} - \frac{x}{3} \\
&= \frac{1}{2x} - \frac{x}{3}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{x}{3}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{3}\right) + \left(\frac{1}{2x} - \frac{x}{3}\right)^2 - \left(\frac{4x^4 - 24x^2 - 9}{36x^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} - \frac{x}{3}\right) dx} \\
&= \sqrt{x}\, e^{-\frac{x^2}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{6x^3 - 21x}{9x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{6} + \frac{7 \ln(x)}{6}} \\
&= z_1 \left(x^{7/6} e^{-\frac{x^2}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^{5/3}\mathrm{e}^{-\frac{x^2}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x^3-21x}{9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{3}+\frac{7\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{3}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{5/3}\mathrm{e}^{-\frac{x^2}{3}}\right) + c_2 \left(x^{5/3}\mathrm{e}^{-\frac{x^2}{3}}\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{3}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.147.2 Maple step by step solution

Let's solve

$9x^2\left(\frac{d}{dx}y'\right) - 3x(-2x^2+7)\,y' + (2x^2+25)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(2x^2+25)y}{9x^2} - \frac{(2x^2-7)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(2x^2-7)y'}{3x} + \frac{(2x^2+25)y}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{2x^2-7}{3x}, P_3(x) = \frac{2x^2+25}{9x^2}\right]$
- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -\frac{7}{3}$
- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{25}{9}$
- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2\left(\frac{d}{dx}y'\right) + 3x(2x^2-7)\,y' + (2x^2+25)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- ○ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- ○ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- ○ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-5+3r)^2 x^r + a_1(-2+3r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-5)^2 + 2a_{k-2}(3k+3r-5)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-5+3r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = \frac{5}{3}$
- Each term must be 0
  $a_1(-2+3r)^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(3k+3r-5)^2 + 2a_{k-2}(3k+3r-5) = 0$
- Shift index using $k- > k+2$
  $a_{k+2}(3k+3r+1)^2 + 2a_k(3k+3r+1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2a_k}{3k+3r+1}$
- Recursion relation for $r = \frac{5}{3}$
  $a_{k+2} = -\frac{2a_k}{3k+6}$
- Solution for $r = \frac{5}{3}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{3}}, a_{k+2} = -\frac{2a_k}{3k+6}, a_1 = 0\right]$

1.147.3 Maple trace

```
Methods for second order ODEs:
```

1.147.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 25

```
dsolve(9*x^2*diff(diff(y(x),x),x)-3*x*(-2*x^2+7)*diff(y(x),x)+(2*x^2+25)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^{5/3} \mathrm{e}^{-\frac{x^2}{3}} \left(c_1 + \mathrm{Ei}_1\left(-\frac{x^2}{3}\right) c_2\right)$$

1.147.5 Mathematica DSolve solution

Solving time : 0.188 (sec)
Leaf size : 39

```
DSolve[{9*x^2*D[y[x],{x,2}]-3*x*(7-2*x^2)*D[y[x],x]+(25+2*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-\frac{x^2}{3}} x^{5/3} \left(c_2 \operatorname{ExpIntegralEi}\left(\frac{x^2}{3}\right) + 2c_1 \right)$$

1.148 problem 150

Internal problem ID [8286]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 150
Date solved : `Monday, October 21, 2024 at 05:05:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x\left(-x^2+1\right)y' + \left(x^2+1\right)y = 0$$

1.148.1 Solved as second order ode using Kovacic algorithm

Time used: 0.304 (sec)

Writing the ode as

$$x^2y'' + \left(x^3-x\right)y' + \left(x^2+1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 - x \\ C &= x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 4x^2 - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 4x^2 - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 280: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{1}{x} - \frac{5}{4x^3} - \frac{5}{2x^5} - \frac{105}{16x^7} - \frac{155}{8x^9} - \frac{1965}{32x^{11}} - \frac{3265}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 4x^2 - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{x^2}{4} - 1\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{4} - 1 - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) &&= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 4x^2 - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{x}{2}\right) \\
&= \frac{1}{2x} - \frac{x}{2} \\
&= \frac{1}{2x} - \frac{x}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{2}\right) + \left(\frac{1}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{x}{2}\right) dx} \\
&= \sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3 - x}{x^2}\, dx} \\
&= z_1 e^{-\frac{x^2}{4} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.148.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x(-x^2+1)\,y' + (x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+1)y}{x^2} - \frac{(x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(x^2-1)y'}{x} + \frac{(x^2+1)y}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x(x^2-1)\,y' + (x^2+1)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

    Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k+r-1)^2 + a_{k-2}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 1$
- Each term must be 0
  $a_1 r^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_k(k+r-1) + a_{k-2}\right) = 0$
- Shift index using $k- >k+2$
  $(k+r+1)\left(a_{k+2}(k+r+1) + a_k\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{k+r+1}$
- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{a_k}{k+2}$
- Solution for $r = 1$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$

1.148.3 Maple trace

```
Methods for second order ODEs:
```

1.148.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-x^2+1)*diff(y(x),x)+(x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x\,\mathrm{e}^{-\frac{x^2}{2}} \left(c_1 + \mathrm{Ei}_1\left(-\frac{x^2}{2}\right) c_2\right)$$

1.148.5 Mathematica DSolve solution

Solving time : 0.019 (sec)
Leaf size : 35

```
DSolve[{x^2*D[y[x],{x,2}]-x*(1-x^2)*D[y[x],x]+(1+x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-\frac{x^2}{2}} x \left(c_1 \operatorname{ExpIntegralEi}\left(\frac{x^2}{2}\right) + 2c_2\right)$$

1.149 problem 151

Internal problem ID [8287]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 151
Date solved : `Monday, October 21, 2024 at 05:05:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-2x)\,y''+3xy'+(1+4x)\,y=0$$

1.149.1 Solved as second order ode using Kovacic algorithm

Time used: 0.276 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right)y''+3xy'+(1+4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2\\ B &= 3x \\ C &= 1+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{32x^2 + 16x - 1}{4(2x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 32x^2 + 16x - 1 \\ t &= 4(2x^2 - x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{32x^2 + 16x - 1}{4(2x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 282: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{x} - \frac{1}{4x^2} + \frac{15}{4\left(x - \frac{1}{2}\right)^2} - \frac{3}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)} + (-)\,(0) \\ &= \frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)} \\ &= \frac{-1-4x}{4x^2-2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{3}{2\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{32x^2+16x-1}{4\left(2x^2-x\right)^2}\right)\right)$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)}\right) dx} \\ &= \frac{\sqrt{x}}{(-1+2x)^{3/2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{-2x^3+x^2}\, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2} + \frac{3\ln(-1+2x)}{2}} \\ &= z_1 \left(\frac{(-1+2x)^{3/2}}{x^{3/2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{-2x^3+x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-3\ln(x)+3\ln(-1+2x)}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{8x^3}{3} + \frac{1}{2} + 6x - 6x^2 - \ln\left(x\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{8x^3}{3} + \frac{1}{2} + 6x - 6x^2 - \ln\left(x\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.149.2 Maple step by step solution

Let's solve

$x^2(1-2x)\left(\frac{d}{dx}y'\right) + 3xy' + (1+4x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(1+4x)y}{x^2(-1+2x)} + \frac{3y'}{x(-1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{3y'}{x(-1+2x)} - \frac{(1+4x)y}{x^2(-1+2x)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{3}{x(-1+2x)}, P_3(x) = -\frac{1+4x}{x^2(-1+2x)}\right]$
- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$
- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$
- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(-1+2x)\left(\frac{d}{dx}y'\right) - 3xy' + (-1-4x)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- ○ Shift index using $k- > k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
- ○ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
- ○ Shift index using $k- > k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r+1)^2 + 2a_{k-1}(k+r)\,(k-3+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = -1$
- Each term in the series must be 0, giving the recursion relation
  $-a_k(k+r+1)^2 + 2a_{k-1}(k+r)(k-3+r) = 0$
- Shift index using $k->k+1$
  $-a_{k+1}(k+2+r)^2 + 2a_k(k+r+1)(k+r-2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{2a_k(k+r+1)(k+r-2)}{(k+2+r)^2}$
- Recursion relation for $r = -1$; series terminates at $k = 3$
  $a_{k+1} = \frac{2a_k k(k-3)}{(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = 0$
- Apply recursion relation for $k = 1$
  $a_2 = -a_1$
- Express in terms of a_0
  $a_2 = 0$
- Apply recursion relation for $k = 2$
  $a_3 = -\frac{4a_2}{9}$
- Express in terms of a_0
  $a_3 = 0$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second
  $y = a_0 \cdot 0$

1.149.3 Maple trace

```
Methods for second order ODEs:
```

1.149.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 31

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)+3*x*diff(y(x),x)+(1+4*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{3 \ln (x) c_2 + (-8x^3 + 18x^2 - 18x) c_2 + c_1}{x}$$

1.149.5 Mathematica DSolve solution

Solving time : 0.078 (sec)
Leaf size : 36

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]+3*x*D[y[x],x]+(1+4*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{2}{3} c_2 (4x^2 - 9x + 9) + \frac{c_1}{x} + \frac{c_2 \log(x)}{x}$$

1.150 problem 152

Internal problem ID [8288]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 152
Date solved : `Monday, October 21, 2024 at 05:05:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(1+x)\,y''+(1-x)\,y'+y=0$$

1.150.1 Solved as second order ode using Kovacic algorithm

Time used: 0.275 (sec)

Writing the ode as

$$\left(x^2+x\right)y''+(1-x)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2+x\\ B&=1-x\\ C&=1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -x^2 - 10x - 1 \\
t &= 4\left(x^2 + x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 284: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{2}{1+x} + \frac{2}{(1+x)^2} - \frac{2}{x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4(x^2 + x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-x^2-10x-1}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{1+x} + \frac{1}{2x} + (-)(0) \\
&= -\frac{1}{1+x} + \frac{1}{2x} \\
&= -\frac{x-1}{2x(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{1+x} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{1+x} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2 - 10x - 1}{4(x^2 + x)^2}\right)\right) = 0
$$

$$
\frac{1 + a_0}{x(1+x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x - 1) \, e^{\int \left(-\frac{1}{1+x} + \frac{1}{2x}\right) dx} \\
&= (x - 1) \, e^{\frac{\ln(x)}{2} - \ln(1+x)} \\
&= \frac{(x - 1)\sqrt{x}}{1 + x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-x}{x^2+x}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}+\ln(1+x)} \\
&= z_1 \left(\frac{1+x}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x - 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-x}{x^2+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)+2\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{4}{x-1} + \ln(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x-1) + c_2\left(x - 1\left(-\frac{4}{x-1} + \ln(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.150.2 Maple step by step solution

Let's solve

$x(1+x)\left(\frac{d}{dx}y'\right)+(1-x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{y}{x(1+x)}+\frac{(x-1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{(x-1)y'}{x(1+x)}+\frac{y}{x(1+x)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=-\frac{x-1}{x(1+x)}, P_3(x)=\frac{1}{x(1+x)}\right]$

- ○ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

$\left.((1+x)\cdot P_2(x))\right|_{x=-1}=-2$

- ○ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left.\left((1+x)^2\cdot P_3(x)\right)\right|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$x(1+x)\left(\frac{d}{dx}y'\right)+(1-x)\,y'+y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2-u)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(k-2+r) + a_k(k+r-1)^2\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-r(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 3\}$
- Each term in the series must be 0, giving the recursion relation
  $-a_{k+1}(k+1+r)(k-2+r) + a_k(k+r-1)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+r-1)^2}{(k+1+r)(k-2+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{a_k(k-1)^2}{(k+1)(k-2)}$
- Apply recursion relation for $k = 0$
  $a_1 = -\frac{a_0}{2}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y(u) = a_0 \cdot \left(1 - \frac{u}{2}\right)$
- Revert the change of variables $u = 1 + x$
  $\left[y = a_0\left(-\frac{x}{2} + \frac{1}{2}\right)\right]$
- Recursion relation for $r = 3$
  $a_{k+1} = \frac{a_k(k+2)^2}{(k+4)(k+1)}$
- Solution for $r = 3$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = \frac{a_k(k+2)^2}{(k+4)(k+1)}\right]$
- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+3}, a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0\left(-\frac{x}{2}+\frac{1}{2}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+3}\right), b_{k+1} = \frac{b_k (k+2)^2}{(k+4)(k+1)}\right]$$

1.150.3 Maple trace

```
Methods for second order ODEs:
```

1.150.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 20

```
dsolve(x*(1+x)*diff(diff(y(x),x),x)+(1-x)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2(x-1)\ln(x) - 4c_2 + c_1(x-1)$$

1.150.5 Mathematica DSolve solution

Solving time : `0.075` (sec)
Leaf size : 23

```
DSolve[{x*(1+x)*D[y[x],{x,2}]+(1-x)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1(x-1) + c_2((x-1)\log(x) - 4)$$

1.151 problem 153

Internal problem ID [8289]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 153
Date solved : `Monday, October 21, 2024 at 05:05:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-x)\,y''-x(3-5x)\,y'+(4-5x)\,y=0$$

1.151.1 Solved as second order ode using Kovacic algorithm

Time used: 0.281 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(5x^2-3x\right)y'+(4-5x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-x^3+x^2\\ B&=5x^2-3x\\ C&=4-5x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2 - 6x - 1 \\ t &= 4\left(x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 286: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(-1+x)^2} - \frac{1}{4x^2} + \frac{2}{-1+x} - \frac{2}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 6x - 1}{4(x^2 - x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{15x^2-6x-1}{4\left(x^2-x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{+}\right) \\
&= \frac{5}{2}-\left(\frac{5}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \frac{2}{-1+x} + (0) \\
&= \frac{1}{2x} + \frac{2}{-1+x} \\
&= \frac{-1+5x}{2x\left(-1+x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x} + \frac{2}{-1+x}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{2}{\left(-1+x\right)^2}\right) + \left(\frac{1}{2x} + \frac{2}{-1+x}\right)^2 - \left(\frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{2}{-1+x}\right) dx} \\
&= \sqrt{x}\left(-1+x\right)^2
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{5x^2 - 3x}{-x^3 + x^2} \, dx} \\
&= z_1 e^{\frac{3\ln(x)}{2} + \ln(-1+x)} \\
&= z_1 \left(x^{3/2}(-1+x)\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2(-1+x)^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2-3x}{-x^3+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{3\ln(x)+2\ln(-1+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\ln(x) - \frac{1}{3(-1+x)^3} - \frac{1}{-1+x} + \frac{1}{2(-1+x)^2} - \ln(-1+x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^2(-1+x)^3\right) \\ &\quad + c_2\left(x^2(-1+x)^3\left(\ln(x) - \frac{1}{3(-1+x)^3} - \frac{1}{-1+x} + \frac{1}{2(-1+x)^2} - \ln(-1+x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.151.2 Maple step by step solution

Let's solve

$x^2(1-x)\left(\frac{d}{dx}y'\right) - x(3-5x)\,y' + (4-5x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(-4+5x)y}{x^2(-1+x)} + \frac{(5x-3)y'}{x(-1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{(5x-3)y'}{x(-1+x)} + \frac{(-4+5x)y}{x^2(-1+x)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{5x-3}{x(-1+x)}, P_3(x) = \frac{-4+5x}{x^2(-1+x)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -3$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2(-1+x)\left(\frac{d}{dx}y'\right) - x(5x-3)\,y' + (-4+5x)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k- >k+2-m$

$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$-a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-2)^2 + a_{k-1}(k+r-2)(k-6+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r-2)^2 + a_{k-1}(k+r-2)(k-6+r) = 0$
- Shift index using $k- >k+1$

  $-a_{k+1}(k+r-1)^2 + a_k(k+r-1)(k+r-5) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-5)}{k+r-1}$
- Recursion relation for $r = 2$; series terminates at $k = 3$

  $a_{k+1} = \frac{a_k(k-3)}{k+1}$
- Apply recursion relation for $k = 0$

  $a_1 = -3a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -a_1$
- Express in terms of a_0

  $a_2 = 3a_0$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2}{3}$
- Express in terms of a_0

  $a_3 = -a_0$
- Terminating series solution of the ODE for $r = 2$. Use reduction of order to find the second li

  $y = a_0 \cdot (-x^3 + 3x^2 - 3x + 1)$

1.151.3 Maple trace

```
Methods for second order ODEs:
```

1.151.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 47

```
dsolve(x^2*(1-x)*diff(diff(y(x),x),x)-x*(3-5*x)*diff(y(x),x)+(4-5*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^2 \left(c_1 (-1+x)^3 + c_2 \left(-(-1+x)^3 \ln(-1+x) + (-1+x)^3 \ln(x) - x^2 + \frac{5x}{2} - \frac{11}{6}\right)\right)$$

1.151.5 Mathematica DSolve solution

Solving time : 0.121 (sec)
Leaf size : 76

```
DSolve[{x^2*(1-x)*D[y[x],{x,2}]-x*(3-5*x)*D[y[x],x]+(4-5*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{6}x^2 \big(6c_1x^3 - 18c_1x^2 - 6c_2x^2 + 18c_1x + 15c_2x - 6c_2(x-1)^3 \log(x-1) + 6c_2(x-1)^3 \log(x) - 6c_1 - 11c_2\big)$$

1.152 problem 154

Internal problem ID [8290]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 154
Date solved : `Monday, October 21, 2024 at 05:05:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2 (x^2+1) y'' - x(9x^2+1) y' + (25x^2+1) y = 0$$

1.152.1 Solved as second order ode using Kovacic algorithm

Time used: 0.385 (sec)

Writing the ode as

$$(x^4+x^2) y'' + (-9x^3-x) y' + (25x^2+1) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= -9x^3-x \\ C &= 25x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 - 98x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^4 - 98x^2 - 1$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 - 98x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 288: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{6}{(x-i)^2} + \frac{6}{(x+i)^2} + \frac{6i}{x-i} - \frac{6i}{x+i}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=3\\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-2
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{-x^4-98x^2-1}{4\left(x^3+x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^4-98x^2-1}{4\left(x^3+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	3	-2
$-i$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{2} - \left(-\frac{7}{2}\right) \\
&= 4
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i} + (-)\,(0) \\
&= \frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i} \\
&= \frac{1}{2x} - \frac{4x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i}\right)\left(4x^3 + 3x^2a_3 + 2a_2x + a_1\right) + \left(\left(-\frac{1}{2x^2} + \frac{2}{(x-i)^2} + \frac{}{(x-} \right.\right.
$$

$$
\underline{(x^2+1)\,(x^4a}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1, a_1 = 0, a_2 = -4, a_3 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 4x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 - 4x^2 + 1\right) \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i}\right) dx} \\
&= \left(x^4 - 4x^2 + 1\right) \mathrm{e}^{-2\ln(x^2+1) + \frac{\ln(x)}{2}} \\
&= \frac{\left(x^4 - 4x^2 + 1\right) \sqrt{x}}{\left(x^2+1\right)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-9x^3 - x}{x^4 + x^2} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{2} + 2\ln(x^2+1)} \\
&= z_1 \left(\sqrt{x}\left(x^2+1\right)^2\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^5 - 4x^3 + x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-9x^3 - x}{x^4 + x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\ln(x) + 4\ln(x^2+1)}}{(y_1)^2} \, dx \\
&= y_1 \left(\ln(x) + \frac{-6x^2 + 3}{x^4 - 4x^2 + 1}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^5 - 4x^3 + x\right) + c_2\left(x^5 - 4x^3 + x\left(\ln(x) + \frac{-6x^2 + 3}{x^4 - 4x^2 + 1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.152.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right) - x(9x^2+1)\,y' + (25x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(25x^2+1)y}{x^2(x^2+1)} + \frac{(9x^2+1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(9x^2+1)y'}{x(x^2+1)} + \frac{(25x^2+1)y}{x^2(x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{9x^2+1}{x(x^2+1)}, P_3(x) = \frac{25x^2+1}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d}{dx}y'\right) - x(9x^2+1)\,y' + (25x^2+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx} y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k+r-1)^2 + a_{k-2} (k-7+r)^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 1$
- Each term must be 0

  $a_1 r^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k (k+r-1)^2 + a_{k-2} (k-7+r)^2 = 0$
- Shift index using $k- >k + 2$

  $a_{k+2} (k+1+r)^2 + a_k (k+r-5)^2 = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k (k+r-5)^2}{(k+1+r)^2}$

- Recursion relation for $r = 1$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k (k-4)^2}{(k+2)^2}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k (k-4)^2}{(k+2)^2}, a_1 = 0\right]$

1.152.3 Maple trace

```
Methods for second order ODEs:
```

1.152.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 41

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(9*x^2+1)*diff(y(x),x)+(25*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \left(c_2 \left(x^4 - 4x^2 + 1\right) \ln (x) + c_1 x^4 + \left(-4c_1 - 6c_2\right) x^2 + c_1 + 3c_2\right) x$$

1.152.5 Mathematica DSolve solution

Solving time : 0.135 (sec)
Leaf size : 43

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(1+9*x^2)*D[y[x],x]+(1+25*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 \left(x^5 - 4x^3 + x\right) + c_2 x \left(-6x^2 + \left(x^4 - 4x^2 + 1\right) \log(x) + 3\right)$$

1.153 problem 155

Internal problem ID [8291]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 155
Date solved : `Monday, October 21, 2024 at 05:05:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' + 3x(-x^2+1)\,y' + (7x^2+1)\,y = 0$$

1.153.1 Solved as second order ode using Kovacic algorithm

Time used: 1.281 (sec)

Writing the ode as

$$9x^2y'' + (-3x^3+3x)\,y' + (7x^2+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= -3x^3+3x \\ C &= 7x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 36x^2 - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 36x^2 - 9 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 36x^2 - 9}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 290: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{36} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} - \frac{3}{x} - \frac{111}{4x^3} - \frac{999}{2x^5} - \frac{180819}{16x^7} - \frac{2292705}{8x^9} - \frac{249239511}{32x^{11}} - \frac{3548540907}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 36x^2 - 9}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{x^2}{36} - 1\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{36} - 1 - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{6} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{6}} - 1\right) &&= -\frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{6}} - 1\right) &&= \frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 36x^2 - 9}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{6}$	$-\frac{7}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{1}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{x}{6}\right) \\
&= \frac{1}{2x} - \frac{x}{6} \\
&= \frac{1}{2x} - \frac{x}{6}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2x} - \frac{x}{6}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2} - \frac{1}{6}\right) + \left(\frac{1}{2x} - \frac{x}{6}\right)^2 - \left(\frac{x^4 - 36x^2 - 9}{36x^2}\right)\right) = 0$$

$$\frac{x^2 a_1 + 2(6 + a_0)\, x + 3a_1}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -6, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 6$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 6\right) \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{x}{6}\right) dx} \\
&= \left(x^2 - 6\right) \mathrm{e}^{-\frac{x^2}{12} + \frac{\ln(x)}{2}} \\
&= \left(x^2 - 6\right) \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{12}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x^3+3x}{9x^2}\,dx} \\
&= z_1 e^{\frac{x^2}{12}-\frac{\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x^2}{12}}}{x^{1/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{1/3}(x^2-6)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+3x}{9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{6}-\frac{\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{6}-\frac{\ln(x)}{3}}}{x^{2/3}(x^2-6)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{1/3}(x^2-6)\right) + c_2\left(x^{1/3}(x^2-6)\left(\int \frac{\mathrm{e}^{\frac{x^2}{6}-\frac{\ln(x)}{3}}}{x^{2/3}(x^2-6)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.153.2 Maple step by step solution

Let's solve

$9x^2\left(\frac{d}{dx}y'\right)+3x(-x^2+1)\,y'+(7x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(7x^2+1)y}{9x^2}+\frac{(x^2-1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{(x^2-1)y'}{3x}+\frac{(7x^2+1)y}{9x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{x^2-1}{3x}, P_3(x)=\frac{7x^2+1}{9x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{1}{3}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=\frac{1}{9}$

- ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$9x^2\left(\frac{d}{dx}y'\right)-3x(x^2-1)\,y'+(7x^2+1)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)^2\, x^r + a_1(2+3r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 - a_{k-2}(3k-13+3r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{3}$

- Each term must be 0

  $a_1(2+3r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r-1)^2 + (-3k+13-3r)\, a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(3k+5+3r)^2 + a_k(-3k-3r+7) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k(3k+3r-7)}{(3k+5+3r)^2}$

- Recursion relation for $r = \frac{1}{3}$; series terminates at $k = 2$

  $a_{k+2} = \frac{a_k(3k-6)}{(3k+6)^2}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{a_k(3k-6)}{(3k+6)^2}, a_1 = 0\right]$

1.153.3 Maple trace

```
Methods for second order ODEs:
```

1.153.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 19

```
dsolve(9*x^2*diff(diff(y(x),x),x)+3*x*(-x^2+1)*diff(y(x),x)+(7*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{x^{1/3}(x^2-6)\,(c_1-c_2)}{6}$$

1.153.5 Mathematica DSolve solution

Solving time : 4.83 (sec)
Leaf size : 53

```
DSolve[{9*x^2*D[y[x],{x,2}]+3*x*(1-x^2)*D[y[x],x]+(1+7*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{72}\sqrt[3]{x}\left(c_2(x^2-6)\,\text{ExpIntegralEi}\left(\frac{x^2}{6}\right)+72c_1(x^2-6)-6c_2e^{\frac{x^2}{6}}\right)$$

1.154 problem 156

Internal problem ID [8292]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 156
Date solved : `Monday, October 21, 2024 at 05:05:23 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x\left(x^{2}+1\right) y''+\left(-x^{2}+1\right) y'-8 x y=0$$

1.154.1 Solved as second order ode using Kovacic algorithm

Time used: 0.423 (sec)

Writing the ode as

$$\left(x^{3}+x\right) y''+\left(-x^{2}+1\right) y'-8 x y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{3}+x \\ B &= -x^{2}+1 \\ C &= -8 x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A} d x}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35x^4 + 22x^2 - 1$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 292: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} - \frac{15i}{4(x-i)} + \frac{15i}{4(x+i)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\ &= \frac{7}{2} - \left(\frac{7}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} + (0) \\ &= \frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} \\ &= \frac{1}{2x} + \frac{3x}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{3}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(\frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \left(x^2+1\right)^{3/2}\sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+1}{x^3+x}\,dx} \\
&= z_1 e^{\frac{\ln\left(x^2+1\right)}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\sqrt{x^2+1}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2+1\right)^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+1}{x^3+x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x^2+1)-\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{1}{4\left(x^2+1\right)^2}+\frac{1}{2x^2+2}-\frac{\ln\left(x^2+1\right)}{2}+\ln\left(x\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2+1\right)^2\right) + c_2\left(\left(x^2+1\right)^2\left(\frac{1}{4\left(x^2+1\right)^2}+\frac{1}{2x^2+2}-\frac{\ln\left(x^2+1\right)}{2}+\ln\left(x\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.154.2 Maple step by step solution

Let's solve

$x(x^2+1)\left(\frac{d}{dx}y'\right)+(-x^2+1)\,y'-8xy=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{8y}{x^2+1}+\frac{(x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{(x^2-1)y'}{x(x^2+1)}-\frac{8y}{x^2+1}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{x^2-1}{x(x^2+1)}, P_3(x)=-\frac{8}{x^2+1}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x^2+1)\left(\frac{d}{dx}y'\right)+(-x^2+1)\,y'-8xy=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x\cdot y$ to series expansion

    $x\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+1}$

  - Shift index using $k->k-1$

    $x\cdot y=\sum_{k=1}^{\infty}a_{k-1}x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=0..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + a_1 (1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 + a_{k-1}(k+r+1)\,(k-5+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term must be 0

  $a_1 (1+r)^2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $((a_{k-1} + a_{k+1})\, k - 5a_{k-1} + a_{k+1})\,(k+1) = 0$
- Shift index using $k->k+1$

  $((a_k + a_{k+2})\,(k+1) - 5a_k + a_{k+2})\,(k+2) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k (k-4)}{k+2}$
- Recursion relation for $r = 0$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k (k-4)}{k+2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k (k-4)}{k+2}, a_1 = 0\right]$

1.154.3 Maple trace

```
Methods for second order ODEs:
```

1.154.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 48

```
dsolve(x*(x^2+1)*diff(diff(y(x),x),x)+(-x^2+1)*diff(y(x),x)-8*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 (x^2 + 1)^2 + c_2 \left(-\frac{(x^2+1)^2 \ln(x^2+1)}{2} + (x^2+1)^2 \ln(x) + \frac{x^2}{2} + \frac{3}{4} \right)$$

1.154.5 Mathematica DSolve solution

Solving time : 0.119 (sec)
Leaf size : 55

```
DSolve[{x*(1+x^2)*D[y[x],{x,2}]+(1-x^2)*D[y[x],x]-8*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 (x^2+1)^2 + \frac{1}{4} c_2 \left(2x^2 + 4(x^2+1)^2 \log(x) - 2(x^2+1)^2 \log(x^2+1) + 3 \right)$$

1.155 problem 157

Internal problem ID [8293]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 157
Date solved : `Monday, October 21, 2024 at 05:05:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 2x(-x^2+4)\,y' + (7x^2+1)\,y = 0$$

1.155.1 Solved as second order ode using Kovacic algorithm

Time used: 0.770 (sec)

Writing the ode as

$$4x^2y'' + (-2x^3+8x)\,y' + (7x^2+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -2x^3+8x \\ C &= 7x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 40x^2 - 4}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 40x^2 - 4 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 40x^2 - 4}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 294: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} - \frac{5}{2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{5}{x} - \frac{101}{2x^3} - \frac{1010}{x^5} - \frac{50601}{2x^7} - \frac{710030}{x^9} - \frac{21351501}{x^{11}} - \frac{672670100}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 40x^2 - 4}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{x^2}{16} - \frac{5}{2}\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{16} - \frac{5}{2} - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{4}\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{4}} - 1\right) &&= -\frac{11}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{4}} - 1\right) &&= \frac{9}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 40x^2 - 4}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{4}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{9}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right)\\
&= \frac{9}{2} - \left(\frac{1}{2}\right)\\
&= 4
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} + (-)\left(\frac{x}{4}\right) \\ &= \frac{1}{2x} - \frac{x}{4} \\ &= \frac{1}{2x} - \frac{x}{4}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(12x^2 + 6xa_3 + 2a_2) + 2\left(\frac{1}{2x} - \frac{x}{4}\right)(4x^3 + 3a_3x^2 + 2a_2x + a_1) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{x}{4}\right)^2 - \left(\frac{x^4 \cdots}{\cdots}\right.\right.$$

$$\frac{x^4 a_3 + 2(16 + a_2)x^3 + 3(a_1 + 6a_3)x^2 + 4(a_0 + 2a \cdots}{2x}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 32, a_1 = 0, a_2 = -16, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 16x^2 + 32$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^4 - 16x^2 + 32\right) e^{\int \left(\frac{1}{2x} - \frac{x}{4}\right) dx} \\ &= \left(x^4 - 16x^2 + 32\right) e^{-\frac{x^2}{8} + \frac{\ln(x)}{2}} \\ &= \left(x^4 - 16x^2 + 32\right) \sqrt{x}\, e^{-\frac{x^2}{8}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^3+8x}{4x^2}\,dx} \\
&= z_1 e^{\frac{x^2}{8}-\ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x^2}{8}}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^4 - 16x^2 + 32}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+8x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{4}-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}-2\ln(x)}x}{(x^4-16x^2+32)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^4-16x^2+32}{\sqrt{x}}\right) + c_2\left(\frac{x^4-16x^2+32}{\sqrt{x}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{4}-2\ln(x)}x}{(x^4-16x^2+32)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.155.2 Maple step by step solution

Let's solve

$4x^2 \left(\frac{d}{dx}y'\right) + 2x(-x^2+4)\, y' + (7x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(7x^2+1)y}{4x^2} + \frac{(x^2-4)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x^2-4)y'}{2x} + \frac{(7x^2+1)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{x^2-4}{2x}, P_3(x) = \frac{7x^2+1}{4x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 2$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{4}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 \left(\frac{d}{dx}y'\right) - 2(x^2-4)\, xy' + (7x^2+1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (1+2r)^2\, x^r + a_1 (3+2r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (2k+2r+1)^2 - a_{k-2}(2k-11+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = -\frac{1}{2}$
- Each term must be 0
  $a_1 (3+2r)^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k (2k+2r+1)^2 + (-2k+11-2r)\, a_{k-2} = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(2k+5+2r)^2 + a_k(-2k-2r+7) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_k (2k+2r-7)}{(2k+5+2r)^2}$
- Recursion relation for $r = -\frac{1}{2}$; series terminates at $k = 4$
  $a_{k+2} = \frac{a_k (2k-8)}{(2k+4)^2}$
- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{a_k (2k-8)}{(2k+4)^2}, a_1 = 0\right]$$

1.155.3 Maple trace

```
Methods for second order ODEs:
```

1.155.4 Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 24

```
dsolve(4*x^2*diff(diff(y(x),x),x)+2*x*(-x^2+4)*diff(y(x),x)+(7*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(x^4 - 16x^2 + 32)(c_1 + 2c_2)}{32\sqrt{x}}$$

1.155.5 Mathematica DSolve solution

Solving time : 0.683 (sec)
Leaf size : 68

```
DSolve[{4*x^2*D[y[x],{x,2}]+2*x*(4-x^2)*D[y[x],x]+(1+7*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2(x^4 - 16x^2 + 32)\,\text{ExpIntegralEi}\left(\frac{x^2}{4}\right) - 4c_2e^{\frac{x^2}{4}}(x^2 - 12) + 2048c_1(x^4 - 16x^2 + 32)}{2048\sqrt{x}}$$

1.156 problem 158

Internal problem ID [8294]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 158
Date solved : `Monday, October 21, 2024 at 05:05:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y''+8x^2y'+(1+x)\,y=0$$

1.156.1 Solved as second order ode using Kovacic algorithm

Time used: 0.191 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+8x^2y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^3+4x^2\\B&=8x^2\\C&=1+x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 296: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx}\\ &= e^{\int \frac{1}{2x} dx}\\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^2}{4x^3+4x^2}\, dx}\\ &= z_1 e^{-\ln(1+x)}\\ &= z_1\left(\frac{1}{1+x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{1+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2}{4x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(1+x)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{1+x}\right) + c_2 \left(\frac{\sqrt{x}}{1+x}(\ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.156.2 Maple step by step solution

Let's solve

$4x^2(1+x)\left(\frac{d}{dx}y'\right) + 8x^2y' + (1+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{4x^2} - \frac{2y'}{1+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2y'}{1+x} + \frac{y}{4x^2} = 0$

- ☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{1+x}, P_3(x) = \frac{1}{4x^2}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.\left((1+x)\cdot P_2(x)\right)\right|_{x=-1} = 2$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left.\left((1+x)^2\cdot P_3(x)\right)\right|_{x=-1} = 0$

- $x = -1$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0 = -1$
- Multiply by denominators
  $4x^2(1+x)\left(\frac{d}{dx}y'\right) + 8x^2y' + (1+x)\,y = 0$
- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$
  $(4u^3 - 8u^2 + 4u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (8u^2 - 16u + 8)\left(\frac{d}{du}y(u)\right) + uy(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $u \cdot y(u)$ to series expansion
    $u \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+1}$
  - Shift index using $k- > k-1$
    $u \cdot y(u) = \sum_{k=1}^{\infty} a_{k-1} u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k- > k+1-m$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$
    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k- > k+2-m$
    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions
  $4a_0r(1+r)\,u^{-1+r} + (4a_1(1+r)\,(2+r) - 8a_0r(1+r))\,u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)\,(k+2+r) -\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r(1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 0\}$

- Each term must be 0

  $4a_1(1+r)(2+r) - 8a_0 r(1+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k-1}(2k-1+2r)^2 - 8\left(\left(-\frac{k}{2} - \frac{r}{2} - 1\right) a_{k+1} + a_k(k+r)\right)(k+r+1) = 0$
- Shift index using $k- > k+1$

  $a_k(2k+2r+1)^2 - 8\left(\left(-\frac{k}{2} - \frac{3}{2} - \frac{r}{2}\right) a_{k+2} + a_{k+1}(k+r+1)\right)(k+2+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 8kra_k - 16kra_{k+1} + 4r^2a_k - 8r^2a_{k+1} + 4ka_k - 24ka_{k+1} + 4ra_k - 24ra_{k+1} + a_k - 16a_{k+1}}{4(k+3+r)(k+2+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+2)(k+1)}$
- Solution for $r = -1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+2)(k+1)}, 0 = 0\right]$
- Revert the change of variables $u = 1+x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-1}, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+2)(k+1)}, 0 = 0\right]$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+3)(k+2)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k}, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+3)(k+2)}, 8a_1 = 0\right]$
- Revert the change of variables $u = 1+x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k}, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+3)(k+2)}, 8a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k}\right), a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+2)(k+1)}, 0 = 0, b_{k+2}\right.$

1.156.3 Maple trace

```
Methods for second order ODEs:
```

1.156.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 19

```
dsolve(4*x^2*(1+x)*diff(diff(y(x),x),x)+8*x^2*diff(y(x),x)+(1+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\sqrt{x}\,(c_2 \ln(x) + c_1)}{1 + x}$$

1.156.5 Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 24

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+8*x^2*D[y[x],x]+(1+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}(c_2 \log(x) + c_1)}{x + 1}$$

1.157 problem 159

Internal problem ID [8295]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 159
Date solved : `Monday, October 21, 2024 at 05:05:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2(3+x)\,y''+3x(3+7x)\,y'+(3+4x)\,y=0$$

1.157.1 Solved as second order ode using Kovacic algorithm

Time used: 0.262 (sec)

Writing the ode as

$$\left(9x^3+27x^2\right)y''+\left(21x^2+9x\right)y'+(3+4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^3+27x^2 \\ B &= 21x^2+9x \\ C &= 3+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 298: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{21x^2+9x}{9x^3+27x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{6} - \ln(3+x)} \\ &= z_1 \left(\frac{1}{x^{1/6}\,(3+x)}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{3+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{21x^2+9x}{9x^3+27x^2} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-\frac{\ln(x)}{3} - 2\ln(3+x)}}{(y_1)^2} dx$$
$$= y_1 \left(\frac{x^2}{9+3x} + \frac{2x}{3+x} + \frac{3}{3+x} - \frac{2\ln(3+x)\,x}{3} - 2\ln(3+x) + \ln(x) + \frac{2\ln(3+x)\,(3+x)}{3} - \frac{x}{3} - 2 \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x^{1/3}}{3+x} \right) + c_2 \left(\frac{x^{1/3}}{3+x} \left(\frac{x^2}{9+3x} + \frac{2x}{3+x} + \frac{3}{3+x} - \frac{2\ln(3+x)\,x}{3} - 2\ln(3+x) + \ln(x) + \frac{2\ln(3+x)\,(3+x)}{3} - \frac{x}{3} - 2 \right) \right)$$

Will add steps showing solving for IC soon.

1.157.2 Maple step by step solution

Let's solve
$9x^2(3+x)\left(\frac{d}{dx}y'\right) + 3x(3+7x)\,y' + (3+4x)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{(3+4x)y}{9x^2(3+x)} - \frac{(3+7x)y'}{3x(3+x)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + \frac{(3+7x)y'}{3x(3+x)} + \frac{(3+4x)y}{9x^2(3+x)} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{3+7x}{3x(3+x)}, P_3(x) = \frac{3+4x}{9x^2(3+x)}\right]$

- $(3+x)\cdot P_2(x)$ is analytic at $x=-3$

  $$((3+x)\cdot P_2(x))\Big|_{x=-3}=2$$

- $(3+x)^2\cdot P_3(x)$ is analytic at $x=-3$

  $$\left((3+x)^2\cdot P_3(x)\right)\Big|_{x=-3}=0$$

- $x=-3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $9x^2(3+x)\left(\frac{d}{dx}y'\right)+3x(3+7x)\,y'+(3+4x)\,y=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $(9u^3-54u^2+81u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(21u^2-117u+162)\left(\frac{d}{du}y(u)\right)+(-9+4u)\,y(u)=0$

- Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..3$

  $$u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$81a_0r(1+r)\,u^{-1+r}+\left(81a_1(1+r)\,(2+r)-9a_0(1+r)\,(1+6r)\right)u^r+\left(\sum_{k=1}^{\infty}\left(81a_{k+1}(k+r+1)\,(k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $81r(1+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{-1,0\}$
- Each term must be 0
  $81a_1(1+r)\,(2+r)-9a_0(1+r)\,(1+6r)=0$
- Each term in the series must be 0, giving the recursion relation
  $81a_{k+1}(k+r+1)\,(k+2+r)-54(k+r+1)\left(k+r+\frac{1}{6}\right)a_k+a_{k-1}(3k-1+3r)^2=0$
- Shift index using $k->k+1$
  $81a_{k+2}(k+2+r)\,(k+3+r)-54(k+2+r)\left(k+\frac{7}{6}+r\right)a_{k+1}+a_k(3k+3r+2)^2=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+18kra_k-108kra_{k+1}+9r^2a_k-54r^2a_{k+1}+12ka_k-171ka_{k+1}+12ra_k-171ra_{k+1}+4a_k-126a_{k+1}}{81(k+2+r)(k+3+r)}$
- Recursion relation for $r=-1$
  $a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)}$
- Solution for $r=-1$
  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-1},a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)},0=0\right]$
- Revert the change of variables $u=3+x$
  $\left[y=\sum_{k=0}^{\infty}a_k(3+x)^{k-1},a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)},0=0\right]$
- Recursion relation for $r=0$
  $a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)},162a_1-9a_0=0\right]$
- Revert the change of variables $u=3+x$
  $\left[y=\sum_{k=0}^{\infty}a_k(3+x)^k,a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)},162a_1-9a_0=0\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty}a_k(3+x)^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k(3+x)^k\right),a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)},0=\right.$

1.157.3 Maple trace

```
Methods for second order ODEs:
```

1.157.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 19

```
dsolve(9*x^2*(3+x)*diff(diff(y(x),x),x)+3*x*(3+7*x)*diff(y(x),x)+(3+4*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^{1/3}(\ln(x)\, c_2 + c_1)}{3+x}$$

1.157.5 Mathematica DSolve solution

Solving time : 0.063 (sec)
Leaf size : 24

```
DSolve[{9*x^2*(3+x)*D[y[x],{x,2}]+3*x*(3+7*x)*D[y[x],x]+(3+4*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[3]{x}(c_2 \log(x) + c_1)}{x+3}$$

1.158 problem 160

Internal problem ID [8296]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 160
Date solved : `Monday, October 21, 2024 at 05:05:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-x^2+2\right)y''-x\left(3x^2+2\right)y'+\left(-x^2+2\right)y=0$$

1.158.1 Solved as second order ode using Kovacic algorithm

Time used: 0.214 (sec)

Writing the ode as

$$\left(-x^4+2x^2\right)y''+\left(-3x^3-2x\right)y'+\left(-x^2+2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^4+2x^2\\B&=-3x^3-2x\\C&=-x^2+2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 300: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3x^3-2x}{-x^4+2x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \ln(x^2-2)} \\ &= z_1 \left(\frac{\sqrt{x}}{x^2-2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{x^2-2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3-2x}{-x^4+2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)-2\ln(x^2-2)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{x^2-2}\right) + c_2\left(\frac{x}{x^2-2}(\ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.158.2 Maple step by step solution

Let's solve

$x^2(-x^2+2)\left(\frac{d}{dx}y'\right) - x(3x^2+2)\,y' + (-x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x^2} - \frac{(3x^2+2)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3x^2+2)y'}{x(x^2-2)} + \frac{y}{x^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3x^2+2}{x(x^2-2)}, P_3(x) = \frac{1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2-2)\left(\frac{d}{dx}y'\right) + x(3x^2+2)\,y' + (x^2-2)\,y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ◦ Shift index using $k->k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - ◦ Shift index using $k->k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

    $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - ◦ Shift index using $k->k+2-m$

    $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0(-1+r)^2\,x^r - 2a_1r^2x^{1+r} + \left(\sum_{k=2}^{\infty}\left(-2a_k(k+r-1)^2 + a_{k-2}(k+r-1)^2\right)x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0
  $-2a_1r^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2\left(a_k - \frac{a_{k-2}}{2}\right)(k+r-1)^2 = 0$
- Shift index using $k- >k+2$
  $-2\left(a_{k+2} - \frac{a_k}{2}\right)(k+r+1)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_k}{2}$
- Recursion relation for $r = 1$
  $a_{k+2} = \frac{a_k}{2}$
- Solution for $r = 1$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$

1.158.3 Maple trace

```
Methods for second order ODEs:
```

1.158.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 19

```
dsolve(x^2*(-x^2+2)*diff(diff(y(x),x),x)-x*(3*x^2+2)*diff(y(x),x)+(-x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x(c_1 + c_2 \ln(x))}{x^2 - 2}$$

1.158.5 Mathematica DSolve solution

Solving time : 0.058 (sec)
Leaf size : 23

```
DSolve[{x^2*(2-x^2)*D[y[x],{x,2}]-x*(2+3*x^2)*D[y[x],x]+(2-x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{x(c_2 \log(x) + c_1)}{x^2 - 2}$$

1.159 problem 161

Internal problem ID [8297]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 161
Date solved : `Monday, October 21, 2024 at 05:05:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$16x^2(x^2+1)\,y''+8x(9x^2+1)\,y'+(49x^2+1)\,y=0$$

1.159.1 Solved as second order ode using Kovacic algorithm

Time used: 0.266 (sec)

Writing the ode as

$$(16x^4+16x^2)\,y''+(72x^3+8x)\,y'+(49x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=16x^4+16x^2\\ B&=72x^3+8x\\ C&=49x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 302: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{72x^3+8x}{16x^4+16x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{4} - \ln(x^2+1)} \\ &= z_1 \left(\frac{1}{x^{1/4}(x^2+1)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}}{x^2+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{72x^3+8x}{16x^4+16x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2}-2\ln(x^2+1)}}{(y_1)^2} dx \\
&= y_1 \left(\ln(x) + \frac{x^4}{2x^2+2} + \frac{x^2}{x^2+1} + \frac{1}{2x^2+2} - \ln(x^2+1)\,x^2 - \ln(x^2+1) \right. \\
&\quad \left. + \ln(x^2+1)\,(x^2+1) - \frac{x^2}{2} - 1 \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/4}}{x^2+1} \right) \\
&\quad + c_2 \left(\frac{x^{1/4}}{x^2+1} \left(\ln(x) + \frac{x^4}{2x^2+2} + \frac{x^2}{x^2+1} + \frac{1}{2x^2+2} - \ln(x^2+1)\,x^2 - \ln(x^2+1) + \ln(x^2+1)\,(x^2+1) - \frac{x^2}{2} \right.\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.159.2 Maple step by step solution

Let's solve

$16x^2(x^2+1)\left(\frac{d}{dx}y'\right) + 8x(9x^2+1)\,y' + (49x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(49x^2+1)y}{16x^2(x^2+1)} - \frac{(9x^2+1)y'}{2x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(9x^2+1)y'}{2x(x^2+1)} + \frac{(49x^2+1)y}{16x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x)=\frac{9x^2+1}{2x(x^2+1)}, P_3(x)=\frac{49x^2+1}{16x^2(x^2+1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x=0$

  $\left.(x \cdot P_2(x))\right|_{x=0}=\frac{1}{2}$
- $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0}=\frac{1}{16}$
- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $16x^2(x^2+1)\left(\frac{d}{dx}y'\right)+8x(9x^2+1)y'+(49x^2+1)y=0$
- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m=0..2$

  $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k->k-m$

  $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot y'$ to series expansion for $m=1..3$

  $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m=2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+4r)^2 x^r + a_1(3+4r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r-1)^2 + a_{k-2}(4k+4r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+4r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = \frac{1}{4}$
- Each term must be 0

  $a_1(3+4r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(4k+4r-1)^2 (a_k + a_{k-2}) = 0$
- Shift index using $k->k+2$

  $(4k+4r+7)^2 (a_{k+2} + a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -a_k$
- Recursion relation for $r = \frac{1}{4}$

  $a_{k+2} = -a_k$
- Solution for $r = \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -a_k, a_1 = 0\right]$

1.159.3 Maple trace

```
Methods for second order ODEs:
```

1.159.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 21

```
dsolve(16*x^2*(x^2+1)*diff(diff(y(x),x),x)+8*x*(9*x^2+1)*diff(y(x),x)+(49*x^2+1)*y(x)
       y(x),singsol=all)
```

$$y = \frac{x^{1/4}(\ln(x)\, c_2 + c_1)}{x^2+1}$$

1.159.5 Mathematica DSolve solution

Solving time : 0.063 (sec)
Leaf size : 26

```
DSolve[{16*x^2*(1+x^2)*D[y[x],{x,2}]+8*x*(1+9*x^2)*D[y[x],x]+(1+49*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[4]{x}(c_2 \log(x) + c_1)}{x^2 + 1}$$

1.160 problem 162

Internal problem ID [8298]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 162
Date solved : `Monday, October 21, 2024 at 05:05:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(4+3x)\,y''-x(4-3x)\,y'+4y=0$$

1.160.1 Solved as second order ode using Kovacic algorithm

Time used: 0.218 (sec)

Writing the ode as

$$\left(3x^3+4x^2\right)y''+\left(3x^2-4x\right)y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^3+4x^2 \\ B &= 3x^2-4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 304: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^2-4x}{3x^3+4x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \ln(4+3x)} \\ &= z_1 \left(\frac{\sqrt{x}}{4+3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{4+3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2-4x}{3x^3+4x^2}\,dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)-2\ln(4+3x)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{4+3x}\right) + c_2\left(\frac{x}{4+3x}(\ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.160.2 Maple step by step solution

Let's solve

$x^2(4+3x)\left(\frac{d}{dx}y'\right) - x(4-3x)\,y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{4y}{x^2(4+3x)} - \frac{(3x-4)y'}{x(4+3x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3x-4)y'}{x(4+3x)} + \frac{4y}{x^2(4+3x)} = 0$

☐ Check to see if x_0 is a regular singular point

- ∘ Define functions

  $\left[P_2(x) = \frac{3x-4}{x(4+3x)}, P_3(x) = \frac{4}{x^2(4+3x)}\right]$

- ∘ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

- ∘ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(4+3x)\left(\frac{d}{dx}y'\right) + x(3x-4)\,y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- > k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $4a_0(-1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(4a_k(k+r-1)^2 + 3a_{k-1}(k+r-1)^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)^2(4a_k + 3a_{k-1}) = 0$

- Shift index using $k- > k+1$

  $(k+r)^2(4a_{k+1} + 3a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{3a_k}{4}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{3a_k}{4}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{3a_k}{4}\right]$

1.160.3 Maple trace

```
Methods for second order ODEs:
```

1.160.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 19

```
dsolve(x^2*(4+3*x)*diff(diff(y(x),x),x)-x*(4-3*x)*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x(c_2 \ln(x) + c_1)}{4 + 3x}$$

1.160.5 Mathematica DSolve solution

Solving time : `0.054` (sec)
Leaf size : 22

```
DSolve[{x^2*(4+3*x)*D[y[x],{x,2}]-x*(4-3*x)*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x(c_2 \log(x) + c_1)}{3x + 4}$$

1.161 problem 163

Internal problem ID [8299]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 163
Date solved : `Monday, October 21, 2024 at 05:05:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+3x+1\right)y''+8x^2(3+2x)\,y'+\left(9x^2+3x+1\right)y=0$$

1.161.1 Solved as second order ode using Kovacic algorithm

Time used: 0.195 (sec)

Writing the ode as

$$\left(4x^4+12x^3+4x^2\right)y''+\left(16x^3+24x^2\right)y'+\left(9x^2+3x+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+12x^3+4x^2 \\ B &= 16x^3+24x^2 \\ C &= 9x^2+3x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 306: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{16x^3 + 24x^2}{4x^4 + 12x^3 + 4x^2} \, dx} \\ &= z_1 e^{-\ln(x^2 + 3x + 1)} \\ &= z_1 \left(\frac{1}{x^2 + 3x + 1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{x^2 + 3x + 1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{16x^3+24x^2}{4x^4+12x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x^2+3x+1)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{x^2+3x+1}\right) + c_2\left(\frac{\sqrt{x}}{x^2+3x+1}(\ln(x))\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.161.2 Maple step by step solution

Let's solve

$4x^2(x^2+3x+1)\left(\frac{d}{dx}y'\right) + 8x^2(3+2x)\,y' + (9x^2+3x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$
- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(9x^2+3x+1)y}{4x^2(x^2+3x+1)} - \frac{2(3+2x)y'}{x^2+3x+1}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2(3+2x)y'}{x^2+3x+1} + \frac{(9x^2+3x+1)y}{4x^2(x^2+3x+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2(3+2x)}{x^2+3x+1}, P_3(x) = \frac{9x^2+3x+1}{4x^2(x^2+3x+1)}\right]$
- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$
- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+3x+1)\left(\frac{d}{dx}y'\right) + 8x^2(3+2x)\,y' + (9x^2+3x+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 2..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- > k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- > k + 2 - m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)^2\,x^r + \left(a_1(1+2r)^2 + 3a_0(1+2r)^2\right)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + 3a_{k-1}(2k+2r\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$

- Each term must be 0

  $a_1(1+2r)^2 + 3a_0(1+2r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = -3a_0$
- Each term in the series must be 0, giving the recursion relation

  $(2k+2r-1)^2 (a_k + 3a_{k-1} + a_{k-2}) = 0$
- Shift index using $k->k+2$

  $(2k+2r+3)^2 (a_{k+2} + 3a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -3a_{k+1} - a_k$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -3a_{k+1} - a_k$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$

1.161.3 Maple trace

```
Methods for second order ODEs:
```

1.161.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 24

```
dsolve(4*x^2*(x^2+3*x+1)*diff(diff(y(x),x),x)+8*x^2*(3+2*x)*diff(y(x),x)+(9*x^2+3*x+1)
       y(x),singsol=all)
```

$$y = \frac{\sqrt{x}\,(c_2 \ln(x) + c_1)}{x^2 + 3x + 1}$$

1.161.5 Mathematica DSolve solution

Solving time : 0.083 (sec)
Leaf size : 29

```
DSolve[{4*x^2*(1+3*x+x^2)*D[y[x],{x,2}]+8*x^2*(3+2*x)*D[y[x],x]+(1+3*x+9*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{\sqrt{x}(c_2\log(x)+c_1)}{x^2+3x+1}$$

1.162 problem 164

Internal problem ID [8300]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 164
Date solved : `Monday, October 21, 2024 at 05:05:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-x)^2 y'' - x(-3x^2+2x+1)\, y' + (x^2+1)\, y = 0$$

1.162.1 Solved as second order ode using Kovacic algorithm

Time used: 0.207 (sec)

Writing the ode as

$$x^2(-1+x)^2 y'' + (3x^3-2x^2-x)\, y' + (x^2+1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(-1+x)^2 \\ B &= 3x^3-2x^2-x \\ C &= x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 308: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3x^3 - 2x^2 - x}{x^2(-1+x)^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - 2\ln(-1+x)} \\ &= z_1 \left(\frac{\sqrt{x}}{(-1+x)^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{(-1+x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3-2x^2-x}{x^2(-1+x)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)-4\ln(-1+x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{(-1+x)^2}\right) + c_2\left(\frac{x}{(-1+x)^2}(\ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.162.2 Maple step by step solution

Let's solve

$x^2(1-x)^2\left(\frac{d}{dx}y'\right) - x(-3x^2+2x+1)\,y' + (x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+1)y}{x^2(-1+x)^2} - \frac{y'(3x+1)}{x(-1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'(3x+1)}{x(-1+x)} + \frac{(x^2+1)y}{x^2(-1+x)^2} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{3x+1}{x(-1+x)}, P_3(x) = \frac{x^2+1}{x^2(-1+x)^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

- $x = 0$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0 = 0$
- Multiply by denominators
  $x^2(-1+x)^2 \left(\frac{d}{dx}y'\right) + x(-1+x)(3x+1)y' + (x^2+1)y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$
    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$
    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$
    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$
    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$
    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k- >k+2-m$
    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions
  $a_0(-1+r)^2 x^r + \left(-2a_0r^2 + a_1r^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 - 2a_{k-1}(k+r-1)^2 + a_{k-2}(k+r\right.\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 1$

- Each term must be 0
  $-2a_0r^2 + a_1r^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 2a_0$
- Each term in the series must be 0, giving the recursion relation
  $(k + r - 1)^2 (a_k - 2a_{k-1} + a_{k-2}) = 0$
- Shift index using $k- >k + 2$
  $(k + r + 1)^2 (a_{k+2} - 2a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = 2a_{k+1} - a_k$
- Recursion relation for $r = 1$
  $a_{k+2} = 2a_{k+1} - a_k$
- Solution for $r = 1$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = 2a_{k+1} - a_k, a_1 = 2a_0\right]$

1.162.3 Maple trace

```
Methods for second order ODEs:
```

1.162.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 17

```
dsolve(x^2*(1-x)^2*diff(diff(y(x),x),x)-x*(-3*x^2+2*x+1)*diff(y(x),x)+(x^2+1)*y(x) = (
       y(x),singsol=all)
```

$$y = \frac{x(c_1 + c_2 \ln(x))}{(-1 + x)^2}$$

1.162.5 Mathematica DSolve solution

Solving time : 0.055 (sec)
Leaf size : 20

```
DSolve[{x^2*(1-x)^2*D[y[x],{x,2}]-x*(1+2*x-3*x^2)*D[y[x],x]+(1+x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x(c_2 \log(x) + c_1)}{(x-1)^2}$$

1.163 problem 165

Internal problem ID [8301]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 165
Date solved : `Monday, October 21, 2024 at 05:05:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2\left(x^2+x+1\right)y''+3x\left(13x^2+7x+1\right)y'+\left(25x^2+4x+1\right)y=0$$

1.163.1 Solved as second order ode using Kovacic algorithm

Time used: 0.320 (sec)

Writing the ode as

$$\left(9x^4+9x^3+9x^2\right)y''+\left(39x^3+21x^2+3x\right)y'+\left(25x^2+4x+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=9x^4+9x^3+9x^2\\B&=39x^3+21x^2+3x\\C&=25x^2+4x+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 310: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{39x^3+21x^2+3x}{9x^4+9x^3+9x^2} \, dx} \\ &= z_1 e^{-\ln(x^2+x+1) - \frac{\ln(x)}{6}} \\ &= z_1 \left(\frac{1}{(x^2+x+1)\, x^{1/6}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{x^2 + x + 1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{39x^3+21x^2+3x}{9x^4+9x^3+9x^2}dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-2\ln(x^2+x+1)-\frac{\ln(x)}{3}}}{(y_1)^2} dx$$
$$= y_1 \left(2x - \frac{19}{24} + (x-1)^2 + \frac{x}{3x^2+3x+3} - \frac{x^5}{3(x^2+x+1)} - \frac{x^4}{3(x^2+x+1)} - \frac{x^3}{3(x^2+x+1)} + \frac{x^2}{3x^2+3x+3} + \frac{1}{3x^2+3x+3} + \frac{x^3}{3} - x^2 + \ln(x)\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{x^{1/3}}{x^2+x+1}\right)$$
$$+ c_2\left(\frac{x^{1/3}}{x^2+x+1}\left(2x - \frac{19}{24} + (x-1)^2 + \frac{x}{3x^2+3x+3} - \frac{x^5}{3(x^2+x+1)} - \frac{x^4}{3(x^2+x+1)} - \frac{x^3}{3(x^2+x+1)} +\right.\right.$$

Will add steps showing solving for IC soon.

1.163.2 Maple step by step solution

Let's solve

$9x^2(x^2+x+1)\left(\frac{d}{dx}y'\right) + 3x(13x^2+7x+1)y' + (25x^2+4x+1)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(25x^2+4x+1)y}{9x^2(x^2+x+1)} - \frac{(13x^2+7x+1)y'}{3x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(13x^2+7x+1)y'}{3x(x^2+x+1)} + \frac{(25x^2+4x+1)y}{9x^2(x^2+x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{13x^2+7x+1}{3x(x^2+x+1)}, P_3(x) = \frac{25x^2+4x+1}{9x^2(x^2+x+1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{3}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{9}$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- • Multiply by denominators

  $9x^2(x^2+x+1)\left(\frac{d}{dx}y'\right) + 3x(13x^2+7x+1)\,y' + (25x^2+4x+1)\,y = 0$

- • Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ○ Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ○ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - ○ Shift index using $k- >k+2-m$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+3r)^2 x^r + \left(a_1(2+3r)^2 + a_0(2+3r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 + a_{k-1}(3k+3r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+3r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = \frac{1}{3}$
- Each term must be 0
  $a_1(2+3r)^2 + a_0(2+3r)^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = -a_0$
- Each term in the series must be 0, giving the recursion relation
  $(3k+3r-1)^2 (a_k + a_{k-1} + a_{k-2}) = 0$
- Shift index using $k- >k+2$
  $(3k+3r+5)^2 (a_{k+2} + a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -a_{k+1} - a_k$
- Recursion relation for $r = \frac{1}{3}$
  $a_{k+2} = -a_{k+1} - a_k$
- Solution for $r = \frac{1}{3}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -a_{k+1} - a_k, a_1 = -a_0\right]$

1.163.3 Maple trace

```
Methods for second order ODEs:
```

1.163.4 Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 22

```
dsolve(9*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+3*x*(13*x^2+7*x+1)*diff(y(x),x)+(25*x^2+4*
       y(x),singsol=all)
```

$$y = \frac{x^{1/3}(\ln(x) c_2 + c_1)}{x^2 + x + 1}$$

1.163.5 Mathematica DSolve solution

Solving time : 0.075 (sec)
Leaf size : 27

```
DSolve[{9*x^2*(1+x+x^2)*D[y[x],{x,2}]+3*x*(1+7*x+13*x^2)*D[y[x],x]+(1+4*x+25*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[3]{x}(c_2 \log(x) + c_1)}{x^2 + x + 1}$$

1.164 problem 166

Internal problem ID [8302]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 166
Date solved : `Monday, October 21, 2024 at 05:05:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''-x(4-7x)\,y'-(5-3x)\,y=0$$

1.164.1 Solved as second order ode using Kovacic algorithm

Time used: 0.296 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y''+\left(7x^2-4x\right)y'+(3x-5)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+4x^2\\B&=7x^2-4x\\C&=3x-5\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 32x + 128}{16\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 - 32x + 128$$
$$t = 16\left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 32x + 128}{16\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 312: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{2x} + \frac{5}{2(2+x)} + \frac{45}{16(2+x)^2} + \frac{2}{x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 32x + 128}{16(x^2 + 2x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2-32x+128}{16\left(x^2+2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{9}{4}$	$-\frac{5}{4}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{5}{4\left(2+x\right)} + \frac{2}{x} + (0) \\
&= -\frac{5}{4\left(2+x\right)} + \frac{2}{x} \\
&= \frac{3x+16}{4x\left(2+x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{5}{4\left(2+x\right)} + \frac{2}{x}\right)(0) + \left(\left(\frac{5}{4\left(2+x\right)^2} - \frac{2}{x^2}\right) + \left(-\frac{5}{4\left(2+x\right)} + \frac{2}{x}\right)^2 - \left(\frac{-3x^2-32x+128}{16\left(x^2+2x\right)^2}\right)\right) = 0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{4(2+x)} + \frac{2}{x}\right) dx} \\
&= \frac{x^2}{\left(2+x\right)^{5/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^2-4x}{2x^3+4x^2}\, dx} \\
&= z_1 e^{\frac{\ln(x)}{2} - \frac{9\ln(2+x)}{4}} \\
&= z_1 \left(\frac{\sqrt{x}}{\left(2+x\right)^{9/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x^{5/2}}{(2+x)^{7/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2-4x}{2x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{9\ln(2+x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-\frac{11(2+x)^{5/2}}{8}+\frac{10(2+x)^{3/2}}{3}-\frac{5\sqrt{2+x}}{2}}{x^3} - \frac{5\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{16}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{5/2}}{(2+x)^{7/2}}\right) \\
&\quad + c_2\left(\frac{x^{5/2}}{(2+x)^{7/2}}\left(\frac{-\frac{11(2+x)^{5/2}}{8}+\frac{10(2+x)^{3/2}}{3}-\frac{5\sqrt{2+x}}{2}}{x^3} - \frac{5\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{16}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.164.2 Maple step by step solution

Let's solve

$2x^2(2+x)\left(\frac{d}{dx}y'\right)-x(4-7x)\,y'-(5-3x)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(3x-5)y}{2x^2(2+x)}-\frac{(7x-4)y'}{2x(2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(7x-4)y'}{2x(2+x)}+\frac{(3x-5)y}{2x^2(2+x)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{7x-4}{2x(2+x)},P_3(x)=\frac{3x-5}{2x^2(2+x)}\right]$

- ◦ $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

$\left((2+x)\cdot P_2(x)\right)\bigg|_{x=-2}=\frac{9}{2}$

- ◦ $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

$\left((2+x)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

- ◦ $x=-2$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-2$

- Multiply by denominators

$2x^2(2+x)\left(\frac{d}{dx}y'\right)+x(7x-4)\,y'+(3x-5)\,y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$(2u^3-8u^2+8u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(7u^2-32u+36)\left(\frac{d}{du}y(u)\right)+(3u-11)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ◦ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(7+2r)\, u^{-1+r} + \left(4a_1(1+r)\,(9+2r) - a_0(8r^2+24r+11)\right) u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)\,(2k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(7+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{7}{2}\right\}$$

- Each term must be 0

$$4a_1(1+r)\,(9+2r) - a_0(8r^2+24r+11) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + \left(4(-4a_k + a_{k-1} + 4a_{k+1})\, r - 24a_k + a_{k-1} + 44a_{k+1}\right) k + 2(-4a_k +$$

- Shift index using $k- >k+1$

$$2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + \left(4(-4a_{k+1} + a_k + 4a_{k+2})\, r - 24a_{k+1} + a_k + 44a_{k+2}\right)(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} + 5ka_k - 40ka_{k+1} + 5ra_k - 40ra_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+4kr+2r^2+15k+15r+22)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}, 36a_1 - 11a_0 = 0\right]$$

- Revert the change of variables $u = 2+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 5k a_k - 40k a_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}, 36a_1 - 11a_0 = 0\right]$$

- Recursion relation for $r = -\frac{7}{2}$

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 9k a_k + 16k a_{k+1} + 10a_k - a_{k+1}}{4(2k^2+k-6)}$$

- Solution for $r = -\frac{7}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{7}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 9k a_k + 16k a_{k+1} + 10a_k - a_{k+1}}{4(2k^2+k-6)}, -20a_1 - 25a_0 = 0\right]$$

- Revert the change of variables $u = 2 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k-\frac{7}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 9k a_k + 16k a_{k+1} + 10a_k - a_{k+1}}{4(2k^2+k-6)}, -20a_1 - 25a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (2+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (2+x)^{k-\frac{7}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 5k a_k - 40k a_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}, 36a\right.$$

1.164.3 Maple trace

```
Methods for second order ODEs:
```

1.164.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 55

```
dsolve(2*x^2*(2+x)*diff(diff(y(x),x),x)-x*(4-7*x)*diff(y(x),x)-(5-3*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{15 \operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right) c_2 x^3 + 33\sqrt{2}\left(x^2 + \frac{52}{33}x + \frac{32}{33}\right) c_2 \sqrt{2+x} + c_1 x^3}{(2+x)^{7/2} \sqrt{x}}$$

1.164.5 Mathematica DSolve solution

Solving time : 0.36 (sec)
Leaf size : 92

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}]-x*(4-7*x)*D[y[x],x]-(5-3*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{15\sqrt{2}c_2x^3\operatorname{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right) - 48c_1x^3 + 66c_2\sqrt{x+2}x^2 + 104c_2\sqrt{x+2}x + 64c_2\sqrt{x+2}}{48\sqrt{x}(x+2)^{7/2}}$$

1.165 problem 167

Internal problem ID [8303]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 167
Date solved : `Monday, October 21, 2024 at 05:05:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-2x)\,y''+x(8-9x)\,y'+(6-3x)\,y=0$$

1.165.1 Solved as second order ode using Kovacic algorithm

Time used: 0.263 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(-9x^2+8x\right)y'+(6-3x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= -9x^2+8x \\ C &= 6-3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 21x^2 - 20x + 24 \\
t &= 4\left(2x^2 - x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 314: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{77}{16\left(x - \frac{1}{2}\right)^2} - \frac{19}{x - \frac{1}{2}} + \frac{6}{x^2} + \frac{19}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{11}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{7}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^2 - 20x + 24}{4(2x^2 - x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21x^2 - 20x + 24}{4(2x^2 - x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2
$\frac{1}{2}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - (\alpha_{c_1}^- + \alpha_{c_2}^+)\\ &= \frac{7}{4} - \left(\frac{3}{4}\right)\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)} + (0) \\
&= -\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)} \\
&= \frac{4+3x}{4x^2-2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)}\right)(1) + \left(\left(\frac{2}{x^2} - \frac{11}{4\left(x-\frac{1}{2}\right)^2}\right) + \left(-\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{21x^2-20x+24}{4\left(2x^2-x\right)^2}\right)\right)
$$

$$
\frac{4-3a_0}{x\left(-1+2x\right)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{4}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{4}{3}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{4}{3}\right) e^{\int \left(-\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)}\right) dx} \\
&= \left(x + \frac{4}{3}\right) e^{-2\ln(x) + \frac{11\ln(-1+2x)}{4}} \\
&= \frac{\left(x+\frac{4}{3}\right)\left(-1+2x\right)^{11/4}}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-9x^2+8x}{-2x^3+x^2}\,dx} \\
&= z_1 e^{-4\ln(x)+\frac{7\ln(-1+2x)}{4}} \\
&= z_1\left(\frac{(-1+2x)^{7/4}}{x^4}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(-1+2x)^{9/2}\,(4+3x)}{3x^6}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-9x^2+8x}{-2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-8\ln(x)+\frac{7\ln(-1+2x)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{(231x^3-198x^2+66x-8)\,x^8 \mathrm{e}^{-8\ln(x)+\frac{7\ln(-1+2x)}{2}}}{385\,(4+3x)\,(-1+2x)^8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(-1+2x)^{9/2}\,(4+3x)}{3x^6}\right) \\
&\quad + c_2\left(\frac{(-1+2x)^{9/2}\,(4+3x)}{3x^6}\left(-\frac{(231x^3-198x^2+66x-8)\,x^8 \mathrm{e}^{-8\ln(x)+\frac{7\ln(-1+2x)}{2}}}{385\,(4+3x)\,(-1+2x)^8}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.165.2 Maple step by step solution

Let's solve

$x^2(1-2x)\left(\frac{d}{dx}y'\right)+x(8-9x)\,y'+(6-3x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{3(-2+x)y}{x^2(-1+2x)}-\frac{(9x-8)y'}{x(-1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(9x-8)y'}{x(-1+2x)}+\frac{3(-2+x)y}{x^2(-1+2x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{9x-8}{x(-1+2x)},P_3(x)=\frac{3(-2+x)}{x^2(-1+2x)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=8$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=6$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(-1+2x)\left(\frac{d}{dx}y'\right)+x(9x-8)\,y'+(3x-6)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(6+r)\,(1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r+6)\,(k+r+1) + a_{k-1}(k+2+r)\,(2k-1+2r)\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(6+r)\,(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-6, -1\}$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+2+r)\left(k-\frac{1}{2}+r\right)a_{k-1} - a_k(k+r+6)\,(k+r+1) = 0$$

- Shift index using $k->k+1$

$$2(k+r+3)\left(k+\frac{1}{2}+r\right)a_k - a_{k+1}(k+7+r)\,(k+2+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(k+r+3)(2k+2r+1)a_k}{(k+7+r)(k+2+r)}$$

- Recursion relation for $r = -6$; series terminates at $k = 3$

$$a_{k+1} = \frac{(k-3)(2k-11)a_k}{(k+1)(k-4)}$$

- Apply recursion relation for $k = 0$

$$a_1 = -\frac{33a_0}{4}$$

- Apply recursion relation for $k = 1$

$$a_2 = -3a_1$$

- Express in terms of a_0

$$a_2 = \frac{99a_0}{4}$$

- Apply recursion relation for $k = 2$

$$a_3 = -\frac{7a_2}{6}$$

- Express in terms of a_0

$a_3 = -\frac{231 a_0}{8}$

- Terminating series solution of the ODE for $r = -6$. Use reduction of order to find the second

  $y = a_0 \cdot \left(-\frac{231}{8}x^3 + \frac{99}{4}x^2 - \frac{33}{4}x + 1\right)$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{(k+2)(2k-1)a_k}{(k+6)(k+1)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{(k+2)(2k-1)a_k}{(k+6)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(-\frac{231}{8}x^3 + \frac{99}{4}x^2 - \frac{33}{4}x + 1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), b_{k+1} = \frac{(k+2)(2k-1)b_k}{(k+6)(k+1)}\right]$

1.165.3 Maple trace

```
Methods for second order ODEs:
```

1.165.4 Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 43

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)+x*(8-9*x)*diff(y(x),x)+(6-3*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{48\left(x-\frac{1}{2}\right)^4 \left(x+\frac{4}{3}\right) c_1 \sqrt{-1+2x} + 231\left(x^3 - \frac{6}{7}x^2 + \frac{2}{7}x - \frac{8}{231}\right) c_2}{x^6}$$

1.165.5 Mathematica DSolve solution

Solving time : `0.277` (sec)
Leaf size : 49

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]+x*(8-9*x)*D[y[x],x]+(6-3*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2(231x^3 - 198x^2 + 66x - 8) + 385c_1(3x+4)(1-2x)^{9/2}}{1155x^6}$$

1.166 problem 168

Internal problem ID [8304]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 168
Date solved : `Monday, October 21, 2024 at 05:05:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''+x\left(10x^2+3\right)y'-\left(-14x^2+15\right)y=0$$

1.166.1 Solved as second order ode using Kovacic algorithm

Time used: 0.392 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(10x^3+3x\right)y'+\left(14x^2-15\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 10x^3+3x \\ C &= 14x^2-15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{24x^4 + 66x^2 + 63}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 24x^4 + 66x^2 + 63$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{24x^4 + 66x^2 + 63}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 316: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16(x-i)^2} + \frac{21}{16(x+i)^2} + \frac{99i}{16(x-i)} - \frac{99i}{16(x+i)} + \frac{63}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{24x^4 + 66x^2 + 63}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = 6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{24x^4 + 66x^2 + 63}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
i	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 3 - (3) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{9}{2x} - \frac{3}{4\,(x-i)} - \frac{3}{4\,(x+i)} + (0) \\
&= \frac{9}{2x} - \frac{3}{4\,(x-i)} - \frac{3}{4\,(x+i)} \\
&= \frac{9}{2x} - \frac{3x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{9}{2x} - \frac{3}{4\,(x-i)} - \frac{3}{4\,(x+i)}\right)(0) + \left(\left(-\frac{9}{2x^2} + \frac{3}{4\,(x-i)^2} + \frac{3}{4\,(x+i)^2}\right) + \left(\frac{9}{2x} - \frac{3}{4\,(x-i)} - \frac{}{4(}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)}\right) dx} \\
&= \frac{x^{9/2}}{\left(x^2+1\right)^{3/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{10x^3+3x}{x^4+x^2}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(x^2+1\right)}{4}-\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\left(x^2+1\right)^{7/4} x^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^3}{\left(x^2+1\right)^{5/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{10x^3+3x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(x^2+1\right)}{2}-3\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{\left(x^2+1\right)^{5/2}}{8x^8}+\frac{\left(x^2+1\right)^{5/2}}{16x^6}-\frac{\left(x^2+1\right)^{5/2}}{64x^4}-\frac{\left(x^2+1\right)^{5/2}}{128x^2}+\frac{\left(x^2+1\right)^{3/2}}{128}\right. \\
&\qquad \left.+\frac{3\sqrt{x^2+1}}{128}-\frac{3\,\mathrm{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{128}\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x^3}{(x^2+1)^{5/2}}\right)$$
$$+ c_2 \left(\frac{x^3}{(x^2+1)^{5/2}} \left(-\frac{(x^2+1)^{5/2}}{8x^8} + \frac{(x^2+1)^{5/2}}{16x^6} - \frac{(x^2+1)^{5/2}}{64x^4} - \frac{(x^2+1)^{5/2}}{128x^2} + \frac{(x^2+1)^{3/2}}{128} + \frac{3\sqrt{x^2+1}}{128} - \frac{3}{}\right.\right.$$

Will add steps showing solving for IC soon.

1.166.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right) + x(10x^2+3)\,y' - (-14x^2+15)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(14x^2-15)y}{x^2(x^2+1)} - \frac{(10x^2+3)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(10x^2+3)y'}{x(x^2+1)} + \frac{(14x^2-15)y}{x^2(x^2+1)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{10x^2+3}{x(x^2+1)}, P_3(x) = \frac{14x^2-15}{x^2(x^2+1)}\right]$

  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 3$

  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -15$

  - ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d}{dx}y'\right) + x(10x^2+3)\,y' + (14x^2-15)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- > k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

  - Shift index using $k- > k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - Convert $x^m \cdot \left(\frac{d}{dx} y'\right)$ to series expansion for $m = 2..4$

    $$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k- > k + 2 - m$

    $$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(5+r)\,(-3+r)\, x^r + a_1(6+r)\,(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+5)\,(k+r-3) + a_{k-2}(k+r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+r)\,(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-5, 3\}$
- Each term must be 0

  $a_1(6+r)\,(-2+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r+5)\,(a_k(k+r-3) + a_{k-2}(k+r)) = 0$
- Shift index using $k- > k + 2$

  $(k+r+7)\,(a_{k+2}(k+r-1) + a_k(k+r+2)) = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+2)}{k+r-1}$$

- Recursion relation for $r = -5$

$$a_{k+2} = -\frac{a_k(k-3)}{k-6}$$

- Series not valid for $r = -5$, division by 0 in the recursion relation at $k = 6$

$$a_{k+2} = -\frac{a_k(k-3)}{k-6}$$

- Recursion relation for $r = 3$

$$a_{k+2} = -\frac{a_k(k+5)}{k+2}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k(k+5)}{k+2}, a_1 = 0\right]$$

1.166.3 Maple trace

```
Methods for second order ODEs:
```

1.166.4 Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 59

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(10*x^2+3)*diff(y(x),x)-(-14*x^2+15)*y(x) =
       y(x),singsol=all)
```

$$y = \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) c_2 x^8 - c_2\left(x^4 - \frac{8}{3}x^2 - \frac{8}{3}\right)\left(x^2+2\right)\sqrt{x^2+1} + c_1 x^8}{\left(x^2+1\right)^{5/2} x^5}$$

1.166.5 Mathematica DSolve solution

Solving time : 0.26 (sec)
Leaf size : 75

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]+x*(3+10*x^2)*D[y[x],x]-(15-14*x^2)*y[x]==0,{}},
      y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2\left(\sqrt{x^2+1}(3x^6 - 2x^4 - 24x^2 - 16) - 3x^8 \operatorname{arctanh}\left(\sqrt{x^2+1}\right)\right) + 128c_1x^8}{128x^5\left(x^2+1\right)^{5/2}}$$

1.167 problem 169

Internal problem ID [8305]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 169
Date solved : `Monday, October 21, 2024 at 05:05:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(-2x^2+1)\,y''+x(-13x^2+7)\,y'-14x^2y=0$$

1.167.1 Solved as second order ode using Kovacic algorithm

Time used: 0.360 (sec)

Writing the ode as

$$(-2x^4+x^2)\,y''+(-13x^3+7x)\,y'-14x^2y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-2x^4+x^2\\B&=-13x^3+7x\\C&=-14x^2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^4 - 68x^2 + 35}{4(2x^3 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^4 - 68x^2 + 35$$
$$t = 4(2x^3 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^4 - 68x^2 + 35}{4(2x^3 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 318: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4x^2} + \frac{9}{64\left(x - \frac{\sqrt{2}}{2}\right)^2} + \frac{9}{64\left(x + \frac{\sqrt{2}}{2}\right)^2} - \frac{279\sqrt{2}}{64\left(x - \frac{\sqrt{2}}{2}\right)} + \frac{279\sqrt{2}}{64\left(x + \frac{\sqrt{2}}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = \frac{\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8} \end{aligned}$$

For the pole at $x = -\frac{\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$\frac{\sqrt{2}}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$
$-\frac{\sqrt{2}}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\ &= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} + (-)\,(0) \\ &= -\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} \\ &= \frac{-x^2+5}{4x^3-2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} \right)(0) + \left(\left(\frac{5}{2x^2} - \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)^2} - \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)^2} \right) + \left(-\frac{5}{2x} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{2x}+\frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)}+\frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)}\right)dx} \\
&= \frac{\left(2x-\sqrt{2}\right)^{9/8}\left(2x+\sqrt{2}\right)^{9/8}}{x^{5/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-13x^3+7x}{-2x^4+x^2}\, dx} \\
&= z_1 e^{-\frac{7\ln(x)}{2}+\frac{\ln\left(2x^2-1\right)}{8}} \\
&= z_1 \left(\frac{\left(2x^2-1\right)^{1/8}}{x^{7/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2\left(2x^2-1\right)^{5/4} 2^{1/8}}{x^6}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-13x^3+7x}{-2x^4+x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-7\ln(x)+\frac{\ln\left(2x^2-1\right)}{4}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{\left(5x^4-20x^2+8\right) x^7 \mathrm{e}^{-7\ln(x)+\frac{\ln\left(2x^2-1\right)}{4}} 2^{3/4}}{120\left(2x^2-1\right)^{3/2}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2(2x^2-1)^{5/4} 2^{1/8}}{x^6}\right) \\
&\quad + c_2 \left(\frac{2(2x^2-1)^{5/4} 2^{1/8}}{x^6} \left(\frac{(5x^4-20x^2+8)\, x^7 \mathrm{e}^{-7\ln(x)+\frac{\ln(2x^2-1)}{4}} 2^{3/4}}{120\,(2x^2-1)^{3/2}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.167.2 Maple step by step solution

Let's solve

$x^2(-2x^2+1)\left(\frac{d}{dx}y'\right) + x(-13x^2+7)\,y' - 14x^2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{14y}{2x^2-1} - \frac{(13x^2-7)y'}{x(2x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(13x^2-7)y'}{x(2x^2-1)} + \frac{14y}{2x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{13x^2-7}{x(2x^2-1)}, P_3(x) = \frac{14}{2x^2-1}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 7$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

$$x(2x^2-1)\left(\frac{d}{dx}y'\right)+(13x^2-7)\,y'+14yx=0$$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

  ◦ Convert $x\cdot y$ to series expansion

$$x\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  ◦ Shift index using $k->k-1$

$$x\cdot y=\sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..2$

$$x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  ◦ Shift index using $k->k+1-m$

$$x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=1..3$

$$x^m\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  ◦ Shift index using $k->k+2-m$

$$x^m\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(6+r)\,x^{-1+r}-a_1(1+r)\,(7+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(-a_{k+1}(k+r+1)\,(k+7+r)+a_{k-1}(2k+5+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(6+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{-6,0\}$$

- Each term must be 0

$$-a_1(1+r)\,(7+r)=0$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(\left(k+r+\frac{5}{2}\right)a_{k-1}-\frac{a_{k+1}(k+7+r)}{2}\right)(k+r+1)=0$$

- Shift index using $k->k+1$

$$2\left(\left(k+\frac{7}{2}+r\right)a_k-\frac{a_{k+2}(k+8+r)}{2}\right)(k+r+2)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=\frac{(2k+2r+7)a_k}{k+8+r}$$

- Recursion relation for $r=-6$

$$a_{k+2}=\frac{(2k-5)a_k}{k+2}$$

- Solution for $r=-6$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-6}, a_{k+2}=\frac{(2k-5)a_k}{k+2}, 5a_1=0\right]$$

- Recursion relation for $r=0$

$$a_{k+2}=\frac{(2k+7)a_k}{k+8}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k}, a_{k+2}=\frac{(2k+7)a_k}{k+8}, -7a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-6}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k}\right), a_{k+2}=\frac{(2k-5)a_k}{k+2}, 5a_1=0, b_{k+2}=\frac{(2k+7)b_k}{k+8}, -7b_1=0\right]$$

1.167.3 Maple trace

```
Methods for second order ODEs:
```

1.167.4 Maple dsolve solution

Solving time : `0.011` (sec)
Leaf size : 35

```
dsolve(x^2*(-2*x^2+1)*diff(diff(y(x),x),x)+x*(-13*x^2+7)*diff(y(x),x)-14*x^2*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1(2x^2-1)^{5/4}+5c_2\,x^4-20c_2\,x^2+8c_2}{x^6}$$

1.167.5 Mathematica DSolve solution

Solving time : 0.2 (sec)
Leaf size : 43

```
DSolve[{x^2*(1-2*x^2)*D[y[x],{x,2}]+x*(7-13*x^2)*D[y[x],x]-14*x^2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{15c_1(1-2x^2)^{5/4} + c_2(-5x^4+20x^2-8)}{15x^6}$$

1.168 problem 170

Internal problem ID [8306]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 170
Date solved : `Monday, October 21, 2024 at 05:05:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y''+4x(1+2x)\,y'-(1+3x)\,y=0$$

1.168.1 Solved as second order ode using Kovacic algorithm

Time used: 0.263 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(8x^2+4x\right)y'+(-3x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+4x^2 \\ B &= 8x^2+4x \\ C &= -3x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x + 4}{4x(1 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x + 4$$
$$t = 4x(1 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x + 4}{4x(1 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 320: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 3 - 1 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{1}{4(1+x)^2} - \frac{1}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x+4}{4x(1+x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\ \alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{3x+4}{4x\,(1+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{3}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right)\\ &= \frac{3}{2}-\left(\frac{3}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2} \right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} + \frac{1}{2 + 2x} + (0) \\
&= \frac{1}{x} + \frac{1}{2 + 2x} \\
&= \frac{1}{x} + \frac{1}{2 + 2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{x} + \frac{1}{2 + 2x}\right)(0) + \left(\left(-\frac{1}{x^2} - \frac{1}{2(1 + x)^2}\right) + \left(\frac{1}{x} + \frac{1}{2 + 2x}\right)^2 - \left(\frac{3x + 4}{4x(1 + x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{x} + \frac{1}{2 + 2x}\right) dx} \\
&= \sqrt{1 + x}\, x
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{8x^2 + 4x}{4x^3 + 4x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x(1 + x))}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x(1 + x)}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\sqrt{1+x}\, x}{\sqrt{x\,(1+x)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2+4x}{4x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x(1+x))}}{(y_1)^2}\, dx \\ &= y_1 \left(\ln(1+x) - \frac{1}{x} - \ln(x) \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{1+x}\, x}{\sqrt{x\,(1+x)}} \right) + c_2 \left(\frac{\sqrt{1+x}\, x}{\sqrt{x\,(1+x)}} \left(\ln(1+x) - \frac{1}{x} - \ln(x) \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.168.2 Maple step by step solution

Let's solve

$4x^2(1+x)\left(\frac{d}{dx}y'\right) + 4x(1+2x)\, y' - (1+3x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(1+3x)y}{4x^2(1+x)} - \frac{(1+2x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(1+2x)y'}{x(1+x)} - \frac{(1+3x)y}{4x^2(1+x)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{1+2x}{x(1+x)}, P_3(x)=-\frac{1+3x}{4x^2(1+x)}\right]$
  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=1$
  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$
  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$
- Multiply by denominators

  $4x^2(1+x)\left(\frac{d}{dx}y'\right)+4x(1+2x)\,y'+(-3x-1)\,y=0$
- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(4u^3-8u^2+4u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(8u^2-12u+4)\left(\frac{d}{du}y(u)\right)+(-3u+2)\,y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $$u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$4a_0r^2u^{-1+r}+\left(4a_1(1+r)^2-2a_0(4r^2+2r-1)\right)u^r+\left(\sum_{k=1}^{\infty}\left(4a_{k+1}(k+1+r)^2-2a_k(4k^2+8kr\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4r^2=0$$

- Values of r that satisfy the indicial equation

  $$r=0$$

- Each term must be 0

  $$4a_1(1+r)^2-2a_0(4r^2+2r-1)=0$$

- Each term in the series must be 0, giving the recursion relation

  $$(4k^2-4k-3)\,a_{k-1}+(-8k^2-4k+2)\,a_k+4a_{k+1}(k+1)^2=0$$

- Shift index using $k- >k+1$

  $$\left(4(k+1)^2-4k-7\right)a_k+\left(-8(k+1)^2-4k-2\right)a_{k+1}+4a_{k+2}(k+2)^2=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2}$$

- Recursion relation for $r=0$

  $$a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2}$$

- Solution for $r=0$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2},4a_1+2a_0=0\right]$$

- Revert the change of variables $u=1+x$

  $$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k,a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2},4a_1+2a_0=0\right]$$

1.168.3 Maple trace

```
Methods for second order ODEs:
```

1.168.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 28

```
dsolve(4*x^2*(1+x)*diff(diff(y(x),x),x)+4*x*(1+2*x)*diff(y(x),x)-(1+3*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 x + \ln(1+x) c_2 x - \ln(x) c_2 x - c_2}{\sqrt{x}}$$

1.168.5 Mathematica DSolve solution

Solving time : `0.076` (sec)
Leaf size : 32

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+4*x*(1+2*x)*D[y[x],x]-(1+3*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1 x + c_2(-x \log(x) + x \log(x+1) - 1)}{\sqrt{x}}$$

1.169 problem 171

Internal problem ID [8307]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 171
Date solved : `Monday, October 21, 2024 at 05:05:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+3x)\,y''+x(4+21x)\,y'-(1-9x)\,y=0$$

1.169.1 Solved as second order ode using Kovacic algorithm

Time used: 0.269 (sec)

Writing the ode as

$$\left(6x^3+4x^2\right)y''+\left(21x^2+4x\right)y'+(9x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=6x^3+4x^2\\B&=21x^2+4x\\C&=9x-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-27x - 48}{16x\left(2 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -27x - 48$$
$$t = 16x(2 + 3x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-27x - 48}{16x\left(2 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 322: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x(2+3x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = -\frac{2}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\left(x+\frac{2}{3}\right)^2} + \frac{3}{4\left(x+\frac{2}{3}\right)} - \frac{3}{4x}$$

For the pole at $x = -\frac{2}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-27x - 48}{16x\left(2+3x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-27x-48}{16x\left(2+3x\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
$-\frac{2}{3}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{3}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{3}{4}-\left(\frac{3}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)} + (0) \\
&= \frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)} \\
&= \frac{8+9x}{12x^2+8x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{4\left(x+\frac{2}{3}\right)^2}\right) + \left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right)^2 - \left(\frac{-27x-48}{16x\left(2+3x\right)^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right) dx} \\
&= \frac{x}{\left(2+3x\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{21x^2+4x}{6x^3+4x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} - \frac{5\ln(2+3x)}{4}} \\
&= z_1 \left(\frac{1}{\sqrt{x}\,\left(2+3x\right)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(2+3x)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{21x^2+4x}{6x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)-\frac{5\ln(2+3x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\sqrt{2+3x}}{x} - \frac{3\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2+3x}\,\sqrt{2}}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{(2+3x)^{3/2}}\right) + c_2 \left(\frac{\sqrt{x}}{(2+3x)^{3/2}} \left(-\frac{\sqrt{2+3x}}{x} - \frac{3\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2+3x}\,\sqrt{2}}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.169.2 Maple step by step solution

Let's solve

$2x^2(2+3x)\left(\frac{d}{dx}y'\right) + x(4+21x)\, y' - (1-9x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = -\frac{(9x-1)y}{2x^2(2+3x)} - \frac{(4+21x)y'}{2x(2+3x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(4+21x)y'}{2x(2+3x)} + \frac{(9x-1)y}{2x^2(2+3x)} = 0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{4+21x}{2x(2+3x)}, P_3(x) = \frac{9x-1}{2x^2(2+3x)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{4}$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2(2+3x)\left(\frac{d}{dx}y'\right) + x(4+21x)\,y' + (9x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 3a_{k-1}(2k+2r+1)\,(k+r)\right) x^{k+}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r+\frac{1}{2}\right)\left(\left(k+r-\frac{1}{2}\right)a_k + \frac{3a_{k-1}(k+r)}{2}\right) = 0$$

- Shift index using $k->k+1$

$$4\left(k+\frac{3}{2}+r\right)\left(\left(k+r+\frac{1}{2}\right)a_{k+1} + \frac{3a_k(k+r+1)}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{3a_k(k+r+1)}{2k+2r+1}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{3a_k\left(k+\frac{3}{2}\right)}{2k+2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{3a_k\left(k+\frac{3}{2}\right)}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}, b_{k+1} = -\frac{3b_k\left(k+\frac{3}{2}\right)}{2k+2}\right]$$

1.169.3 Maple trace

```
Methods for second order ODEs:
```

1.169.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 48

```
dsolve(2*x^2*(2+3*x)*diff(diff(y(x),x),x)+x*(4+21*x)*diff(y(x),x)-(1-9*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{3 \operatorname{arctanh}\left(\frac{\sqrt{2+3x}\,\sqrt{2}}{2}\right) c_2 x + \sqrt{2}\,\sqrt{2+3x}\,c_2 + c_1 x}{\sqrt{x}\,\left(2+3x\right)^{3/2}}$$

1.169.5 Mathematica DSolve solution

Solving time : 0.228 (sec)
Leaf size : 64

```
DSolve[{2*x^2*(2+3*x)*D[y[x],{x,2}]+x*(4+21*x)*D[y[x],x]-(1-9*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{3\sqrt{2}c_2 x \operatorname{arctanh}\left(\sqrt{\frac{3x}{2}+1}\right) - 2c_1 x + 2c_2\sqrt{3x+2}}{2\sqrt{x}(3x+2)^{3/2}}$$

1.170 problem 172

Internal problem ID [8308]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 172
Date solved : `Monday, October 21, 2024 at 05:05:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(2 + x)\,y' - (2 - 3x)\,y = 0$$

1.170.1 Solved as second order ode using Kovacic algorithm

Time used: 0.294 (sec)

Writing the ode as

$$x^2y'' + \left(x^2 + 2x\right) y' + (3x - 2)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 + 2x \\ C &= 3x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 8x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 8x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 8x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 324: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{2}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{2}{x} - \frac{2}{x^2} - \frac{8}{x^3} - \frac{36}{x^4} - \frac{176}{x^5} - \frac{912}{x^6} - \frac{4928}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 8x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-8x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-8x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -8. Dividing this by leading coefficient in t which is 4 gives -2. Now b can be found.

$$
\begin{aligned}
b &= (-2) - (0) \\
&= -2
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 0\right) &= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 0\right) &= 2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 8x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= 2 - (2) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{2}{x} + (-) \left(\frac{1}{2} \right) \\
&= \frac{2}{x} - \frac{1}{2} \\
&= -\frac{x - 4}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r \right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{2}{x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{2}{x^2} \right) + \left(\frac{2}{x} - \frac{1}{2} \right)^2 - \left(\frac{x^2 - 8x + 8}{4x^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{x} - \frac{1}{2}\right) dx} \\
&= x^2 \mathrm{e}^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 + 2x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{x} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+2x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x-2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^x}{3x^3} - \frac{\mathrm{e}^x}{6x^2} - \frac{\mathrm{e}^x}{6x} - \frac{\mathrm{Ei}_1(-x)}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x\,\mathrm{e}^{-x}\right) + c_2 \left(x\,\mathrm{e}^{-x} \left(-\frac{\mathrm{e}^x}{3x^3} - \frac{\mathrm{e}^x}{6x^2} - \frac{\mathrm{e}^x}{6x} - \frac{\mathrm{Ei}_1(-x)}{6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.170.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + x(2+x)\,y' - (2-3x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(3x-2)y}{x^2} - \frac{(2+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(2+x)y'}{x} + \frac{(3x-2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x)=\frac{2+x}{x}, P_3(x)=\frac{3x-2}{x^2}\right]$
- $x\cdot P_2(x)$ is analytic at $x=0$

  $$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=2$$
- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-2$$
- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

• Multiply by denominators

$$x^2\left(\frac{d}{dx}y'\right)+x(2+x)\,y'+(3x-2)\,y=0$$

• Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $$x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$
- Shift index using $k->k-m$

  $$x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
- Convert $x^m\cdot y'$ to series expansion for $m=1..2$

  $$x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$
- Shift index using $k->k+1-m$

  $$x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$
- Convert $x^2\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(-1+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+2)\,(k+r-1)+a_{k-1}(k+r+2)\right)x^{k+r}\right)=0$$

• $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r+2)(a_k(k+r-1) + a_{k-1}) = 0$
- Shift index using $k- >k+1$

  $(k+r+3)(a_{k+1}(k+r) + a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k+r}$
- Recursion relation for $r = -2$

  $a_{k+1} = -\frac{a_k}{k-2}$
- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = -\frac{a_k}{k-2}$
- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{k+1}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{k+1}\right]$

1.170.3 Maple trace

```
Methods for second order ODEs:
```

1.170.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 40

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(2+x)*diff(y(x),x)-(2-3*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\mathrm{Ei}_1(-x)\,\mathrm{e}^{-x} c_2\, x^3 + c_1\,\mathrm{e}^{-x} x^3 + c_2(x^2+x+2)}{x^2}$$

1.170.5 Mathematica DSolve solution

Solving time : `0.158` (sec)
Leaf size : 46

```
DSolve[{x^2*D[y[x],{x,2}]+x*(2+x)*D[y[x],x]-(2-3*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x}(c_2(x^3 \operatorname{ExpIntegralEi}(x) - e^x(x^2 + x + 2)) + 6c_1x^3)}{6x^2}$$

1.171 problem 173

Internal problem ID [8309]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 173
Date solved : `Monday, October 21, 2024 at 05:05:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y''+4x(3+8x)\,y'-(5-49x)\,y=0$$

1.171.1 Solved as second order ode using Kovacic algorithm

Time used: 0.290 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(32x^2+12x\right)y'+(49x-5)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+4x^2 \\ B &= 32x^2+12x \\ C &= 49x-5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-x^2 - 8x + 8}{4\left(x^2 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -x^2 - 8x + 8 \\
t &= 4\left(x^2 + x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-x^2 - 8x + 8}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 326: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4(1+x)^2} - \frac{6}{x} + \frac{6}{1+x} + \frac{2}{x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 8x + 8}{4(x^2+x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\ \alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-x^2-8x+8}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right)\\ &= \frac{1}{2}-\left(\frac{1}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{2\left(1+x\right)} + \frac{2}{x} + (-)\left(0\right) \\
&= -\frac{3}{2\left(1+x\right)} + \frac{2}{x} \\
&= \frac{x+4}{2x\left(1+x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{3}{2\left(1+x\right)} + \frac{2}{x}\right)(0) + \left(\left(\frac{3}{2\left(1+x\right)^2} - \frac{2}{x^2}\right) + \left(-\frac{3}{2\left(1+x\right)} + \frac{2}{x}\right)^2 - \left(\frac{-x^2-8x+8}{4\left(x^2+x\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{3}{2(1+x)} + \frac{2}{x}\right) dx} \\
&= \frac{x^2}{\left(1+x\right)^{3/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{32x^2+12x}{4x^3+4x^2}\, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2} - \frac{5\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{x^{3/2}\left(1+x\right)^{5/2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(1+x)^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{32x^2+12x}{4x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3\ln(x)-5\ln(1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{1}{3x^3} + \ln(x) - \frac{3}{2x^2} - \frac{3}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{(1+x)^4}\right) + c_2 \left(\frac{\sqrt{x}}{(1+x)^4}\left(-\frac{1}{3x^3} + \ln(x) - \frac{3}{2x^2} - \frac{3}{x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.171.2 Maple step by step solution

Let's solve

$4x^2(1+x)\left(\frac{d}{dx}y'\right) + 4x(3+8x)\, y' - (5-49x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(49x-5)y}{4x^2(1+x)} - \frac{(3+8x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3+8x)y'}{x(1+x)} + \frac{(49x-5)y}{4x^2(1+x)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{3+8x}{x(1+x)}, P_3(x) = \frac{49x-5}{4x^2(1+x)}\right]$
  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$
    $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = 5$
  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$
    $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$
  - $x = -1$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = -1$
- Multiply by denominators
  $4x^2(1+x)\left(\frac{d}{dx}y'\right) + 4x(3+8x)\,y' + (49x-5)\,y = 0$
- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$
  $(4u^3 - 8u^2 + 4u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (32u^2 - 52u + 20)\left(\frac{d}{du}y(u)\right) + (49u - 54)\,y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$
    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k- >k-m$
    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$
  - Shift index using $k- >k+1-m$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$
    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $$u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$4a_{0}r(4+r)\,u^{-1+r}+\left(4a_{1}(1+r)\,(5+r)-2a_{0}\left(4r^{2}+22r+27\right)\right)u^{r}+\left(\sum_{k=1}^{\infty}\left(4a_{k+1}(k+1+r)\,(k+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4r(4+r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\{-4,0\}$$

- Each term must be 0

  $$4a_{1}(1+r)\,(5+r)-2a_{0}\left(4r^{2}+22r+27\right)=0$$

- Each term in the series must be 0, giving the recursion relation

  $$4a_{k+1}(k+1+r)\,(k+5+r)-2a_{k}\left(4k^{2}+8kr+4r^{2}+22k+22r+27\right)+a_{k-1}(2k+5+2r)^{2}=($$

- Shift index using $k- >k+1$

  $$4a_{k+2}(k+2+r)\,(k+6+r)-2a_{k+1}\left(4(k+1)^{2}+8(k+1)\,r+4r^{2}+22k+49+22r\right)+a_{k}(2k+$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{4k^{2}a_{k}-8k^{2}a_{k+1}+8kra_{k}-16kra_{k+1}+4r^{2}a_{k}-8r^{2}a_{k+1}+28ka_{k}-60ka_{k+1}+28ra_{k}-60ra_{k+1}+49a_{k}-106a_{k+1}}{4(k+2+r)(k+6+r)}$$

- Recursion relation for $r=-4$

  $$a_{k+2}=-\frac{4k^{2}a_{k}-8k^{2}a_{k+1}-4ka_{k}+4ka_{k+1}+a_{k}+6a_{k+1}}{4(k-2)(k+2)}$$

- Series not valid for $r=-4$, division by 0 in the recursion relation at $k=2$

  $$a_{k+2}=-\frac{4k^{2}a_{k}-8k^{2}a_{k+1}-4ka_{k}+4ka_{k+1}+a_{k}+6a_{k+1}}{4(k-2)(k+2)}$$

- Recursion relation for $r=0$

  $$a_{k+2}=-\frac{4k^{2}a_{k}-8k^{2}a_{k+1}+28ka_{k}-60ka_{k+1}+49a_{k}-106a_{k+1}}{4(k+2)(k+6)}$$

- Solution for $r=0$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k},a_{k+2}=-\frac{4k^{2}a_{k}-8k^{2}a_{k+1}+28ka_{k}-60ka_{k+1}+49a_{k}-106a_{k+1}}{4(k+2)(k+6)},20a_{1}-54a_{0}=0\right]$$

- Revert the change of variables $u=1+x$

  $$\left[y=\sum_{k=0}^{\infty}a_{k}(1+x)^{k},a_{k+2}=-\frac{4k^{2}a_{k}-8k^{2}a_{k+1}+28ka_{k}-60ka_{k+1}+49a_{k}-106a_{k+1}}{4(k+2)(k+6)},20a_{1}-54a_{0}=0\right]$$

1.171.3 Maple trace

```
Methods for second order ODEs:
```

1.171.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 40

```
dsolve(4*x^2*(1+x)*diff(diff(y(x),x),x)+4*x*(3+8*x)*diff(y(x),x)-(5-49*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 x^3 + 6\ln(x) c_2 x^3 - 18c_2 x^2 - 9c_2 x - 2c_2}{x^{5/2} (1+x)^4}$$

1.171.5 Mathematica DSolve solution

Solving time : 0.093 (sec)
Leaf size : 52

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+4*x*(3+8*x)*D[y[x],x]-(5-49*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{6c_1x^3 + 6c_2x^3 \log(x) - 18c_2x^2 - 9c_2x - 2c_2}{6x^{5/2}(x+1)^4}$$

1.172 problem 174

Internal problem ID [8310]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 174
Date solved : `Monday, October 21, 2024 at 05:05:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y'' - x(3+10x)\,y' + 30xy = 0$$

1.172.1 Solved as second order ode using Kovacic algorithm

Time used: 0.293 (sec)

Writing the ode as

$$x^2(1+x)\,y'' + \left(-10x^2 - 3x\right)y' + 30xy = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= -10x^2 - 3x \\ C &= 30x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-48x + 15}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -48x + 15$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-48x + 15}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 328: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 1\\ &= 3\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2} + \frac{39}{2(1+x)} + \frac{63}{4(1+x)^2} - \frac{39}{2x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-48x+15}{4\left(x^{2}+x\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=0$ then

$$\begin{aligned}
d&=\alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^+\right)\\
&=0-(-1)\\
&=1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+\left((+)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^+}{x-c_2}\right)+(+)[\sqrt{r}]_\infty\\
&=-\frac{7}{2\left(1+x\right)}+\frac{5}{2x}+(0)\\
&=-\frac{7}{2\left(1+x\right)}+\frac{5}{2x}\\
&=-\frac{2x-5}{2x\left(1+x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{7}{2(1+x)} + \frac{5}{2x}\right)(1) + \left(\left(\frac{7}{2(1+x)^2} - \frac{5}{2x^2}\right) + \left(-\frac{7}{2(1+x)} + \frac{5}{2x}\right)^2 - \left(\frac{-48x+15}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{5 + 2a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{5}{2}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{5}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{5}{2}\right) e^{\int \left(-\frac{7}{2(1+x)} + \frac{5}{2x}\right) dx} \\
&= \left(x - \frac{5}{2}\right) e^{-\frac{7\ln(1+x)}{2} + \frac{5\ln(x)}{2}} \\
&= \frac{\left(x - \frac{5}{2}\right) x^{5/2}}{(1+x)^{7/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10x^2-3x}{x^2(1+x)} \, dx} \\
&= z_1 e^{\frac{7\ln(1+x)}{2} + \frac{3\ln(x)}{2}} \\
&= z_1 \left((1+x)^{7/2} x^{3/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^5 - \frac{5}{2}x^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x^2-3x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{7\ln(1+x)+3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(x - \frac{823543}{6250\,(2x-5)} - \frac{1}{25x^4} - \frac{52}{125x^3} - \frac{1354}{625x^2} - \frac{27708}{3125x} + 12\ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x^5 - \frac{5}{2}x^4\right) \\
&\quad + c_2\left(x^5 - \frac{5}{2}x^4\left(x - \frac{823543}{6250\,(2x-5)} - \frac{1}{25x^4} - \frac{52}{125x^3} - \frac{1354}{625x^2} - \frac{27708}{3125x} + 12\ln(x)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.172.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right) - x(3+10x)\,y' + 30xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{30y}{x(1+x)} + \frac{(3+10x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{(3+10x)y'}{x(1+x)} + \frac{30y}{x(1+x)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{3+10x}{x(1+x)}, P_3(x) = \frac{30}{x(1+x)}\right]$

  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = -7$

  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $x(1+x)\left(\frac{d}{dx}y'\right) + (-3-10x)\,y' + 30y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (7-10u)\left(\frac{d}{du}y(u)\right) + 30y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-a_0 r(-8+r)\, u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(k-7+r)+a_k(k+r-5)\,(k+r-6)\right) u^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-r(-8+r)=0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 8\}$
- Each term in the series must be 0, giving the recursion relation
  $-a_{k+1}(k+1+r)\,(k-7+r)+a_k(k+r-5)\,(k+r-6)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k(k+r-5)(k+r-6)}{(k+1+r)(k-7+r)}$
- Recursion relation for $r=0$; series terminates at $k=5$
  $a_{k+1}=\frac{a_k(k-5)(k-6)}{(k+1)(k-7)}$
- Apply recursion relation for $k=0$
  $a_1=-\frac{30a_0}{7}$
- Apply recursion relation for $k=1$
  $a_2=-\frac{5a_1}{3}$
- Express in terms of a_0
  $a_2=\frac{50a_0}{7}$
- Apply recursion relation for $k=2$
  $a_3=-\frac{4a_2}{5}$
- Express in terms of a_0
  $a_3=-\frac{40a_0}{7}$
- Apply recursion relation for $k=3$
  $a_4=-\frac{3a_3}{8}$
- Express in terms of a_0
  $a_4=\frac{15a_0}{7}$
- Apply recursion relation for $k=4$
  $a_5=-\frac{2a_4}{15}$
- Express in terms of a_0
  $a_5=-\frac{2a_0}{7}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li
  $y(u)=a_0\cdot\left(1-\frac{30}{7}u+\frac{50}{7}u^2-\frac{40}{7}u^3+\frac{15}{7}u^4-\frac{2}{7}u^5\right)$
- Revert the change of variables $u=1+x$

$$\left[y = a_0\left(\frac{5}{7}x^4 - \frac{2}{7}x^5\right)\right]$$

- Recursion relation for $r = 8$

$$a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}$$

- Solution for $r = 8$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+8}, a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+8}, a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0\left(\frac{5}{7}x^4 - \frac{2}{7}x^5\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k+8}\right), b_{k+1} = \frac{b_k(k+3)(k+2)}{(k+9)(k+1)}\right]$$

1.172.3 Maple trace

```
Methods for second order ODEs:
```

1.172.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 65

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)-x*(3+10*x)*diff(y(x),x)+30*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = 3c_2\left(x - \frac{5}{2}\right)x^4 \ln(x) + \frac{c_2 x^6}{4} + \frac{(16c_1 - 5c_2)x^5}{8} + \frac{(-80c_1 - 299c_2)x^4}{16} + 5c_2 x^3 + \frac{5c_2 x^2}{4} + \frac{c_2 x}{4} + \frac{c_2}{40}$$

1.172.5 Mathematica DSolve solution

Solving time : 0.109 (sec)
Leaf size : 68

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-x*(3+10*x)*D[y[x],x]+30*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1\left(x^5 - \frac{5x^4}{2}\right) + \frac{1}{20}c_2\left(20x^6 - 50x^5 - 1495x^4 + 120(2x-5)x^4\log(x) + 400x^3 + 100x^2 + 20x + 2\right)$$

1.173 problem 175

Internal problem ID [8311]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 175
Date solved : `Monday, October 21, 2024 at 05:05:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(1+x)\,y' - 3(3+x)\,y = 0$$

1.173.1 Solved as second order ode using Kovacic algorithm

Time used: 0.400 (sec)

Writing the ode as

$$x^2y'' + \left(x^2+x\right)y' + (-3x-9)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 + x \\ C &= -3x - 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 14x + 35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 14x + 35 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 14x + 35}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 330: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{7}{2x} + \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{7}{2x} - \frac{7}{2x^2} + \frac{49}{2x^3} - \frac{735}{4x^4} + \frac{5831}{4x^5} - \frac{48363}{4x^6} + \frac{415373}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 14x + 35}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{14x + 35}{4x^2}\right) \\ &= \frac{1}{4} + \frac{14x + 35}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 14. Dividing this by leading coefficient in t which is 4 gives $\frac{7}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{7}{2}\right) - (0)\\ &= \frac{7}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2}\\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{7}{2}}{\frac{1}{2}} - 0\right) &&= \frac{7}{2}\\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{7}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{7}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 14x + 35}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{7}{2}$	$-\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - (\alpha_{c_1}^+)\\ &= \frac{7}{2} - \left(\frac{7}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{2x} + \left(\frac{1}{2} \right) \\
&= \frac{1}{2} + \frac{7}{2x} \\
&= \frac{x + 7}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r \right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2} + \frac{7}{2x} \right)(0) + \left(\left(-\frac{7}{2x^2} \right) + \left(\frac{1}{2} + \frac{7}{2x} \right)^2 - \left(\frac{x^2 + 14x + 35}{4x^2} \right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{7}{2x} \right) dx} \\
&= x^{7/2} \mathrm{e}^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 + x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-x}}{6x^6} + \frac{\mathrm{e}^{-x}}{30x^5} - \frac{\mathrm{e}^{-x}}{120x^4} + \frac{\mathrm{e}^{-x}}{360x^3} - \frac{\mathrm{e}^{-x}}{720x^2} + \frac{\mathrm{e}^{-x}}{720x} - \frac{\mathrm{Ei}_1(x)}{720}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^3) + c_2 \left(x^3 \left(-\frac{\mathrm{e}^{-x}}{6x^6} + \frac{\mathrm{e}^{-x}}{30x^5} - \frac{\mathrm{e}^{-x}}{120x^4} + \frac{\mathrm{e}^{-x}}{360x^3} - \frac{\mathrm{e}^{-x}}{720x^2} + \frac{\mathrm{e}^{-x}}{720x} - \frac{\mathrm{Ei}_1(x)}{720}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.173.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + x(1+x)\, y' - 3(3+x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{3(3+x)y}{x^2} - \frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{(1+x)y'}{x} - \frac{3(3+x)y}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{1+x}{x}, P_3(x) = -\frac{3(3+x)}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -9$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x(1+x)\,y' + (-3x-9)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

    Rewrite ODE with series expansions

$$a_0(3+r)(-3+r)x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+3)(k+r-3)+a_{k-1}(k-4+r)\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(3+r)(-3+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{-3,3\}$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+3)(k+r-3)+a_{k-1}(k-4+r)=0$
- Shift index using $k->k+1$
  $a_{k+1}(k+4+r)(k-2+r)+a_k(k+r-3)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k(k+r-3)}{(k+4+r)(k-2+r)}$
- Recursion relation for $r=-3$; series terminates at $k=6$
  $a_{k+1}=-\frac{a_k(k-6)}{(k+1)(k-5)}$
- Series not valid for $r=-3$, division by 0 in the recursion relation at $k=5$
  $a_{k+1}=-\frac{a_k(k-6)}{(k+1)(k-5)}$
- Recursion relation for $r=3$
  $a_{k+1}=-\frac{a_k k}{(k+7)(k+1)}$
- Solution for $r=3$
  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+3},a_{k+1}=-\frac{a_k k}{(k+7)(k+1)}\right]$

1.173.3 Maple trace

```
Methods for second order ODEs:
```

1.173.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 50

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(1+x)*diff(y(x),x)-3*(3+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{(x^5-x^4+2x^3-6x^2+24x-120)\,c_2\,\mathrm{e}^{-x}+x^6(-c_2\,\mathrm{Ei}_1(x)+c_1)}{x^3}$$

1.173.5 Mathematica DSolve solution

Solving time : `0.222` (sec)
Leaf size : 60

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1+x)*D[y[x],x]-3*(3+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 e^{-x}(e^x x^6 \,\text{ExpIntegralEi}(-x) + x^5 - x^4 + 2x^3 - 6x^2 + 24x - 120)}{720x^3} + c_1 x^3$$

1.174 problem 176

Internal problem ID [8312]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 176
Date solved : `Monday, October 21, 2024 at 05:05:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+2x)\,y''+x(9+13x)\,y'+(7+5x)\,y=0$$

1.174.1 Solved as second order ode using Kovacic algorithm

Time used: 0.297 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(13x^2+9x\right)y'+(7+5x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+x^2\\ B&=13x^2+9x\\ C&=7+5x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 77x^2 + 86x + 35$$
$$t = 4\left(2x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 332: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{45}{16\left(x + \frac{1}{2}\right)^2} + \frac{27}{2\left(x + \frac{1}{2}\right)} - \frac{27}{2x} + \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{77}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-\frac{1}{2}$	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{7}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\ &= -\frac{7}{4} - \left(-\frac{15}{4}\right)\\ &= 2\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)} + (-)\,(0) \\
&= -\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)} \\
&= \frac{-5-15x}{4x^2+2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right)(2x+a_1) + \left(\left(\frac{5}{2x^2} + \frac{5}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{77x^2+86x \cdots}{4\left(2x^2+\cdots\right.}\right.\right.
$$

$$
\frac{(11a_1-8)\,x + 26a_0 \cdots}{2x^2+x}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{20}{143}, a_1 = \frac{8}{11}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + \frac{8}{11}x + \frac{20}{143}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + \frac{8}{11}x + \frac{20}{143}\right) \mathrm{e}^{\int \left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right) dx} \\
&= \left(x^2 + \frac{8}{11}x + \frac{20}{143}\right) \mathrm{e}^{-\frac{5\ln(x)}{2} - \frac{5\ln(1+2x)}{4}} \\
&= \frac{x^2 + \frac{8}{11}x + \frac{20}{143}}{x^{5/2}\left(1+2x\right)^{5/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{13x^2+9x}{2x^3+x^2}\,dx} \\
&= z_1 e^{-\frac{9\ln(x)}{2}+\frac{5\ln(1+2x)}{4}} \\
&= z_1 \left(\frac{(1+2x)^{5/4}}{x^{9/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2+\frac{8}{11}x+\frac{20}{143}}{x^7}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{13x^2+9x}{2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-9\ln(x)+\frac{5\ln(1+2x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{143(1+2x)\left(35x^3-45x^2+36x-20\right)x^9 \mathrm{e}^{-9\ln(x)+\frac{5\ln(1+2x)}{2}}}{315\left(143x^2+104x+20\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2+\frac{8}{11}x+\frac{20}{143}}{x^7}\right) \\
&\quad + c_2\left(\frac{x^2+\frac{8}{11}x+\frac{20}{143}}{x^7}\left(\frac{143(1+2x)\left(35x^3-45x^2+36x-20\right)x^9 \mathrm{e}^{-9\ln(x)+\frac{5\ln(1+2x)}{2}}}{315\left(143x^2+104x+20\right)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.174.2 Maple step by step solution

Let's solve

$x^2(1+2x)\left(\frac{d}{dx}y'\right)+x(9+13x)\,y'+(7+5x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(7+5x)y}{x^2(1+2x)}-\frac{(9+13x)y'}{x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(9+13x)y'}{x(1+2x)}+\frac{(7+5x)y}{x^2(1+2x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{9+13x}{x(1+2x)},P_3(x)=\frac{7+5x}{x^2(1+2x)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=9$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=7$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(1+2x)\left(\frac{d}{dx}y'\right)+x(9+13x)\,y'+(7+5x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$x^{m} \cdot y' = \sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m} \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m} \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

  $x^{m} \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m} \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(7+r)\,(1+r)\, x^{r} + \left(\sum_{k=1}^{\infty} \left(a_{k}(k+r+7)\,(k+r+1) + a_{k-1}(k+4+r)\,(2k-1+2r)\right) x^{k+r}\right) =$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(7+r)\,(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-7, -1\}$
- Each term in the series must be 0, giving the recursion relation

  $2\left(k-\frac{1}{2}+r\right)(k+4+r)\, a_{k-1} + a_{k}(k+r+7)\,(k+r+1) = 0$
- Shift index using $k->k+1$

  $2\left(k+\frac{1}{2}+r\right)(k+r+5)\, a_{k} + a_{k+1}(k+8+r)\,(k+2+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{(2k+2r+1)(k+r+5)a_{k}}{(k+8+r)(k+2+r)}$
- Recursion relation for $r = -7$; series terminates at $k = 2$

  $a_{k+1} = -\frac{(2k-13)(k-2)a_{k}}{(k+1)(k-5)}$
- Apply recursion relation for $k = 0$

  $a_{1} = \frac{26a_{0}}{5}$
- Apply recursion relation for $k = 1$

  $a_{2} = \frac{11a_{1}}{8}$
- Express in terms of a_{0}

  $a_{2} = \frac{143a_{0}}{20}$
- Terminating series solution of the ODE for $r = -7$. Use reduction of order to find the second

  $y = a_{0} \cdot \left(\frac{143}{20}x^{2} + \frac{26}{5}x + 1\right)$

- Recursion relation for $r=-1$
  $a_{k+1}=-\frac{(2k-1)(k+4)a_k}{(k+7)(k+1)}$
- Solution for $r=-1$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1}=-\frac{(2k-1)(k+4)a_k}{(k+7)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y=a_0\cdot\left(\frac{143}{20}x^2+\frac{26}{5}x+1\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), b_{k+1}=-\frac{(2k-1)(k+4)b_k}{(k+7)(k+1)}\right]$

1.174.3 Maple trace

```
Methods for second order ODEs:
```

1.174.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 50

```
dsolve(x^2*(1+2*x)*diff(diff(y(x),x),x)+x*(9+13*x)*diff(y(x),x)+(7+5*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{280\left(x^3-\frac{9}{7}x^2+\frac{36}{35}x-\frac{4}{7}\right)\left(x+\frac{1}{2}\right)^3 c_2\sqrt{1+2x}+143c_1x^2+104c_1x+20c_1}{x^7}$$

1.174.5 Mathematica DSolve solution

Solving time : 1.643 (sec)
Leaf size : 58

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]+x*(9+13*x)*D[y[x],x]+(7+5*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{c_1(13x(11x+8)+20)}{143x^7}+\frac{c_2(35x^3-45x^2+36x-20)(2x+1)^{7/2}}{315x^7}$$

1.175 problem 177

Internal problem ID [8313]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 177
Date solved : `Monday, October 21, 2024 at 05:05:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+2x)\,y''-2x(4-x)\,y'-(7+5x)\,y=0$$

1.175.1 Solved as second order ode using Kovacic algorithm

Time used: 0.255 (sec)

Writing the ode as

$$\left(8x^3+4x^2\right)y''+\left(2x^2-8x\right)y'+(-5x-7)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=8x^3+4x^2\\B&=2x^2-8x\\C&=-5x-7\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{33x^2 + 132x + 60}{16(2x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 33x^2 + 132x + 60$$
$$t = 16(2x^2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{33x^2 + 132x + 60}{16(2x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 334: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{27}{4x} + \frac{15}{4x^2} + \frac{9}{64\left(x + \frac{1}{2}\right)^2} + \frac{27}{4\left(x + \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}
\end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{9}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{33x^2 + 132x + 60}{16 \left(2x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{11}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{33x^2 + 132x + 60}{16 \left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
$-\frac{1}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= -\frac{3}{8} - \left(-\frac{3}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)} + (-)(0) \\
&= -\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)} \\
&= -\frac{3(x+2)}{4x(1+2x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{9}{8\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{33x^2 + 132x + 6}{16\left(2x^2+x\right)^2}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right) dx} \\
&= \frac{(1+2x)^{9/8}}{x^{3/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2-8x}{8x^3+4x^2} \, dx} \\
&= z_1 e^{-\frac{9\ln(1+2x)}{8} + \ln(x)} \\
&= z_1 \left(\frac{x}{(1+2x)^{9/8}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2-8x}{8x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{9\ln(1+2x)}{4}+2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2(1+2x)\left(5x^3-10x^2-40x-16\right) e^{-\frac{9\ln(1+2x)}{4}+2\ln(x)}}{35x^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{\sqrt{x}}\right) + c_2\left(\frac{1}{\sqrt{x}}\left(\frac{2(1+2x)\left(5x^3-10x^2-40x-16\right) e^{-\frac{9\ln(1+2x)}{4}+2\ln(x)}}{35x^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.175.2 Maple step by step solution

Let's solve

$4x^2(1+2x)\left(\frac{d}{dx}y'\right) - 2x(4-x)\,y' - (7+5x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(7+5x)y}{4x^2(1+2x)} - \frac{(x-4)y'}{2x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(x-4)y'}{2x(1+2x)} - \frac{(7+5x)y}{4x^2(1+2x)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{x-4}{2x(1+2x)}, P_3(x) = -\frac{7+5x}{4x^2(1+2x)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{7}{4}$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $4x^2(1+2x)\left(\frac{d}{dx}y'\right) + 2x(x-4)\,y' + (-5x-7)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)(-7+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)(2k+2r-7) + a_{k-1}(2k-1+2r)(4k-9+4r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-7+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{7}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $8\left(k-\frac{9}{4}+r\right)\left(k-\frac{1}{2}+r\right) a_{k-1} + 4\left(k+r+\frac{1}{2}\right) a_k \left(k+r-\frac{7}{2}\right) = 0$
- Shift index using $k- >k+1$

  $8\left(k-\frac{5}{4}+r\right)\left(k+r+\frac{1}{2}\right) a_k + 4\left(k+\frac{3}{2}+r\right) a_{k+1}\left(k-\frac{5}{2}+r\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{(4k+4r-5)(2k+2r+1)a_k}{(2k+3+2r)(2k-5+2r)}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{2(4k-7)ka_k}{(2k+2)(2k-6)}$
- Series not valid for $r = -\frac{1}{2}$, division by 0 in the recursion relation at $k = 3$

  $a_{k+1} = -\frac{2(4k-7)ka_k}{(2k+2)(2k-6)}$
- Recursion relation for $r = \frac{7}{2}$

  $a_{k+1} = -\frac{(4k+9)(2k+8)a_k}{(2k+10)(2k+2)}$
- Solution for $r = \frac{7}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{7}{2}}, a_{k+1} = -\frac{(4k+9)(2k+8)a_k}{(2k+10)(2k+2)}\right]$

1.175.3 Maple trace

```
Methods for second order ODEs:
```

1.175.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 34

```
dsolve(4*x^2*(1+2*x)*diff(diff(y(x),x),x)-2*x*(4-x)*diff(y(x),x)-(7+5*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 + \frac{c_2(5x^3-10x^2-40x-16)}{(1+2x)^{5/4}}}{\sqrt{x}}$$

1.175.5 Mathematica DSolve solution

Solving time : 0.132 (sec)
Leaf size : 47

```
DSolve[{4*x^2*(1+2*x)*D[y[x],{x,2}]-2*x*(4-x)*D[y[x],x]-(7+5*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\frac{2c_2(5x^3-10x^2-40x-16)}{(2x+1)^{5/4}} + 35c_1}{35\sqrt{x}}$$

1.176 problem 178

Internal problem ID [8314]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 178
Date solved : `Monday, October 21, 2024 at 05:05:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2(3+x)\,y''-x(15+x)\,y'-20y=0$$

1.176.1 Solved as second order ode using Kovacic algorithm

Time used: 0.273 (sec)

Writing the ode as

$$\left(3x^3+9x^2\right)y''+\left(-x^2-15x\right)y'-20y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=3x^3+9x^2\\B&=-x^2-15x\\C&=-20\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 450x + 1215}{36\left(x^2 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 7x^2 + 450x + 1215$$
$$t = 36\left(x^2 + 3x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 450x + 1215}{36\left(x^2 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 336: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9(3+x)^2} - \frac{10}{9x} + \frac{10}{9(3+x)} + \frac{15}{4x^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(3+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7x^2 + 450x + 1215}{36(x^2 + 3x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{6}\\ \alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{6}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{7x^2+450x+1215}{36\left(x^2+3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-3	2	0	$\frac{2}{3}$	$\frac{1}{3}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-\frac{1}{6}$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\ &= -\frac{1}{6}-\left(-\frac{7}{6}\right)\\ &= 1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{9+3x} - \frac{3}{2x} + (-)\,(0) \\
&= \frac{1}{9+3x} - \frac{3}{2x} \\
&= -\frac{7x+27}{6x\,(3+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{9+3x} - \frac{3}{2x}\right)(1) + \left(\left(-\frac{1}{3\,(3+x)^2} + \frac{3}{2x^2}\right) + \left(\frac{1}{9+3x} - \frac{3}{2x}\right)^2 - \left(\frac{7x^2+450x+1215}{36\,(x^2+3x)^2}\right)\right) = \frac{-27+7a_0}{3x\,(3+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{27}{7}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{27}{7}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x + \frac{27}{7}\right) e^{\int \left(\frac{1}{9+3x} - \frac{3}{2x}\right) dx} \\
&= \left(x + \frac{27}{7}\right) e^{\frac{\ln(3+x)}{3} - \frac{3\ln(x)}{2}} \\
&= \frac{\left(x + \frac{27}{7}\right)(3+x)^{1/3}}{x^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-15x}{3x^3+9x^2}\,dx} \\
&= z_1 e^{-\frac{2\ln(3+x)}{3}+\frac{5\ln(x)}{6}} \\
&= z_1\left(\frac{x^{5/6}}{(3+x)^{2/3}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{7x+27}{7\,(3+x)^{1/3}\,x^{2/3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-15x}{3x^3+9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{4\ln(3+x)}{3}+\frac{5\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{21(3+x)^{5/3}\,(x^2-36x-243)\,\mathrm{e}^{-\frac{4\ln(3+x)}{3}+\frac{5\ln(x)}{3}}}{4\,(7x+27)\,x^{5/3}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{7x+27}{7\,(3+x)^{1/3}\,x^{2/3}}\right) \\
&\quad + c_2\left(\frac{7x+27}{7\,(3+x)^{1/3}\,x^{2/3}}\left(\frac{21(3+x)^{5/3}\,(x^2-36x-243)\,\mathrm{e}^{-\frac{4\ln(3+x)}{3}+\frac{5\ln(x)}{3}}}{4\,(7x+27)\,x^{5/3}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.176.2 Maple step by step solution

Let's solve

$3x^2(3+x)\left(\frac{d}{dx}y'\right)-x(15+x)\,y'-20y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{20y}{3x^2(3+x)}+\frac{(15+x)y'}{3x(3+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{(15+x)y'}{3x(3+x)}-\frac{20y}{3x^2(3+x)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{15+x}{3x(3+x)},P_3(x)=-\frac{20}{3x^2(3+x)}\right]$

- ◦ $(3+x)\cdot P_2(x)$ is analytic at $x=-3$

$\left((3+x)\cdot P_2(x)\right)\bigg|_{x=-3}=\frac{4}{3}$

- ◦ $(3+x)^2\cdot P_3(x)$ is analytic at $x=-3$

$\left((3+x)^2\cdot P_3(x)\right)\bigg|_{x=-3}=0$

- ◦ $x=-3$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-3$

- Multiply by denominators

$3x^2(3+x)\left(\frac{d}{dx}y'\right)-x(15+x)\,y'-20y=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

$(3u^3-18u^2+27u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-u^2-9u+36)\left(\frac{d}{du}y(u)\right)-20y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..3$

$$u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$9a_{0}r(1+3r)\,u^{-1+r}+\left(9a_{1}(1+r)\,(4+3r)-a_{0}(18r^{2}-9r+20)\right)u^{r}+\left(\sum_{k=1}^{\infty}(9a_{k+1}(k+1+r)\,(3k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$9r(1+3r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{1}{3}\right\}$$

- Each term must be 0

$$9a_{1}(1+r)\,(4+3r)-a_{0}(18r^{2}-9r+20)=0$$

- Each term in the series must be 0, giving the recursion relation

$$3(-6a_{k}+a_{k-1}+9a_{k+1})\,k^{2}+\left(6(-6a_{k}+a_{k-1}+9a_{k+1})\,r+9a_{k}-10a_{k-1}+63a_{k+1}\right)k+3(-6a_{k}+$$

- Shift index using $k->k+1$

$$3(-6a_{k+1}+a_{k}+9a_{k+2})\,(k+1)^{2}+\left(6(-6a_{k+1}+a_{k}+9a_{k+2})\,r+9a_{k+1}-10a_{k}+63a_{k+2}\right)(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{3k^{2}a_{k}-18k^{2}a_{k+1}+6kra_{k}-36kra_{k+1}+3r^{2}a_{k}-18r^{2}a_{k+1}-4ka_{k}-27ka_{k+1}-4ra_{k}-27ra_{k+1}-29a_{k+1}}{9(3k^{2}+6kr+3r^{2}+13k+13r+14)}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{3k^{2}a_{k}-18k^{2}a_{k+1}-4ka_{k}-27ka_{k+1}-29a_{k+1}}{9(3k^{2}+13k+14)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k},a_{k+2}=-\frac{3k^{2}a_{k}-18k^{2}a_{k+1}-4ka_{k}-27ka_{k+1}-29a_{k+1}}{9(3k^{2}+13k+14)},36a_{1}-20a_{0}=0\right]$$

- Revert the change of variables $u=3+x$

$$\left[y=\sum_{k=0}^{\infty}a_{k}(3+x)^{k},a_{k+2}=-\frac{3k^{2}a_{k}-18k^{2}a_{k+1}-4ka_{k}-27ka_{k+1}-29a_{k+1}}{9(3k^{2}+13k+14)},36a_{1}-20a_{0}=0\right]$$

- Recursion relation for $r=-\frac{1}{3}$

$$a_{k+2}=-\frac{3k^{2}a_{k}-18k^{2}a_{k+1}-6ka_{k}-15ka_{k+1}+\frac{5}{3}a_{k}-22a_{k+1}}{9(3k^{2}+11k+10)}$$

- Solution for $r=-\frac{1}{3}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{1}{3}}, a_{k+2}=-\frac{3k^2 a_k-18k^2 a_{k+1}-6k a_k-15k a_{k+1}+\frac{5}{3}a_k-22a_{k+1}}{9(3k^2+11k+10)}, 18a_1-25a_0=0\right]$$

- Revert the change of variables $u=3+x$

$$\left[y=\sum_{k=0}^{\infty} a_k(3+x)^{k-\frac{1}{3}}, a_{k+2}=-\frac{3k^2 a_k-18k^2 a_{k+1}-6k a_k-15k a_{k+1}+\frac{5}{3}a_k-22a_{k+1}}{9(3k^2+11k+10)}, 18a_1-25a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k(3+x)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k(3+x)^{k-\frac{1}{3}}\right), a_{k+2}=-\frac{3k^2 a_k-18k^2 a_{k+1}-4k a_k-27k a_{k+1}-29a_{k+1}}{9(3k^2+13k+14)}, 36a_1 -$$

1.176.3 Maple trace

```
Methods for second order ODEs:
```

1.176.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 31

```
dsolve(3*x^2*(3+x)*diff(diff(y(x),x),x)-x*(15+x)*diff(y(x),x)-20*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1(x^2-36x-243)+\frac{c_2(7x+27)}{(3+x)^{1/3}}}{x^{2/3}}$$

1.176.5 Mathematica DSolve solution

Solving time : 0.287 (sec)
Leaf size : 43

```
DSolve[{3*x^2*(3+x)*D[y[x],{x,2}]-x*(15+x)*D[y[x],x]-20*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{21c_2(x^2-36x-243)+\frac{4c_1(7x+27)}{\sqrt[3]{x+3}}}{28x^{2/3}}$$

1.177 problem 179

Internal problem ID [8315]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 179
Date solved : `Monday, October 21, 2024 at 05:05:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+x(1-10x)\,y'-(9-10x)\,y=0$$

1.177.1 Solved as second order ode using Kovacic algorithm

Time used: 0.297 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(-10x^2+x\right)y'+(10x-9)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\B&=-10x^2+x\\C&=10x-9\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{80x^2 - 28x + 35}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 80x^2 - 28x + 35$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{80x^2 - 28x + 35}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 338: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{49}{2x} + \frac{49}{2(1+x)} + \frac{35}{4x^2} + \frac{143}{4(1+x)^2}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{143}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{11}{2}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{80x^2 - 28x + 35}{4(x^2 + x)^2}
$$

Since the gcd$(s,t)=1$. This gives $b=20$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 5 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -4
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{80x^2 - 28x + 35}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{13}{2}$	$-\frac{11}{2}$
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	5	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=5$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-}\right) \\
&= 5 - (4) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{13}{2(1+x)} - \frac{5}{2x} + (0) \\
&= \frac{13}{2(1+x)} - \frac{5}{2x} \\
&= \frac{8x - 5}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{13}{2(1+x)} - \frac{5}{2x}\right)(1) + \left(\left(-\frac{13}{2(1+x)^2} + \frac{5}{2x^2}\right) + \left(\frac{13}{2(1+x)} - \frac{5}{2x}\right)^2 - \left(\frac{80x^2 - 28x + 35}{4(x^2+x)^2}\right)\right) = \frac{-5 - 8a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{5}{8}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{5}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{5}{8}\right) e^{\int \left(\frac{13}{2(1+x)} - \frac{5}{2x}\right) dx} \\
&= \left(x - \frac{5}{8}\right) e^{\frac{13\ln(1+x)}{2} - \frac{5\ln(x)}{2}} \\
&= \frac{\left(x - \frac{5}{8}\right)(1+x)^{13/2}}{x^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10x^2+x}{x^2(1+x)}\,dx} \\
&= z_1 e^{\frac{11\ln(1+x)}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{(1+x)^{11/2}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(1+x)^{12}\left(x-\frac{5}{8}\right)}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x^2+x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{11\ln(1+x)-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{8\,\mathrm{e}^{11\ln(1+x)-\ln(x)} x(715x^4+572x^3+234x^2+52x+5)}{6435\,(8x-5)\,(1+x)^{23}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(1+x)^{12}\left(x-\frac{5}{8}\right)}{x^3}\right) \\
&\quad + c_2 \left(\frac{(1+x)^{12}\left(x-\frac{5}{8}\right)}{x^3}\left(-\frac{8\,\mathrm{e}^{11\ln(1+x)-\ln(x)} x(715x^4+572x^3+234x^2+52x+5)}{6435\,(8x-5)\,(1+x)^{23}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.177.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right)+x(1-10x)\,y'-(9-10x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(10x-9)y}{x^2(1+x)}+\frac{(10x-1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{(10x-1)y'}{x(1+x)}+\frac{(10x-9)y}{x^2(1+x)}=0$

- ☐ Check to see if x_0 is a regular singular point
  - ○ Define functions

    $\left[P_2(x)=-\frac{10x-1}{x(1+x)},P_3(x)=\frac{10x-9}{x^2(1+x)}\right]$

  - ○ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=-11$

  - ○ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  - ○ $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\left(\frac{d}{dx}y'\right)-x(10x-1)\,y'+(10x-9)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-10u^2+21u-11)\left(\frac{d}{du}y(u)\right)+(10u-19)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  - ○ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-12+r)\, u^{-1+r} + \left(a_1(1+r)\,(-11+r) - a_0(2r^2-23r+19)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-12+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 12\}$

- Each term must be 0

  $a_1(1+r)\,(-11+r) - a_0(2r^2-23r+19) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-2a_k + a_{k-1} + a_{k+1})\, k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1})\, r + 23a_k - 13a_{k-1} - 10a_{k+1})\, k + (-2a_k + a$

- Shift index using $k- >k+1$

  $(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2})\, r + 23a_{k+1} - 13a_k - 10a_{k+2})\,(k+1)$ -

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2kra_k - 4kra_{k+1} + r^2a_k - 2r^2a_{k+1} - 11ka_k + 19ka_{k+1} - 11ra_k + 19ra_{k+1} + 10a_k + 2a_{k+1}}{k^2+2kr+r^2-8k-8r-20}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 11ka_k + 19ka_{k+1} + 10a_k + 2a_{k+1}}{k^2-8k-20}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 10$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 11ka_k + 19ka_{k+1} + 10a_k + 2a_{k+1}}{k^2-8k-20}$

- Recursion relation for $r = 12$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 13k a_k - 29k a_{k+1} + 22a_k - 58a_{k+1}}{k^2+16k+28}$$

- Solution for $r = 12$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+12}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 13k a_k - 29k a_{k+1} + 22a_k - 58a_{k+1}}{k^2+16k+28}, 13a_1 - 31a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+12}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 13k a_k - 29k a_{k+1} + 22a_k - 58a_{k+1}}{k^2+16k+28}, 13a_1 - 31a_0 = 0\right]$$

1.177.3 Maple trace

```
Methods for second order ODEs:
```

1.177.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 82

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)+x*(1-10*x)*diff(y(x),x)-(9-10*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{8c_2 x^{13} + 91c_2 x^{12} + 468c_2 x^{11} + 1430c_2 x^{10} + 2860c_2 x^9 + 3861c_2 x^8 + 3432c_2 x^7 + 1716c_2 x^6 + 715c_1 x^4}{x^3}$$

1.177.5 Mathematica DSolve solution

Solving time : `0.136` (sec)
Leaf size : 51

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+x*(1-10*x)*D[y[x],x]-(9-10*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{6435c_1(x+1)^{12}(8x-5) - 8c_2(715x^4 + 572x^3 + 234x^2 + 52x + 5)}{51480x^3}$$

1.178 problem 180

Internal problem ID [8316]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 180
Date solved : `Monday, October 21, 2024 at 05:05:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y'' + 3x^2y' - (6-x)\,y = 0$$

1.178.1 Solved as second order ode using Kovacic algorithm

Time used: 0.258 (sec)

Writing the ode as

$$x^2(1+x)\,y'' + 3x^2y' + (x-6)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= 3x^2 \\ C &= x-6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 20x + 24}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 20x + 24$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 20x + 24}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 340: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{7}{1+x} + \frac{3}{4(1+x)^2} - \frac{7}{x} + \frac{6}{x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 20x + 24}{4(x^2+x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^2+20x+24}{4\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{3}{2(1+x)} - \frac{2}{x} + (-)(0) \\
&= \frac{3}{2(1+x)} - \frac{2}{x} \\
&= -\frac{x+4}{2x(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{2(1+x)} - \frac{2}{x}\right)(1) + \left(\left(-\frac{3}{2(1+x)^2} + \frac{2}{x^2}\right) + \left(\frac{3}{2(1+x)} - \frac{2}{x}\right)^2 - \left(\frac{-x^2+20x+24}{4(x^2+x)^2}\right)\right) = 0
$$

$$
\frac{-4+a_0}{x(1+x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 4\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 4
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x+4)\, e^{\int \left(\frac{3}{2(1+x)} - \frac{2}{x}\right) dx} \\
&= (x+4)\, e^{\frac{3\ln(1+x)}{2} - 2\ln(x)} \\
&= \frac{(x+4)(1+x)^{3/2}}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^2}{x^2(1+x)}\,dx} \\
&= z_1 e^{-\frac{3\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{(1+x)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x+4}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{256}{27\,(x+4)} + \ln(1+x) - \frac{1}{18\,(1+x)^2} + \frac{14}{27\,(1+x)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x+4}{x^2}\right) + c_2\left(\frac{x+4}{x^2}\left(\frac{256}{27\,(x+4)} + \ln(1+x) - \frac{1}{18\,(1+x)^2} + \frac{14}{27\,(1+x)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.178.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right)+3x^2y'-(6-x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(x-6)y}{x^2(1+x)}-\frac{3y'}{1+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{3y'}{1+x}+\frac{(x-6)y}{x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{3}{1+x},P_3(x)=\frac{x-6}{x^2(1+x)}\right]$

- ○ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=3$

- ○ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\left(\frac{d}{dx}y'\right)+3x^2y'+(x-6)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(3u^2-6u+3)\left(\frac{d}{du}y(u)\right)+(u-7)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_ku^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_ku^{k+r+m}$

- ○ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r)\, u^{-1+r} + \left(a_1(1+r)\,(3+r) - a_0(2r^2+4r+7)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+3+r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-2, 0\}$$

- Each term must be 0

$$a_1(1+r)\,(3+r) - a_0(2r^2+4r+7) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k-1}(k+r)^2 + a_{k+1}(k+r+1)\,(k+3+r) - 2\left(k^2+(2r+2)\,k+r^2+2r+\frac{7}{2}\right) a_k = 0$$

- Shift index using $k- >k+1$

$$a_k(k+r+1)^2 + a_{k+2}(k+r+2)\,(k+4+r) - 2\left((k+1)^2+(2r+2)\,(k+1)+r^2+2r+\frac{7}{2}\right) a_{k+1}$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}+2kra_k-4kra_{k+1}+r^2a_k-2r^2a_{k+1}+2ka_k-8ka_{k+1}+2ra_k-8ra_{k+1}+a_k-13a_{k+1}}{(k+r+2)\,(k+4+r)}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-2ka_k+a_k-5a_{k+1}}{k(k+2)}$$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-2ka_k+a_k-5a_{k+1}}{k(k+2)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2k a_k - 8k a_{k+1} + a_k - 13 a_{k+1}}{(k+2)(k+4)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2k a_k - 8k a_{k+1} + a_k - 13 a_{k+1}}{(k+2)(k+4)}, 3a_1 - 7a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2k a_k - 8k a_{k+1} + a_k - 13 a_{k+1}}{(k+2)(k+4)}, 3a_1 - 7a_0 = 0\right]$$

1.178.3 Maple trace

```
Methods for second order ODEs:
```

1.178.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 45

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)+3*x^2*diff(y(x),x)-(6-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1(x+4) + \frac{c_2\left(6(x+4)(1+x)^2 \ln(1+x) + 60x^2 + 129x + 68\right)}{(1+x)^2}}{x^2}$$

1.178.5 Mathematica DSolve solution

Solving time : `0.118` (sec)
Leaf size : 49

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+3*x^2*D[y[x],x]-(6-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\frac{c_2(60x^2+129x+68)}{(x+1)^2} + 6c_1(x+4) + 6c_2(x+4)\log(x+1)}{6x^2}$$

1.179 problem 181

Internal problem ID [8317]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 181
Date solved : `Monday, October 21, 2024 at 05:05:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+2x)\,y''-2x(3+14x)\,y'+(6+100x)\,y=0$$

1.179.1 Solved as second order ode using Kovacic algorithm

Time used: 0.257 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(-28x^2-6x\right)y'+(6+100x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+x^2\\B&=-28x^2-6x\\C&=6+100x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{24x^2 - 16x + 6}{(2x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 24x^2 - 16x + 6 \\ t &= (2x^2 + x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{24x^2 - 16x + 6}{(2x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 342: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{40}{x} + \frac{6}{x^2} + \frac{20}{\left(x + \frac{1}{2}\right)^2} + \frac{40}{x + \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2\end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 5\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -4\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{24x^2 - 16x + 6}{\left(2x^2 + x\right)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = 6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{24x^2 - 16x + 6}{\left(2x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2
$-\frac{1}{2}$	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 3$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 3 - (3) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + \frac{5}{x + \frac{1}{2}} + (0) \\
&= -\frac{2}{x} + \frac{5}{x + \frac{1}{2}} \\
&= \frac{-2 + 6x}{2x^2 + x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{2}{x} + \frac{5}{x + \frac{1}{2}}\right)(0) + \left(\left(\frac{2}{x^2} - \frac{5}{\left(x + \frac{1}{2}\right)^2}\right) + \left(-\frac{2}{x} + \frac{5}{x + \frac{1}{2}}\right)^2 - \left(\frac{24x^2 - 16x + 6}{\left(2x^2 + x\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{2}{x} + \frac{5}{x + \frac{1}{2}}\right) dx} \\
&= \frac{(1 + 2x)^5}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-28x^2 - 6x}{2x^3 + x^2} \, dx} \\
&= z_1 e^{3 \ln(x) + 4 \ln(1 + 2x)} \\
&= z_1 \left(x^3 (1 + 2x)^4\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x(1+2x)^9$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-28x^2-6x}{2x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{6\ln(x)+8\ln(1+2x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{(2016x^4+672x^3+144x^2+18x+1)\, e^{6\ln(x)+8\ln(1+2x)}}{20160\,(1+2x)^{17}\, x^6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x(1+2x)^9\right) \\
&\quad + c_2 \left(x(1+2x)^9 \left(-\frac{(2016x^4+672x^3+144x^2+18x+1)\, e^{6\ln(x)+8\ln(1+2x)}}{20160\,(1+2x)^{17}\, x^6}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.179.2 Maple step by step solution

Let's solve

$x^2(1+2x)\left(\frac{d}{dx}y'\right) - 2x(3+14x)\, y' + (6+100x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2(3+50x)y}{x^2(1+2x)} + \frac{2(3+14x)y'}{x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{2(3+14x)y'}{x(1+2x)} + \frac{2(3+50x)y}{x^2(1+2x)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2(3+14x)}{x(1+2x)}, P_3(x) = \frac{2(3+50x)}{x^2(1+2x)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -6$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 6$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2(1+2x)\left(\frac{d}{dx}y'\right) - 2x(3+14x)\,y' + (6+100x)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+r)(-6+r)\, x^r + \left(\sum_{k=1}^{\infty} (a_k(k+r-1)(k+r-6) + 2a_{k-1}(k+r-6)(k-11+r))\, x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)(-6+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 6\}$
- Each term in the series must be 0, giving the recursion relation

  $((2k+2r-22)\, a_{k-1} + a_k(k+r-1))(k+r-6) = 0$
- Shift index using $k- >k+1$

  $((2k+2r-20)\, a_k + a_{k+1}(k+r))(k+r-5) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2(k+r-10)a_k}{k+r}$
- Recursion relation for $r = 1$; series terminates at $k = 9$

  $a_{k+1} = -\frac{2(k-9)a_k}{k+1}$
- Recursion relation that defines the terminating series solution of the ODE for $r = 1$

  $\left[y = \sum_{k=0}^{8} a_k x^{k+1}, a_{k+1} = -\frac{2(k-9)a_k}{k+1}\right]$
- Recursion relation for $r = 6$; series terminates at $k = 4$

  $a_{k+1} = -\frac{2(k-4)a_k}{k+6}$
- Apply recursion relation for $k = 0$

  $a_1 = \frac{4a_0}{3}$
- Apply recursion relation for $k = 1$

  $a_2 = \frac{6a_1}{7}$
- Express in terms of a_0

  $a_2 = \frac{8a_0}{7}$
- Apply recursion relation for $k = 2$

  $a_3 = \frac{a_2}{2}$
- Express in terms of a_0

  $a_3 = \frac{4a_0}{7}$
- Apply recursion relation for $k = 3$

  $a_4 = \frac{2a_3}{9}$

- Express in terms of a_0

  $a_4 = \frac{8a_0}{63}$
- Terminating series solution of the ODE for $r = 6$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 + \frac{4}{3}x + \frac{8}{7}x^2 + \frac{4}{7}x^3 + \frac{8}{63}x^4\right)$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{8} a_k x^{k+1}\right) + b_0 \cdot \left(1 + \frac{4}{3}x + \frac{8}{7}x^2 + \frac{4}{7}x^3 + \frac{8}{63}x^4\right), a_{k+1} = -\frac{2(k-9)a_k}{k+1}\right]$

1.179.3 Maple trace

```
Methods for second order ODEs:
```

1.179.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 62

```
dsolve(x^2*(1+2*x)*diff(diff(y(x),x),x)-2*x*(3+14*x)*diff(y(x),x)+(6+100*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = 8c_2\,x^{10} + 36c_2\,x^9 + 72c_2\,x^8 + 84c_2\,x^7 + 63c_2\,x^6 + 2016c_1\,x^5 + 672c_1\,x^4 + 144c_1\,x^3 + 18c_1\,x^2 + c_1 x$$

1.179.5 Mathematica DSolve solution

Solving time : `0.109` (sec)
Leaf size : 44

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]-2*x*(3+14*x)*D[y[x],x]+(6+100*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x(2x+1)^9 - \frac{c_2 x(2016x^4 + 672x^3 + 144x^2 + 18x + 1)}{20160}$$

1.180 problem 182

Internal problem ID [8318]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 182
Date solved : `Monday, October 21, 2024 at 05:05:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y'' - x(6+11x)\,y' + (6+32x)\,y = 0$$

1.180.1 Solved as second order ode using Kovacic algorithm

Time used: 0.277 (sec)

Writing the ode as

$$x^2(1+x)\,y'' + \left(-11x^2-6x\right)y' + (6+32x)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= -11x^2-6x \\ C &= 6+32x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{15x^2 + 4x + 24}{4(x^2 + x)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 15x^2 + 4x + 24 \\
t &= 4(x^2 + x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{15x^2 + 4x + 24}{4(x^2 + x)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 344: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2} + \frac{35}{4(1+x)^2} - \frac{11}{x} + \frac{11}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 + 4x + 24}{4(x^2 + x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{15x^2+4x+24}{4\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}\right) \\
&= \frac{5}{2}-\left(\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{7}{2(1+x)} - \frac{2}{x} + (0) \\
&= \frac{7}{2(1+x)} - \frac{2}{x} \\
&= \frac{3x - 4}{2x(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{7}{2(1+x)} - \frac{2}{x}\right)(1) + \left(\left(-\frac{7}{2(1+x)^2} + \frac{2}{x^2}\right) + \left(\frac{7}{2(1+x)} - \frac{2}{x}\right)^2 - \left(\frac{15x^2 + 4x + 24}{4(x^2 + x)^2}\right)\right) = 0
$$

$$
\frac{-4 - 3a_0}{x(1+x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{4}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{4}{3}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{7}{2(1+x)} - \frac{2}{x}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{\frac{7 \ln(1+x)}{2} - 2\ln(x)} \\
&= \frac{\left(x - \frac{4}{3}\right)(1+x)^{7/2}}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-11x^2-6x}{x^2(1+x)}\,dx} \\
&= z_1 e^{\frac{5\ln(1+x)}{2}+3\ln(x)} \\
&= z_1 \left((1+x)^{5/2}\, x^3\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (1+x)^6\, x \left(x - \frac{4}{3}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-11x^2-6x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(1+x)+6\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{3\,\mathrm{e}^{5\ln(1+x)+6\ln(x)}(35x^3+42x^2+21x+4)}{140\,(3x-4)\,x^6\,(1+x)^{11}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((1+x)^6\, x \left(x-\frac{4}{3}\right)\right) \\
&\quad + c_2\left((1+x)^6\, x\left(x-\frac{4}{3}\right)\left(-\frac{3\,\mathrm{e}^{5\ln(1+x)+6\ln(x)}(35x^3+42x^2+21x+4)}{140\,(3x-4)\,x^6\,(1+x)^{11}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.180.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right)-x(6+11x)\,y'+(6+32x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{2(3+16x)y}{x^2(1+x)}+\frac{(6+11x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{(6+11x)y'}{x(1+x)}+\frac{2(3+16x)y}{x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{6+11x}{x(1+x)},P_3(x)=\frac{2(3+16x)}{x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=-5$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\left(\frac{d}{dx}y'\right)-x(6+11x)\,y'+(6+32x)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-11u^2+16u-5)\left(\frac{d}{du}y(u)\right)+(-26+32u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-6+r)\, u^{-1+r} + \left(a_1(1+r)\,(-5+r) - 2a_0(r^2-9r+13)\right) u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-5\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-6+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 6\}$
- Each term must be 0
  $a_1(1+r)\,(-5+r) - 2a_0(r^2-9r+13) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(-2a_k + a_{k-1} + a_{k+1})\, k^2 + 2((-2a_k + a_{k-1} + a_{k+1})\, r + 9a_k - 7a_{k-1} - 2a_{k+1})\, k + (-2a_k + a_{k-1}$
- Shift index using $k->k+1$
  $(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + 2((-2a_{k+1} + a_k + a_{k+2})\, r + 9a_{k+1} - 7a_k - 2a_{k+2})\,(k+1) + (-$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kr a_k - 4kr a_{k+1} + r^2 a_k - 2r^2 a_{k+1} - 12k a_k + 14k a_{k+1} - 12r a_k + 14r a_{k+1} + 32a_k - 10a_{k+1}}{k^2 + 2kr + r^2 - 2k - 2r - 8}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 12k a_k + 14k a_{k+1} + 32a_k - 10a_{k+1}}{k^2 - 2k - 8}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 4$
  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 12k a_k + 14k a_{k+1} + 32a_k - 10a_{k+1}}{k^2 - 2k - 8}$
- Recursion relation for $r = 6$

$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10k a_{k+1} - 4a_k + 2a_{k+1}}{k^2+10k+16}$

- Solution for $r = 6$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10k a_{k+1} - 4a_k + 2a_{k+1}}{k^2+10k+16}, 7a_1 + 10a_0 = 0\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+6}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10k a_{k+1} - 4a_k + 2a_{k+1}}{k^2+10k+16}, 7a_1 + 10a_0 = 0\right]$

1.180.3 Maple trace

```
Methods for second order ODEs:
```

1.180.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 45

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)-x*(6+11*x)*diff(y(x),x)+(6+32*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = 3c_1 x^8 + 14c_1 x^7 + 21c_1 x^6 + 35c_2 x^4 + 42c_2 x^3 + 21c_2 x^2 + 4c_2 x$$

1.180.5 Mathematica DSolve solution

Solving time : `0.118` (sec)
Leaf size : 45

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-x*(6+11*x)*D[y[x],x]+(6+32*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3}c_1 x(x+1)^6(3x-4) - \frac{1}{140}c_2 x\left(35x^3 + 42x^2 + 21x + 4\right)$$

1.181 problem 183

Internal problem ID [8319]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 183
Date solved : `Monday, October 21, 2024 at 05:05:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y'' + 4x(1+4x)\,y' - (49+27x)\,y = 0$$

1.181.1 Solved as second order ode using Kovacic algorithm

Time used: 0.270 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y'' + \left(16x^2+4x\right)y' + (-27x-49)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+4x^2 \\ B &= 16x^2+4x \\ C &= -27x-49 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35x^2 + 80x + 48}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35x^2 + 80x + 48$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35x^2 + 80x + 48}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 346: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4(1+x)^2} - \frac{4}{x} + \frac{4}{1+x} + \frac{12}{x^2}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{35x^2 + 80x + 48}{4(x^2+x)^2}
$$

Since the gcd$(s,t)=1$. This gives $b=\frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{35x^2+80x+48}{4\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{7}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= \frac{7}{2}-\left(\frac{7}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2(1+x)} + \frac{4}{x} + (0) \\
&= -\frac{1}{2(1+x)} + \frac{4}{x} \\
&= \frac{7x+8}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(1+x)} + \frac{4}{x}\right)(0) + \left(\left(\frac{1}{2(1+x)^2} - \frac{4}{x^2}\right) + \left(-\frac{1}{2(1+x)} + \frac{4}{x}\right)^2 - \left(\frac{35x^2+80x+48}{4(x^2+x)^2}\right)\right) =$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2(1+x)} + \frac{4}{x}\right) dx} \\
&= \frac{x^4}{\sqrt{1+x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{16x^2+4x}{4x^3+4x^2} \, dx} \\
&= z_1 e^{-\frac{3\ln(1+x)}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{(1+x)^{3/2}\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{7/2}}{(1+x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{16x^2+4x}{4x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(1+x)-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(7x+6)(1+x)^3 \mathrm{e}^{-3\ln(1+x)-\ln(x)}}{42x^6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{7/2}}{(1+x)^2}\right) + c_2 \left(\frac{x^{7/2}}{(1+x)^2}\left(-\frac{(7x+6)(1+x)^3 \mathrm{e}^{-3\ln(1+x)-\ln(x)}}{42x^6}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.181.2 Maple step by step solution

Let's solve

$4x^2(1+x)\left(\frac{d}{dx}y'\right) + 4x(1+4x)\,y' - (49+27x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(49+27x)y}{4x^2(1+x)} - \frac{(1+4x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(1+4x)y'}{x(1+x)} - \frac{(49+27x)y}{4x^2(1+x)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{1+4x}{x(1+x)}, P_3(x) = -\frac{49+27x}{4x^2(1+x)}\right]$
  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Bigg|_{x=-1} = 3$
  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\Bigg|_{x=-1} = 0$
  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$
- Multiply by denominators

  $4x^2(1+x)\left(\frac{d}{dx}y'\right) + 4x(1+4x)\,y' + (-27x-49)\,y = 0$
- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^3 - 8u^2 + 4u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (16u^2 - 28u + 12)\left(\frac{d}{du}y(u)\right) + (-27u - 22)\,y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k- >k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^{m} \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m} \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_{0}r(2+r)\, u^{-1+r} + \left(4a_{1}(1+r)(3+r) - 2a_{0}(4r^{2}+10r+11)\right) u^{r} + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)(k+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-2, 0\}$$

- Each term must be 0

$$4a_{1}(1+r)(3+r) - 2a_{0}(4r^{2}+10r+11) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4(-2a_{k} + a_{k-1} + a_{k+1})\, k^{2} + 4(2(-2a_{k} + a_{k-1} + a_{k+1})\, r - 5a_{k} + a_{k-1} + 4a_{k+1})\, k + 4(-2a_{k} + a_{k-}$$

- Shift index using $k->k+1$

$$4(-2a_{k+1} + a_{k} + a_{k+2})\, (k+1)^{2} + 4(2(-2a_{k+1} + a_{k} + a_{k+2})\, r - 5a_{k+1} + a_{k} + 4a_{k+2})\, (k+1) + 4($$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4k^{2}a_{k} - 8k^{2}a_{k+1} + 8kra_{k} - 16kra_{k+1} + 4r^{2}a_{k} - 8r^{2}a_{k+1} + 12ka_{k} - 36ka_{k+1} + 12ra_{k} - 36ra_{k+1} - 27a_{k} - 50a_{k+1}}{4(k^{2}+2kr+r^{2}+6k+6r+8)}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{4k^{2}a_{k} - 8k^{2}a_{k+1} - 4ka_{k} - 4ka_{k+1} - 35a_{k} - 10a_{k+1}}{4(k^{2}+2k)}$$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = -\frac{4k^{2}a_{k} - 8k^{2}a_{k+1} - 4ka_{k} - 4ka_{k+1} - 35a_{k} - 10a_{k+1}}{4(k^{2}+2k)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{4k^{2}a_{k} - 8k^{2}a_{k+1} + 12ka_{k} - 36ka_{k+1} - 27a_{k} - 50a_{k+1}}{4(k^{2}+6k+8)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_{k}u^{k}, a_{k+2} = -\frac{4k^{2}a_{k} - 8k^{2}a_{k+1} + 12ka_{k} - 36ka_{k+1} - 27a_{k} - 50a_{k+1}}{4(k^{2}+6k+8)}, 12a_{1} - 22a_{0} = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_{k}(1+x)^{k}, a_{k+2} = -\frac{4k^{2}a_{k} - 8k^{2}a_{k+1} + 12ka_{k} - 36ka_{k+1} - 27a_{k} - 50a_{k+1}}{4(k^{2}+6k+8)}, 12a_{1} - 22a_{0} = 0\right]$$

1.181.3 Maple trace

```
Methods for second order ODEs:
```

1.181.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 26

```
dsolve(4*x^2*(1+x)*diff(diff(y(x),x),x)+4*x*(1+4*x)*diff(y(x),x)-(49+27*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 x^7 + 7c_2 x + 6c_2}{(1+x)^2 x^{7/2}}$$

1.181.5 Mathematica DSolve solution

Solving time : 0.084 (sec)
Leaf size : 36

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+4*x*(1+4*x)*D[y[x],x]-(49+27*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{42c_1 x^7 - 7c_2 x - 6c_2}{42x^{7/2}(x+1)^2}$$

1.182 problem 184

Internal problem ID [8320]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 184
Date solved : `Monday, October 21, 2024 at 05:05:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''-x\left(-2x^2+7\right)y'+12y=0$$

1.182.1 Solved as second order ode using Kovacic algorithm

Time used: 0.359 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(2x^3-7x\right)y'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^4+x^2\\B&=2x^3-7x\\C&=12\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-30x^2 + 15}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -30x^2 + 15$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-30x^2 + 15}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 348: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2} + \frac{45}{16(x-i)^2} + \frac{45}{16(x+i)^2} + \frac{75i}{16(x-i)} - \frac{75i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-30x^2 + 15}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{9}{4}$	$-\frac{5}{4}$
$-i$	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)} + (0) \\
&= \frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)} \\
&= \frac{5}{2x(x^2+1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)} \right)(0) + \left(\left(-\frac{5}{2x^2} + \frac{5}{4(x-i)^2} + \frac{5}{4(x+i)^2} \right) + \left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)} \right) dx} \\
&= \frac{x^{5/2}}{(x^2+1)^{5/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3-7x}{x^4+x^2}\,dx} \\
&= z_1 e^{-\frac{9\ln\left(x^2+1\right)}{4}+\frac{7\ln(x)}{2}} \\
&= z_1\left(\frac{x^{7/2}}{(x^2+1)^{9/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^6}{(x^2+1)^{7/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-7x}{x^4+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{9\ln\left(x^2+1\right)}{2}+7\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{(x^2+1)^{7/2}}{4x^4}-\frac{3(x^2+1)^{7/2}}{8x^2}+\frac{3(x^2+1)^{5/2}}{8}+\frac{5(x^2+1)^{3/2}}{8}+\frac{15\sqrt{x^2+1}}{8}\right. \\
&\qquad \left. -\frac{15\,\mathrm{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{8}\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1\left(\frac{x^6}{(x^2+1)^{7/2}}\right)$$
$$+c_2\left(\frac{x^6}{(x^2+1)^{7/2}}\left(-\frac{(x^2+1)^{7/2}}{4x^4}-\frac{3(x^2+1)^{7/2}}{8x^2}+\frac{3(x^2+1)^{5/2}}{8}+\frac{5(x^2+1)^{3/2}}{8}+\frac{15\sqrt{x^2+1}}{8}-\frac{15 \operatorname{arctan}}{8}\right.\right.$$

Will add steps showing solving for IC soon.

1.182.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right) - x(-2x^2+7)\,y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{12y}{x^2(x^2+1)} - \frac{(2x^2-7)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(2x^2-7)y'}{x(x^2+1)} + \frac{12y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2x^2-7}{x(x^2+1)}, P_3(x) = \frac{12}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -7$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 12$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d}{dx}y'\right) + x(2x^2-7)\,y' + 12y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

  - ◦ Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

  - ◦ Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(-2+r)(-6+r)\, x^r + a_1(-1+r)(-5+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)(k+r-6) + a_{k-2}(k+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-2+r)(-6+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{2, 6\}$$

- Each term must be 0

$$a_1(-1+r)(-5+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-2)(a_k(k+r-6) + a_{k-2}(k+r-1)) = 0$$

- Shift index using $k->k+2$

$$(k+r)(a_{k+2}(k-4+r) + a_k(k+r+1)) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+1)}{k-4+r}$$

- Recursion relation for $r = 2$

$$a_{k+2} = -\frac{a_k(k+3)}{k-2}$$

- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 2$

$a_{k+2} = -\frac{a_k(k+3)}{k-2}$

- Recursion relation for $r = 6$

  $a_{k+2} = -\frac{a_k(k+7)}{k+2}$

- Solution for $r = 6$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+2} = -\frac{a_k(k+7)}{k+2}, a_1 = 0\right]$

1.182.3 Maple trace

```
Methods for second order ODEs:
```

1.182.4 Maple dsolve solution

Solving time : 0.013 (sec)
Leaf size : 56

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-2*x^2+7)*diff(y(x),x)+12*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\left(-15 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) c_2 x^4 + c_2(8x^4 - 9x^2 - 2)\sqrt{x^2+1} + c_1 x^4\right) x^2}{(x^2+1)^{7/2}}$$

1.182.5 Mathematica DSolve solution

Solving time : 0.224 (sec)
Leaf size : 88

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(7-2*x^2)*D[y[x],x]+12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-15c_2x^6 \operatorname{arctanh}\left(\sqrt{x^2+1}\right) - 2c_2\sqrt{x^2+1}x^2 + 8x^6\left(c_2\sqrt{x^2+1} + c_1\right) - 9c_2\sqrt{x^2+1}x^4}{8\left(x^2+1\right)^{7/2}}$$

1.183 problem 185

Internal problem ID [8321]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 185
Date solved : `Monday, October 21, 2024 at 05:05:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x\left(-x^2+7\right)y' + 12y = 0$$

1.183.1 Solved as second order ode using Kovacic algorithm

Time used: 0.313 (sec)

Writing the ode as

$$x^2y'' + \left(x^3-7x\right)y' + 12y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3-7x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^4 - 12x^2 + 15}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^4 - 12x^2 + 15 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^4 - 12x^2 + 15}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 350: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 3 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{x} - \frac{21}{4x^3} - \frac{63}{2x^5} - \frac{3465}{16x^7} - \frac{13041}{8x^9} - \frac{417501}{32x^{11}} - \frac{1744659}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 12x^2 + 15}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{x^2}{4} - 3\right) + \left(\frac{15}{4x^2}\right) \\
&= \frac{x^2}{4} - 3 + \frac{15}{4x^2}
\end{aligned}$$

We see that the coefficient of the term x in the quotient is -3. Now b can be found.

$$\begin{aligned}
b &= (-3) - (0) \\
&= -3
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{\frac{1}{2}} - 1\right) &&= -\frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{\frac{1}{2}} - 1\right) &&= \frac{5}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 12x^2 + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{7}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{5}{2} - \left(\frac{5}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{5}{2x} + (-)\left(\frac{x}{2}\right) \\
&= \frac{5}{2x} - \frac{x}{2} \\
&= \frac{5}{2x} - \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{5}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{5}{2x^2} - \frac{1}{2}\right) + \left(\frac{5}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 12x^2 + 15}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{5}{2x} - \frac{x}{2}\right) dx} \\
&= x^{5/2} e^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^3 - 7x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{4} + \frac{7 \ln(x)}{2}} \\
&= z_1 \left(x^{7/2} e^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^6 \mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-7x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}+7\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{4x^4} - \frac{\mathrm{e}^{\frac{x^2}{2}}}{8x^2} - \frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{16} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^6 \mathrm{e}^{-\frac{x^2}{2}} \right) + c_2 \left(x^6 \mathrm{e}^{-\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{4x^4} - \frac{\mathrm{e}^{\frac{x^2}{2}}}{8x^2} - \frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{16} \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.183.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) - x(-x^2 + 7)\, y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{12y}{x^2} - \frac{(x^2-7)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(x^2-7)y'}{x} + \frac{12y}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{x^2-7}{x}, P_3(x) = \frac{12}{x^2}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -7$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 12$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- • Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x(x^2-7)\,y' + 12y = 0$
- • Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - ◦ Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(-2+r)\,(-6+r)\,x^r + a_1(-1+r)\,(-5+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-6) + a_{k-2}(k +$
- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-6+r) = 0$
- • Values of r that satisfy the indicial equation

  $r \in \{2, 6\}$

- Each term must be 0
  $a_1(-1+r)(-5+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-2)(a_k(k+r-6)+a_{k-2}) = 0$
- Shift index using $k->k+2$
  $(k+r)(a_{k+2}(k-4+r)+a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{k-4+r}$
- Recursion relation for $r = 2$
  $a_{k+2} = -\frac{a_k}{k-2}$
- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 2$
  $a_{k+2} = -\frac{a_k}{k-2}$
- Recursion relation for $r = 6$
  $a_{k+2} = -\frac{a_k}{k+2}$
- Solution for $r = 6$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$

1.183.3 Maple trace

```
Methods for second order ODEs:
```

1.183.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 47

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-x^2+7)*diff(y(x),x)+12*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^2 \left(\mathrm{Ei}_1\left(-\frac{x^2}{2}\right) \mathrm{e}^{-\frac{x^2}{2}} c_2\, x^4 + c_1\, x^4 \mathrm{e}^{-\frac{x^2}{2}} + 2c_2\, x^2 + 4c_2\right)$$

1.183.5 Mathematica DSolve solution

Solving time : `0.244` (sec)
Leaf size : 61

```
DSolve[{x^2*D[y[x],{x,2}]-x*(7-x^2)*D[y[x],x]+12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{1}{16}c_2e^{-\frac{x^2}{2}}x^6\,\text{ExpIntegralEi}\left(\frac{x^2}{2}\right) - \frac{1}{8}c_2(x^2+2)\,x^2 + c_1e^{-\frac{x^2}{2}}x^6$$

1.184 problem 186

Internal problem ID [8322]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 186
Date solved : `Monday, October 21, 2024 at 05:05:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x\left(2x^2+1\right)y' - \left(-10x^2+1\right)y = 0$$

1.184.1 Solved as second order ode using Kovacic algorithm

Time used: 0.388 (sec)

Writing the ode as

$$x^2y'' + \left(2x^3+x\right)y' + \left(10x^2-1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^3+x \\ C &= 10x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 32x^2 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^4 - 32x^2 + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 32x^2 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 352: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = x^2 - 8 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{4}{x} - \frac{61}{8x^3} - \frac{61}{2x^5} - \frac{19337}{128x^7} - \frac{26779}{32x^9} - \frac{5083557}{1024x^{11}} - \frac{7896633}{256x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 32x^2 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(x^2 - 8\right) + \left(\frac{3}{4x^2}\right) \\ &= x^2 - 8 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -8. Now b can be found.

$$\begin{aligned} b &= (-8) - (0) \\ &= -8 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= x \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) \quad = \frac{1}{2}\left(\frac{-8}{1} - 1\right) \quad = -\frac{9}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-8}{1} - 1\right) = \frac{7}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4 - 32x^2 + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	$-\frac{9}{2}$	$\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{7}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{7}{2} - \left(\frac{3}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{3}{2x} + (-)(x) \\
&= \frac{3}{2x} - x \\
&= \frac{3}{2x} - x
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{3}{2x} - x\right)(2x + a_1) + \left(\left(-\frac{3}{2x^2} - 1\right) + \left(\frac{3}{2x} - x\right)^2 - \left(\frac{4x^4 - 32x^2 + 3}{4x^2}\right)\right) = 0
$$

$$
\frac{2x^2 a_1 + (4a_0 + 8)\, x + 3a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -2, a_1 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 2\right) e^{\int \left(\frac{3}{2x} - x\right) dx} \\
&= \left(x^2 - 2\right) e^{-\frac{x^2}{2} + \frac{3\ln(x)}{2}} \\
&= \left(x^2 - 2\right) x^{3/2} e^{-\frac{x^2}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x^2}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\,\mathrm{e}^{-x^2}\left(x^2-2\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x^2-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-x^2-\ln(x)}\mathrm{e}^{2x^2}}{x^2\left(x^2-2\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-x^2}\left(x^2-2\right)\right) + c_2\left(x\,\mathrm{e}^{-x^2}\left(x^2-2\right)\left(\int \frac{\mathrm{e}^{-x^2-\ln(x)}\mathrm{e}^{2x^2}}{x^2\left(x^2-2\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.184.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+x(2x^2+1)\,y'-(-10x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(10x^2-1)y}{x^2}-\frac{(2x^2+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(2x^2+1)y'}{x}+\frac{(10x^2-1)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{2x^2+1}{x},P_3(x)=\frac{10x^2-1}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=-1$

  ◦ $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right)+x(2x^2+1)\,y'+(10x^2-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+r)\, x^r + a_1(2+r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+1)\,(k+r-1) + 2a_{k-2}(k+3+r))\, x^{k}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 1\}$$

- Each term must be 0

$$a_1(2+r)\, r = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+1)\,(k+r-1) + 2a_{k-2}(k+3+r) = 0$$

- Shift index using $k->k+2$

$$a_{k+2}(k+3+r)\,(k+r+1) + 2a_k(k+r+5) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2a_k(k+r+5)}{(k+3+r)(k+r+1)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}, a_1 = 0\right]$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{2a_k(k+6)}{(k+4)(k+2)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_k(k+6)}{(k+4)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}, a_1 = 0, b_{k+2} = -\frac{2b_k(k+6)}{(k+4)(k+2)}, b_1 = 0\right]$$

1.184.3 Maple trace

```
Methods for second order ODEs:
```

1.184.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(2*x^2+1)*diff(y(x),x)-(-10*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{x\,\mathrm{e}^{-x^2}(x^2-2)(c_1-2c_2)}{2}$$

1.184.5 Mathematica DSolve solution

Solving time : `0.362` (sec)
Leaf size : 68

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1+2*x^2)*D[y[x],x]-(1-10*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x^2}\left(c_2(x^2-2)\,x^2\,\text{ExpIntegralEi}\,(x^2) + 4c_1x^4 - x^2\left(c_2e^{x^2}+8c_1\right) + c_2e^{x^2}\right)}{4x}$$

1.185 problem 187

Internal problem ID [8323]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 187
Date solved : `Monday, October 21, 2024 at 05:05:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(-2x^2 + 1)\,y' - 4(2x^2 + 1)\,y = 0$$

1.185.1 Solved as second order ode using Kovacic algorithm

Time used: 0.323 (sec)

Writing the ode as

$$x^2y'' + (-2x^3 + x)\,y' + (-8x^2 - 4)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^3 + x \\ C &= -8x^2 - 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4x^4 + 24x^2 + 15}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4x^4 + 24x^2 + 15 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{4x^4 + 24x^2 + 15}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 354: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = x^2 + 6 + \frac{15}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{x} - \frac{21}{8x^3} + \frac{63}{8x^5} - \frac{3465}{128x^7} + \frac{13041}{128x^9} - \frac{417501}{1024x^{11}} + \frac{1744659}{1024x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 + 24x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (x^2 + 6) + \left(\frac{15}{4x^2}\right) \\ &= x^2 + 6 + \frac{15}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 6. Now b can be found.

$$\begin{aligned} b &= (6) - (0) \\ &= 6 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= x \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{6}{1} - 1\right) &= \frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{6}{1} - 1\right) &= -\frac{7}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4 + 24x^2 + 15}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	$\frac{5}{2}$	$-\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\
&= \frac{5}{2} - \left(\frac{5}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{5}{2x} + (x) \\
&= \frac{5}{2x} + x \\
&= \frac{5}{2x} + x
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{2x} + x\right)(0) + \left(\left(-\frac{5}{2x^2} + 1\right) + \left(\frac{5}{2x} + x\right)^2 - \left(\frac{4x^4 + 24x^2 + 15}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{5}{2x} + x\right) dx} \\
&= x^{5/2} e^{\frac{x^2}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^3 + x}{x^2} \, dx} \\
&= z_1 e^{\frac{x^2}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{\frac{x^2}{2}}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2 \mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x^2-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-x^2}}{4x^4} + \frac{\mathrm{e}^{-x^2}}{4x^2} - \frac{\mathrm{Ei}_1(x^2)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^2 \mathrm{e}^{x^2}\right) + c_2 \left(x^2 \mathrm{e}^{x^2} \left(-\frac{\mathrm{e}^{-x^2}}{4x^4} + \frac{\mathrm{e}^{-x^2}}{4x^2} - \frac{\mathrm{Ei}_1(x^2)}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.185.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + x(-2x^2+1)\, y' - 4(2x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{4(2x^2+1)y}{x^2} + \frac{(2x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' - \frac{(2x^2-1)y'}{x} - \frac{4(2x^2+1)y}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2x^2-1}{x}, P_3(x) = -\frac{4(2x^2+1)}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -4$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $x^2\left(\frac{d}{dx}y'\right) - x(2x^2 - 1)\, y' + (-8x^2 - 4)\, y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$
    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$
    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$
    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$
    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$
    Rewrite ODE with series expansions

$$a_0(2+r)(-2+r)x^r + a_1(3+r)(-1+r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)(k+r-2) - 2a_{k-2}(k+r\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(2+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 2\}$
- Each term must be 0
  $a_1(3+r)(-1+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r+2)(a_k(k+r-2) - 2a_{k-2}) = 0$
- Shift index using $k- >k+2$
  $(k+r+4)(a_{k+2}(k+r) - 2a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2a_k}{k+r}$
- Recursion relation for $r = -2$
  $a_{k+2} = \frac{2a_k}{k-2}$
- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$
  $a_{k+2} = \frac{2a_k}{k-2}$
- Recursion relation for $r = 2$
  $a_{k+2} = \frac{2a_k}{k+2}$
- Solution for $r = 2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2a_k}{k+2}, a_1 = 0\right]$

1.185.3 Maple trace

```
Methods for second order ODEs:
```

1.185.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 41

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(-2*x^2+1)*diff(y(x),x)-4*(2*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-\operatorname{Ei}_1\left(x^2\right) \mathrm{e}^{x^2} c_2\, x^4 + c_1\, x^4 \mathrm{e}^{x^2} + c_2\, x^2 - c_2}{x^2}$$

1.185.5 Mathematica DSolve solution

Solving time : `0.168` (sec)
Leaf size : 46

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1-2*x^2)*D[y[x],x]-4*(1+2*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2\left(e^{x^2} x^4 \operatorname{ExpIntegralEi}\left(-x^2\right) + x^2 - 1\right)}{4x^2} + c_1 e^{x^2} x^2$$

1.186 problem 188

Internal problem ID [8324]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 188
Date solved : `Monday, October 21, 2024 at 05:05:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x\left(-3x^2 + 1\right) y' - 4\left(-3x^2 + 1\right) y = 0$$

1.186.1 Solved as second order ode using Kovacic algorithm

Time used: 0.666 (sec)

Writing the ode as

$$x^2y'' + \left(-3x^3 + x\right) y' + \left(12x^2 - 4\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x^3 + x \\ C &= 12x^2 - 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9x^4 - 60x^2 + 15}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9x^4 - 60x^2 + 15 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{9x^4 - 60x^2 + 15}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 356: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9x^2}{4} - 15 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3x}{2} - \frac{5}{x} - \frac{85}{12x^3} - \frac{425}{18x^5} - \frac{41225}{432x^7} - \frac{278375}{648x^9} - \frac{1787125}{864x^{11}} - \frac{40534375}{3888x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{3x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{9x^4 - 60x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{9x^2}{4} - 15\right) + \left(\frac{15}{4x^2}\right) \\ &= \frac{9x^2}{4} - 15 + \frac{15}{4x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is -15. Now b can be found.

$$\begin{aligned}b &= (-15) - (0) \\ &= -15\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-15}{\frac{3}{2}} - 1\right) &&= -\frac{11}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-15}{\frac{3}{2}} - 1\right) &&= \frac{9}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{9x^4 - 60x^2 + 15}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{3x}{2}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{9}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{9}{2} - \left(\frac{5}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{5}{2x} + (-)\left(\frac{3x}{2}\right) \\
&= \frac{5}{2x} - \frac{3x}{2} \\
&= \frac{5}{2x} - \frac{3x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{5}{2x} - \frac{3x}{2}\right)(2x + a_1) + \left(\left(-\frac{5}{2x^2} - \frac{3}{2}\right) + \left(\frac{5}{2x} - \frac{3x}{2}\right)^2 - \left(\frac{9x^4 - 60x^2 + 15}{4x^2}\right)\right) = 0
$$

$$
\frac{3x^2 a_1 + 6(2 + a_0)\, x + 5a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -2, a_1 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 2\right) e^{\int \left(\frac{5}{2x} - \frac{3x}{2}\right) dx} \\
&= \left(x^2 - 2\right) e^{-\frac{3x^2}{4} + \frac{5\ln(x)}{2}} \\
&= \left(x^2 - 2\right) x^{5/2} e^{-\frac{3x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x^3+x}{x^2}\,dx} \\
&= z_1 e^{\frac{3x^2}{4}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{3x^2}{4}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2-2\right)x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+x}{x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{3x^2}{2}-\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}-\ln(x)}}{\left(x^2-2\right)^2 x^4}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2-2\right)x^2\right) + c_2\left(\left(x^2-2\right)x^2\left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}-\ln(x)}}{\left(x^2-2\right)^2 x^4}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.186.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+x(-3x^2+1)\,y'-4(-3x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{4(3x^2-1)y}{x^2}+\frac{(3x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{(3x^2-1)y'}{x}+\frac{4(3x^2-1)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{3x^2-1}{x}, P_3(x)=\frac{4(3x^2-1)}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-4$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right)-(3x^2-1)\,xy'+(12x^2-4)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)\,(-2+r)\, x^r + a_1(3+r)\,(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(k+r-2) - 3a_{k-2}(k-6$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(2+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 2\}$
- Each term must be 0
  $a_1(3+r)\,(-1+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+2)\,(k+r-2) - 3a_{k-2}(k-6+r) = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(k+4+r)\,(k+r) - 3a_k(k+r-4) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{3a_k(k+r-4)}{(k+4+r)(k+r)}$
- Recursion relation for $r = -2$; series terminates at $k = 6$
  $a_{k+2} = \frac{3a_k(k-6)}{(k+2)(k-2)}$
- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$
  $a_{k+2} = \frac{3a_k(k-6)}{(k+2)(k-2)}$
- Recursion relation for $r = 2$; series terminates at $k = 2$
  $a_{k+2} = \frac{3a_k(k-2)}{(k+6)(k+2)}$
- Solution for $r = 2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{3a_k(k-2)}{(k+6)(k+2)}, a_1 = 0\right]$

1.186.3 Maple trace

```
Methods for second order ODEs:
```

1.186.4 Maple dsolve solution

Solving time : 0.011 (sec)
Leaf size : 19

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(-3*x^2+1)*diff(y(x),x)-4*(-3*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{x^2(x^2-2)\,(c_1-c_2)}{2}$$

1.186.5 Mathematica DSolve solution

Solving time : 0.406 (sec)
Leaf size : 89

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1-3*x^2)*D[y[x],x]-4*(1-3*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{64}\left(27c_2\left(x^2-2\right)x^2\,\text{ExpIntegralEi}\left(\frac{3x^2}{2}\right)+64c_1x^4-2x^2\left(9c_2e^{\frac{3x^2}{2}}+64c_1\right)\right.$$
$$\left.+24c_2e^{\frac{3x^2}{2}}+\frac{8c_2e^{\frac{3x^2}{2}}}{x^2}\right)$$

1.187 problem 189

Internal problem ID [8325]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 189
Date solved : `Monday, October 21, 2024 at 05:05:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+1)\,y''+x(11x^2+5)\,y'+24x^2y=0$$

1.187.1 Solved as second order ode using Kovacic algorithm

Time used: 0.445 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(11x^3+5x)\,y'+24x^2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 11x^3+5x \\ C &= 24x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^4 + 6x^2 + 15$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 358: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 4 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\,(x-i)^2} + \frac{3}{4\,(x+i)^2} + \frac{9i}{4\,(x-i)} - \frac{9i}{4\,(x+i)} + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\ &= \frac{3}{2} - \left(\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} + (0) \\ &= -\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} \\ &= -\frac{3}{2x} + \frac{3x}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{3}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{3/2}}{x^{3/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^3+5x}{x^4+x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{2}-\frac{3\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{1}{x^{5/2}\left(x^2+1\right)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^4}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+5x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-5\ln(x)-3\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{\left(x^2+1\right)\left(2x^2+1\right)x^5 \mathrm{e}^{-5\ln(x)-3\ln\left(x^2+1\right)}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^4}\right) + c_2\left(\frac{1}{x^4}\left(-\frac{\left(x^2+1\right)\left(2x^2+1\right)x^5 \mathrm{e}^{-5\ln(x)-3\ln\left(x^2+1\right)}}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.187.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(11x^2+5)\,y'+24x^2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{24y}{x^2+1}-\frac{(11x^2+5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(11x^2+5)y'}{x(x^2+1)}+\frac{24y}{x^2+1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{11x^2+5}{x(x^2+1)},P_3(x)=\frac{24}{x^2+1}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=5$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x^2+1)\left(\frac{d}{dx}y'\right)+(11x^2+5)\,y'+24yx=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x\cdot y$ to series expansion

  $x\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+1}$

  ◦ Shift index using $k->k-1$

  $x\cdot y=\sum_{k=1}^{\infty}a_{k-1}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(4+r)\, x^{-1+r} + a_1(1+r)\,(5+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+5+r) + a_{k-1}(k+5+r)\,(k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$r(4+r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \{-4, 0\}$$
- Each term must be 0
$$a_1(1+r)\,(5+r) = 0$$
- Each term in the series must be 0, giving the recursion relation
$$(k+5+r)\,(a_{k+1}(k+r+1) + a_{k-1}(k+3+r)) = 0$$
- Shift index using $k- >k+1$
$$(k+r+6)\,(a_{k+2}(k+2+r) + a_k(k+r+4)) = 0$$
- Recursion relation that defines series solution to ODE
$$a_{k+2} = -\frac{a_k(k+r+4)}{k+2+r}$$
- Recursion relation for $r = -4$
$$a_{k+2} = -\frac{a_k k}{k-2}$$
- Series not valid for $r = -4$, division by 0 in the recursion relation at $k = 2$
$$a_{k+2} = -\frac{a_k k}{k-2}$$
- Recursion relation for $r = 0$
$$a_{k+2} = -\frac{a_k(k+4)}{k+2}$$
- Solution for $r = 0$
$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{k+2}, 5a_1 = 0\right]$$

1.187.3 Maple trace

```
Methods for second order ODEs:
```

1.187.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 28

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(11*x^2+5)*diff(y(x),x)+24*x^2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 x^4 + 2c_2 x^2 + c_2}{\left(x^2+1\right)^2 x^4}$$

1.187.5 Mathematica DSolve solution

Solving time : 0.082 (sec)
Leaf size : 36

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]+x*(5+11*x^2)*D[y[x],x]+24*x^2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{-4c_1x^4 + 2c_2x^2 + c_2}{4x^4 \left(x^2+1\right)^2}$$

1.188 problem 190

Internal problem ID [8326]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 190
Date solved : `Monday, October 21, 2024 at 05:06:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+1\right)y''+8xy'-\left(-x^2+35\right)y=0$$

1.188.1 Solved as second order ode using Kovacic algorithm

Time used: 0.392 (sec)

Writing the ode as

$$\left(4x^4+4x^2\right)y''+8xy'+\left(x^2-35\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 8x \\ C &= x^2-35 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 + 22x^2 + 35}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^4 + 22x^2 + 35$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 + 22x^2 + 35}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 360: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{21i}{4(x-i)} - \frac{21i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-x^4 + 22x^2 + 35}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^4 + 22x^2 + 35}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + \frac{3}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{5}{2x} + \frac{3}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\
&= -\frac{5}{2x} + \frac{3x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{5}{2x} + \frac{3}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)}\right)(0) + \left(\left(\frac{5}{2x^2} - \frac{3}{2\left(x-i\right)^2} - \frac{3}{2\left(x+i\right)^2}\right) + \left(-\frac{5}{2x} + \frac{3}{2\left(x-i\right)} +
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right)dx} \\
&= \frac{\left(x^2+1\right)^{3/2}}{x^{5/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x}{4x^4+4x^2}\,dx} \\
&= z_1 e^{-\ln(x)+\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{\sqrt{x^2+1}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+1\right)^2}{x^{7/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x}{4x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)+\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{1}{4\left(x^2+1\right)^2} + \frac{\ln\left(x^2+1\right)}{2} + \frac{1}{x^2+1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x^2+1\right)^2}{x^{7/2}}\right) + c_2 \left(\frac{\left(x^2+1\right)^2}{x^{7/2}} \left(-\frac{1}{4\left(x^2+1\right)^2} + \frac{\ln\left(x^2+1\right)}{2} + \frac{1}{x^2+1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.188.2 Maple step by step solution

Let's solve

$4x^2(x^2+1)\left(\frac{d}{dx}y'\right)+8xy'-(-x^2+35)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(x^2-35)y}{4x^2(x^2+1)}-\frac{2y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{2y'}{x(x^2+1)}+\frac{(x^2-35)y}{4x^2(x^2+1)}=0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{2}{x(x^2+1)},P_3(x)=\frac{x^2-35}{4x^2(x^2+1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\Big|_{x=0}=2$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=-\frac{35}{4}$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$4x^2(x^2+1)\left(\frac{d}{dx}y'\right)+8xy'+(x^2-35)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x\cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(7+2r)\,(-5+2r)\, x^r + a_1(9+2r)\,(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+7)\,(2k+2r-5) + a_k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(7+2r)\,(-5+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{7}{2}, \frac{5}{2}\right\}$

- Each term must be 0

  $a_1(9+2r)\,(-3+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{5}{2}\right)\left(\left(k+r-\frac{5}{2}\right)a_{k-2} + a_k\left(k+r+\frac{7}{2}\right)\right) = 0$

- Shift index using $k- >k+2$

  $4\left(k-\frac{1}{2}+r\right)\left(\left(k-\frac{1}{2}+r\right)a_k + a_{k+2}\left(k+\frac{11}{2}+r\right)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{(2k+2r-1)a_k}{2k+11+2r}$

- Recursion relation for $r = -\frac{7}{2}$; series terminates at $k = 4$

  $a_{k+2} = -\frac{(2k-8)a_k}{2k+4}$

- Solution for $r = -\frac{7}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{7}{2}}, a_{k+2} = -\frac{(2k-8)a_k}{2k+4}, a_1 = 0\right]$

- Recursion relation for $r = \frac{5}{2}$

  $a_{k+2} = -\frac{(2k+4)a_k}{2k+16}$

- Solution for $r = \frac{5}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{(2k+4)a_k}{2k+16}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{7}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = -\frac{(2k-8)a_k}{2k+4}, a_1 = 0, b_{k+2} = -\frac{(2k+4)b_k}{2k+16}, b_1 = 0\right]$$

1.188.3 Maple trace

```
Methods for second order ODEs:
```

1.188.4 Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 42

```
dsolve(4*x^2*(x^2+1)*diff(diff(y(x),x),x)+8*x*diff(y(x),x)-(-x^2+35)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\left(x^2+1\right)^2 c_2 \ln\left(x^2+1\right) + \left(2x^2+\frac{3}{2}\right) c_2 + c_1 \left(x^2+1\right)^2}{x^{7/2}}$$

1.188.5 Mathematica DSolve solution

Solving time : `0.117` (sec)
Leaf size : 53

```
DSolve[{4*x^2*(1+x^2)*D[y[x],{x,2}]+8*x*D[y[x],x]-(35-x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1\left(x^2+1\right)^2 + c_2\left(4x^2+3\right) + 2c_2\left(x^2+1\right)^2 \log\left(x^2+1\right)}{4x^{7/2}}$$

1.189 problem 191

Internal problem ID [8327]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 191
Date solved : `Monday, October 21, 2024 at 05:06:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''-x\left(-x^2+5\right)y'-\left(25x^2+7\right)y=0$$

1.189.1 Solved as second order ode using Kovacic algorithm

Time used: 0.402 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(x^3-5x\right)y'+\left(-25x^2-7\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^4+x^2\\B&=x^3-5x\\C&=-25x^2-7\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 99x^4 + 150x^2 + 63 \\
t &= 4\left(x^3 + x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 362: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{63}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} - \frac{15i}{4(x-i)} + \frac{15i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2}\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{99}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{2}$	$-\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{9}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= -\frac{9}{2} - \left(-\frac{9}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} + (-)(0) \\ &= -\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} \\ &= -\frac{7}{2x} - \frac{x}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right)(0) + \left(\left(\frac{7}{2x^2} + \frac{1}{2(x-i)^2} + \frac{1}{2(x+i)^2}\right) + \left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right)dx} \\ &= \frac{1}{x^{7/2}\sqrt{x^2+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3-5x}{x^4+x^2}\,dx} \\
&= z_1 e^{\frac{5\ln(x)}{2}-\frac{3\ln\left(x^2+1\right)}{2}} \\
&= z_1\left(\frac{x^{5/2}}{\left(x^2+1\right)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x\left(x^2+1\right)^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-5x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(x)-3\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{x^3\left(4x^2+5\right)\left(x^2+1\right)^3 \mathrm{e}^{5\ln(x)-3\ln\left(x^2+1\right)}}{40}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x\left(x^2+1\right)^2}\right) + c_2\left(\frac{1}{x\left(x^2+1\right)^2}\left(\frac{x^3\left(4x^2+5\right)\left(x^2+1\right)^3 \mathrm{e}^{5\ln(x)-3\ln\left(x^2+1\right)}}{40}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.189.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)-x(-x^2+5)\,y'-(25x^2+7)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{(25x^2+7)y}{x^2(x^2+1)}-\frac{(x^2-5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(x^2-5)y'}{x(x^2+1)}-\frac{(25x^2+7)y}{x^2(x^2+1)}=0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{x^2-5}{x(x^2+1)}, P_3(x)=-\frac{25x^2+7}{x^2(x^2+1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=-5$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=-7$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(x^2-5)\,y'+(-25x^2-7)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-7+r)\, x^r + a_1(2+r)\,(-6+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+1)\,(k+r-7) + a_{k-2}(k+3-$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-7+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 7\}$$

- Each term must be 0

$$a_1(2+r)\,(-6+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-7)\,(a_k(k+r+1) + a_{k-2}(k+3+r)) = 0$$

- Shift index using $k->k+2$

$$(k+r-5)\,(a_{k+2}(k+3+r) + a_k(k+r+5)) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+5)}{k+3+r}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{a_k(k+4)}{k+2}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0\right]$$

- Recursion relation for $r = 7$

$$a_{k+2} = -\frac{a_k(k+12)}{k+10}$$

- Solution for $r = 7$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+7}, a_{k+2} = -\frac{a_k(k+12)}{k+10}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+7}\right), a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0, b_{k+2} = -\frac{b_k(k+12)}{k+10}, b_1 = 0\right]$$

1.189.3 Maple trace

```
Methods for second order ODEs:
```

1.189.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 29

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-x^2+5)*diff(y(x),x)-(25*x^2+7)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{4c_2 x^{10} + 5c_2 x^8 + c_1}{x(x^2+1)^2}$$

1.189.5 Mathematica DSolve solution

Solving time : 0.093 (sec)
Leaf size : 37

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(5-x^2)*D[y[x],x]-(7+25*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2(4x^2+5)x^8 + 40c_1}{40x(x^2+1)^2}$$

1.190 problem 192

Internal problem ID [8328]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 192
Date solved : `Monday, October 21, 2024 at 05:06:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+1)\,y''+x(2x^2+5)\,y'-21y=0$$

1.190.1 Solved as second order ode using Kovacic algorithm

Time used: 0.362 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(2x^3+5x)\,y'-21y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^4+x^2\\B&=2x^3+5x\\C&=-21\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{78x^2 + 99}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 78x^2 + 99$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{78x^2 + 99}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 364: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{99}{4x^2} + \frac{21}{16(x-i)^2} + \frac{21}{16(x+i)^2} + \frac{219i}{16(x-i)} - \frac{219i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{99}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{4} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{78x^2+99}{4\left(x^3+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{11}{2}$	$-\frac{9}{2}$
i	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}+\alpha_{c_3}^{+}\right) \\
&= 1-(-1) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)} + (-)(0) \\
&= -\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)} \\
&= -\frac{9}{2x} + \frac{7x}{2x^2+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)} \right)(2x + a_1) + \left(\left(\frac{9}{2x^2} - \frac{7}{4(x-i)^2} - \frac{7}{4(x+i)^2} \right) + \left(-\frac{9}{2x} + \frac{7}{4(x-\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^2 + 8)\, e^{\int \left(-\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)} \right) dx} \\
&= (x^2 + 8)\, e^{-\frac{9\ln(x)}{2} + \frac{7\ln(x^2+1)}{4}} \\
&= \frac{(x^2 + 8)(x^2 + 1)^{7/4}}{x^{9/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^3+5x}{x^4+x^2}\,dx} \\ &= z_1 e^{-\frac{5\ln(x)}{2}+\frac{3\ln\left(x^2+1\right)}{4}} \\ &= z_1\left(\frac{\left(x^2+1\right)^{3/4}}{x^{5/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2+1\right)^{5/2}\left(x^2+8\right)}{x^7}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+5x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\ &= y_1 \int \frac{e^{-5\ln(x)+\frac{3\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\ &= y_1\left(-\frac{\left(35x^6+140x^4+168x^2+64\right)x^5 \mathrm{e}^{-5\ln(x)+\frac{3\ln\left(x^2+1\right)}{2}}}{35\left(x^2+1\right)^4\left(x^2+8\right)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\left(x^2+1\right)^{5/2}\left(x^2+8\right)}{x^7}\right) \\ &\quad + c_2\left(\frac{\left(x^2+1\right)^{5/2}\left(x^2+8\right)}{x^7}\left(-\frac{\left(35x^6+140x^4+168x^2+64\right)x^5 \mathrm{e}^{-5\ln(x)+\frac{3\ln\left(x^2+1\right)}{2}}}{35\left(x^2+1\right)^4\left(x^2+8\right)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.190.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(2x^2+5)\,y'-21y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{21y}{x^2(x^2+1)}-\frac{(2x^2+5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(2x^2+5)y'}{x(x^2+1)}-\frac{21y}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{2x^2+5}{x(x^2+1)},P_3(x)=-\frac{21}{x^2(x^2+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=5$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-21$

- ○ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(2x^2+5)\,y'-21y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(7+r)\,(-3+r)\,x^r + a_1(8+r)\,(-2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+7)\,(k+r-3) + a_{k-2}(k-2\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(7+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-7, 3\}$

- Each term must be 0

  $a_1(8+r)\,(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+7)\,(k+r-3) + a_{k-2}(k-2+r)\,(k+r-1) = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+9+r)\,(k+r-1) + a_k(k+r)\,(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r)(k+r+1)}{(k+9+r)(k+r-1)}$

- Recursion relation for $r = -7$; series terminates at $k = 6$

  $a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}$

- Solution for $r = -7$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-7}, a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}, a_1 = 0\right]$$

- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{a_k(k+3)(k+4)}{(k+12)(k+2)}$

- Solution for $r = 3$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k(k+3)(k+4)}{(k+12)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-7}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+3}\right), a_{k+2}=-\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}, a_1=0, b_{k+2}=-\frac{b_k(k+3)(k+4)}{(k+12)(k+2)}, b_1=0\right]$$

1.190.3 Maple trace

```
Methods for second order ODEs:
```

1.190.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 41

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(2*x^2+5)*diff(y(x),x)-21*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1(x^2+1)^{5/2}(x^2+8)+35\left(x^6+4x^4+\frac{24}{5}x^2+\frac{64}{35}\right)c_2}{x^7}$$

1.190.5 Mathematica DSolve solution

Solving time : `0.255` (sec)
Leaf size : 52

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]+x*(5+2*x^2)*D[y[x],x]-21*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{35c_1(x^2+1)^{5/2}(x^2+8)-c_2(35x^6+140x^4+168x^2+64)}{35x^7}$$

1.191 problem 193

Internal problem ID [8329]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 193
Date solved : `Monday, October 21, 2024 at 05:06:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+1\right)y''+4x\left(x^2+2\right)y'-\left(x^2+15\right)y=0$$

1.191.1 Solved as second order ode using Kovacic algorithm

Time used: 0.297 (sec)

Writing the ode as

$$\left(4x^4+4x^2\right)y''+\left(4x^3+8x\right)y'+\left(-x^2-15\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 4x^3+8x \\ C &= -x^2-15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{10x^2 + 15}{4\left(x^3 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 10x^2 + 15 \\
t &= 4\left(x^3 + x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{10x^2 + 15}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 366: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 2 \\
&= 4
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16 (x - i)^2} + \frac{5}{16 (x + i)^2} + \frac{35i}{16 (x - i)} - \frac{35i}{16 (x + i)} + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{10x^2 + 15}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + \frac{5}{4(x - i)} + \frac{5}{4(x + i)} + (-)(0) \\
&= -\frac{3}{2x} + \frac{5}{4(x - i)} + \frac{5}{4(x + i)} \\
&= -\frac{3}{2x} + \frac{5x}{2x^2 + 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x} + \frac{5}{4(x - i)} + \frac{5}{4(x + i)} \right)(0) + \left(\left(\frac{3}{2x^2} - \frac{5}{4(x - i)^2} - \frac{5}{4(x + i)^2} \right) + \left(-\frac{3}{2x} + \frac{5}{4(x - i)} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{3}{2x} + \frac{5}{4(x - i)} + \frac{5}{4(x + i)} \right) dx} \\
&= \frac{(x^2 + 1)^{5/4}}{x^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^3+8x}{4x^4+4x^2}\,dx} \\
&= z_1 e^{-\ln(x)+\frac{\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{\left(x^2+1\right)^{1/4}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+1\right)^{3/2}}{x^{5/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^3+8x}{4x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)+\frac{\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{\left(3x^2+2\right)x^2 \mathrm{e}^{-2\ln(x)+\frac{\ln\left(x^2+1\right)}{2}}}{3\left(x^2+1\right)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x^2+1\right)^{3/2}}{x^{5/2}}\right) + c_2 \left(\frac{\left(x^2+1\right)^{3/2}}{x^{5/2}} \left(-\frac{\left(3x^2+2\right)x^2 \mathrm{e}^{-2\ln(x)+\frac{\ln\left(x^2+1\right)}{2}}}{3\left(x^2+1\right)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.191.2 Maple step by step solution

Let's solve

$4x^2(x^2+1)\left(\frac{d}{dx}y'\right)+4x(x^2+2)\,y'-(x^2+15)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{(x^2+15)y}{4x^2(x^2+1)}-\frac{(x^2+2)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(x^2+2)y'}{x(x^2+1)}-\frac{(x^2+15)y}{4x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{x^2+2}{x(x^2+1)},P_3(x)=-\frac{x^2+15}{4x^2(x^2+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=2$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{15}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+1)\left(\frac{d}{dx}y'\right)+4x(x^2+2)\,y'+(-x^2-15)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(5+2r)\,(-3+2r)\, x^r + a_1(7+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+5)\,(2k+2r-3) + a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(5+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{5}{2}, \frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(7+2r)\,(-1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{3}{2}\right)\left(\left(k+r-\frac{5}{2}\right)a_{k-2} + a_k\left(k+r+\frac{5}{2}\right)\right) = 0$$

- Shift index using $k->k+2$

$$4\left(k+\frac{1}{2}+r\right)\left(\left(k-\frac{1}{2}+r\right)a_k + a_{k+2}\left(k+\frac{9}{2}+r\right)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{(2k+2r-1)a_k}{2k+9+2r}$$

- Recursion relation for $r = -\frac{5}{2}$

$$a_{k+2} = -\frac{(2k-6)a_k}{2k+4}$$

- Solution for $r = -\frac{5}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{(2k-6)a_k}{2k+4}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{3}{2}$

$a_{k+2} = -\frac{(2k+2)a_k}{2k+12}$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{(2k+2)a_k}{2k+12}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -\frac{(2k-6)a_k}{2k+4}, a_1 = 0, b_{k+2} = -\frac{(2k+2)b_k}{2k+12}, b_1 = 0\right]$$

1.191.3 Maple trace

```
Methods for second order ODEs:
```

1.191.4 Maple dsolve solution

Solving time : `0.014` (sec)
Leaf size : 27

```
dsolve(4*x^2*(x^2+1)*diff(diff(y(x),x),x)+4*x*(x^2+2)*diff(y(x),x)-(x^2+15)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 (x^2+1)^{3/2} + 3c_1 x^2 + 2c_1}{x^{5/2}}$$

1.191.5 Mathematica DSolve solution

Solving time : `0.145` (sec)
Leaf size : 39

```
DSolve[{4*x^2*(1+x^2)*D[y[x],{x,2}]+4*x*(2+x^2)*D[y[x],x]-(15+x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3c_1 (x^2+1)^{3/2} - c_2 (3x^2+2)}{3x^{5/2}}$$

1.192 problem 194

Internal problem ID [8330]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 194
Date solved : `Monday, October 21, 2024 at 05:06:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - \frac{2(t+1)\,y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0$$

1.192.1 Solved as second order ode using Kovacic algorithm

Time used: 0.297 (sec)

Writing the ode as

$$y'' + \frac{(-2t-2)\,y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{-2t-2}{t^2+2t-1} \\ C &= \frac{2}{t^2+2t-1} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{(t^2 + 2t - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 6$$
$$t = (t^2 + 2t - 1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{6}{(t^2 + 2t - 1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 368: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2 + 2t - 1)^2$. There is a pole at $t = \sqrt{2} - 1$ of order 2. There is a pole at $t = -1 - \sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(t - \sqrt{2} + 1\right)^2} + \frac{3}{4\left(t + 1 + \sqrt{2}\right)^2} - \frac{3\sqrt{2}}{8\left(t - \sqrt{2} + 1\right)} + \frac{3\sqrt{2}}{8\left(t + 1 + \sqrt{2}\right)}$$

For the pole at $t = \sqrt{2} - 1$ let b be the coefficient of $\frac{1}{\left(t - \sqrt{2} + 1\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $t = -1 - \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(t + 1 + \sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{6}{(t^2 + 2t - 1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\sqrt{2} - 1$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-1 - \sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{t-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)} + (-)\,(0) \\
&= -\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)} \\
&= \frac{t+1-2\sqrt{2}}{t^2+2t-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)}\right)(0) + \left(\left(\frac{1}{2\left(t-\sqrt{2}+1\right)^2} - \frac{3}{2\left(t+1+\sqrt{2}\right)^2}\right) + \left(-\frac{}{2\left(t-\right.}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)}\right) dt} \\
&= \frac{\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{-2t-2}{t^2+2t-1}}{1} \, dt} \\
&= z_1 e^{\frac{\ln\left(t^2+2t-1\right)}{2}} \\
&= z_1\left(\sqrt{t^2+2t-1}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\sqrt{t^2+2t-1}\,\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{-2t-2}{t^2+2t-1}}{1}\,dt}}{\left(y_1\right)^2}\,dt \\
&= y_1 \int \frac{e^{\ln\left(t^2+2t-1\right)}}{\left(y_1\right)^2}\,dt \\
&= y_1 \left(-\frac{1}{t+1+\sqrt{2}} + \frac{\sqrt{2}}{\left(t+1+\sqrt{2}\right)^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{t^2+2t-1}\,\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}}\right) \\
&\quad + c_2 \left(\frac{\sqrt{t^2+2t-1}\,\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}} \left(-\frac{1}{t+1+\sqrt{2}} + \frac{\sqrt{2}}{\left(t+1+\sqrt{2}\right)^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.192.2 Maple step by step solution

Let's solve

$\frac{d}{dt}y' - \frac{2(t+1)y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

□ Check to see if t_0 is a regular singular point

- ○ Define functions

  $\left[P_2(t) = -\frac{2(t+1)}{t^2+2t-1}, P_3(t) = \frac{2}{t^2+2t-1}\right]$

- ○ $(t+1+\sqrt{2}) \cdot P_2(t)$ is analytic at $t = -1-\sqrt{2}$

  $\left((t+1+\sqrt{2}) \cdot P_2(t)\right)\Big|_{t=-1-\sqrt{2}} = 0$

- ○ $(t+1+\sqrt{2})^2 \cdot P_3(t)$ is analytic at $t = -1-\sqrt{2}$

  $\left((t+1+\sqrt{2})^2 \cdot P_3(t)\right)\Big|_{t=-1-\sqrt{2}} = 0$

- ○ $t = -1-\sqrt{2}$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = -1-\sqrt{2}$

- Multiply by denominators

  $(t^2+2t-1)\left(\frac{d}{dt}y'\right) + (-2t-2)\,y' + 2y = 0$

- Change variables using $t = u-1-\sqrt{2}$ so that the regular singular point is at $u = 0$

  $(u^2-2u\sqrt{2})\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-2u+2\sqrt{2})\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2\sqrt{2}\,(r-2)\,r a_{0} u^{r-1}+\left(\sum_{k=0}^{\infty}\left(-2\sqrt{2}\,(k+r-1)\,(k+1+r)\,a_{k+1}+a_{k}(k+r-1)\,(k+r-2)\right) u$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-2\sqrt{2}\,(r-2)\,r=0$
- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(-2a_{k+1}(k+1+r)\sqrt{2}+a_{k}(k+r-2)\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_{k}(k+r-2)\sqrt{2}}{4(k+1+r)}$
- Recursion relation for $r=0$; series terminates at $k=2$

  $a_{k+1}=\frac{a_{k}(k-2)\sqrt{2}}{4(k+1)}$
- Apply recursion relation for $k=0$

  $a_{1}=-\frac{a_{0}\sqrt{2}}{2}$
- Apply recursion relation for $k=1$

  $a_{2}=-\frac{a_{1}\sqrt{2}}{8}$
- Express in terms of a_{0}

  $a_{2}=\frac{a_{0}}{8}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_{0}\cdot\left(1-\frac{u\sqrt{2}}{2}+\frac{u^{2}}{8}\right)$
- Revert the change of variables $u=t+1+\sqrt{2}$

  $\left[y=a_{0}\left(\frac{(-2t-2)\sqrt{2}}{8}+\frac{t^{2}}{8}+\frac{t}{4}+\frac{3}{8}\right)\right]$
- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_{k}k\sqrt{2}}{4(k+3)}$
- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty} a_{k}u^{k+2}, a_{k+1}=\frac{a_{k}k\sqrt{2}}{4(k+3)}\right]$

- Revert the change of variables $u = t + 1 + \sqrt{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(t + 1 + \sqrt{2}\right)^{k+2}, a_{k+1} = \frac{a_k k \sqrt{2}}{4(k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \left(\frac{(-2t-2)\sqrt{2}}{8} + \frac{t^2}{8} + \frac{t}{4} + \frac{3}{8}\right) + \left(\sum_{k=0}^{\infty} b_k \left(t + 1 + \sqrt{2}\right)^{k+2}\right), b_{k+1} = \frac{b_k k \sqrt{2}}{4(k+3)}\right]$$

1.192.3 Maple trace

```
Methods for second order ODEs:
```

1.192.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 15

```
dsolve(diff(diff(y(t),t),t)-2*(t+1)/(t^2+2*t-1)*diff(y(t),t)+2/(t^2+2*t-1)*y(t) = 0,
       y(t),singsol=all)
```

$$y = c_2 t^2 + c_1 t + c_1 + c_2$$

1.192.5 Mathematica DSolve solution

Solving time : `0.31` (sec)
Leaf size : 64

```
DSolve[{D[y[t],{t,2}]-2*(t+1)/(t^2+2*t-1)*D[y[t],t]+2/(t^2+2*t-1)*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{\sqrt{t^2 + 2t - 1}\left(c_1 \left(t^2 - 2\left(\sqrt{2} - 1\right) t - 2\sqrt{2} + 3\right) + c_2(t + 1)\right)}{\sqrt{-t^2 - 2t + 1}}$$

1.193 problem 195

Internal problem ID [8331]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 195
Date solved : `Monday, October 21, 2024 at 05:06:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4ty' + \left(4t^2 - 2\right) y = 0$$

1.193.1 Solved as second order ode using Kovacic algorithm

Time used: 0.084 (sec)

Writing the ode as

$$y'' - 4ty' + \left(4t^2 - 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4t \\ C &= 4t^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 370: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4t}{1}\, dt} \\
&= z_1 e^{t^2} \\
&= z_1 \left(\mathrm{e}^{t^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{t^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4t}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{2t^2}}{(y_1)^2}\, dt \\
&= y_1(t)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{t^2}\right) + c_2\left(\mathrm{e}^{t^2}(t)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.193.2 Maple step by step solution

Let's solve

$\frac{d}{dt}y' - 4ty' + (4t^2 - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^k$

☐ Rewrite ODE with series expansions

- ◦ Convert $t^m \cdot y$ to series expansion for $m = 0..2$

  $t^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k t^{k+m}$

- ◦ Shift index using $k- >k - m$

  $t^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} t^k$

- ◦ Convert $t \cdot y'$ to series expansion

  $t \cdot y' = \sum_{k=0}^{\infty} a_k k\, t^k$

- ◦ Convert $\frac{d}{dt}y'$ to series expansion

  $\frac{d}{dt}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, t^{k-2}$

- ◦ Shift index using $k- >k + 2$

  $\frac{d}{dt}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, t^k$

Rewrite ODE with series expansions

$2a_2 - 2a_0 + (6a_3 - 6a_1)\, t + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2})\, t^k\right) = 0$

- The coefficients of each power of t must be 0

  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = a_0, a_3 = a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+4} = \frac{2(2ka_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$

1.193.3 Maple trace

```
Methods for second order ODEs:
```

1.193.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 14

```
dsolve(diff(diff(y(t),t),t)-4*t*diff(y(t),t)+(4*t^2-2)*y(t) = 0,
       y(t),singsol=all)
```

$$y = e^{t^2}(tc_2 + c_1)$$

1.193.5 Mathematica DSolve solution

Solving time : `0.031` (sec)
Leaf size : 18

```
DSolve[{D[y[t],{t,2}]-4*t*D[y[t],t]+(4*t^2-2)*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^{t^2}(c_2 t + c_1)$$

1.194 problem 196

Internal problem ID [8332]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 196
Date solved : `Monday, October 21, 2024 at 05:06:07 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-t^2+1\right) y''-2ty'+2y=0$$

1.194.1 Solved as second order ode using Kovacic algorithm

Time used: 0.280 (sec)

Writing the ode as

$$\left(-t^2+1\right) y''-2ty'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -t^2+1 \\ B &= -2t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2t^2 - 3}{(t^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2t^2 - 3$$
$$t = (t^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{2t^2 - 3}{(t^2 - 1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 372: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2-1)^2$. There is a pole at $t = 1$ of order 2. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4(t+1)} + \frac{5}{4(t-1)} - \frac{1}{4(t+1)^2} - \frac{1}{4(t-1)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(t+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2t^2 - 3}{(t^2-1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2t^2-3}{(t^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{t - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2t-2} + \frac{1}{2t+2} + (0) \\
&= \frac{1}{2t-2} + \frac{1}{2t+2} \\
&= \frac{t}{t^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = t + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)(1) + \left(\left(-\frac{1}{2(t-1)^2} - \frac{1}{2(t+1)^2}\right) + \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)^2 - \left(\frac{2t^2-3}{(t^2-1)^2}\right)\right) = -\frac{2a_0}{t^2-1} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$
p(t) = t
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= (t)\, e^{\int \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right) dt} \\
&= (t) \sqrt{(t-1)(t+1)} \\
&= t\sqrt{t^2-1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2t}{-t^2+1}\, dt} \\
&= z_1 e^{-\frac{\ln(t-1)}{2}-\frac{\ln(t+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t-1}\,\sqrt{t+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{-t^2+1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-\ln(t-1)-\ln(t+1)}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{1}{t}+\frac{\ln(t-1)}{2}-\frac{\ln(t+1)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}\right) + c_2 \left(\frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}\left(\frac{1}{t}+\frac{\ln(t-1)}{2}-\frac{\ln(t+1)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.194.2 Maple step by step solution

Let's solve

$(-t^2+1)\left(\frac{d}{dt}y'\right)-2ty'+2y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dt}y'$

- Isolate 2nd derivative

$\frac{d}{dt}y'=\frac{2y}{t^2-1}-\frac{2ty'}{t^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y'+\frac{2ty'}{t^2-1}-\frac{2y}{t^2-1}=0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(t)=\frac{2t}{t^2-1}, P_3(t)=-\frac{2}{t^2-1}\right]$

  ◦ $(t+1)\cdot P_2(t)$ is analytic at $t=-1$

  $\left((t+1)\cdot P_2(t)\right)\Big|_{t=-1}=1$

  ◦ $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$

  $\left((t+1)^2\cdot P_3(t)\right)\Big|_{t=-1}=0$

  ◦ $t=-1$is a regular singular point

Check to see if t_0 is a regular singular point

$t_0=-1$

- Multiply by denominators

$(t^2-1)\left(\frac{d}{dt}y'\right)+2ty'-2y=0$

- Change variables using $t=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+2)\,(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2a_{k+1}(k+1)^2 + a_k(k+2)\,(k-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y(u) = a_0 \cdot (-u+1)$
- Revert the change of variables $u = t+1$
  $[y = -a_0 t]$

1.194.3 Maple trace

```
Methods for second order ODEs:
```

1.194.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve((-t^2+1)*diff(diff(y(t),t),t)-2*t*diff(y(t),t)+2*y(t) = 0,
       y(t),singsol=all)
```

$$y = \frac{\ln (t-1) c_2 t}{2} - \frac{\ln (t+1) c_2 t}{2} + c_1 t + c_2$$

1.194.5 Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 33

```
DSolve[{(1-t^2)*D[y[t],{t,2}]-2*t*D[y[t],t]+2*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_1 t - \frac{1}{2} c_2 (t \log(1-t) - t \log(t+1) + 2)$$

1.195 problem 197

Internal problem ID [8333]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 197
Date solved : `Monday, October 21, 2024 at 05:06:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(t^{2}+1\right) y''-2 t y'+2 y = 0$$

1.195.1 Solved as second order ode using Kovacic algorithm

Time used: 0.297 (sec)

Writing the ode as

$$\left(t^{2}+1\right) y''-2 t y'+2 y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^{2}+1 \\ B &= -2 t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2 A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(t^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (t^2+1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{3}{(t^2+1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 374: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2+1)^2$. There is a pole at $t = i$ of order 2. There is a pole at $t = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4(t-i)^2} + \frac{3}{4(t+i)^2} + \frac{3i}{4(t-i)} - \frac{3i}{4(t+i)}
$$

For the pole at $t = i$ let b be the coefficient of $\frac{1}{(t-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $t = -i$ let b be the coefficient of $\frac{1}{(t+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(t^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(t-i\right)} + \frac{3}{2\left(t+i\right)} + (-)\left(0\right) \\ &= -\frac{1}{2\left(t-i\right)} + \frac{3}{2\left(t+i\right)} \\ &= \frac{t-2i}{t^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)(0) + \left(\left(\frac{1}{2(t-i)^2} - \frac{3}{2(t+i)^2}\right) + \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)^2 - \left(-\frac{3}{(t^2 -\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega\, dt} \\ &= e^{\int \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right) dt} \\ &= \frac{(t^2+1)^{3/2}}{(it+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2t}{t^2+1}\, dt} \\ &= z_1 e^{\frac{\ln(t^2+1)}{2}} \\ &= z_1 \left(\sqrt{t^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(t^2+1)^2}{(it+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2t}{t^2+1}\,dt}}{(y_1)^2}\,dt$$

$$= y_1 \int \frac{e^{\ln(t^2+1)}}{(y_1)^2}\,dt$$

$$= y_1\left(-\frac{t}{(t+i)^2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{(t^2+1)^2}{(it+1)^2}\right) + c_2\left(\frac{(t^2+1)^2}{(it+1)^2}\left(-\frac{t}{(t+i)^2}\right)\right)$$

Will add steps showing solving for IC soon.

1.195.2 Maple step by step solution

1.195.3 Maple trace

```
Methods for second order ODEs:
```

1.195.4 Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 16

```
dsolve((t^2+1)*diff(diff(y(t),t),t)-2*t*diff(y(t),t)+2*y(t) = 0,
       y(t),singsol=all)
```

$$y = c_2 t^2 + c_1 t - c_2$$

1.195.5 Mathematica DSolve solution

Solving time : 0.075 (sec)
Leaf size : 21

```
DSolve[{(1+t^2)*D[y[t],{t,2}]-2*t*D[y[t],t]+2*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_2 t - c_1 (t - i)^2$$

1.196 problem 198

Internal problem ID [8334]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 198
Date solved : `Monday, October 21, 2024 at 05:06:08 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-t^2+1\right) y''-2ty'+6y=0$$

1.196.1 Solved as second order ode using Kovacic algorithm

Time used: 0.300 (sec)

Writing the ode as

$$\left(-t^2+1\right) y''-2ty'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -t^2+1 \\ B &= -2t \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6t^2 - 7}{(t^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 6t^2 - 7$$
$$t = (t^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{6t^2 - 7}{(t^2 - 1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 375: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2 - 1)^2$. There is a pole at $t = 1$ of order 2. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(t-1)^2} - \frac{1}{4(t+1)^2} + \frac{13}{4(t-1)} - \frac{13}{4(t+1)}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(t+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6t^2 - 7}{(t^2 - 1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6t^2 - 7}{(t^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 3 - (1) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{t-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2t-2} + \frac{1}{2t+2} + (0) \\ &= \frac{1}{2t-2} + \frac{1}{2t+2} \\ &= \frac{t}{t^2-1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)(2t + a_1) + \left(\left(-\frac{1}{2(t-1)^2} - \frac{1}{2(t+1)^2}\right) + \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)^2 - \left(\frac{6t^2-7}{(t^2-1)^2}\right.\right.$$

$$\frac{-4a_1 t - 6a_0 - 2}{t^2-1}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{1}{3}, a_1 = 0\right\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 - \frac{1}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega\, dt} \\ &= \left(t^2 - \frac{1}{3}\right) e^{\int \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right) dt} \\ &= \left(t^2 - \frac{1}{3}\right) \sqrt{(t-1)(t+1)} \\ &= \frac{(3t^2-1)\sqrt{t^2-1}}{3}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2t}{-t^2+1}\,dt} \\
&= z_1 e^{-\frac{\ln(t-1)}{2}-\frac{\ln(t+1)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{t-1}\,\sqrt{t+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(3t^2-1)\sqrt{t^2-1}}{3\sqrt{t-1}\,\sqrt{t+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{-t^2+1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\ln(t-1)-\ln(t+1)}}{(y_1)^2}\,dt \\
&= y_1\left(\frac{9t}{4\left(t^2-\frac{1}{3}\right)} - \frac{9\ln(t+1)}{8} + \frac{9\ln(t-1)}{8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(3t^2-1)\sqrt{t^2-1}}{3\sqrt{t-1}\,\sqrt{t+1}}\right) \\
&\quad + c_2\left(\frac{(3t^2-1)\sqrt{t^2-1}}{3\sqrt{t-1}\,\sqrt{t+1}}\left(\frac{9t}{4\left(t^2-\frac{1}{3}\right)} - \frac{9\ln(t+1)}{8} + \frac{9\ln(t-1)}{8}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.196.2 Maple step by step solution

Let's solve

$(-t^2+1)\left(\frac{d}{dt}y'\right)-2ty'+6y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y'=\frac{6y}{t^2-1}-\frac{2ty'}{t^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y'+\frac{2ty'}{t^2-1}-\frac{6y}{t^2-1}=0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(t)=\frac{2t}{t^2-1}, P_3(t)=-\frac{6}{t^2-1}\right]$

  ◦ $(t+1)\cdot P_2(t)$ is analytic at $t=-1$

  $\left((t+1)\cdot P_2(t)\right)\Big|_{t=-1}=1$

  ◦ $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$

  $\left((t+1)^2\cdot P_3(t)\right)\Big|_{t=-1}=0$

  ◦ $t=-1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0=-1$

- Multiply by denominators

  $(t^2-1)\left(\frac{d}{dt}y'\right)+2ty'-6y=0$

- Change variables using $t=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-6y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+3)\,(k+r-2)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(k+3)\,(k-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+3)(k-2)}{2(k+1)^2}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k+3)(k-2)}{2(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = -3a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{2}$

- Express in terms of a_0

  $a_2 = \frac{3a_0}{2}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 3u + \frac{3}{2}u^2\right)$

- Revert the change of variables $u = t + 1$

  $\left[y = a_0\left(\frac{3t^2}{2} - \frac{1}{2}\right)\right]$

1.196.3 Maple trace

```
Methods for second order ODEs:
```

1.196.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 44

```
dsolve((-t^2+1)*diff(diff(y(t),t),t)-2*t*diff(y(t),t)+6*y(t) = 0,
       y(t),singsol=all)
```

$$y = \frac{c_2(3t^2-1)\ln(t-1)}{2} + \frac{(-3t^2+1)\, c_2 \ln(t+1)}{2} - 3c_1\, t^2 + 3c_2 t + c_1$$

1.196.5 Mathematica DSolve solution

Solving time : 0.035 (sec)
Leaf size : 55

```
DSolve[{(1-t^2)*D[y[t],{t,2}]-2*t*D[y[t],t]+6*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{1}{2}c_1(3t^2-1) - \frac{1}{4}c_2\left((3t^2-1)\log(1-t) + (1-3t^2)\log(t+1) + 6t\right)$$

1.197 problem 199

Internal problem ID [8335]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 199
Date solved : `Monday, October 21, 2024 at 05:06:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2t+1)\,y''-4(t+1)\,y'+4y=0$$

1.197.1 Solved as second order ode using Kovacic algorithm

Time used: 0.270 (sec)

Writing the ode as

$$(2t+1)\,y''+(-4t-4)\,y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t+1 \\ B &= -4t-4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4t^2 + 2}{(2t + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4t^2 + 2 \\ t &= (2t + 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{4t^2 + 2}{(2t + 1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 377: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2t+1)^2$. There is a pole at $t = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(t+\frac{1}{2}\right)^2} - \frac{1}{t+\frac{1}{2}}$$

For the pole at $t = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(t+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2t} + \frac{1}{2t^2} - \frac{1}{4t^3} + \frac{3}{32t^4} - \frac{3}{64t^5} + \frac{1}{32t^6} - \frac{1}{64t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= 1
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4t^2 + 2}{4t^2 + 4t + 1} \\
&= Q + \frac{R}{4t^2 + 4t + 1} \\
&= (1) + \left(\frac{-4t + 1}{4t^2 + 4t + 1}\right) \\
&= 1 + \frac{-4t + 1}{4t^2 + 4t + 1}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4t^2 + 2}{(2t+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(t + \frac{1}{2}\right)} + (1) \\
&= -\frac{1}{2\left(t + \frac{1}{2}\right)} + 1 \\
&= \frac{2t}{2t + 1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(t + \frac{1}{2}\right)} + 1\right)(0) + \left(\left(\frac{1}{2\left(t + \frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(t + \frac{1}{2}\right)} + 1\right)^2 - \left(\frac{4t^2 + 2}{(2t + 1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(t + \frac{1}{2}\right)} + 1\right) dt} \\
&= \frac{\mathrm{e}^t}{\sqrt{2t + 1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4t-4}{2t+1} \, dt} \\
&= z_1 e^{t + \frac{\ln(2t+1)}{2}} \\
&= z_1 \left(\sqrt{2t + 1}\, \mathrm{e}^t\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4t-4}{2t+1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{2t+\ln(2t+1)}}{(y_1)^2}\, dt \\ &= y_1 \left(-\frac{(t+1)\, e^{2t+\ln(2t+1)} e^{-4t}}{2t+1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{2t}\right) + c_2 \left(e^{2t} \left(-\frac{(t+1)\, e^{2t+\ln(2t+1)} e^{-4t}}{2t+1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.197.2 Maple step by step solution

Let's solve

$(2t+1)\left(\frac{d}{dt}y'\right) - 4(t+1)\, y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{4y}{2t+1} + \frac{4(t+1)y'}{2t+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' - \frac{4(t+1)y'}{2t+1} + \frac{4y}{2t+1} = 0$

☐ Check to see if $t_0 = -\frac{1}{2}$ is a regular singular point

- Define functions

  $\left[P_2(t) = -\frac{4(t+1)}{2t+1}, P_3(t) = \frac{4}{2t+1}\right]$

- $\left(t+\frac{1}{2}\right) \cdot P_2(t)$ is analytic at $t = -\frac{1}{2}$

  $\left(\left(t+\frac{1}{2}\right) \cdot P_2(t)\right)\bigg|_{t=-\frac{1}{2}} = -1$

- $\left(t+\frac{1}{2}\right)^2 \cdot P_3(t)$ is analytic at $t = -\frac{1}{2}$

  $\left(\left(t+\frac{1}{2}\right)^2 \cdot P_3(t)\right)\bigg|_{t=-\frac{1}{2}} = 0$

- $t = -\frac{1}{2}$is a regular singular point

  Check to see if $t_0 = -\frac{1}{2}$ is a regular singular point

  $t_0 = -\frac{1}{2}$

- Multiply by denominators

  $(2t+1)\left(\frac{d}{dt}y'\right) + (-4t-4)\, y' + 4y = 0$

- Change variables using $t = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $2u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-4u-2)\left(\frac{d}{du}y(u)\right) + 4y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- >k+1$

  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

$$2a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)\,(k+r-1) - 4a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $2r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $2(a_{k+1}(k+1+r) - 2a_k)\,(k+r-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{2a_k}{k+1+r}$
- Recursion relation for $r = 0$
  $a_{k+1} = \frac{2a_k}{k+1}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{2a_k}{k+1}\right]$
- Revert the change of variables $u = t + \frac{1}{2}$
  $\left[y = \sum_{k=0}^{\infty} a_k \left(t+\frac{1}{2}\right)^k, a_{k+1} = \frac{2a_k}{k+1}\right]$
- Recursion relation for $r = 2$
  $a_{k+1} = \frac{2a_k}{k+3}$
- Solution for $r = 2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{2a_k}{k+3}\right]$
- Revert the change of variables $u = t + \frac{1}{2}$
  $\left[y = \sum_{k=0}^{\infty} a_k \left(t+\frac{1}{2}\right)^{k+2}, a_{k+1} = \frac{2a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k \left(t+\frac{1}{2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(t+\frac{1}{2}\right)^{k+2}\right), a_{k+1} = \frac{2a_k}{k+1}, b_{k+1} = \frac{2b_k}{k+3}\right]$

1.197.3 Maple trace

```
Methods for second order ODEs:
```

1.197.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 15

```
dsolve((2*t+1)*diff(diff(y(t),t),t)-4*(t+1)*diff(y(t),t)+4*y(t) = 0,
       y(t),singsol=all)
```

$$y = c_2 \, \mathrm{e}^{2t} + c_1 t + c_1$$

1.197.5 Mathematica DSolve solution

Solving time : 0.21 (sec)
Leaf size : 23

```
DSolve[{(2*t+1)*D[y[t],{t,2}]-4*(t+1)*D[y[t],t]+4*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_1 e^{2t+1} - c_2(t+1)$$

1.198 problem 200

Internal problem ID [8336]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 200
Date solved : `Monday, October 21, 2024 at 05:06:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right) y = 0$$

1.198.1 Solved as second order ode using Kovacic algorithm

Time used: 0.172 (sec)

Writing the ode as

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t \\ C &= t^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 379: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{t}{t^2}\, dt} \\ &= z_1 e^{-\frac{\ln(t)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{t}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(t)}{\sqrt{t}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{t^2}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-\ln(t)}}{(y_1)^2}\, dt \\ &= y_1(\tan(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(t)}{\sqrt{t}}\right) + c_2 \left(\frac{\cos(t)}{\sqrt{t}} (\tan(t))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.198.2 Maple step by step solution

Let's solve

$t^2 \left(\frac{d}{dt} y'\right) + t y' + \left(t^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt} y'$

- Isolate 2nd derivative

  $\frac{d}{dt} y' = -\frac{(4t^2-1) y}{4t^2} - \frac{y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt} y' + \frac{y'}{t} + \frac{(4t^2-1) y}{4t^2} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(t) = \frac{1}{t}, P_3(t) = \frac{4t^2-1}{4t^2}\right]$

- ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left.(t \cdot P_2(t))\right|_{t=0} = 1$

- ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left.(t^2 \cdot P_3(t))\right|_{t=0} = -\frac{1}{4}$

- ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $4t^2 \left(\frac{d}{dt} y'\right) + 4t y' + (4t^2 - 1) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k t^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $t^m \cdot y$ to series expansion for $m = 0..2$

$$t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$$

- Shift index using $k- >k-m$

$$t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$$

- Convert $t \cdot y'$ to series expansion

$$t \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r}$$

- Convert $t^2 \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

$$t^2 \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\, t^r + a_1(3+2r)\,(1+2r)\, t^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-2}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

$$a_1(3+2r)\,(1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k t^{k-\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k t^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

1.198.3 Maple trace

```
Methods for second order ODEs:
```

1.198.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 17

```
dsolve(t^2*diff(diff(y(t),t),t)+t*diff(y(t),t)+(t^2-1/4)*y(t) = 0,
       y(t),singsol=all)
```

$$y=\frac{c_1 \sin(t)+c_2 \cos(t)}{\sqrt{t}}$$

1.198.5 Mathematica DSolve solution

Solving time : 0.061 (sec)
Leaf size : 39

```
DSolve[{t^2*D[y[t],{t,2}]+t*D[y[t],t]+(t^2-1/4)*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t)\to\frac{e^{-it}(2c_1-ic_2e^{2it})}{2\sqrt{t}}$$

1.199 problem 201

Internal problem ID [8337]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 201
Date solved : `Monday, October 21, 2024 at 05:06:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - \frac{2ty'}{t^2+1} + \frac{2y}{t^2+1} = 0$$

1.199.1 Solved as second order ode using Kovacic algorithm

Time used: 0.302 (sec)

Writing the ode as

$$y'' - \frac{2ty'}{t^2+1} + \frac{2y}{t^2+1} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{2t}{t^2+1} \\ C &= \frac{2}{t^2+1} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(t^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (t^2+1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{3}{(t^2+1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 381: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2+1)^2$. There is a pole at $t = i$ of order 2. There is a pole at $t = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(t-i)^2} + \frac{3}{4(t+i)^2} + \frac{3i}{4(t-i)} - \frac{3i}{4(t+i)}$$

For the pole at $t = i$ let b be the coefficient of $\frac{1}{(t-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $t = -i$ let b be the coefficient of $\frac{1}{(t+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(t^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(t-i)} + \frac{3}{2(t+i)} + (-)(0) \\
&= -\frac{1}{2(t-i)} + \frac{3}{2(t+i)} \\
&= \frac{t-2i}{t^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)(0) + \left(\left(\frac{1}{2(t-i)^2} - \frac{3}{2(t+i)^2}\right) + \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)^2 - \left(-\frac{}{(t^2 -}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right) dt} \\
&= \frac{\left(t^2+1\right)^{3/2}}{\left(it+1\right)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-\frac{2t}{t^2+1}}{1} \, dt} \\
&= z_1 e^{\frac{\ln\left(t^2+1\right)}{2}} \\
&= z_1 \left(\sqrt{t^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(t^2+1\right)^2}{\left(it+1\right)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{2t}{t^2+1}}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{\ln(t^2+1)}}{(y_1)^2}\, dt \\
&= y_1 \left(-\frac{t}{(t+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(t^2+1)^2}{(it+1)^2}\right) + c_2 \left(\frac{(t^2+1)^2}{(it+1)^2} \left(-\frac{t}{(t+i)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.199.2 Maple step by step solution

1.199.3 Maple trace

```
Methods for second order ODEs:
```

1.199.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(t),t),t)-2*t/(t^2+1)*diff(y(t),t)+2/(t^2+1)*y(t) = 0,
       y(t),singsol=all)
```

$$
y = c_2 t^2 + c_1 t - c_2
$$

1.199.5 Mathematica DSolve solution

Solving time : 0.057 (sec)
Leaf size : 21

```
DSolve[{D[y[t],{t,2}]-2*t/(1+t^2)*D[y[t],t]+2/(1+t^2)*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_2 t - c_1 (t - i)^2$$

1.200 problem 202

Internal problem ID [8338]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 202
Date solved : `Monday, October 21, 2024 at 05:06:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + \left(t^2 + 2t + 1\right) y' - (4 + 4t)\, y = 0$$

1.200.1 Solved as second order ode using Kovacic algorithm

Time used: 0.312 (sec)

Writing the ode as

$$y'' + (1 + t)^2\, y' + (-4 - 4t)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= (1 + t)^2 \\ C &= -4 - 4t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 + 4t^3 + 6t^2 + 24t + 21}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^4 + 4t^3 + 6t^2 + 24t + 21 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 382: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{2} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t^2}{2} + t + \frac{1}{2} + \frac{5}{t} - \frac{5}{t^2} + \frac{5}{t^3} - \frac{30}{t^4} + \frac{105}{t^5} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i t^i \\ &= \frac{1}{2}t^2 + t + \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^1 = t$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2 + t + \frac{1}{4}$$

This shows that the coefficient of t in the above is 1. Now we need to find the coefficient of t in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of t in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^4 + 4t^3 + 6t^2 + 24t + 21}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2\right) + (0) \\
&= \frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{t}$ in the quotient is 6. Now b can be found.

$$\begin{aligned}
b &= (6) - (1) \\
&= 5
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}t^2 + t + \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{5}{\frac{1}{2}} - 2\right) &&= 4 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{5}{\frac{1}{2}} - 2\right) &&= -6
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{1}{2}t^2+t+\frac{1}{2}$	4	-6

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right) \\ &= \frac{1}{2}t^2 + t + \frac{1}{2} \\ &= \frac{(1+t)^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 4$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(12t^2 + 6ta_3 + 2a_2) + 2\left(\frac{1}{2}t^2 + t + \frac{1}{2}\right)(4t^3 + 3t^2a_3 + 2ta_2 + a_1) + \left((1+t) + \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right)^2 - \left(\frac{21}{4} + \right.\right.$$

$$(-a_3 + 4)\, t^4 + 2(2 - a_2 + a_3)\, t^3 + 3(4 - a_1 + a_3)\, t^2 + 2(-2a_0 - a_1 + a_2 + 3a_3)$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 5, a_1 = 8, a_2 = 6, a_3 = 4\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^4 + 4t^3 + 6t^2 + 8t + 5$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^4 + 4t^3 + 6t^2 + 8t + 5\right) \mathrm{e}^{\int \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right) dt} \\
&= \left(t^4 + 4t^3 + 6t^2 + 8t + 5\right) \mathrm{e}^{\frac{(1+t)^3}{6}} \\
&= (1+t)\left(t^3 + 3t^2 + 3t + 5\right) \mathrm{e}^{\frac{(1+t)^3}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{(1+t)^2}{1} \, dt} \\
&= z_1 e^{-\frac{(1+t)^3}{6}} \\
&= z_1 \left(\mathrm{e}^{-\frac{(1+t)^3}{6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (1+t)\left(t^3 + 3t^2 + 3t + 5\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{(1+t)^2}{1} \, dt}}{(y_1)^2} \, dt \\
&= y_1 \int \frac{e^{-\frac{(1+t)^3}{3}}}{(y_1)^2} \, dt \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{(1+t)^3}{3}}}{(1+t)^2 \left(t^3 + 3t^2 + 3t + 5\right)^2} dt\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((1+t)\left(t^3+3t^2+3t+5\right)\right) \\
&\quad + c_2\left((1+t)\left(t^3+3t^2+3t+5\right)\left(\int \frac{\mathrm{e}^{-\frac{(1+t)^3}{3}}}{(1+t)^2\left(t^3+3t^2+3t+5\right)^2}dt\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.200.2 Maple step by step solution

Let's solve

$\frac{d}{dt}y' + (t^2+2t+1)\,y' - (4+4t)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -(t^2+2t+1)\,y' + (4+4t)\,y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' + (t^2+2t+1)\,y' + (-4-4t)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^k$

□ Rewrite ODE with series expansions

- ○ Convert $t^m \cdot y$ to series expansion for $m = 0..1$

  $t^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k t^{k+m}$

- ○ Shift index using $k- >k-m$

  $t^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} t^k$

- ○ Convert $t^m \cdot y'$ to series expansion for $m = 0..2$

  $t^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, t^{k-1+m}$

- ○ Shift index using $k- >k+1-m$

  $t^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, t^k$

- Convert $\frac{d}{dt}y'$ to series expansion

  $$\frac{d}{dt}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, t^{k-2}$$

- Shift index using $k- >k+2$

  $$\frac{d}{dt}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, t^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + a_1 - 4a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k+1}(k+1) + 2a_k(k-2) + a_{k-1}(k-5)\right) t^k\right) = 0$$

- Each term must be 0

  $$2a_2 + a_1 - 4a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$k^2 a_{k+2} + \left(2a_k + a_{k-1} + a_{k+1} + 3a_{k+2}\right) k - 4a_k - 5a_{k-1} + a_{k+1} + 2a_{k+2} = 0$$

- Shift index using $k- >k+1$

  $$(k+1)^2 a_{k+3} + \left(2a_{k+1} + a_k + a_{k+2} + 3a_{k+3}\right)(k+1) - 4a_{k+1} - 5a_k + a_{k+2} + 2a_{k+3} = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+3} = -\frac{a_k k + 2a_{k+1}k + k a_{k+2} - 4a_k - 2a_{k+1} + 2a_{k+2}}{k^2+5k+6}, 2a_2 + a_1 - 4a_0 = 0\right]$$

1.200.3 Maple trace

```
Methods for second order ODEs:
```

1.200.4 Maple dsolve solution

Solving time : `0.037` (sec)
Leaf size : 60

```
dsolve(diff(diff(y(t),t),t)+(t^2+2*t+1)*diff(y(t),t)-(4+4*t)*y(t) = 0,
       y(t),singsol=all)
```

$$y = (1+t)\left(t^3+3t^2+3t+5\right)\left(\left(\int \frac{\mathrm{e}^{-\frac{t\left(t^2+3t+3\right)}{3}}}{(1+t)^2\left(t^3+3t^2+3t+5\right)^2}dt\right)c_2 + c_1\right)$$

1.200.5 Mathematica DSolve solution

Solving time : 2.655 (sec)
Leaf size : 132

```
DSolve[{D[y[t],{t,2}]+(t^2+2*t+1)*D[y[t],t]-(4+4*t)*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{1}{36}e^{-\frac{1}{3}t(t^2+3t+3)}\left(-3c_2(t^3+3t^2+3t+4) + 3^{2/3}c_2e^{\frac{1}{3}(t+1)^3}\sqrt[3]{(t+1)^3}(t^3+3t^2+3t+5)\,\Gamma\left(\frac{2}{3},\frac{1}{3}(t+1)^3\right) + 36c_1e^{\frac{t^3}{3}+t^2+t}(t^4+4t^3+6t^2+8t+5)\right)$$

1.201 problem 204

Internal problem ID [8339]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 204
Date solved : `Monday, October 21, 2024 at 05:06:14 PM`
CAS classification : `[_Laguerre]`

Solve

$$2ty'' + (1-2t)\,y' - y = 0$$

1.201.1 Solved as second order ode using Kovacic algorithm

Time used: 0.249 (sec)

Writing the ode as

$$2ty'' + (1-2t)\,y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t \\ B &= 1-2t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4t^2 + 4t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4t^2 + 4t - 3$$
$$t = 16t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{4t^2 + 4t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 384: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{16t^2} + \frac{1}{4t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4t} - \frac{1}{4t^2} + \frac{1}{8t^3} - \frac{1}{8t^4} + \frac{1}{8t^5} - \frac{9}{64t^6} + \frac{21}{128t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4t^2 + 4t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4t - 3}{16t^2}\right) \\ &= \frac{1}{4} + \frac{4t - 3}{16t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4t^2 + 4t - 3}{16t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4t} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{1}{4t} \\ &= \frac{1}{2} + \frac{1}{4t}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2} + \frac{1}{4t}\right)(0) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{2} + \frac{1}{4t}\right)^2 - \left(\frac{4t^2 + 4t - 3}{16t^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(\frac{1}{2} + \frac{1}{4t}\right) dt} \\ &= t^{1/4} e^{\frac{t}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{1-2t}{2t} \, dt} \\ &= z_1 e^{\frac{t}{2} - \frac{\ln(t)}{4}} \\ &= z_1 \left(\frac{e^{\frac{t}{2}}}{t^{1/4}} \right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1-2t}{2t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t-\frac{\ln(t)}{2}}}{(y_1)^2}\, dt \\ &= y_1 \left(\sqrt{\pi}\ \mathrm{erf}\left(\sqrt{t}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{t}\right) + c_2 \left(\mathrm{e}^{t} \left(\sqrt{\pi}\ \mathrm{erf}\left(\sqrt{t}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.201.2 Maple step by step solution

Let's solve

$2t\left(\frac{d}{dt}y'\right) + (1-2t)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = \frac{y}{2t} + \frac{(2t-1)y'}{2t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' - \frac{(2t-1)y'}{2t} - \frac{y}{2t} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(t) = -\frac{2t-1}{2t}, P_3(t) = -\frac{1}{2t}\right]$

- $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\Big|_{t=0} = \frac{1}{2}$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 0$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $2t\left(\frac{d}{dt}y'\right) + (1-2t)\, y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

- Convert $t^m \cdot y'$ to series expansion for $m = 0..1$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

  $t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$

- Shift index using $k- >k+1$

  $t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+2r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+2r+1) - a_k(2k+2r+1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $2(a_{k+1}(k+1+r)-a_k)\left(k+r+\frac{1}{2}\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y=\sum_{k=0}^{\infty} a_k t^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1}=\frac{a_k}{k+\frac{3}{2}}$
- Solution for $r=\frac{1}{2}$
  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1}=\frac{a_k}{k+\frac{3}{2}}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k t^k\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+\frac{3}{2}}\right]$

1.201.3 Maple trace

```
Methods for second order ODEs:
```

1.201.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 15

```
dsolve(2*t*diff(diff(y(t),t),t)+(1-2*t)*diff(y(t),t)-y(t) = 0,
       y(t),singsol=all)
```

$$y=\mathrm{e}^{t}\left(\operatorname{erf}\left(\sqrt{t}\right) c_1+c_2\right)$$

1.201.5 Mathematica DSolve solution

Solving time : 0.179 (sec)
Leaf size : 21

```
DSolve[{2*t*D[y[t],{t,2}]+(1-2*t)*D[y[t],t]-y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^t \left(c_1 - c_2 \Gamma\left(\frac{1}{2}, t\right)\right)$$

1.202 problem 205

Internal problem ID [8340]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 205
Date solved : `Monday, October 21, 2024 at 05:06:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2ty'' + (1+t)\,y' - 2y = 0$$

1.202.1 Solved as second order ode using Kovacic algorithm

Time used: 0.483 (sec)

Writing the ode as

$$2ty'' + (1+t)\,y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t \\ B &= 1+t \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 18t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 + 18t - 3$$
$$t = 16t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 18t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 386: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{9}{8t} - \frac{3}{16t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{9}{4t} - \frac{21}{2t^2} + \frac{189}{2t^3} - \frac{1071}{t^4} + \frac{13608}{t^5} - \frac{370629}{2t^6} + \frac{5288409}{2t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 + 18t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{18t - 3}{16t^2}\right) \\ &= \frac{1}{16} + \frac{18t - 3}{16t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 18. Dividing this by leading coefficient in t which is 16 gives $\frac{9}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{9}{8}\right) - (0) \\ &= \frac{9}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{9}{8}}{\frac{1}{4}} - 0\right) &&= \frac{9}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{9}{8}}{\frac{1}{4}} - 0\right) &&= -\frac{9}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 18t - 3}{16t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$\frac{9}{4}$	$-\frac{9}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{9}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= \frac{9}{4} - \left(\frac{1}{4}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{4t} + \left(\frac{1}{4}\right) \\
&= \frac{1}{4t} + \frac{1}{4} \\
&= \frac{1+t}{4t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{4t} + \frac{1}{4}\right)(2t + a_1) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{4t} + \frac{1}{4}\right)^2 - \left(\frac{t^2 + 18t - 3}{16t^2}\right) \right) = 0$$

$$\frac{(-a_1 + 6)\,t - 2a_0 + a_1}{2t} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = 6\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 + 6t + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^2 + 6t + 3\right) e^{\int \left(\frac{1}{4t} + \frac{1}{4}\right) dt} \\
&= \left(t^2 + 6t + 3\right) e^{\frac{t}{4} + \frac{\ln(t)}{4}} \\
&= \left(t^2 + 6t + 3\right) t^{1/4} e^{\frac{t}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+t}{2t}\,dt} \\
&= z_1 e^{-\frac{t}{4}-\frac{\ln(t)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{t}{4}}}{t^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^2 + 6t + 3
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+t}{2t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\frac{t}{2}-\frac{\ln(t)}{2}}}{(y_1)^2}\,dt \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{t}{2}-\frac{\ln(t)}{2}}}{(t^2+6t+3)^2}dt\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^2+6t+3\right) + c_2 \left(t^2+6t+3 \left(\int \frac{\mathrm{e}^{-\frac{t}{2}-\frac{\ln(t)}{2}}}{(t^2+6t+3)^2}dt\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.202.2 Maple step by step solution

Let's solve

$2t\left(\frac{d}{dt}y'\right)+(1+t)\,y'-2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y'=\frac{y}{t}-\frac{(1+t)y'}{2t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y'+\frac{(1+t)y'}{2t}-\frac{y}{t}=0$

- ☐ Check to see if $t_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(t)=\frac{1+t}{2t}, P_3(t)=-\frac{1}{t}\right]$

  - $t\cdot P_2(t)$ is analytic at $t=0$

    $\left(t\cdot P_2(t)\right)\bigg|_{t=0}=\frac{1}{2}$

  - $t^2\cdot P_3(t)$ is analytic at $t=0$

    $\left(t^2\cdot P_3(t)\right)\bigg|_{t=0}=0$

  - $t=0$is a regular singular point

    Check to see if $t_0=0$ is a regular singular point

    $t_0=0$

- Multiply by denominators

  $2t\left(\frac{d}{dt}y'\right)+(1+t)\,y'-2y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k t^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $t^m\cdot y'$ to series expansion for $m=0..1$

    $t^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,t^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $t^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,t^{k+r}$

  - Convert $t\cdot\left(\frac{d}{dt}y'\right)$ to series expansion

$$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r-1}$$

- Shift index using $k->k+1$

$$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+2r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(2k+1+2r) + a_k(k+r-2)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $2(k+1+r)\left(k+\frac{1}{2}+r\right) a_{k+1} + a_k(k+r-2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k(k+r-2)}{(k+1+r)(2k+1+2r)}$
- Recursion relation for $r = 0$; series terminates at $k = 2$
  $a_{k+1} = -\frac{a_k(k-2)}{(k+1)(2k+1)}$
- Apply recursion relation for $k = 0$
  $a_1 = 2a_0$
- Apply recursion relation for $k = 1$
  $a_2 = \frac{a_1}{6}$
- Express in terms of a_0
  $a_2 = \frac{a_0}{3}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y = a_0 \cdot \left(1 + 2t + \frac{1}{3}t^2\right)$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+1} = -\frac{a_k\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$
- Solution for $r = \frac{1}{2}$
  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y = a_0 \cdot \left(1 + 2t + \frac{1}{3}t^2\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), b_{k+1} = -\frac{b_k\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

1.202.3 Maple trace

```
Methods for second order ODEs:
```

1.202.4 Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 56

```
dsolve(2*t*diff(diff(y(t),t),t)+(1+t)*diff(y(t),t)-2*y(t) = 0,
       y(t),singsol=all)
```

$$y = c_1\sqrt{\pi}\,(t^2+6t+3)\,\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{t}}{2}\right)+5c_1\sqrt{2}\left(\sqrt{t}+\frac{t^{3/2}}{5}\right)\mathrm{e}^{-\frac{t}{2}}+c_2(t^2+6t+3)$$

1.202.5 Mathematica DSolve solution

Solving time : 11.023 (sec)
Leaf size : 71

```
DSolve[{2*t*D[y[t],{t,2}]+(1+t)*D[y[t],t]-2*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{1}{24}\left(\sqrt{2\pi}c_2(t^2+6t+3)\,\mathrm{erf}\left(\frac{\sqrt{t}}{\sqrt{2}}\right)+24c_1(t^2+6t+3)+2c_2e^{-t/2}\sqrt{t}(t+5)\right)$$

1.203 problem 206

Internal problem ID [8341]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 206
Date solved : `Monday, October 21, 2024 at 05:06:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2t^2y'' - ty' + (1+t)\,y = 0$$

1.203.1 Solved as second order ode using Kovacic algorithm

Time used: 0.242 (sec)

Writing the ode as

$$2t^2y'' - ty' + (1+t)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2 \\ B &= -t \\ C &= 1+t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3 - 8t}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3 - 8t$$
$$t = 16t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{-3 - 8t}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 388: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{2t} - \frac{3}{16t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(t)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{t-c} \\ &= \frac{1}{2}\left(\frac{1}{(t-(0))}\right) \\ &= \frac{1}{2t} \end{aligned}$$

Now we search for a monic polynomial $p(t)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(t)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2t} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2t} + \frac{1+8t}{16t^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\,\sqrt{-t}}{4t}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= e^{\int \omega\, dt} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{2}\,\sqrt{-t}}{4t} dt} \\ &= t^{1/4} \mathrm{e}^{\sqrt{2}\,\sqrt{-t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t}{2t^2}\, dt} \\ &= z_1 e^{\frac{\ln(t)}{4}} \\ &= z_1 \left(t^{1/4}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t}{2t^2}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{\frac{\ln(t)}{2}}}{(y_1)^2}\, dt \\ &= y_1 \left(-\frac{\sqrt{2}\,\sqrt{-t}\left(1 - \mathrm{e}^{-2\sqrt{2}\,\sqrt{-t}}\right)}{2\sqrt{t}} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}}\right) + c_2\left(\sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}}\left(-\frac{\sqrt{2}\,\sqrt{-t}\left(1-\mathrm{e}^{-2\sqrt{2}\,\sqrt{-t}}\right)}{2\sqrt{t}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.203.2 Maple step by step solution

Let's solve
$2t^2\left(\frac{d}{dt}y'\right) - ty' + (1+t)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dt}y'$
- Isolate 2nd derivative
  $\frac{d}{dt}y' = -\frac{(1+t)y}{2t^2} + \frac{y'}{2t}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dt}y' - \frac{y'}{2t} + \frac{(1+t)y}{2t^2} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = -\frac{1}{2t}, P_3(t) = \frac{1+t}{2t^2}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$
    $\left.(t \cdot P_2(t))\right|_{t=0} = -\frac{1}{2}$
  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$
    $\left.(t^2 \cdot P_3(t))\right|_{t=0} = \frac{1}{2}$
  - $t = 0$is a regular singular point
    Check to see if $t_0 = 0$ is a regular singular point
    $t_0 = 0$
- Multiply by denominators
  $2t^2\left(\frac{d}{dt}y'\right) - ty' + (1+t)\,y = 0$
- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k t^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $t^m \cdot y$ to series expansion for $m = 0..1$

    $t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$
  - Shift index using $k- >k - m$

    $t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$
  - Convert $t \cdot y'$ to series expansion

    $t \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r}$
  - Convert $t^2 \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

    $t^2 \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) + a_{k-1}\right) t^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k+r-1)\left(k+r-\frac{1}{2}\right) a_k + a_{k-1} = 0$
- Shift index using $k- >k + 1$

  $2(k+r)\left(k+\frac{1}{2}+r\right) a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+r)(2k+1+2r)}$
- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{(k+1)(2k+3)}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{(k+1)(2k+3)}\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{\left(k+\frac{1}{2}\right)(2k+2)}$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{(k+1)(2k+3)}, b_{k+1} = -\frac{b_k}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

1.203.3 Maple trace

```
Methods for second order ODEs:
```

1.203.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 29

```
dsolve(2*t^2*diff(diff(y(t),t),t)-t*diff(y(t),t)+(1+t)*y(t) = 0,
       y(t),singsol=all)
```

$$y = \sqrt{t}\left(c_1 \sin\left(\sqrt{t}\sqrt{2}\right) + c_2 \cos\left(\sqrt{t}\sqrt{2}\right)\right)$$

1.203.5 Mathematica DSolve solution

Solving time : `0.119` (sec)
Leaf size : 62

```
DSolve[{2*t^2*D[y[t],{t,2}]-t*D[y[t],t]+(1+t)*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{1}{2} e^{-i\sqrt{2}\sqrt{t}}\sqrt{t}\left(2c_1 e^{2i\sqrt{2}\sqrt{t}} + i\sqrt{2}c_2\right)$$

1.204 problem 207

Internal problem ID [8342]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 207
Date solved : `Monday, October 21, 2024 at 05:06:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2t^2y'' + \left(t^2 - t\right)y' + y = 0$$

1.204.1 Solved as second order ode using Kovacic algorithm

Time used: 0.340 (sec)

Writing the ode as

$$2t^2y'' + \left(t^2 - t\right)y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2 \\ B &= t^2 - t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 2t - 3$$
$$t = 16t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t - 3}{16t^2} \right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 390: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} - \frac{3}{16t^2} - \frac{1}{8t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i\\ &= \sum_{i=0}^{0} a_i t^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{1}{4t} - \frac{1}{2t^2} - \frac{1}{2t^3} - \frac{1}{t^4} - \frac{2}{t^5} - \frac{9}{2t^6} - \frac{21}{2t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 - 2t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-2t - 3}{16t^2}\right) \\ &= \frac{1}{16} + \frac{-2t - 3}{16t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 16 gives $-\frac{1}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{8}\right) - (0) \\ &= -\frac{1}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{8}}{\frac{1}{4}} - 0\right) &= -\frac{1}{4} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{8}}{\frac{1}{4}} - 0\right) &= \frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t - 3}{16t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{4t} + (-)\left(\frac{1}{4}\right) \\
&= \frac{1}{4t} - \frac{1}{4} \\
&= -\frac{t-1}{4t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{4t} - \frac{1}{4}\right)(0) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{4t} - \frac{1}{4}\right)^2 - \left(\frac{t^2 - 2t - 3}{16t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{4t} - \frac{1}{4}\right) dt} \\
&= t^{1/4} \mathrm{e}^{-\frac{t}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t^2 - t}{2t^2} \, dt} \\
&= z_1 e^{-\frac{t}{4} + \frac{\ln(t)}{4}} \\
&= z_1 \left(t^{1/4} \mathrm{e}^{-\frac{t}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{t}\, \mathrm{e}^{-\frac{t}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-t}{2t^2}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-\frac{t}{2}+\frac{\ln(t)}{2}}}{(y_1)^2}\, dt \\ &= y_1 \left(-i\sqrt{\pi}\,\sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\sqrt{t}\, \mathrm{e}^{-\frac{t}{2}}\right) + c_2 \left(\sqrt{t}\, \mathrm{e}^{-\frac{t}{2}} \left(-i\sqrt{\pi}\,\sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.204.2 Maple step by step solution

Let's solve

$2t^2\left(\frac{d}{dt}y'\right) + (t^2 - t)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{y}{2t^2} - \frac{(t-1)y'}{2t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' + \frac{(t-1)y'}{2t} + \frac{y}{2t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(t) = \frac{t-1}{2t}, P_3(t) = \frac{1}{2t^2}\right]$

- ○ $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -\frac{1}{2}$

- ○ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = \frac{1}{2}$

- ○ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $2t^2\left(\frac{d}{dt}y'\right) + t(t-1)\, y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- ○ Convert $t^2 \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

  $t^2 \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) + a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$2\left(\left(k+r-\frac{1}{2}\right) a_k+\frac{a_{k-1}}{2}\right)(k+r-1)=0$

- Shift index using $k->k+1$

  $2\left(\left(k+\frac{1}{2}+r\right) a_{k+1}+\frac{a_k}{2}\right)(k+r)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k}{2k+1+2r}$

- Recursion relation for $r=1$

  $a_{k+1}=-\frac{a_k}{2k+3}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1}=-\frac{a_k}{2k+3}\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=-\frac{a_k}{2k+2}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1}=-\frac{a_k}{2k+2}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k t^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{a_k}{2k+3}, b_{k+1}=-\frac{b_k}{2k+2}\right]$

1.204.3 Maple trace

```
Methods for second order ODEs:
```

1.204.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 47

```
dsolve(2*t^2*diff(diff(y(t),t),t)+(t^2-t)*diff(y(t),t)+y(t) = 0,
       y(t),singsol=all)
```

$$y=\frac{\mathrm{e}^{-\frac{t}{2}}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{-t}}{2}\right) 2^{3/4}\sqrt{\pi}\, c_1 t+4\sqrt{-t}\,\sqrt{t}\, c_2\right)}{4\sqrt{-t}}$$

1.204.5 Mathematica DSolve solution

Solving time : 0.088 (sec)
Leaf size : 46

```
DSolve[{2*t^2*D[y[t],{t,2}]+(t^2-t)*D[y[t],t]+y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^{-t/2}\left(c_2\sqrt{t} + \sqrt{2}c_1\sqrt{-t}\Gamma\left(\frac{1}{2}, -\frac{t}{2}\right)\right)$$

1.205 problem 208

Internal problem ID [8343]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 208
Date solved : `Monday, October 21, 2024 at 05:06:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2 y'' + \left(-t^2 + t\right) y' - y = 0$$

1.205.1 Solved as second order ode using Kovacic algorithm

Time used: 0.278 (sec)

Writing the ode as

$$t^2 y'' + \left(-t^2 + t\right) y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 + t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 2t + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 392: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i\\ &= \sum_{i=0}^{0} a_i t^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2t} \\ &= \frac{t-1}{2t}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2-2t+3}{4t^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\ &= \frac{e^{\frac{t}{2}}}{\sqrt{t}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t^2+t}{t^2}\,dt} \\
&= z_1 e^{\frac{t}{2}-\frac{\ln(t)}{2}} \\
&= z_1\left(\frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^t}{t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2+t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{t-\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(-(1+t)\,t\,\mathrm{e}^{t-\ln(t)}\mathrm{e}^{-2t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^t}{t}\right) + c_2\left(\frac{\mathrm{e}^t}{t}\left(-(1+t)\,t\,\mathrm{e}^{t-\ln(t)}\mathrm{e}^{-2t}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.205.2 Maple step by step solution

Let's solve

$t^2\left(\frac{d}{dt}y'\right)+\left(-t^2+t\right)y'-y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dt}y'$

- Isolate 2nd derivative

$\frac{d}{dt}y'=\frac{y}{t^2}+\frac{(t-1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y'-\frac{(t-1)y'}{t}-\frac{y}{t^2}=0$

☐ Check to see if $t_0=0$ is a regular singular point

- ○ Define functions

$\left[P_2(t)=-\frac{t-1}{t},P_3(t)=-\frac{1}{t^2}\right]$

- ○ $t\cdot P_2(t)$ is analytic at $t=0$

$\left(t\cdot P_2(t)\right)\bigg|_{t=0}=1$

- ○ $t^2\cdot P_3(t)$ is analytic at $t=0$

$\left(t^2\cdot P_3(t)\right)\bigg|_{t=0}=-1$

- ○ $t=0$is a regular singular point

Check to see if $t_0=0$ is a regular singular point

$t_0=0$

- Multiply by denominators

$t^2\left(\frac{d}{dt}y'\right)-t(t-1)y'-y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $t^m\cdot y'$ to series expansion for $m=1..2$

$t^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,t^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$t^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,t^{k+r}$

- ○ Convert $t^2\cdot\left(\frac{d}{dt}y'\right)$ to series expansion

$t^2 \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,t^{k+r}$

Rewrite ODE with series expansions

$a_0(1+r)\,(-1+r)\,t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) - a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r+1) - a_{k-1}) = 0$
- Shift index using $k- >k+1$

  $(k+r)\,(a_{k+1}(k+2+r) - a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+2+r}$
- Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+3}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

1.205.3 Maple trace

```
Methods for second order ODEs:
```

1.205.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 17

```
dsolve(t^2*diff(diff(y(t),t),t)+(-t^2+t)*diff(y(t),t)-y(t) = 0,
       y(t),singsol=all)
```

$$y = \frac{c_2 \, \mathrm{e}^t + c_1 t + c_1}{t}$$

1.205.5 Mathematica DSolve solution

Solving time : 0.019 (sec)
Leaf size : 23

```
DSolve[{t^2*D[y[t],{t,2}]+(t-t^2)*D[y[t],t]-y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{c_2 e^t - c_1(t+1)}{t}$$

1.206 problem 209

Internal problem ID [8344]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 209
Date solved : `Monday, October 21, 2024 at 05:06:19 PM`
CAS classification : `[_Lienard]`

Solve

$$ty'' - \left(t^2+2\right) y' + ty = 0$$

1.206.1 Solved as second order ode using Kovacic algorithm

Time used: 0.315 (sec)

Writing the ode as

$$ty'' + \left(-t^2-2\right) y' + ty = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t^2-2 \\ C &= t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 - 2t^2 + 8}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^4 - 2t^2 + 8 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^4 - 2t^2 + 8}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 394: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - \frac{1}{2} + \frac{2}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{1} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - \frac{1}{2t} + \frac{7}{4t^3} + \frac{7}{4t^5} - \frac{21}{16t^7} - \frac{119}{16t^9} - \frac{189}{32t^{11}} + \frac{791}{32t^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= \frac{t}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^4 - 2t^2 + 8}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{t^2}{4} - \frac{1}{2}\right) + \left(\frac{2}{t^2}\right) \\ &= \frac{t^2}{4} - \frac{1}{2} + \frac{2}{t^2} \end{aligned}$$

We see that the coefficient of the term t in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{t}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^4 - 2t^2 + 8}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{t}{2}$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{t} + \left(\frac{t}{2}\right) \\
&= -\frac{1}{t} + \frac{t}{2} \\
&= -\frac{1}{t} + \frac{t}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{t} + \frac{t}{2}\right)(0) + \left(\left(\frac{1}{t^2} + \frac{1}{2}\right) + \left(-\frac{1}{t} + \frac{t}{2}\right)^2 - \left(\frac{t^4 - 2t^2 + 8}{4t^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{t} + \frac{t}{2}\right) dt} \\
&= \frac{\mathrm{e}^{\frac{t^2}{4}}}{t}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-t^2-2}{t} \, dt} \\
&= z_1 e^{\frac{t^2}{4} + \ln(t)} \\
&= z_1 \left(t \, \mathrm{e}^{\frac{t^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{t^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-2}{t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{\frac{t^2}{2}+2\ln(t)}}{(y_1)^2}\, dt \\
&= y_1 \left(-t\,\mathrm{e}^{-\frac{t^2}{2}} + \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{t^2}{2}}\right) + c_2\left(\mathrm{e}^{\frac{t^2}{2}}\left(-t\,\mathrm{e}^{-\frac{t^2}{2}} + \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.206.2 Maple step by step solution

Let's solve

$t\left(\frac{d}{dt}y'\right) - (t^2+2)\,y' + ty = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -y + \frac{(t^2+2)\,y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y' - \frac{(t^2+2)y'}{t} + y = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(t) = -\frac{t^2+2}{t}, P_3(t) = 1\right]$

- ○ $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -2$

- ○ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 0$

- ○ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $t\left(\frac{d}{dt}y'\right) + (-t^2 - 2)\, y' + ty = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $t \cdot y$ to series expansion

  $t \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+1}$

- ○ Shift index using $k- >k - 1$

  $t \cdot y = \sum_{k=1}^{\infty} a_{k-1} t^{k+r}$

- ○ Convert $t^m \cdot y'$ to series expansion for $m = 0..2$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$

- ○ Shift index using $k- >k + 1 - m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, t^{k+r}$

- ○ Convert $t \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

  $t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r-1}$

- ○ Shift index using $k- >k + 1$

$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\,t^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-3+r)\,t^{-1+r} + a_1(1+r)\,(-2+r)\,t^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)\,(k-2+r) - a_{k-1}(k-2+r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 3\}$
- Each term must be 0
  $a_1(1+r)\,(-2+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k-2+r)\,(a_{k+1}(k+r+1) - a_{k-1}) = 0$
- Shift index using $k- > k+1$
  $(k+r-1)\,(a_{k+2}(k+2+r) - a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_k}{k+2+r}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{a_k}{k+2}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = \frac{a_k}{k+2}, -2a_1 = 0\right]$
- Recursion relation for $r = 3$
  $a_{k+2} = \frac{a_k}{k+5}$
- Solution for $r = 3$
  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+3}, a_{k+2} = \frac{a_k}{k+5}, 4a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+3}\right), a_{k+2} = \frac{a_k}{k+2}, -2a_1 = 0, b_{k+2} = \frac{b_k}{k+5}, 4b_1 = 0\right]$

1.206.3 Maple trace

```
Methods for second order ODEs:
```

1.206.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 34

```
dsolve(t*diff(diff(y(t),t),t)-(t^2+2)*diff(y(t),t)+t*y(t) = 0,
       y(t),singsol=all)
```

$$y = \left(c_2\pi \operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right)+c_1\right) e^{\frac{t^2}{2}}-\sqrt{\pi}\,\sqrt{2}\,c_2 t$$

1.206.5 Mathematica DSolve solution

Solving time : 0.187 (sec)
Leaf size : 52

```
DSolve[{t*D[y[t],{t,2}]-(t^2+2)*D[y[t],t]+t*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t)\to \sqrt{\frac{\pi}{2}}c_2 e^{\frac{t^2}{2}}\operatorname{erf}\left(\frac{t}{\sqrt{2}}\right)+c_1 e^{\frac{t^2}{2}}-c_2 t$$

1.207 problem 210

Internal problem ID [8345]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 210
Date solved : `Monday, October 21, 2024 at 05:06:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' + t(t+1)\,y' - y = 0$$

1.207.1 Solved as second order ode using Kovacic algorithm

Time used: 0.236 (sec)

Writing the ode as

$$t^2y'' + \left(t^2+t\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t^2+t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{t^2 + 2t + 3}{4t^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= t^2 + 2t + 3 \\
t &= 4t^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{t^2 + 2t + 3}{4t^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 396: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2t} + \frac{1}{2t^2} - \frac{1}{2t^3} + \frac{1}{4t^4} + \frac{1}{4t^5} - \frac{3}{4t^6} + \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 + 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2t} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{2t} - \frac{1}{2} \\
&= -\frac{t+1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2t} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(-\frac{1}{2t} - \frac{1}{2}\right)^2 - \left(\frac{t^2 + 2t + 3}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(-\frac{1}{2t} - \frac{1}{2}\right) dt} \\
&= \frac{e^{-\frac{t}{2}}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2+t}{t^2}\,dt} \\
&= z_1 e^{-\frac{t}{2}-\frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{t}{2}}}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-t}}{t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2+t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-t-\ln(t)}}{(y_1)^2}\,dt \\
&= y_1 \left((-1+t)\, t\, \mathrm{e}^{-t-\ln(t)} \mathrm{e}^{2t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-t}}{t}\right) + c_2 \left(\frac{\mathrm{e}^{-t}}{t}\left((-1+t)\, t\, \mathrm{e}^{-t-\ln(t)} \mathrm{e}^{2t}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.207.2 Maple step by step solution

Let's solve

$t^2\left(\frac{d}{dt}y'\right) + t(t+1)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dt}y'$

- Isolate 2nd derivative

$\frac{d}{dt}y' = \frac{y}{t^2} - \frac{(t+1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y' + \frac{(t+1)y'}{t} - \frac{y}{t^2} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(t) = \frac{t+1}{t}, P_3(t) = -\frac{1}{t^2}\right]$

- ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\bigg|_{t=0} = 1$

- ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = -1$

- ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

$t^2\left(\frac{d}{dt}y'\right) + t(t+1)\,y' - y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,t^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,t^{k+r}$

- ◦ Convert $t^2 \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

$t^2 \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, t^{k+r}$

Rewrite ODE with series expansions

$a_0(1+r)(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)(k+r-1) + a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_k(k+r+1) + a_{k-1}\right) = 0$
- Shift index using $k- >k+1$

  $(k+r)\left(a_{k+1}(k+2+r) + a_k\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k+2+r}$
- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k}{k+1}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$
- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{k+3}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = -\frac{a_k}{k+1}, b_{k+1} = -\frac{b_k}{k+3}\right]$

1.207.3 Maple trace

```
Methods for second order ODEs:
```

1.207.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 20

```
dsolve(t^2*diff(diff(y(t),t),t)+t*(t+1)*diff(y(t),t)-y(t) = 0,
       y(t),singsol=all)
```

$$y = \frac{c_2 \,\mathrm{e}^{-t} + c_1(-1+t)}{t}$$

1.207.5 Mathematica DSolve solution

Solving time : 0.018 (sec)
Leaf size : 26

```
DSolve[{t^2*D[y[t],{t,2}]+t*(t+1)*D[y[t],t]-y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{e^{-t}(c_1 e^t (t-1) + c_2)}{t}$$

1.208 problem 211

Internal problem ID [8346]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 211
Date solved : `Monday, October 21, 2024 at 05:06:21 PM`
CAS classification : `[_Laguerre]`

Solve

$$ty'' - (4+t)\,y' + 2y = 0$$

1.208.1 Solved as second order ode using Kovacic algorithm

Time used: 0.276 (sec)

Writing the ode as

$$ty'' + (-4-t)\,y' + 2y = 0 \quad (1)$$
$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -4-t \\ C &= 2 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \quad (4)$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 24}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 + 24$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 24}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 398: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{6}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{6}{t^2} - \frac{36}{t^4} + \frac{432}{t^6} - \frac{6480}{t^8} + \frac{108864}{t^{10}} - \frac{1959552}{t^{12}} + \frac{36951552}{t^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 + 24}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{6}{t^2}\right) \\ &= \frac{1}{4} + \frac{6}{t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &&= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 24}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{t} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{2}{t} - \frac{1}{2} \\ &= -\frac{4+t}{2t}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(2) + 2\left(-\frac{2}{t} - \frac{1}{2}\right)(2t + a_1) + \left(\left(\frac{2}{t^2}\right) + \left(-\frac{2}{t} - \frac{1}{2}\right)^2 - \left(\frac{t^2+24}{4t^2}\right)\right) &= 0 \\ \frac{(a_1 - 6)\, t + 2a_0 - 4a_1}{t} &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 12, a_1 = 6\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 + 6t + 12$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega\, dt} \\ &= (t^2 + 6t + 12)\, e^{\int \left(-\frac{2}{t} - \frac{1}{2}\right) dt} \\ &= (t^2 + 6t + 12)\, e^{-\frac{t}{2} - 2\ln(t)} \\ &= \frac{(t^2 + 6t + 12)\, e^{-\frac{t}{2}}}{t^2}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4-t}{t}\, dt} \\
&= z_1 e^{\frac{t}{2}+2\ln(t)} \\
&= z_1 \left(t^2 \mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^2 + 6t + 12
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4-t}{t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{t+4\ln(t)}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{(t^2-6t+12)\, \mathrm{e}^{t+4\ln(t)}}{(t^2+6t+12)\, t^4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^2+6t+12\right) + c_2 \left(t^2+6t+12 \left(\frac{(t^2-6t+12)\, \mathrm{e}^{t+4\ln(t)}}{(t^2+6t+12)\, t^4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.208.2 Maple step by step solution

Let's solve

$t\left(\frac{d}{dt}y'\right)-(4+t)\,y'+2y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dt}y'$

- Isolate 2nd derivative

$\frac{d}{dt}y'=-\frac{2y}{t}+\frac{(4+t)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y'-\frac{(4+t)y'}{t}+\frac{2y}{t}=0$

□ Check to see if $t_0=0$ is a regular singular point

- ◦ Define functions

$\left[P_2(t)=-\frac{4+t}{t}, P_3(t)=\frac{2}{t}\right]$

- ◦ $t\cdot P_2(t)$ is analytic at $t=0$

$\left(t\cdot P_2(t)\right)\bigg|_{t=0}=-4$

- ◦ $t^2\cdot P_3(t)$ is analytic at $t=0$

$\left(t^2\cdot P_3(t)\right)\bigg|_{t=0}=0$

- ◦ $t=0$is a regular singular point

Check to see if $t_0=0$ is a regular singular point

$t_0=0$

- Multiply by denominators

$t\left(\frac{d}{dt}y'\right)+(-4-t)\,y'+2y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k t^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $t^m\cdot y'$ to series expansion for $m=0..1$

$t^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,t^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$t^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,t^{k+r}$

- ◦ Convert $t\cdot\left(\frac{d}{dt}y'\right)$ to series expansion

$t\cdot\left(\frac{d}{dt}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,t^{k+r-1}$

◦ Shift index using $k->k+1$

$t\cdot\left(\frac{d}{dt}y'\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,t^{k+r}$

Rewrite ODE with series expansions

$a_0r(-5+r)\,t^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k-4+r)-a_k(k+r-2)\right)t^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-5+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{0,5\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k-4+r)-a_k(k+r-2)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k(k+r-2)}{(k+1+r)(k-4+r)}$
- Recursion relation for $r=0$; series terminates at $k=2$
  $a_{k+1}=\frac{a_k(k-2)}{(k+1)(k-4)}$
- Apply recursion relation for $k=0$
  $a_1=\frac{a_0}{2}$
- Apply recursion relation for $k=1$
  $a_2=\frac{a_1}{6}$
- Express in terms of a_0
  $a_2=\frac{a_0}{12}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li
  $y=a_0\cdot\left(1+\frac{1}{2}t+\frac{1}{12}t^2\right)$
- Recursion relation for $r=5$
  $a_{k+1}=\frac{a_k(k+3)}{(k+6)(k+1)}$
- Solution for $r=5$
  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+5}, a_{k+1}=\frac{a_k(k+3)}{(k+6)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y=a_0\cdot\left(1+\frac{1}{2}t+\frac{1}{12}t^2\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+5}\right), b_{k+1}=\frac{b_k(k+3)}{(k+6)(k+1)}\right]$

1.208.3 Maple trace

```
Methods for second order ODEs:
```

1.208.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 27

```
dsolve(t*diff(diff(y(t),t),t)-(4+t)*diff(y(t),t)+2*y(t) = 0,
       y(t),singsol=all)
```

$$y = c_1\left(t^2 + 6t + 12\right) + c_2\left(t^2 - 6t + 12\right) \mathrm{e}^t$$

1.208.5 Mathematica DSolve solution

Solving time : 0.094 (sec)
Leaf size : 85

```
DSolve[{t*D[y[t],{t,2}]-(4+t)*D[y[t],t]+2*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{2e^{t/2}\sqrt{t}\left((c_2t^2 - 6ic_1t + 12c_2)\cosh\left(\frac{t}{2}\right) + i(c_1(t^2 + 12) + 6ic_2t)\sinh\left(\frac{t}{2}\right)\right)}{\sqrt{\pi}\sqrt{-it}}$$

1.209 problem 212

Internal problem ID [8347]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 212
Date solved : `Monday, October 21, 2024 at 05:06:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' + \left(t^2 - 3t\right)y' + 3y = 0$$

1.209.1 Solved as second order ode using Kovacic algorithm

Time used: 0.272 (sec)

Writing the ode as

$$t^2y'' + \left(t^2 - 3t\right)y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t^2 - 3t \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 6t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 6t + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 6t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 400: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4t^2} - \frac{3}{2t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i\\ &= \sum_{i=0}^{0} a_i t^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2t} - \frac{3}{2t^2} - \frac{9}{2t^3} - \frac{63}{4t^4} - \frac{243}{4t^5} - \frac{999}{4t^6} - \frac{4293}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 6t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-6t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 6t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{3}{2} - \left(\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{2t} + (-)\left(\frac{1}{2}\right) \\
&= \frac{3}{2t} - \frac{1}{2} \\
&= -\frac{t-3}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{3}{2t} - \frac{1}{2}\right)(0) + \left(\left(-\frac{3}{2t^2}\right) + \left(\frac{3}{2t} - \frac{1}{2}\right)^2 - \left(\frac{t^2-6t+3}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{3}{2t} - \frac{1}{2}\right) dt} \\
&= t^{3/2} \mathrm{e}^{-\frac{t}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2-3t}{t^2} \, dt} \\
&= z_1 e^{-\frac{t}{2} + \frac{3\ln(t)}{2}} \\
&= z_1 \left(t^{3/2} \mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = t^3 \mathrm{e}^{-t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-3t}{t^2}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-t+3\ln(t)}}{(y_1)^2}\, dt \\ &= y_1 \left(-\frac{\mathrm{e}^t}{2t^2} - \frac{\mathrm{e}^t}{2t} - \frac{\mathrm{Ei}_1(-t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(t^3 \mathrm{e}^{-t}\right) + c_2 \left(t^3 \mathrm{e}^{-t} \left(-\frac{\mathrm{e}^t}{2t^2} - \frac{\mathrm{e}^t}{2t} - \frac{\mathrm{Ei}_1(-t)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.209.2 Maple step by step solution

Let's solve

$t^2 \left(\frac{d}{dt} y'\right) + (t^2 - 3t)\, y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt} y'$

- Isolate 2nd derivative

  $\frac{d}{dt} y' = -\frac{3y}{t^2} - \frac{(t-3)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt} y' + \frac{(t-3)y'}{t} + \frac{3y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

- Define functions
  $\left[P_2(t) = \frac{t-3}{t}, P_3(t) = \frac{3}{t^2}\right]$
- $t \cdot P_2(t)$ is analytic at $t = 0$
  $$\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -3$$
- $t^2 \cdot P_3(t)$ is analytic at $t = 0$
  $$\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 3$$
- $t = 0$is a regular singular point
  Check to see if $t_0 = 0$ is a regular singular point
  $t_0 = 0$

- Multiply by denominators
  $t^2\left(\frac{d}{dt}y'\right) + t(t-3)\, y' + 3y = 0$
- Assume series solution for y
  $$y = \sum_{k=0}^{\infty} a_k t^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $t^m \cdot y'$ to series expansion for $m = 1..2$
  $$t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$
- Convert $t^2 \cdot \left(\frac{d}{dt}y'\right)$ to series expansion
  $$t^2 \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0(-1+r)\,(-3+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-3) + a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)\,(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{1, 3\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_k(k+r-3) + a_{k-1}\right) = 0$
- Shift index using $k->k+1$

$(k+r)\,(a_{k+1}(k-2+r)+a_k)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k}{k-2+r}$

- Recursion relation for $r=1$

  $a_{k+1}=-\frac{a_k}{k-1}$

- Series not valid for $r=1$, division by 0 in the recursion relation at $k=1$

  $a_{k+1}=-\frac{a_k}{k-1}$

- Recursion relation for $r=3$

  $a_{k+1}=-\frac{a_k}{k+1}$

- Solution for $r=3$

  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+3}, a_{k+1}=-\frac{a_k}{k+1}\right]$

1.209.3 Maple trace

```
Methods for second order ODEs:
```

1.209.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 34

```
dsolve(t^2*diff(diff(y(t),t),t)+(t^2-3*t)*diff(y(t),t)+3*y(t) = 0,
       y(t),singsol=all)
```

$$y=t\left(\mathrm{Ei}_1\left(-t\right)\mathrm{e}^{-t}c_2\,t^2+c_1\,t^2\mathrm{e}^{-t}+c_2t+c_2\right)$$

1.209.5 Mathematica DSolve solution

Solving time : `0.061` (sec)
Leaf size : 41

```
DSolve[{t^2*D[y[t],{t,2}]+(t^2-3*t)*D[y[t],t]+3*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t)\to\frac{1}{2}e^{-t}\left(c_1t^3\,\mathrm{ExpIntegralEi}(t)+2c_2t^3-c_1e^t(t+1)t\right)$$

1.210 problem 213

Internal problem ID [8348]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 213
Date solved : `Monday, October 21, 2024 at 05:06:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$ty'' + ty' + 2y = 0$$

1.210.1 Solved as second order ode using Kovacic algorithm

Time used: 0.246 (sec)

Writing the ode as

$$ty'' + ty' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{t - 8}{4t} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= t - 8 \\
t &= 4t
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{t-8}{4t}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 402: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t$. There is a pole at $t = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $t = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{2}{t} - \frac{4}{t^2} - \frac{16}{t^3} - \frac{80}{t^4} - \frac{448}{t^5} - \frac{2688}{t^6} - \frac{16896}{t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t-8}{4t} \\
&= Q + \frac{R}{4t} \\
&= \left(\frac{1}{4}\right) + \left(-\frac{2}{t}\right) \\
&= \frac{1}{4} - \frac{2}{t}
\end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is -8. Dividing this by leading coefficient in t which is 4 gives -2. Now b can be found.

$$\begin{aligned}
b &= (-2) - (0) \\
&= -2
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 0\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 0\right) &&= 2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t-8}{4t}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{t} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{t} - \frac{1}{2} \\
&= \frac{1}{t} - \frac{1}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = t + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{t} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{t^2}\right) + \left(\frac{1}{t} - \frac{1}{2}\right)^2 - \left(\frac{t-8}{4t}\right)\right) &= 0 \\
\frac{2 + a_0}{t} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -2\}
$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$
p(t) = -2 + t
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= (-2 + t)\, \mathrm{e}^{\int \left(\frac{1}{t} - \frac{1}{2}\right) dt} \\
&= (-2 + t)\, \mathrm{e}^{-\frac{t}{2} + \ln(t)} \\
&= (-2 + t)\, t\, \mathrm{e}^{-\frac{t}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t}{t}\,dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-t}(-2+t)\,t
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-t}}{(y_1)^2}\,dt \\
&= y_1 \left(-\frac{\mathrm{e}^{t}(-t+1)}{2\,(2-t)\,t} - \frac{\mathrm{Ei}_1\,(-t)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-t}(-2+t)\,t\right) + c_2\left(\mathrm{e}^{-t}(-2+t)\,t\left(-\frac{\mathrm{e}^{t}(-t+1)}{2\,(2-t)\,t} - \frac{\mathrm{Ei}_1\,(-t)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.210.2 Maple step by step solution

Let's solve

$t\left(\frac{d}{dt}y'\right)+ty'+2y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dt}y'$

- Isolate 2nd derivative

$\frac{d}{dt}y'=-\frac{2y}{t}-y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y'+y'+\frac{2y}{t}=0$

□ Check to see if $t_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(t)=1, P_3(t)=\frac{2}{t}\right]$

  ◦ $t\cdot P_2(t)$ is analytic at $t=0$

  $\left(t\cdot P_2(t)\right)\Big|_{t=0}=0$

  ◦ $t^2\cdot P_3(t)$ is analytic at $t=0$

  $\left(t^2\cdot P_3(t)\right)\Big|_{t=0}=0$

  ◦ $t=0$is a regular singular point

Check to see if $t_0=0$ is a regular singular point

$t_0=0$

- Multiply by denominators

$t\left(\frac{d}{dt}y'\right)+ty'+2y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t\cdot y'$ to series expansion

  $t\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r}$

  ◦ Convert $t\cdot\left(\frac{d}{dt}y'\right)$ to series expansion

  $t\cdot\left(\frac{d}{dt}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$

  ◦ Shift index using $k->k+1$

$$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r) + a_k(k+r+2)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 1\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k+r) + a_k(k+r+2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k(k+r+2)}{(k+1+r)(k+r)}$
- Recursion relation for $r = 0$
  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}\right]$
- Recursion relation for $r = 1$
  $a_{k+1} = -\frac{a_k(k+3)}{(k+2)(k+1)}$
- Solution for $r = 1$
  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k(k+3)}{(k+2)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}, b_{k+1} = -\frac{b_k(k+3)}{(k+2)(k+1)}\right]$

1.210.3 Maple trace

```
Methods for second order ODEs:
```

1.210.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 35

```
dsolve(t*diff(diff(y(t),t),t)+t*diff(y(t),t)+2*y(t) = 0,
       y(t),singsol=all)
```

$$y = tc_2 \,\mathrm{e}^{-t}(-2+t)\,\mathrm{Ei}_1\left(-t\right) + c_1 \,\mathrm{e}^{-t}(-2+t)\,t + c_2(t-1)$$

1.210.5 Mathematica DSolve solution

Solving time : 0.228 (sec)
Leaf size : 51

```
DSolve[{t*D[y[t],{t,2}]+t*D[y[t],t]+2*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{1}{2}e^{-t}\left(c_2(t-2)t\,\text{ExpIntegralEi}(t) + 2c_1t^2 - t\left(c_2e^t + 4c_1\right) + c_2e^t\right)$$

1.211 problem 214

Internal problem ID [8349]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 214
Date solved : `Monday, October 21, 2024 at 05:06:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t y''+\left(-t^{2}+1\right) y'+4 t y = 0$$

1.211.1 Solved as second order ode using Kovacic algorithm

Time used: 0.722 (sec)

Writing the ode as

$$t y''+\left(-t^{2}+1\right) y'+4 t y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t^{2}+1 \\ C &= 4 t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \,dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{t^4 - 20t^2 - 1}{4t^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= t^4 - 20t^2 - 1 \\
t &= 4t^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{t^4 - 20t^2 - 1}{4t^2} \right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 404: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - 5 - \frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{1} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - \frac{5}{t} - \frac{101}{4t^3} - \frac{505}{2t^5} - \frac{50601}{16t^7} - \frac{355015}{8t^9} - \frac{21351501}{32t^{11}} - \frac{168167525}{16t^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= \frac{t}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^4 - 20t^2 - 1}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{t^2}{4} - 5\right) + \left(-\frac{1}{4t^2}\right) \\ &= \frac{t^2}{4} - 5 - \frac{1}{4t^2} \end{aligned}$$

We see that the coefficient of the term t in the quotient is -5. Now b can be found.

$$\begin{aligned} b &= (-5) - (0) \\ &= -5 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{t}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-5}{\frac{1}{2}} - 1\right) &&= -\frac{11}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-5}{\frac{1}{2}} - 1\right) &&= \frac{9}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^4 - 20t^2 - 1}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{t}{2}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{9}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{9}{2} - \left(\frac{1}{2}\right) \\
&= 4
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{t-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2t} + (-)\left(\frac{t}{2}\right) \\
&= \frac{1}{2t} - \frac{t}{2} \\
&= \frac{1}{2t} - \frac{t}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 4$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\left(12t^2 + 6ta_3 + 2a_2\right) + 2\left(\frac{1}{2t} - \frac{t}{2}\right)\left(4t^3 + 3a_3 t^2 + 2a_2 t + a_1\right) + \left(\left(-\frac{1}{2t^2} - \frac{1}{2}\right) + \left(\frac{1}{2t} - \frac{t}{2}\right)^2 - \left(\frac{t^4 - 20t}{4t^2}\right.\right.
$$

$$
\frac{t^4 a_3 + 2(8 + a_2)\, t^3 + 3(a_1 + 3a_3)\, t^2 + 4(a_0 + a_2)\, t}{t}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 8, a_1 = 0, a_2 = -8, a_3 = 0\}
$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$
p(t) = t^4 - 8t^2 + 8
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^4 - 8t^2 + 8\right) e^{\int \left(\frac{1}{2t} - \frac{t}{2}\right) dt} \\
&= \left(t^4 - 8t^2 + 8\right) e^{-\frac{t^2}{4} + \frac{\ln(t)}{2}} \\
&= \left(t^4 - 8t^2 + 8\right) \sqrt{t}\, e^{-\frac{t^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t^2+1}{t}\,dt} \\
&= z_1 e^{\frac{t^2}{4}-\frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{e^{\frac{t^2}{4}}}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^4 - 8t^2 + 8
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2+1}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{\frac{t^2}{2}-\ln(t)}}{(y_1)^2}\,dt \\
&= y_1 \left(\int \frac{e^{\frac{t^2}{2}-\ln(t)}}{(t^4-8t^2+8)^2}dt\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^4 - 8t^2 + 8\right) + c_2 \left(t^4 - 8t^2 + 8 \left(\int \frac{e^{\frac{t^2}{2}-\ln(t)}}{(t^4-8t^2+8)^2}dt\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.211.2 Maple step by step solution

Let's solve

$t\left(\frac{d}{dt}y'\right)+\left(-t^{2}+1\right)y'+4ty=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y'=-4y+\frac{(t^{2}-1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y'-\frac{(t^{2}-1)y'}{t}+4y=0$

□ Check to see if $t_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(t)=-\frac{t^{2}-1}{t}, P_3(t)=4\right]$

  ◦ $t\cdot P_2(t)$ is analytic at $t=0$

  $\left(t\cdot P_2(t)\right)\bigg|_{t=0}=1$

  ◦ $t^{2}\cdot P_3(t)$ is analytic at $t=0$

  $\left(t^{2}\cdot P_3(t)\right)\bigg|_{t=0}=0$

  ◦ $t=0$is a regular singular point

  Check to see if $t_0=0$ is a regular singular point

  $t_0=0$

- Multiply by denominators

  $t\left(\frac{d}{dt}y'\right)+\left(-t^{2}+1\right)y'+4ty=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t\cdot y$ to series expansion

  $t\cdot y=\sum_{k=0}^{\infty}a_k t^{k+r+1}$

  ◦ Shift index using $k->k-1$

  $t\cdot y=\sum_{k=1}^{\infty}a_{k-1}t^{k+r}$

  ◦ Convert $t^{m}\cdot y'$ to series expansion for $m=0..2$

$$t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- Convert $t \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

$$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$$

- Shift index using $k->k+1$

$$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 t^{-1+r} + a_1(1+r)^2\, t^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 - a_{k-1}(k-5+r)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term must be 0

$$a_1(1+r)^2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 - a_{k-1}(k-5) = 0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2)^2 - a_k(k-4) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a_k(k-4)}{(k+2)^2}$$

- Recursion relation for $r=0$; series terminates at $k=4$

$$a_{k+2} = \frac{a_k(k-4)}{(k+2)^2}$$

- Solution for $r=0$

$$\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = \frac{a_k(k-4)}{(k+2)^2}, a_1 = 0\right]$$

1.211.3 Maple trace

```
Methods for second order ODEs:
```

1.211.4 Maple dsolve solution

Solving time : 0.012 (sec)
Leaf size : 21

```
dsolve(t*diff(diff(y(t),t),t)+(-t^2+1)*diff(y(t),t)+4*t*y(t) = 0,
       y(t),singsol=all)
```

$$y = \frac{(t^4 - 8t^2 + 8)(c_1 + 2c_2)}{8}$$

1.211.5 Mathematica DSolve solution

Solving time : 0.664 (sec)
Leaf size : 61

```
DSolve[{t*D[y[t],{t,2}]+(1-t^2)*D[y[t],t]+4*t*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{1}{128} c_2 \left((t^4 - 8t^2 + 8) \operatorname{ExpIntegralEi}\left(\frac{t^2}{2}\right) - 2e^{\frac{t^2}{2}}(t^2 - 6) \right) + c_1(t^4 - 8t^2 + 8)$$

1.212 problem 215

Internal problem ID [8350]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 215
Date solved : `Monday, October 21, 2024 at 05:06:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2 y'' - t(1+t)\, y' + y = 0$$

1.212.1 Solved as second order ode using Kovacic algorithm

Time used: 0.246 (sec)

Writing the ode as

$$t^2 y'' + \left(-t^2 - t\right) y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 - t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 2t - 1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 + 2t - 1$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 2t - 1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 406: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4t^2} + \frac{1}{2t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2t} - \frac{1}{2t^2} + \frac{1}{2t^3} - \frac{3}{4t^4} + \frac{5}{4t^5} - \frac{9}{4t^6} + \frac{17}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 + 2t - 1}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2t - 1}{4t^2}\right) \\ &= \frac{1}{4} + \frac{2t - 1}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 2t - 1}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{2t} \\
&= \frac{1+t}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2} + \frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2} + \frac{1}{2t}\right)^2 - \left(\frac{t^2+2t-1}{4t^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(\frac{1}{2} + \frac{1}{2t}\right) dt} \\
&= \sqrt{t}\, e^{\frac{t}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-t^2-t}{t^2} \, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(t)}{2}} \\
&= z_1 \left(\sqrt{t}\, e^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = t\,\mathrm{e}^{t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-t}{t^2}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\,dt \\ &= y_1(-\operatorname{Ei}_1(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(t\,\mathrm{e}^{t}\right) + c_2\left(t\,\mathrm{e}^{t}(-\operatorname{Ei}_1(t))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.212.2 Maple step by step solution

Let's solve
$t^2\left(\frac{d}{dt}y'\right) - t(1+t)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dt}y'$
- Isolate 2nd derivative
  $\frac{d}{dt}y' = -\frac{y}{t^2} + \frac{(1+t)y'}{t}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dt}y' - \frac{(1+t)y'}{t} + \frac{y}{t^2} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = -\frac{1+t}{t}, P_3(t) = \frac{1}{t^2}\right]$

- $t \cdot P_2(t)$ is analytic at $t = 0$

  $$\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -1$$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $$\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 1$$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $t^2\left(\frac{d}{dt}y'\right) - t(1+t)\,y' + y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k t^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

  $$t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- Convert $t^2 \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

  $$t^2 \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)^2\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)^2 - a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_k(k+r-1) - a_{k-1}\right) = 0$

- Shift index using $k- >k+1$

  $(k+r)\left(a_{k+1}(k+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k}{k+r}$

- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$

1.212.3 Maple trace

```
Methods for second order ODEs:
```

1.212.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 15

```
dsolve(t^2*diff(diff(y(t),t),t)-t*(1+t)*diff(y(t),t)+y(t) = 0,
       y(t),singsol=all)
```

$$y = t\,\mathrm{e}^{t}(c_1 + \mathrm{Ei}_1(t)\,c_2)$$

1.212.5 Mathematica DSolve solution

Solving time : `0.035` (sec)
Leaf size : 20

```
DSolve[{t^2*D[y[t],{t,2}]-t*(1+t)*D[y[t],t]+y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^t t(c_1 \,\text{ExpIntegralEi}(-t) + c_2)$$

1.213 problem 216

Internal problem ID [8351]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 216
Date solved : `Monday, October 21, 2024 at 05:06:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + \left(4x^2+6\right) y = 0$$

1.213.1 Solved as second order ode using Kovacic algorithm

Time used: 0.177 (sec)

Writing the ode as

$$y'' + 4xy' + \left(4x^2+6\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 408: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\ &= z_1 e^{-x^2} \\ &= z_1\left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}\cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-x^2} \cos(2x)\right) + c_2 \left(e^{-x^2} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.213.2 Maple step by step solution

Let's solve

$\frac{d}{dx} y' + 4xy' + (4x^2 + 6) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$

- ○ Convert $\frac{d}{dx} y'$ to series expansion

  $\frac{d}{dx} y' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$

- ○ Shift index using $k- > k + 2$

  $\frac{d}{dx} y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$

Rewrite ODE with series expansions

$2a_2 + 6a_0 + (6a_3 + 10a_1) x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) + 2a_k(2k+3) + 4a_{k-2}) x^k\right) = 0$

- The coefficients of each power of x must be 0
  $[2a_2 + 6a_0 = 0, 6a_3 + 10a_1 = 0]$
- Solve for the dependent coefficient(s)
  $\left\{a_2 = -3a_0, a_3 = -\frac{5a_1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 6a_k + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$
  $\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k + 2) + 6a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE
  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2}+2a_k+7a_{k+2})}{k^2+7k+12}, a_2 = -3a_0, a_3 = -\frac{5a_1}{3}\right]$

1.213.3 Maple trace

```
Methods for second order ODEs:
```

1.213.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 24

```
dsolve(diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(4*x^2+6)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{-x^2}(c_1 \cos(2x) + c_2 \sin(2x))$$

1.213.5 Mathematica DSolve solution

Solving time : `0.054` (sec)
Leaf size : 37

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2+6)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4} e^{-x(x+2i)} \left(4c_1 - ic_2 e^{4ix}\right)$$

1.214 problem 217

Internal problem ID [8352]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 217
Date solved : `Monday, October 21, 2024 at 05:06:27 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-z^2+1\right) y''-3zy'+\lambda y=0$$

1.214.1 Solved as second order ode using Kovacic algorithm

Time used: 0.456 (sec)

Writing the ode as

$$\left(-z^2+1\right) y''-3zy'+\lambda y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -z^2+1 \\ B &= -3z \\ C &= \lambda \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z)=ye^{\int \frac{B}{2A}\, dz}$$

Then (2) becomes

$$z''(z)=rz(z) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4\lambda z^2 + 3z^2 - 4\lambda - 6}{4(z^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4\lambda z^2 + 3z^2 - 4\lambda - 6$$
$$t = 4(z^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{4\lambda z^2 + 3z^2 - 4\lambda - 6}{4(z^2 - 1)^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 410: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(z^2 - 1)^2$. There is a pole at $z = 1$ of order 2. There is a pole at $z = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16 (z - 1)^2} + \frac{\frac{9}{16} + \frac{\lambda}{2}}{z - 1} - \frac{3}{16 (z + 1)^2} + \frac{-\frac{\lambda}{2} - \frac{9}{16}}{z + 1}$$

For the pole at $z = 1$ let b be the coefficient of $\frac{1}{(z-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $z = -1$ let b be the coefficient of $\frac{1}{(z+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{z^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4\lambda z^2 + 3z^2 - 4\lambda - 6}{4 (z^2 - 1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=1$. Hence

$$\begin{aligned}E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\ &= \{2\}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1,2,3\}$
-1	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(z)$), which is generated using

$$\begin{aligned}d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\ &= \frac{1}{2}(2-(1+(1)))\\ &= 0\end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{z-c}\\ &= \frac{1}{2}\left(\frac{1}{(z-(1))}+\frac{1}{(z-(-1))}\right)\\ &= \frac{1}{2z-2}+\frac{1}{2z+2}\end{aligned}$$

Now we search for a monic polynomial $p(z)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(z)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2z-2} + \frac{1}{2z+2}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2z-2} + \frac{1}{2z+2}\right) w + \frac{-4\lambda\, z^2 - 3z^2 + 4\lambda + 4}{4\left(z^2-1\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{z + 2\sqrt{\left(z^2-1\right)\left(\lambda+1\right)}}{2\left(z-1\right)\left(z+1\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(z) &= e^{\int \omega\, dz} \\ &= \mathrm{e}^{\int \frac{z+2\sqrt{\left(z^2-1\right)\left(\lambda+1\right)}}{2(z-1)(z+1)} dz} \\ &= \left(z^2-1\right)^{1/4} \left(\frac{\sqrt{\left(z^2-1\right)\left(\lambda+1\right)}\,\sqrt{\lambda+1} + \lambda z + z}{\sqrt{\lambda+1}}\right)^{\sqrt{\lambda+1}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3z}{-z^2+1}\, dz} \\ &= z_1 e^{-\frac{3\ln(z-1)}{4} - \frac{3\ln(z+1)}{4}} \\ &= z_1\left(\frac{1}{\left(z-1\right)^{3/4}\left(z+1\right)^{3/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(z^2-1)^{1/4} \left(\sqrt{\lambda+1}\,\left(z+\sqrt{z^2-1}\right)\right)^{\sqrt{\lambda+1}}}{(z-1)^{3/4}\,(z+1)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dz}}{y_1^2}\,dz$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3z}{-z^2+1}\,dz}}{(y_1)^2}\,dz \\ &= y_1 \int \frac{e^{-\frac{3\ln(z-1)}{2}-\frac{3\ln(z+1)}{2}}}{(y_1)^2}\,dz \\ &= y_1 \left(-\frac{\left(\sqrt{\lambda+1}\,\left(z+\sqrt{z^2-1}\right)\right)^{-2\sqrt{\lambda+1}}}{2\sqrt{\lambda+1}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(z^2-1)^{1/4} \left(\sqrt{\lambda+1}\,\left(z+\sqrt{z^2-1}\right)\right)^{\sqrt{\lambda+1}}}{(z-1)^{3/4}\,(z+1)^{3/4}}\right) + c_2 \left(\frac{(z^2-1)^{1/4} \left(\sqrt{\lambda+1}\,\left(z+\sqrt{z^2-1}\right)\right)^{\sqrt{\lambda+1}}}{(z-1)^{3/4}\,(z+1)^{3/4}}\left(-\frac{(\sqrt{\lambda}\right.\right. \end{aligned}$$

Will add steps showing solving for IC soon.

1.214.2 Maple step by step solution

Let's solve

$(-z^2+1)\left(\frac{d}{dz}y'\right) - 3zy' + \lambda y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dz}y'$

- Isolate 2nd derivative

$\frac{d}{dz}y' = \frac{\lambda y}{z^2-1} - \frac{3zy'}{z^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dz}y' + \frac{3zy'}{z^2-1} - \frac{\lambda y}{z^2-1} = 0$

□ Check to see if z_0 is a regular singular point

- ○ Define functions

$\left[P_2(z) = \frac{3z}{z^2-1}, P_3(z) = -\frac{\lambda}{z^2-1}\right]$

- ○ $(z+1)\cdot P_2(z)$ is analytic at $z=-1$

$$\left((z+1)\cdot P_2(z)\right)\bigg|_{z=-1} = \frac{3}{2}$$

- ○ $(z+1)^2\cdot P_3(z)$ is analytic at $z=-1$

$$\left((z+1)^2\cdot P_3(z)\right)\bigg|_{z=-1} = 0$$

- ○ $z=-1$is a regular singular point

Check to see if z_0 is a regular singular point

$z_0 = -1$

- Multiply by denominators

$(z^2-1)\left(\frac{d}{dz}y'\right) + 3zy' - \lambda y = 0$

- Change variables using $z = u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (3u-3)\left(\frac{d}{du}y(u)\right) - \lambda y(u) = 0$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- ○ Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- ○ Convert $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

$$u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- ○ Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+3+2r) + a_k(k^2+2kr+r^2+2k-\lambda+2r)\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $-2(k+1+r)\left(k+\frac{3}{2}+r\right) a_{k+1} + a_k(k^2+(2r+2)\,k+r^2+2r-\lambda) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k^2+2kr+r^2+2k-\lambda+2r)}{(k+1+r)(2k+3+2r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k^2+2k-\lambda)}{(k+1)(2k+3)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k^2+2k-\lambda)}{(k+1)(2k+3)}\right]$
- Revert the change of variables $u = z+1$

  $\left[y = \sum_{k=0}^{\infty} a_k(z+1)^k, a_{k+1} = \frac{a_k(k^2+2k-\lambda)}{(k+1)(2k+3)}\right]$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = \frac{a_k\left(k^2+k-\lambda-\frac{3}{4}\right)}{\left(k+\frac{1}{2}\right)(2k+2)}$
- Solution for $r = -\frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k^2+k-\lambda-\frac{3}{4}\right)}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$
- Revert the change of variables $u = z+1$

  $\left[y = \sum_{k=0}^{\infty} a_k(z+1)^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k^2+k-\lambda-\frac{3}{4}\right)}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k(z+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(z+1)^{k-\frac{1}{2}}\right), a_{k+1} = \frac{a_k(k^2+2k-\lambda)}{(k+1)(2k+3)}, b_{k+1} = \frac{b_k\left(k^2+k-\lambda-\frac{3}{4}\right)}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$

1.214.3 Maple trace

```
Methods for second order ODEs:
```

1.214.4 Maple dsolve solution

Solving time : 0.025 (sec)
Leaf size : 49

```
dsolve((-z^2+1)*diff(diff(y(z),z),z)-3*z*diff(y(z),z)+lambda*y(z) = 0,
       y(z),singsol=all)
```

$$y = \frac{c_2\left(z+\sqrt{z^2-1}\right)^{-\sqrt{\lambda+1}}+c_1\left(z+\sqrt{z^2-1}\right)^{\sqrt{\lambda+1}}}{\sqrt{z^2-1}}$$

1.214.5 Mathematica DSolve solution

Solving time : 0.061 (sec)
Leaf size : 54

```
DSolve[{(1-z^2)*D[y[z],{z,2}]-3*z*D[y[z],z]+\[Lambda]*y[z]==0,{}},
       y[z],z,IncludeSingularSolutions->True]
```

$$y(z) \to \frac{c_1 P^{\frac{1}{2}}_{\sqrt{\lambda+1}-\frac{1}{2}}(z)+c_2 Q^{\frac{1}{2}}_{\sqrt{\lambda+1}-\frac{1}{2}}(z)}{\sqrt[4]{z^2-1}}$$

1.215 problem 218

Internal problem ID [8353]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 218
Date solved : `Monday, October 21, 2024 at 05:06:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4zy'' + 2(1 - z)\, y' - y = 0$$

1.215.1 Solved as second order ode using Kovacic algorithm

Time used: 0.276 (sec)

Writing the ode as

$$4zy'' + (-2z + 2)\, y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4z \\ B &= -2z + 2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = y e^{\int \frac{B}{2A}\, dz}$$

Then (2) becomes

$$z''(z) = r z(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{z^2 + 2z - 3}{16z^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= z^2 + 2z - 3 \\ t &= 16z^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{z^2 + 2z - 3}{16z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 412: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} - \frac{3}{16z^2} + \frac{1}{8z}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\
&= \sum_{i=0}^{0} a_i z^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{1}{4z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{z^4} + \frac{2}{z^5} - \frac{9}{2z^6} + \frac{21}{2z^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{z^2 + 2z - 3}{16z^2} \\ &= Q + \frac{R}{16z^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{2z - 3}{16z^2}\right) \\ &= \frac{1}{16} + \frac{2z - 3}{16z^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is 2. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{8}\right) - (0) \\ &= \frac{1}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{8}}{\frac{1}{4}} - 0\right) &= \frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{8}}{\frac{1}{4}} - 0\right) &= -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{z^2 + 2z - 3}{16z^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{z-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{4z} + \left(\frac{1}{4}\right) \\
&= \frac{1}{4} + \frac{1}{4z} \\
&= \frac{z+1}{4z}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 0$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(z) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4} + \frac{1}{4z}\right)(0) + \left(\left(-\frac{1}{4z^2}\right) + \left(\frac{1}{4} + \frac{1}{4z}\right)^2 - \left(\frac{z^2+2z-3}{16z^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(z) &= p e^{\int \omega \, dz} \\
&= e^{\int \left(\frac{1}{4} + \frac{1}{4z}\right) dz} \\
&= z^{1/4} e^{\frac{z}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dz} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2z+2}{4z} \, dz} \\
&= z_1 e^{\frac{z}{4} - \frac{\ln(z)}{4}} \\
&= z_1 \left(\frac{e^{\frac{z}{4}}}{z^{1/4}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{z}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{y_1^2}\, dz$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2z+2}{4z}\, dz}}{(y_1)^2}\, dz \\ &= y_1 \int \frac{e^{\frac{z}{2}-\frac{\ln(z)}{2}}}{(y_1)^2}\, dz \\ &= y_1 \left(\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{\frac{z}{2}}\right) + c_2\left(\mathrm{e}^{\frac{z}{2}}\left(\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.215.2 Maple step by step solution

Let's solve

$4z\left(\frac{d}{dz}y'\right) + 2(1-z)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dz}y'$

- Isolate 2nd derivative

  $\frac{d}{dz}y' = \frac{y}{4z} + \frac{(z-1)y'}{2z}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dz}y' - \frac{(z-1)y'}{2z} - \frac{y}{4z} = 0$

□ Check to see if $z_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(z) = -\frac{z-1}{2z}, P_3(z) = -\frac{1}{4z}\right]$

- ◦ $z \cdot P_2(z)$ is analytic at $z = 0$

  $\left(z \cdot P_2(z)\right)\bigg|_{z=0} = \frac{1}{2}$

- ◦ $z^2 \cdot P_3(z)$ is analytic at $z = 0$

  $\left(z^2 \cdot P_3(z)\right)\bigg|_{z=0} = 0$

- ◦ $z = 0$is a regular singular point

  Check to see if $z_0 = 0$ is a regular singular point

  $z_0 = 0$

- Multiply by denominators

  $4z\left(\frac{d}{dz}y'\right) + (-2z+2)\, y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k z^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $z^m \cdot y'$ to series expansion for $m = 0..1$

  $z^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, z^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

  $z^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, z^{k+r}$

- ◦ Convert $z \cdot \left(\frac{d}{dz}y'\right)$ to series expansion

  $z \cdot \left(\frac{d}{dz}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r-1}$

- ◦ Shift index using $k- >k+1$

  $z \cdot \left(\frac{d}{dz}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, z^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(-1+2r)\, z^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)\,(2k+2r+1) - a_k(2k+2r+1)\right) z^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(a_{k+1}(k+1+r)-\frac{a_k}{2}\right)\left(k+r+\frac{1}{2}\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{2(k+1+r)}$
- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{2(k+1)}$
- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k z^k, a_{k+1}=\frac{a_k}{2(k+1)}\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=\frac{a_k}{2\left(k+\frac{3}{2}\right)}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k z^{k+\frac{1}{2}}, a_{k+1}=\frac{a_k}{2\left(k+\frac{3}{2}\right)}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k z^k\right)+\left(\sum_{k=0}^{\infty} b_k z^{k+\frac{1}{2}}\right), a_{k+1}=\frac{a_k}{2(k+1)}, b_{k+1}=\frac{b_k}{2\left(k+\frac{3}{2}\right)}\right]$

1.215.3 Maple trace

```
Methods for second order ODEs:
```

1.215.4 Maple dsolve solution

Solving time : `0.011` (sec)
Leaf size : 22

```
dsolve(4*z*diff(diff(y(z),z),z)+2*(1-z)*diff(y(z),z)-y(z) = 0,
       y(z),singsol=all)
```

$$y = e^{\frac{z}{2}}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right)c_1 + c_2\right)$$

1.215.5 Mathematica DSolve solution

Solving time : 0.203 (sec)
Leaf size : 34

```
DSolve[{4*z*D[y[z],{z,2}]+2*(1-z)*D[y[z],z]-y[z]==0,{}},
       y[z],z,IncludeSingularSolutions->True]
```

$$y(z) \to e^{z/2}\left(c_1 - \sqrt{2}c_2\Gamma\left(\frac{1}{2}, \frac{z}{2}\right)\right)$$

1.216 problem 219

Internal problem ID [8354]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 219
Date solved : `Monday, October 21, 2024 at 05:06:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$f'' + 2(z-1) f' + 4f = 0$$

1.216.1 Solved as second order ode using Kovacic algorithm

Time used: 0.269 (sec)

Writing the ode as

$$f'' + (2z-2) f' + 4f = 0 \tag{1}$$
$$Af'' + Bf' + Cf = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2z-2 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = f e^{\int \frac{B}{2A} dz}$$

Then (2) becomes

$$z''(z) = r z(z) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{z^2 - 2z - 2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = z^2 - 2z - 2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(z) = (z^2 - 2z - 2)\, z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then f is found using the inverse transformation

$$f = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 414: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\
&= \sum_{i=0}^{1} a_i z^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $z^v = z^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx z - 1 - \frac{3}{2z} - \frac{3}{2z^2} - \frac{21}{8z^3} - \frac{39}{8z^4} - \frac{159}{16z^5} - \frac{339}{16z^6} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = 1
$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i z^i \\
&= z - 1
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = z^2 - 2z + 1
$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{z^2-2z-2}{1} \\ &= Q + \frac{R}{1} \\ &= (z^2-2z-2) + (0) \\ &= z^2-2z-2 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{z}$ in the quotient is -2. Now b can be found.

$$\begin{aligned} b &= (-2) - (1) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= z-1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{1} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{1} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = z^2-2z-2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$z-1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)(z - 1) \\ &= 1 - z \\ &= 1 - z \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(z) = z + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2(1 - z)(1) + \left((-1) + (1 - z)^2 - \left(z^2 - 2z - 2\right)\right) &= 0 \\ 2 + 2a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$p(z) = z - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(z) &= p e^{\int \omega \, dz} \\
&= (z-1)\,\mathrm{e}^{\int (1-z) dz} \\
&= (z-1)\,\mathrm{e}^{z-\frac{1}{2}z^2} \\
&= (z-1)\,\mathrm{e}^{-\frac{z(-2+z)}{2}}
\end{aligned}$$

The first solution to the original ode in f is found from

$$\begin{aligned}
f_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dz} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2z-2}{1}\, dz} \\
&= z_1 e^{z-\frac{1}{2}z^2} \\
&= z_1 \left(\mathrm{e}^{-\frac{z(-2+z)}{2}}\right)
\end{aligned}$$

Which simplifies to

$$f_1 = \mathrm{e}^{-z(-2+z)}(z-1)$$

The second solution f_2 to the original ode is found using reduction of order

$$f_2 = f_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{f_1^2}\, dz$$

Substituting gives

$$\begin{aligned}
f_2 &= f_1 \int \frac{e^{\int -\frac{2z-2}{1}\, dz}}{(f_1)^2}\, dz \\
&= f_1 \int \frac{e^{-z^2+2z}}{(f_1)^2}\, dz \\
&= f_1 \left(-\frac{\mathrm{e}^{(z-1)^2-1}}{z-1} - i\sqrt{\pi}\,\mathrm{e}^{-1}\,\mathrm{erf}\left(i(z-1)\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
f &= c_1 f_1 + c_2 f_2 \\
&= c_1\left(\mathrm{e}^{-z(-2+z)}(z-1)\right) + c_2\left(\mathrm{e}^{-z(-2+z)}(z-1)\left(-\frac{\mathrm{e}^{(z-1)^2-1}}{z-1} - i\sqrt{\pi}\,\mathrm{e}^{-1}\,\mathrm{erf}\left(i(z-1)\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.216.2 Maple step by step solution

Let's solve

$\frac{d}{dz}f' + 2(z-1)\, f' + 4f = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dz}f'$

- Isolate 2nd derivative

  $\frac{d}{dz}f' = -2(z-1)\, f' - 4f$

- Group terms with f on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dz}f' + (2z-2)\, f' + 4f = 0$

- Assume series solution for f

  $f = \sum_{k=0}^{\infty} a_k z^k$

□ Rewrite DE with series expansions

  ◦ Convert $z^m \cdot f'$ to series expansion for $m = 0..1$

  $z^m \cdot f' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, z^{k-1+m}$

  ◦ Shift index using $k- > k+1-m$

  $z^m \cdot f' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, z^k$

  ◦ Convert $\frac{d}{dz}f'$ to series expansion

  $\frac{d}{dz}f' = \sum_{k=2}^{\infty} a_k k(k-1)\, z^{k-2}$

  ◦ Shift index using $k- > k+2$

  $\frac{d}{dz}f' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, z^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_{k+1}(k+1) + 2a_k(k+2)\right) z^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $k^2 a_{k+2} + (2a_k - 2a_{k+1} + 3a_{k+2})\, k + 4a_k - 2a_{k+1} + 2a_{k+2} = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[f = \sum_{k=0}^{\infty} a_k z^k, a_{k+2} = -\frac{2(k a_k - k a_{k+1} + 2a_k - a_{k+1})}{k^2+3k+2}\right]$

1.216.3 Maple trace

```
Methods for second order ODEs:
```

1.216.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 42

```
dsolve(diff(diff(f(z),z),z)+2*(z-1)*diff(f(z),z)+4*f(z) = 0,
       f(z),singsol=all)
```

$$f = \sqrt{\pi}\, \operatorname{erf}\left(i(z-1)\right) c_2(z-1)\, \mathrm{e}^{-(z-1)^2} + c_1\, \mathrm{e}^{-z(-2+z)}(z-1) - ic_2$$

1.216.5 Mathematica DSolve solution

Solving time : 0.203 (sec)
Leaf size : 72

```
DSolve[{D[ f[z],{z,2}]+2*(z-a)*D[ f[z],z]+4*f[z]==0,{}},
       f[z],z,IncludeSingularSolutions->True]
```

$$f(z) \to e^{z(2a-z)}\left(-\sqrt{\pi}c_2\sqrt{(a-z)^2}\operatorname{erfi}\left(\sqrt{(a-z)^2}\right) + c_2e^{(a-z)^2} - 2ac_1 + 2c_1z\right)$$

1.217 problem 220

Internal problem ID [8355]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 220
Date solved : `Monday, October 21, 2024 at 05:06:30 PM`
CAS classification : `[_Lienard]`

Solve

$$zy'' - 2y' + zy = 0$$

1.217.1 Solved as second order ode using Kovacic algorithm

Time used: 0.285 (sec)

Writing the ode as

$$zy'' - 2y' + zy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= z \\ B &= -2 \\ C &= z \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\, dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-z^2 + 2}{z^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -z^2 + 2$$
$$t = z^2$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{-z^2 + 2}{z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 416: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{z^2}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{0} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{z^2} - \frac{i}{2z^4} - \frac{i}{2z^6} - \frac{5i}{8z^8} - \frac{7i}{8z^{10}} - \frac{21i}{16z^{12}} - \frac{33i}{16z^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-z^2 + 2}{z^2} \\ &= Q + \frac{R}{z^2} \\ &= (-1) + \left(\frac{2}{z^2}\right) \\ &= -1 + \frac{2}{z^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-z^2 + 2}{z^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{z - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{z} + (-)\,(i) \\
&= -\frac{1}{z} - i \\
&= -\frac{1}{z} - i
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(z) = z + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{z} - i\right)(1) + \left(\left(\frac{1}{z^2}\right) + \left(-\frac{1}{z} - i\right)^2 - \left(\frac{-z^2 + 2}{z^2}\right)\right) = 0
$$

$$
\frac{2ia_0 - 2}{z} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -i\}
$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$
p(z) = z - i
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(z) &= p e^{\int \omega \, dz} \\
&= (z - i)\,\mathrm{e}^{\int \left(-\frac{1}{z} - i\right) dz} \\
&= (z - i)\,\mathrm{e}^{-\ln(z) - iz} \\
&= \frac{(z - i)\,\mathrm{e}^{-iz}}{z}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dz} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{z}\,dz} \\
&= z_1 e^{\ln(z)} \\
&= z_1(z)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (z - i)\,\mathrm{e}^{-iz}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dz}}{y_1^2}\,dz
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{z}\,dz}}{(y_1)^2}\,dz \\
&= y_1 \int \frac{e^{2\ln(z)}}{(y_1)^2}\,dz \\
&= y_1 \left(\frac{(iz-1)\,\mathrm{e}^{2iz}}{-2z+2i}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((z-i)\,\mathrm{e}^{-iz}\right) + c_2\left((z-i)\,\mathrm{e}^{-iz}\left(\frac{(iz-1)\,\mathrm{e}^{2iz}}{-2z+2i}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.217.2 Maple step by step solution

Let's solve

$z\left(\frac{d}{dz}y'\right) - 2y' + zy = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dz}y'$

- Isolate 2nd derivative

$\frac{d}{dz}y' = -y + \frac{2y'}{z}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dz}y' - \frac{2y'}{z} + y = 0$

□ Check to see if $z_0 = 0$ is a regular singular point

- ○ Define functions

$\left[P_2(z) = -\frac{2}{z}, P_3(z) = 1\right]$

- ○ $z \cdot P_2(z)$ is analytic at $z = 0$

$\left.(z \cdot P_2(z))\right|_{z=0} = -2$

- ○ $z^2 \cdot P_3(z)$ is analytic at $z = 0$

$\left.(z^2 \cdot P_3(z))\right|_{z=0} = 0$

- ○ $z = 0$is a regular singular point

Check to see if $z_0 = 0$ is a regular singular point

$z_0 = 0$

- Multiply by denominators

$z\left(\frac{d}{dz}y'\right) - 2y' + zy = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k z^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $z \cdot y$ to series expansion

$z \cdot y = \sum_{k=0}^{\infty} a_k z^{k+r+1}$

- ○ Shift index using $k- >k - 1$

$z \cdot y = \sum_{k=1}^{\infty} a_{k-1} z^{k+r}$

- ○ Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k (k+r)\, z^{k+r-1}$$

- Shift index using $k- >k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, z^{k+r}$$

- Convert $z \cdot \left(\frac{d}{dz}y'\right)$ to series expansion

$$z \cdot \left(\frac{d}{dz}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, z^{k+r-1}$$

- Shift index using $k- >k+1$

$$z \cdot \left(\frac{d}{dz}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, z^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r)\, z^{-1+r} + a_1(1+r)\,(-2+r)\, z^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k-2+r) + a_{k-1}\right) z^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term must be 0

$$a_1(1+r)\,(-2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+r+1)\,(k-2+r) + a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$a_{k+2}(k+2+r)\,(k+r-1) + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{(k+2+r)(k+r-1)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k}{(k+2)(k-1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k z^k, a_{k+2} = -\frac{a_k}{(k+2)(k-1)}, -2a_1 = 0\right]$$

- Recursion relation for $r = 3$

$$a_{k+2} = -\frac{a_k}{(k+5)(k+2)}$$

- Solution for $r = 3$

$$\left[y=\sum_{k=0}^{\infty} a_k z^{k+3}, a_{k+2}=-\frac{a_k}{(k+5)(k+2)}, 4a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k z^{k}\right)+\left(\sum_{k=0}^{\infty} b_k z^{k+3}\right), a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, -2a_1=0, b_{k+2}=-\frac{b_k}{(k+5)(k+2)}, 4b_1=0\right]$$

1.217.3 Maple trace

```
Methods for second order ODEs:
```

1.217.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 23

```
dsolve(z*diff(diff(y(z),z),z)-2*diff(y(z),z)+z*y(z) = 0,
       y(z),singsol=all)
```

$$y=(c_1 z+c_2)\cos(z)+\sin(z)(c_2 z-c_1)$$

1.217.5 Mathematica DSolve solution

Solving time : `0.077` (sec)
Leaf size : 39

```
DSolve[{z*D[y[z],{z,2}]-2*D[y[z],z]+z*y[z]==0,{}},
       y[z],z,IncludeSingularSolutions->True]
```

$$y(z)\to -\sqrt{\frac{2}{\pi}}((c_1 z+c_2)\cos(z)+(c_2 z-c_1)\sin(z))$$

1.218 problem 221

Internal problem ID [8356]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 221
Date solved : `Monday, October 21, 2024 at 05:06:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$zy'' + (2z-3)\,y' + \frac{4y}{z} = 0$$

1.218.1 Solved as second order ode using Kovacic algorithm

Time used: 0.308 (sec)

Writing the ode as

$$zy'' + (2z-3)\,y' + \frac{4y}{z} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= z \\ B &= 2z-3 \\ C &= \frac{4}{z} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4z^2 - 12z - 1}{4z^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4z^2 - 12z - 1$$
$$t = 4z^2$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{4z^2 - 12z - 1}{4z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A} \, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 418: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{1}{4z^2} - \frac{3}{z}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{0} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{3}{2z} - \frac{5}{4z^2} - \frac{15}{8z^3} - \frac{115}{32z^4} - \frac{495}{64z^5} - \frac{2285}{128z^6} - \frac{11055}{256z^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4z^2 - 12z - 1}{4z^2} \\ &= Q + \frac{R}{4z^2} \\ &= (1) + \left(\frac{-12z - 1}{4z^2}\right) \\ &= 1 + \frac{-12z - 1}{4z^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is -12. Dividing this by leading coefficient in t which is 4 gives -3. Now b can be

found.

$$
\begin{aligned}
b &= (-3) - (0)\\
&= -3
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{1} - 0\right) &&= -\frac{3}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{1} - 0\right) &&= \frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4z^2 - 12z - 1}{4z^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right)\\
&= \frac{3}{2} - \left(\frac{1}{2}\right)\\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z - c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{z-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2z} + (-)\,(1) \\
&= \frac{1}{2z} - 1 \\
&= \frac{1}{2z} - 1
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}
$$

Let

$$
p(z) = z + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2z} - 1\right)(1) + \left(\left(-\frac{1}{2z^2}\right) + \left(\frac{1}{2z} - 1\right)^2 - \left(\frac{4z^2 - 12z - 1}{4z^2}\right)\right) &= 0 \\
\frac{1 + 2a_0}{z} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{1}{2}\right\}
$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$
p(z) = z - \frac{1}{2}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(z) &= p e^{\int \omega \, dz} \\
&= \left(z - \frac{1}{2}\right) e^{\int \left(\frac{1}{2z} - 1\right) dz} \\
&= \left(z - \frac{1}{2}\right) e^{-z + \frac{\ln(z)}{2}} \\
&= \frac{(-1 + 2z)\sqrt{z}\, e^{-z}}{2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dz} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2z-3}{z}\,dz} \\
&= z_1 e^{-z+\frac{3\ln(z)}{2}} \\
&= z_1 \left(z^{3/2}\mathrm{e}^{-z}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{z^2 \mathrm{e}^{-2z}(-1+2z)}{2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dz}}{y_1^2}\,dz
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2z-3}{z}\,dz}}{(y_1)^2}\,dz \\
&= y_1 \int \frac{e^{-2z+3\ln(z)}}{(y_1)^2}\,dz \\
&= y_1 \left(-4\ \mathrm{Ei}_1\left(-2z\right) - \frac{4\,\mathrm{e}^{2z}}{-1+2z}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{z^2 \mathrm{e}^{-2z}(-1+2z)}{2}\right) + c_2 \left(\frac{z^2 \mathrm{e}^{-2z}(-1+2z)}{2}\left(-4\ \mathrm{Ei}_1\left(-2z\right) - \frac{4\,\mathrm{e}^{2z}}{-1+2z}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.218.2 Maple step by step solution

Let's solve

$z\left(\frac{d}{dz}y'\right)+\left(2z-3\right)y'+\frac{4y}{z}=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dz}y'$

- Isolate 2nd derivative

$\frac{d}{dz}y'=-\frac{4y}{z^{2}}-\frac{(2z-3)y'}{z}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dz}y'+\frac{(2z-3)y'}{z}+\frac{4y}{z^{2}}=0$

□ Check to see if $z_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(z)=\frac{2z-3}{z}, P_3(z)=\frac{4}{z^{2}}\right]$

  ◦ $z\cdot P_2(z)$ is analytic at $z=0$

  $\left(z\cdot P_2(z)\right)\bigg|_{z=0}=-3$

  ◦ $z^{2}\cdot P_3(z)$ is analytic at $z=0$

  $\left(z^{2}\cdot P_3(z)\right)\bigg|_{z=0}=4$

  ◦ $z=0$is a regular singular point

Check to see if $z_0=0$ is a regular singular point

$z_0=0$

- Multiply by denominators

$z^{2}\left(\frac{d}{dz}y'\right)+z(2z-3)y'+4y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k z^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $z^m\cdot y'$ to series expansion for $m=1..2$

  $z^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,z^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $z^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,z^{k+r}$

  ◦ Convert $z^{2}\cdot\left(\frac{d}{dz}y'\right)$ to series expansion

$$z^2 \cdot \left(\frac{d}{dz}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) z^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)^2 z^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r-2)^2 + 2a_{k-1}(k+r-1)\right) z^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 + 2a_{k-1}(k+r-1) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+r-1)^2 + 2a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k(k+r)}{(k+r-1)^2}$
- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{2a_k(k+2)}{(k+1)^2}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k z^{k+2}, a_{k+1} = -\frac{2a_k(k+2)}{(k+1)^2}\right]$

1.218.3 Maple trace

```
Methods for second order ODEs:
```

1.218.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 36

```
dsolve(z*diff(diff(y(z),z),z)+(2*z-3)*diff(y(z),z)+4/z*y(z) = 0,
       y(z),singsol=all)
```

$$y = 2\left(c_2\, \mathrm{e}^{-2z}\left(z-\frac{1}{2}\right)\mathrm{Ei}_1\left(-2z\right) + c_1\left(z-\frac{1}{2}\right)\mathrm{e}^{-2z} + \frac{c_2}{2}\right) z^2$$

1.218.5 Mathematica DSolve solution

Solving time : `1.054` (sec)
Leaf size : 47

```
DSolve[{z*D[y[z],{z,2}]+(2*z-3)*D[y[z],z]+4/z*y[z]==0,{}},
       y[z],z,IncludeSingularSolutions->True]
```

$$y(z) \to -\frac{1}{2}e^{-2z}z^2\left(4c_2(1-2z)\operatorname{ExpIntegralEi}(2z) - 2c_1z + 4c_2e^{2z} + c_1\right)$$

1.219 problem 222

Internal problem ID [8357]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 222
Date solved : `Monday, October 21, 2024 at 05:06:32 PM`
CAS classification : `[_erf]`

Solve

$$y'' + 2xy' + 4y = 0$$

1.219.1 Solved as second order ode using Kovacic algorithm

Time used: 0.210 (sec)

Writing the ode as

$$y'' + 2xy' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 3}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 3 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(x^2 - 3\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 420: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 2\\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{3}{2x} - \frac{9}{8x^3} - \frac{27}{16x^5} - \frac{405}{128x^7} - \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} - \frac{72171}{2048x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i\\ &= x \qquad (10)\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 3}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2 - 3) + (0) \\ &= x^2 - 3 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -3. Now b can be found.

$$\begin{aligned} b &= (-3) - (0) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-3}{1} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-3}{1} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 - 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\,(x) \\ &= -x \\ &= -x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2(-x)\,(1) + \left((-1) + (-x)^2 - \left(x^2 - 3\right)\right) &= 0 \\ 2a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, \mathrm{e}^{\int -x dx} \\
&= (x) \, \mathrm{e}^{-\frac{x^2}{2}} \\
&= x \, \mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x^2} x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x^2}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{x^2}}{x} + \sqrt{\pi} \, \mathrm{erfi}(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x^2} x\right) + c_2 \left(\mathrm{e}^{-x^2} x \left(-\frac{\mathrm{e}^{x^2}}{x} + \sqrt{\pi} \, \mathrm{erfi}(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.219.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 2xy' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2} + 2a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k}{k+1}\right]$

1.219.3 Maple trace

```
Methods for second order ODEs:
```

1.219.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 25

```
dsolve(diff(diff(y(x),x),x)+2*x*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = x\left(c_2\sqrt{\pi}\,\operatorname{erfi}\left(x\right) + c_1\right) \mathrm{e}^{-x^2} - c_2$$

1.219.5 Mathematica DSolve solution

Solving time : 0.054 (sec)
Leaf size : 51

```
DSolve[{D[y[x],{x,2}]+2*x*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x^2}\left(-\sqrt{\pi}c_2\sqrt{x^2}\text{erfi}\left(\sqrt{x^2}\right) + c_2e^{x^2} + 2c_1x\right)$$

1.220 problem 223

Internal problem ID [8358]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 223
Date solved : `Monday, October 21, 2024 at 05:06:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 3y = 0$$

1.220.1 Solved as second order ode using Kovacic algorithm

Time used: 0.293 (sec)

Writing the ode as

$$y'' + xy' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 10$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 422: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned}d &= \alpha_\infty^- \\ &= 2\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\ a_1 x + 2a_0 + 2 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= (x^2 - 1)\, e^{\int -\frac{x}{2} dx} \\
&= (x^2 - 1)\, e^{-\frac{x^2}{4}} \\
&= (x^2 - 1)\, e^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(e^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{x^2}{2}} (x^2 - 1)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2 - 1)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} \left(x^2-1\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} \left(x^2-1\right) \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^2-1\right)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.220.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+3)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $\left(k^2+3k+2\right) a_{k+2} + a_k(k+3) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+3)}{k^2+3k+2}\right]$

1.220.3 Maple trace

```
Methods for second order ODEs:
```

1.220.4 Maple dsolve solution

Solving time : 0.081 (sec)
Leaf size : 42

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)+3*y(x) = 0,
       y(x),singsol=all)
```

$$
y = -(x-1)(x+1)\left(c_1 \operatorname{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}\,\sqrt{\pi}-c_2\right)\mathrm{e}^{-\frac{x^2}{2}}+2c_1x
$$

1.220.5 Mathematica DSolve solution

Solving time : 0.169 (sec)
Leaf size : 65

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to \frac{1}{4}e^{-\frac{x^2}{2}}\left(\sqrt{2\pi}c_2(x^2-1)\operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right)+4c_1(x^2-1)-2c_2e^{\frac{x^2}{2}}x\right)
$$

1.221 problem 224

Internal problem ID [8359]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 224
Date solved : `Monday, October 21, 2024 at 05:06:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2 y' - 3xy = 0$$

1.221.1 Solved as second order ode using Kovacic algorithm

Time used: 0.265 (sec)

Writing the ode as

$$y'' - x^2 y' - 3xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= -3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3 + 8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x(x^3 + 8)$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3 + 8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 424: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} + \frac{2}{x} - \frac{4}{x^4} + \frac{16}{x^7} - \frac{80}{x^{10}} + \frac{448}{x^{13}} - \frac{2688}{x^{16}} + \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x(x^3+8)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^4 + 2x\right) + (0) \\
&= \frac{1}{4}x^4 + 2x
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 2. Now b can be found.

$$\begin{aligned}
b &= (2) - (0) \\
&= 2
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^2}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 2\right) &&= 1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 2\right) &&= -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3+8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x^2}{2}\right) \\ &= \frac{x^2}{2} \\ &= \frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{x^2}{2}\right)(1) + \left((x) + \left(\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3+8)}{4}\right)\right) &= 0 \\ -xa_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, \mathrm{e}^{\int \frac{x^2}{2} dx} \\ &= (x) \, \mathrm{e}^{\frac{x^3}{6}} \\ &= x \, \mathrm{e}^{\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x^2}{1} \, dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x^3}{3}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x^3}{3}} x\right) + c_2 \left(\mathrm{e}^{\frac{x^3}{3}} x \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.221.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - x^2 y' - 3xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - ○ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - ○ Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  - ○ Shift index using $k- >k-1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  - ○ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - ○ Shift index using $k- >k+2$

$\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_{k-1}(k+2)\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)(k a_{k+2} - a_{k-1} + a_{k+2}) = 0$

- Shift index using $k- > k+1$

  $(k+3)((k+1)\, a_{k+3} - a_k + a_{k+3}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k+2}, 2a_2 = 0\right]$

1.221.3 Maple trace

```
Methods for second order ODEs:
```

1.221.4 Maple dsolve solution

Solving time : `0.063` (sec)
Leaf size : 58

```
dsolve(diff(diff(y(x),x),x)-x^2*diff(y(x),x)-3*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{9\,\mathrm{WhittakerM}\left(\frac{1}{3}, \frac{5}{6}, \frac{x^3}{3}\right) \mathrm{e}^{\frac{x^3}{6}} c_2\, x^3 + 9c_1\, \mathrm{e}^{\frac{x^3}{3}} x^2 + 5\, 3^{2/3} c_2 (x^3)^{1/3} (x^3 + 2)}{9x}$$

1.221.5 Mathematica DSolve solution

Solving time : 0.092 (sec)
Leaf size : 51

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]-3*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{9} e^{\frac{x^3}{3}} \left(9 c_1 x - 3^{2/3} c_2 \sqrt[3]{x^3} \Gamma\left(-\frac{1}{3}, \frac{x^3}{3}\right)\right)$$

1.222 problem 225

Internal problem ID [8360]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 225
Date solved : `Monday, October 21, 2024 at 05:06:35 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-4x^2+1\right)y''-20xy'-16y=0$$

1.222.1 Solved as second order ode using Kovacic algorithm

Time used: 0.278 (sec)

Writing the ode as

$$\left(-4x^2+1\right)y''-20xy'-16y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-4x^2+1\\B&=-20x\\C&=-16\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x^2 + 6}{(4x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4x^2 + 6$$
$$t = (4x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^2 + 6}{(4x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 426: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (4x^2 - 1)^2$. There is a pole at $x = \frac{1}{2}$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\left(x + \frac{1}{2}\right)^2} + \frac{7}{8\left(x + \frac{1}{2}\right)} + \frac{5}{16\left(x - \frac{1}{2}\right)^2} - \frac{7}{8\left(x - \frac{1}{2}\right)}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4x^2+6}{(4x^2-1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4x^2+6}{(4x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)} + (-)\,(0) \\
&= -\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)} \\
&= -\frac{2x}{4x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right)(1) + \left(\left(\frac{1}{4\left(x-\frac{1}{2}\right)^2} + \frac{1}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right)^2 -
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right) dx} \\
&= (x)\, \frac{1}{\left((2x-1)\,(2x+1)\right)^{1/4}} \\
&= \frac{x}{\left(4x^2-1\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-20x}{-4x^2+1}\,dx} \\
&= z_1 e^{-\frac{5\ln\left(4x^2-1\right)}{4}} \\
&= z_1\left(\frac{1}{\left(4x^2-1\right)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{\left(4x^2-1\right)^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-20x}{-4x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(4x^2-1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{\left(4x^2-1\right)^{3/2}}{x} - 4x\sqrt{4x^2-1} + \ln\left(x\sqrt{4}+\sqrt{4x^2-1}\right)\sqrt{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{\left(4x^2-1\right)^{3/2}}\right) \\
&\quad + c_2\left(\frac{x}{\left(4x^2-1\right)^{3/2}}\left(\frac{\left(4x^2-1\right)^{3/2}}{x} - 4x\sqrt{4x^2-1} + \ln\left(x\sqrt{4}+\sqrt{4x^2-1}\right)\sqrt{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.222.2 Maple step by step solution

Let's solve

$\left(-4x^2+1\right)\left(\frac{d}{dx}y'\right)-20xy'-16y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{16y}{4x^2-1}-\frac{20xy'}{4x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{20xy'}{4x^2-1}+\frac{16y}{4x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{20x}{4x^2-1}, P_3(x)=\frac{16}{4x^2-1}\right]$

- ◦ $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}}=\frac{5}{2}$

- ◦ $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}}=0$

- ◦ $x=-\frac{1}{2}$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-\frac{1}{2}$

- Multiply by denominators

$\left(4x^2-1\right)\left(\frac{d}{dx}y'\right)+20xy'+16y=0$

- Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

$\left(4u^2-4u\right)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\left(20u-10\right)\left(\frac{d}{du}y(u)\right)+16y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty}a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)$ to series expansion for $m =1..2$

$$u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- Shift index using $k->k +2-m$

$$u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$-2 a_{0} r \left(3+2 r \right) u^{-1+r}+\left(\sum_{k =0}^{\infty}\left(-2 a_{k +1} \left(k +1+r \right) \left(2 k +5+2 r \right)+4 a_{k} \left(k +r +2\right)^{2}\right) u^{k +r}\right)=0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$-2 r \left(3+2 r \right)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4 a_{k} \left(k +r +2\right)^{2}-4 \left(k +\frac{5}{2}+r \right) a_{k +1} \left(k +1+r \right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k +1}=\frac{2 a_{k} \left(k +r +2\right)^{2}}{\left(2 k +5+2 r \right) \left(k +1+r \right)}$$

- Recursion relation for $r =0$

$$a_{k +1}=\frac{2 a_{k} \left(k +2\right)^{2}}{\left(2 k +5\right) \left(k +1\right)}$$

- Solution for $r =0$

$$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k}, a_{k +1}=\frac{2 a_{k} \left(k +2\right)^{2}}{\left(2 k +5\right) \left(k +1\right)}\right]$$

- Revert the change of variables $u =x +\frac{1}{2}$

$$\left[y =\sum_{k =0}^{\infty}a_{k} \left(x +\frac{1}{2}\right)^{k}, a_{k +1}=\frac{2 a_{k} \left(k +2\right)^{2}}{\left(2 k +5\right) \left(k +1\right)}\right]$$

- Recursion relation for $r =-\frac{3}{2}$

$$a_{k +1}=\frac{2 a_{k} \left(k +\frac{1}{2}\right)^{2}}{\left(2 k +2\right) \left(k -\frac{1}{2}\right)}$$

- Solution for $r =-\frac{3}{2}$

$$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k -\frac{3}{2}}, a_{k +1}=\frac{2 a_{k} \left(k +\frac{1}{2}\right)^{2}}{\left(2 k +2\right) \left(k -\frac{1}{2}\right)}\right]$$

- Revert the change of variables $u =x +\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k-\frac{3}{2}}, a_{k+1}=\frac{2a_k \left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k \left(x+\frac{1}{2}\right)^{k-\frac{3}{2}}\right), a_{k+1}=\frac{2a_k (k+2)^2}{(2k+5)(k+1)}, b_{k+1}=\frac{2b_k \left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$$

1.222.3 Maple trace

```
Methods for second order ODEs:
```

1.222.4 Maple dsolve solution

Solving time : `0.011` (sec)
Leaf size : 48

```
dsolve((-4*x^2+1)*diff(diff(y(x),x),x)-20*x*diff(y(x),x)-16*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{2c_2 \ln\left(2x+\sqrt{4x^2-1}\right)x-\sqrt{4x^2-1}\,c_2+c_1 x}{\left(4x^2-1\right)^{3/2}}$$

1.222.5 Mathematica DSolve solution

Solving time : `0.232` (sec)
Leaf size : 68

```
DSolve[{(1-4*x^2)*D[y[x],{x,2}]-20*x*D[y[x],x]-16*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{-2c_2 x \arctan\left(\frac{2x}{\sqrt{1-4x^2}}\right)-c_2\sqrt{1-4x^2}+c_1 x}{\sqrt[4]{1-4x^2}\left(4x^2-1\right)^{5/4}}$$

1.223 problem 226

Internal problem ID [8361]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 226
Date solved : `Monday, October 21, 2024 at 05:06:36 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^2-1\right) y''-6xy'+12y=0$$

1.223.1 Solved as second order ode using Kovacic algorithm

Time used: 0.218 (sec)

Writing the ode as

$$\left(x^2-1\right) y''-6xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= -6x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 428: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{15}{4(x-1)} + \frac{15}{4(x-1)^2} + \frac{15}{4(x+1)^2} + \frac{15}{4(x+1)}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{\left(x^2-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2\,(x-1)} + \frac{5}{2\,(x+1)} + (-)\,(0) \\ &= -\frac{3}{2\,(x-1)} + \frac{5}{2\,(x+1)} \\ &= \frac{x-4}{x^2-1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2(x-1)} + \frac{5}{2(x+1)}\right)(0) + \left(\left(\frac{3}{2(x-1)^2} - \frac{5}{2(x+1)^2}\right) + \left(-\frac{3}{2(x-1)} + \frac{5}{2(x+1)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{3}{2(x-1)} + \frac{5}{2(x+1)}\right) dx} \\
&= \frac{(x+1)^{5/2}}{(x-1)^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6x}{x^2-1}\, dx} \\
&= z_1 e^{\frac{3\ln(x-1)}{2} + \frac{3\ln(x+1)}{2}} \\
&= z_1 \left((x-1)^{3/2} (x+1)^{3/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (x+1)^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x}{x^2-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x-1)+3\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x(x^2+1)\,\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7\,(x-1)^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x+1)^4\right) + c_2\left((x+1)^4\left(-\frac{x(x^2+1)\,\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7\,(x-1)^3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.223.2 Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d}{dx}y'\right) - 6xy' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{12y}{x^2-1} + \frac{6xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{6xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{6x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = -3$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d}{dx}y'\right) - 6xy' + 12y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-6u+6)\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0 r(-4+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r-3) + a_k(k+r-3)(k+r-4)\right)u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 4\}$

- Each term in the series must be 0, giving the recursion relation

  $\left((-2k-2r-2)\,a_{k+1} + a_k(k+r-4)\right)(k+r-3) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r-4)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 4$

  $a_{k+1} = \frac{a_k(k-4)}{2(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -2a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{3a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{3a_0}{2}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2}{3}$
- Express in terms of a_0

  $a_3 = -\frac{a_0}{2}$
- Apply recursion relation for $k = 3$

  $a_4 = -\frac{a_3}{8}$
- Express in terms of a_0

  $a_4 = \frac{a_0}{16}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 2u + \frac{3}{2}u^2 - \frac{1}{2}u^3 + \frac{1}{16}u^4\right)$
- Revert the change of variables $u = x + 1$

  $\left[y = \frac{a_0(x-1)^4}{16}\right]$
- Recursion relation for $r = 4$

  $a_{k+1} = \frac{a_k k}{2(k+5)}$
- Solution for $r = 4$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+4}, a_{k+1} = \frac{a_k k}{2(k+5)}\right]$
- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+4}, a_{k+1} = \frac{a_k k}{2(k+5)}\right]$
- Combine solutions and rename parameters

  $\left[y = \frac{a_0(x-1)^4}{16} + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+4}\right), b_{k+1} = \frac{b_k k}{2(k+5)}\right]$

1.223.3 Maple trace

```
Methods for second order ODEs:
```

1.223.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 25

```
dsolve((x^2-1)*diff(diff(y(x),x),x)-6*x*diff(y(x),x)+12*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2 x^4 + c_1 x^3 + 6c_2 x^2 + c_1 x + c_2$$

1.223.5 Mathematica DSolve solution

Solving time : 0.179 (sec)
Leaf size : 45

```
DSolve[{(x^2-1)*D[y[x],{x,2}]-6*x*D[y[x],x]+12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{x^2-1}(c_2 x(x^2+1) + c_1(x-1)^4)}{\sqrt{1-x^2}}$$

1.224 problem 227

Internal problem ID [8362]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 227
Date solved : `Monday, October 21, 2024 at 05:06:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + (2 + x)\, y = 0$$

1.224.1 Solved as second order ode using Kovacic algorithm

Time used: 0.297 (sec)

Writing the ode as

$$y'' + xy' + (2 + x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 + x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 4x - 6}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 4x - 6 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 430: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - 1 - \frac{5}{2x} - \frac{5}{x^2} - \frac{65}{4x^3} - \frac{115}{2x^4} - \frac{885}{4x^5} - \frac{1785}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} - 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 - x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right) + (0) \\ &= \frac{1}{4}x^2 - x - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (1) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} - 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 - x - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}-1$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} - 1\right) \\ &= 1 - \frac{x}{2} \\ &= 1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (2) + 2\left(1 - \frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right)\right) = 0 \\ (2 + x)\, a_1 + 4x + 2a_0 + 2 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = -4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 4x + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 4x + 3\right) \mathrm{e}^{\int \left(1 - \frac{x}{2}\right) dx} \\
&= \left(x^2 - 4x + 3\right) \mathrm{e}^{x - \frac{1}{4}x^2} \\
&= \left(x^2 - 4x + 3\right) \mathrm{e}^{-\frac{x(-4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \left(x^2 - 4x + 3\right) \mathrm{e}^{-\frac{x(-2+x)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}} \mathrm{e}^{x(-2+x)}}{\left(x^2 - 4x + 3\right)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2-4x+3\right)\mathrm{e}^{-\frac{x(-2+x)}{2}}\right) + c_2\left(\left(x^2-4x+3\right)\mathrm{e}^{-\frac{x(-2+x)}{2}}\left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}\mathrm{e}^{x(-2+x)}}{\left(x^2-4x+3\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.224.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' + (2+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2) + a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 + 2a_0 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + a_k k + 2a_k + a_{k-1} = 0$
- Shift index using $k- >k + 1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} + a_{k+1}(k+1) + 2a_{k+1} + a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{k a_{k+1} + a_k + 3a_{k+1}}{k^2+5k+6}, 2a_2 + 2a_0 = 0\right]$

1.224.3 Maple trace

```
Methods for second order ODEs:
```

1.224.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 78

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)+(2+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-x} \left((x-3)\, c_2\, \mathrm{e}^{-\frac{(-2+x)^2}{2}} (-1+x) \left(\operatorname{erf}\left(\frac{\sqrt{2}\sqrt{-(-2+x)^2}}{2} \right) - 1 \right) \sqrt{\pi} - \sqrt{2}\sqrt{-(-2+x)^2}\, c_2 - c_1\, \mathrm{e}^{-\frac{(-2+x)^2}{2}} (-1+x)\,(x-3) \right)$$

1.224.5 Mathematica DSolve solution

Solving time : 0.974 (sec)
Leaf size : 94

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+(2+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}e^{-\frac{x^2}{2}+x-\frac{9}{2}}\left(e^{5/2}\sqrt{2\pi}c_2\left(x^2-4x+3\right)\operatorname{erfi}\left(\frac{x-2}{\sqrt{2}}\right) + 4e^{9/2}c_1\left(x^2-4x+3\right) - 2c_2e^{\frac{1}{2}(x-3)^2+x}(x-2)\right)$$

1.225 problem 228

Internal problem ID [8363]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 228
Date solved : `Monday, October 21, 2024 at 05:06:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+1\right)y''+7xy'+2y=0$$

1.225.1 Solved as second order ode using Kovacic algorithm

Time used: 0.354 (sec)

Writing the ode as

$$\left(2x^2+1\right)y''+7xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+1 \\ B &= 7x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 6}{4\left(2x^2 + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 + 6$$
$$t = 4\left(2x^2 + 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 6}{4\left(2x^2 + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 432: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{7}{64\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{17i\sqrt{2}}{64\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{17i\sqrt{2}}{64\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2+6}{4\left(2x^2+1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2+6}{4\left(2x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{5}{4} - \left(\frac{1}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} + (0) \\ &= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} \\ &= \frac{x}{4x^2 + 2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} \right)(1) + \left(\left(-\frac{1}{8\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{8\left(x + \frac{i\sqrt{2}}{2}\right)^2} \right) + \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4 \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} \right) dx} \\ &= (x) \left(\left(i\sqrt{2} - 2x \right) \left(2x + i\sqrt{2} \right) \right)^{1/8} \\ &= x\left(-4x^2 - 2\right)^{1/8}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x}{2x^2+1}\,dx} \\
&= z_1 e^{-\frac{7\ln(2x^2+1)}{8}} \\
&= z_1\left(\frac{1}{(2x^2+1)^{7/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{7/8}x(-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x}{2x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(2x^2+1)}{4}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\,(2x^2+1)^{7/4}\,x^2\,(-4x^2-2)^{1/4}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{2^{7/8}x(-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}}\right) + c_2\left(\frac{2^{7/8}x(-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}}\left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\,(2x^2+1)^{7/4}\,x^2\,(-4x^2-2)^{1/4}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.225.2 Maple step by step solution

1.225.3 Maple trace

```
Methods for second order ODEs:
```

1.225.4 Maple dsolve solution

Solving time : `0.018` (sec)
Leaf size : 37

```
dsolve((2*x^2+1)*diff(diff(y(x),x),x)+7*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \operatorname{LegendreP}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\, x\right) + c_2 \operatorname{LegendreQ}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\, x\right)}{\left(2x^2+1\right)^{3/8}}$$

1.225.5 Mathematica DSolve solution

Solving time : `0.094` (sec)
Leaf size : 66

```
DSolve[{(1+2*x^2)*D[y[x],{x,2}]+7*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 Q_{\frac{1}{4}}^{\frac{3}{4}}\left(i\sqrt{2}x\right)}{\left(2x^2+1\right)^{3/8}} + \frac{2i\sqrt[4]{2}c_1 x}{\left(2x^2+1\right)^{3/4} \operatorname{Gamma}\left(\frac{1}{4}\right)}$$

1.226 problem 229

Internal problem ID [8364]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 229
Date solved : `Monday, October 21, 2024 at 05:06:39 PM`
CAS classification : `[_Lienard]`

Solve

$$4y'' + xy' + 4y = 0$$

1.226.1 Solved as second order ode using Kovacic algorithm

Time used: 0.280 (sec)

Writing the ode as

$$4y'' + xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 56}{64} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 56$$
$$t = 64$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{64} - \frac{7}{8}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 433: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 2\\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{8} - \frac{7}{2x} - \frac{49}{x^3} - \frac{1372}{x^5} - \frac{48020}{x^7} - \frac{1882384}{x^9} - \frac{79060128}{x^{11}} - \frac{3478645632}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i\\ &= \frac{x}{8}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{64}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 56}{64} \\ &= Q + \frac{R}{64} \\ &= \left(\frac{x^2}{64} - \frac{7}{8}\right) + (0) \\ &= \frac{x^2}{64} - \frac{7}{8} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{7}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{8}\right) - (0) \\ &= -\frac{7}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{8} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{7}{8}}{\frac{1}{8}} - 1\right) &= -4 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{7}{8}}{\frac{1}{8}} - 1\right) &= 3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{64} - \frac{7}{8}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{8}$	-4	3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 3$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{8}\right) \\ &= -\frac{x}{8} \\ &= -\frac{x}{8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (6x + 2a_2) + 2\left(-\frac{x}{8}\right)\left(3x^2 + 2xa_2 + a_1\right) + \left(\left(-\frac{1}{8}\right) + \left(-\frac{x}{8}\right)^2 - \left(\frac{x^2}{64} - \frac{7}{8}\right)\right) = 0 \\ 6x + 2a_2 + \frac{1}{4}a_2 x^2 + \frac{1}{2}a_1 x + \frac{3}{4}a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = -12, a_2 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 - 12x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^3 - 12x\right) \mathrm{e}^{\int -\frac{x}{8} dx} \\ &= \left(x^3 - 12x\right) \mathrm{e}^{-\frac{x^2}{16}} \\ &= x\left(x^2 - 12\right) \mathrm{e}^{-\frac{x^2}{16}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{4} \, dx} \\ &= z_1 e^{-\frac{x^2}{16}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{16}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{8}} x\left(x^2 - 12\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{4} \, dx}}{\left(y_1\right)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{8}}}{\left(y_1\right)^2} \, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{8}}}{x^2 \left(x^2 - 12\right)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^2}{8}} x\left(x^2-12\right)\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{8}} x\left(x^2-12\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{8}}}{x^2\left(x^2-12\right)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.226.2 Maple step by step solution

Let's solve

$4\frac{d}{dx}y' + xy' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{xy'}{4} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{xy'}{4} + y = 0$

- Multiply by denominators

  $4\frac{d}{dx}y' + xy' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(4a_{k+2}(k+2)(k+1) + a_k(k+4)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4(k^2+3k+2)\, a_{k+2} + a_k(k+4) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{4(k^2+3k+2)}\right]$$

1.226.3 Maple trace

```
Methods for second order ODEs:
```

1.226.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 34

```
dsolve(4*diff(diff(y(x),x),x)+x*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{\left(-12 \,\mathrm{hypergeom}\left(\left[-\frac{3}{2}\right], \left[\frac{1}{2}\right], \frac{x^2}{8}\right) c_2 + c_1 x(x^2-12)\right) \mathrm{e}^{-\frac{x^2}{8}}}{12}$$

1.226.5 Mathematica DSolve solution

Solving time : `0.149` (sec)
Leaf size : 122

```
DSolve[{4*D[y[x],{x,2}]+x*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{x^2}{8}}\left(\sqrt{2\pi}c_2(x^2-12)\, x^2 \mathrm{erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{2}}\right) + 4\sqrt{x^2}\left(2\sqrt{2}c_1x^3 - c_2e^{\frac{x^2}{8}}x^2 + 8c_2e^{\frac{x^2}{8}} - 24\sqrt{2}c_1x\right)\right)}{32\sqrt{x^2}}$$

1.227 problem 230

Internal problem ID [8365]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 230
Date solved : `Monday, October 21, 2024 at 05:06:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' - 4y = 0$$

1.227.1 Solved as second order ode using Kovacic algorithm

Time used: 0.259 (sec)

Writing the ode as

$$y'' + xy' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 18}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 18 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{9}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 435: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{9}{2x} - \frac{81}{4x^3} + \frac{729}{4x^5} - \frac{32805}{16x^7} + \frac{413343}{16x^9} - \frac{11160261}{32x^{11}} + \frac{157837977}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+18}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} + \frac{9}{2}\right) + (0) \\
&= \frac{x^2}{4} + \frac{9}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{9}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{9}{2}\right) - (0) \\
&= \frac{9}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{9}{2}}{\frac{1}{2}} - 1\right) &= 4 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{9}{2}}{\frac{1}{2}} - 1\right) &= -5
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{9}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	4	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$, and since there are no poles, then

$$
\begin{aligned}
d &= \alpha_\infty^+ \\
&= 4
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= (+)[\sqrt{r}]_\infty \\
&= 0 + \left(\frac{x}{2}\right) \\
&= \frac{x}{2} \\
&= \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{x}{2}\right)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{9}{2}\right)\right) = 0
$$

$$
-a_3x^3 + (-2a_2 + 12)x^2 + (-3a_1 + 6a_3)x - 4a_0 + 2a_2 = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 3, a_1 = 0, a_2 = 6, a_3 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^4 + 6x^2 + 3
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 + 6x^2 + 3\right) \mathrm{e}^{\int \frac{x}{2} dx} \\
&= \left(x^4 + 6x^2 + 3\right) \mathrm{e}^{\frac{x^2}{4}} \\
&= \left(x^4 + 6x^2 + 3\right) \mathrm{e}^{\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 + 6x^2 + 3
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^4 + 6x^2 + 3\right)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 + 6x^2 + 3\right) + c_2 \left(x^4 + 6x^2 + 3 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^4 + 6x^2 + 3\right)^2} dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.227.2 Maple trace

```
Methods for second order ODEs:
```

1.227.3 Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 47

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y = x c_1 (x^2 + 5) \sqrt{2}\, e^{-\frac{x^2}{2}} + (x^4 + 6x^2 + 3) \left(\operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) \sqrt{\pi}\, c_1 + c_2\right)$$

1.227.4 Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 43

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{2}} \operatorname{HermiteH}\left(-5, \frac{x}{\sqrt{2}}\right) + \frac{1}{3} c_2 (x^4 + 6x^2 + 3)$$

1.228 problem 231

Internal problem ID [8366]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 231
Date solved : `Monday, October 21, 2024 at 05:06:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4xy'' - xy' + 2y = 0$$

1.228.1 Solved as second order ode using Kovacic algorithm

Time used: 0.243 (sec)

Writing the ode as

$$4xy'' - xy' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x - 32}{64x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x - 32 \\ t &= 64x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x - 32}{64x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 436: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{8} - \frac{2}{x} - \frac{16}{x^2} - \frac{256}{x^3} - \frac{5120}{x^4} - \frac{114688}{x^5} - \frac{2752512}{x^6} - \frac{69206016}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{8}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x - 32}{64x} \\
&= Q + \frac{R}{64x} \\
&= \left(\frac{1}{64}\right) + \left(-\frac{1}{2x}\right) \\
&= \frac{1}{64} - \frac{1}{2x}
\end{aligned}
$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is -32. Dividing this by leading coefficient in t which is 64 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{8} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{8}} - 0\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{8}} - 0\right) &&= 2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x - 32}{64x}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{8}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} + (-)\left(\frac{1}{8}\right) \\
&= \frac{1}{x} - \frac{1}{8} \\
&= \frac{1}{x} - \frac{1}{8}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{x} - \frac{1}{8}\right)(1) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} - \frac{1}{8}\right)^2 - \left(\frac{x-32}{64x}\right)\right) = 0
$$

$$
\frac{8+a_0}{4x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -8\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = -8 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (-8+x)\, \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{8}\right) dx} \\
&= (-8+x)\, \mathrm{e}^{-\frac{x}{8} + \ln(x)} \\
&= (-8+x)\, x\, \mathrm{e}^{-\frac{x}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{4x}\, dx} \\
&= z_1 e^{\frac{x}{8}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (-8 + x)\, x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{4x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{\frac{x}{4}}}{256\left(-2 + \frac{x}{4}\right)} - \frac{\mathrm{Ei}_1\left(-\frac{x}{4}\right)}{128} - \frac{\mathrm{e}^{\frac{x}{4}}}{64x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1((-8 + x)\, x) + c_2 \left((-8 + x)\, x \left(-\frac{\mathrm{e}^{\frac{x}{4}}}{256\left(-2 + \frac{x}{4}\right)} - \frac{\mathrm{Ei}_1\left(-\frac{x}{4}\right)}{128} - \frac{\mathrm{e}^{\frac{x}{4}}}{64x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.228.2 Maple step by step solution

1.228.3 Maple trace

```
Methods for second order ODEs:
```

1.228.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 33

```
dsolve(4*x*diff(diff(y(x),x),x)-x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 x(-8+x)\,\mathrm{Ei}_1\left(-\frac{x}{4}\right)}{16} + \frac{c_2(x-4)\,\mathrm{e}^{\frac{x}{4}}}{4} + c_1(-8+x)\,x$$

1.228.5 Mathematica DSolve solution

Solving time : `0.227` (sec)
Leaf size : 43

```
DSolve[{4*x*D[y[x],{x,2}]-x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{128}c_2\left((x-8)x\,\text{ExpIntegralEi}\left(\frac{x}{4}\right) - 4e^{x/4}(x-4)\right) + c_1(x-8)x$$

1.229 problem 232

Internal problem ID [8367]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 232
Date solved : `Monday, October 21, 2024 at 05:06:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2y'' + x(1+18x)\,y' + (1+12x)\,y = 0$$

1.229.1 Solved as second order ode using Kovacic algorithm

Time used: 0.305 (sec)

Writing the ode as

$$6x^2y'' + \left(18x^2 + x\right)y' + (1+12x)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= 18x^2 + x \\ C &= 1 + 12x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{324x^2 - 252x - 35}{144x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 324x^2 - 252x - 35 \\
t &= 144x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{324x^2 - 252x - 35}{144x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 437: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} - \frac{7}{4x} - \frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{7}{12x} - \frac{7}{36x^2} - \frac{49}{648x^3} - \frac{245}{5832x^4} - \frac{343}{13122x^5} - \frac{66199}{3779136x^6} - \frac{837949}{68024448x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{324x^2 - 252x - 35}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{9}{4}\right) + \left(\frac{-252x - 35}{144x^2}\right) \\ &= \frac{9}{4} + \frac{-252x - 35}{144x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -252. Dividing this by leading coefficient in t which is 144 gives $-\frac{7}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{4}\right) - (0) \\ &= -\frac{7}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{3}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{4}}{\frac{3}{2}} - 0\right) &&= -\frac{7}{12} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{4}}{\frac{3}{2}} - 0\right) &&= \frac{7}{12} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{324x^2 - 252x - 35}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{3}{2}$	$-\frac{7}{12}$	$\frac{7}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{12}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{7}{12} - \left(\frac{7}{12}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{7}{12x} + (-)\left(\frac{3}{2}\right) \\
&= \frac{7}{12x} - \frac{3}{2} \\
&= \frac{7}{12x} - \frac{3}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{12x} - \frac{3}{2}\right)(0) + \left(\left(-\frac{7}{12x^2}\right) + \left(\frac{7}{12x} - \frac{3}{2}\right)^2 - \left(\frac{324x^2 - 252x - 35}{144x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{7}{12x} - \frac{3}{2}\right) dx} \\
&= x^{7/12} e^{-\frac{3x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{18x^2 + x}{6x^2} \, dx} \\
&= z_1 e^{-\frac{3x}{2} - \frac{\ln(x)}{12}} \\
&= z_1 \left(\frac{e^{-\frac{3x}{2}}}{x^{1/12}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{18x^2+x}{6x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3x-\frac{\ln(x)}{6}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-3x-\frac{\ln(x)}{6}}\,\mathrm{e}^{6x}}{x}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\,\mathrm{e}^{-3x}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-3x}\left(\int \frac{\mathrm{e}^{-3x-\frac{\ln(x)}{6}}\,\mathrm{e}^{6x}}{x}dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.229.2 Maple step by step solution

Let's solve

$6x^2\left(\frac{d}{dx}y'\right) + x(1+18x)\,y' + (1+12x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(1+12x)y}{6x^2} - \frac{(1+18x)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(1+18x)y'}{6x} + \frac{(1+12x)y}{6x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{1+18x}{6x}, P_3(x) = \frac{1+12x}{6x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{6}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{6}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2\left(\frac{d}{dx}y'\right) + x(1 + 18x)\, y' + (1 + 12x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + 3r)\,(-1 + 2r)\, x^r + \left(\sum_{k=1}^{\infty} (a_k(3k + 3r - 1)\,(2k + 2r - 1) + 6a_{k-1}(3k + 3r - 1))\, x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+3r)(-1+2r)=0$
- Values of r that satisfy the indicial equation
  $r\in\left\{\frac{1}{2},\frac{1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $6\left(k+r-\frac{1}{3}\right)\left(\left(k+r-\frac{1}{2}\right)a_k+3a_{k-1}\right)=0$
- Shift index using $k->k+1$
  $6\left(k+\frac{2}{3}+r\right)\left(\left(k+\frac{1}{2}+r\right)a_{k+1}+3a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{6a_k}{2k+1+2r}$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1}=-\frac{6a_k}{2k+2}$
- Solution for $r=\frac{1}{2}$
  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}},a_{k+1}=-\frac{6a_k}{2k+2}\right]$
- Recursion relation for $r=\frac{1}{3}$
  $a_{k+1}=-\frac{6a_k}{2k+\frac{5}{3}}$
- Solution for $r=\frac{1}{3}$
  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{3}},a_{k+1}=-\frac{6a_k}{2k+\frac{5}{3}}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{3}}\right),a_{k+1}=-\frac{6a_k}{2k+2},b_{k+1}=-\frac{6b_k}{2k+\frac{5}{3}}\right]$

1.229.3 Maple trace

```
Methods for second order ODEs:
```

1.229.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 40

```
dsolve(6*x^2*diff(diff(y(x),x),x)+x*(1+18*x)*diff(y(x),x)+(1+12*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-\frac{c_2(-x)^{5/6}3^{5/6}}{3} + x\,\mathrm{e}^{-3x}\left(c_2\Gamma\left(\frac{5}{6}\right) - c_2\Gamma\left(\frac{5}{6}, -3x\right) + c_1\right)}{\sqrt{x}}$$

1.229.5 Mathematica DSolve solution

Solving time : 0.506 (sec)
Leaf size : 47

```
DSolve[{6*x^2*D[y[x],{x,2}]+x*(1+18*x)*D[y[x],x]+(1+12*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-3x}\left(\frac{\sqrt[6]{3}c_2x^{4/3}\Gamma\left(-\frac{1}{6}, -3x\right)}{(-x)^{5/6}} + c_1\sqrt{x}\right)$$

1.230 problem 233

Internal problem ID [8368]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 233
Date solved : `Monday, October 21, 2024 at 05:06:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2y'' - x(x+8)\,y' + 6y = 0$$

1.230.1 Solved as second order ode using Kovacic algorithm

Time used: 3.677 (sec)

Writing the ode as

$$3x^2y'' + \left(-x^2 - 8x\right)y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= -x^2 - 8x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 + 16x + 40}{36x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 16x + 40 \\
t &= 36x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 + 16x + 40}{36x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 439: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{4}{9x} + \frac{10}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{10}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{2}{3} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{4}{3x} - \frac{2}{x^2} + \frac{16}{x^3} - \frac{140}{x^4} + \frac{1312}{x^5} - \frac{12944}{x^6} + \frac{132736}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 16x + 40}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{16x + 40}{36x^2}\right) \\ &= \frac{1}{36} + \frac{16x + 40}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 16. Dividing this by leading coefficient in t which is 36 gives $\frac{4}{9}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{4}{9}\right) - (0) \\ &= \frac{4}{9} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{4}{9}}{\frac{1}{6}} - 0\right) &&= \frac{4}{3} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{4}{9}}{\frac{1}{6}} - 0\right) &&= -\frac{4}{3} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 16x + 40}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{3}$	$-\frac{2}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{4}{3}$	$-\frac{4}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{4}{3}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= \frac{4}{3} - \left(-\frac{2}{3}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{2}{3x} + \left(\frac{1}{6}\right) \\ &= -\frac{2}{3x} + \frac{1}{6} \\ &= \frac{-4 + x}{6x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{3x} + \frac{1}{6}\right)(2x + a_1) + \left(\left(\frac{2}{3x^2}\right) + \left(-\frac{2}{3x} + \frac{1}{6}\right)^2 - \left(\frac{x^2 + 16x + 40}{36x^2}\right) \right) = 0$$

$$\frac{(-a_1 - 2)\, x - 2a_0 - 4a_1}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4, a_1 = -2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2x + 4$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= (x^2 - 2x + 4)\, e^{\int \left(-\frac{2}{3x} + \frac{1}{6}\right) dx} \\ &= (x^2 - 2x + 4)\, e^{\frac{x}{6} - \frac{2 \ln(x)}{3}} \\ &= \frac{(x^2 - 2x + 4)\, e^{\frac{x}{6}}}{x^{2/3}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-8x}{3x^2}\,dx} \\
&= z_1 e^{\frac{x}{6}+\frac{4\ln(x)}{3}} \\
&= z_1\left(x^{4/3}\mathrm{e}^{\frac{x}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{2/3}\mathrm{e}^{\frac{x}{3}}\left(x^2-2x+4\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-8x}{3x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x}{3}+\frac{8\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x}{3}+\frac{8\ln(x)}{3}}\mathrm{e}^{-\frac{2x}{3}}}{x^{4/3}\left(x^2-2x+4\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{2/3}\mathrm{e}^{\frac{x}{3}}\left(x^2-2x+4\right)\right) + c_2\left(x^{2/3}\mathrm{e}^{\frac{x}{3}}\left(x^2-2x+4\right)\left(\int \frac{\mathrm{e}^{\frac{x}{3}+\frac{8\ln(x)}{3}}\mathrm{e}^{-\frac{2x}{3}}}{x^{4/3}\left(x^2-2x+4\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.230.2 Maple step by step solution

Let's solve

$3x^2 \left(\frac{d}{dx}y'\right) - x(x+8)\,y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = -\frac{2y}{x^2} + \frac{(x+8)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{(x+8)y'}{3x} + \frac{2y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

  $\left[P_2(x) = -\frac{x+8}{3x}, P_3(x) = \frac{2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -\frac{8}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

$3x^2 \left(\frac{d}{dx}y'\right) - x(x+8)\,y' + 6y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+3r)\,(-3+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-2)\,(k+r-3) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-2+3r)\,(-3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{3, \frac{2}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$3(k+r-3)\left(k+r-\frac{2}{3}\right) a_k - a_{k-1}(k+r-1) = 0$$

- Shift index using $k- >k+1$

  $$3(k-2+r)\left(k+\frac{1}{3}+r\right) a_{k+1} - a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+r)}{(k-2+r)(3k+1+3r)}$$

- Recursion relation for $r = 3$

  $$a_{k+1} = \frac{a_k(k+3)}{(k+1)(3k+10)}$$

- Solution for $r = 3$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{a_k(k+3)}{(k+1)(3k+10)}\right]$$

- Recursion relation for $r = \frac{2}{3}$

  $$a_{k+1} = \frac{a_k\left(k+\frac{2}{3}\right)}{\left(k-\frac{4}{3}\right)(3k+3)}$$

- Solution for $r = \frac{2}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{2}{3}}, a_{k+1} = \frac{a_k\left(k+\frac{2}{3}\right)}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{2}{3}}\right), a_{k+1} = \frac{a_k(k+3)}{(k+1)(3k+10)}, b_{k+1} = \frac{b_k\left(k+\frac{2}{3}\right)}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$$

1.230.3 Maple trace

```
Methods for second order ODEs:
```

1.230.4 Maple dsolve solution

Solving time : 0.027 (sec)
Leaf size : 38

```
dsolve(3*x^2*diff(diff(y(x),x),x)-x*(x+8)*diff(y(x),x)+6*y(x) = 0,
       y(x),singsol=all)
```

$$y = \left(x^{2/3} - \frac{x^{5/3}}{2} + \frac{x^{8/3}}{4}\right) c_2 \, \mathrm{e}^{\frac{x}{3}} + c_1 \, \mathrm{hypergeom}\left([3], \left[\frac{10}{3}\right], \frac{x}{3}\right) x^3$$

1.230.5 Mathematica DSolve solution

Solving time : 1.772 (sec)
Leaf size : 79

```
DSolve[{3*x^2*D[y[x],{x,2}]-x*(x+8)*D[y[x],x]+6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{x/3} x^{2/3} (x^2 - 2x + 4) - \frac{c_2 e^{x/3} x^{2/3} (x^2 - 2x + 4)\, \Gamma\left(\frac{1}{3}, \frac{x}{3}\right)}{6\ 3^{2/3}} + \frac{1}{6} c_2 (x - 4) x$$

1.231 problem 234

Internal problem ID [8369]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 234
Date solved : `Monday, October 21, 2024 at 05:06:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' - x(1+2x)\,y' + 2(4x-1)\,y = 0$$

1.231.1 Solved as second order ode using Kovacic algorithm

Time used: 0.693 (sec)

Writing the ode as

$$2x^2y'' + \left(-2x^2 - x\right)y' + (8x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -2x^2 - x \\ C &= 8x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 60x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 - 60x + 21$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 60x + 21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 441: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{21}{16x^2} - \frac{15}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{15}{4x} - \frac{51}{4x^2} - \frac{765}{8x^3} - \frac{3519}{4x^4} - \frac{144585}{16x^5} - \frac{6358527}{64x^6} - \frac{146409525}{128x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 60x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-60x + 21}{16x^2}\right) \\ &= \frac{1}{4} + \frac{-60x + 21}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -60. Dividing this by leading coefficient in t which is 16 gives $-\frac{15}{4}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{15}{4}\right) - (0) \\
&= -\frac{15}{4}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{15}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{15}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{15}{4}}{\frac{1}{2}} - 0\right) &&= \frac{15}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 - 60x + 21}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{15}{4}$	$\frac{15}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{15}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{15}{4} - \left(\frac{7}{4}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{7}{4x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{7}{4x} - \frac{1}{2} \\
&= \frac{7}{4x} - \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{7}{4x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{7}{4x^2}\right) + \left(\frac{7}{4x} - \frac{1}{2}\right)^2 - \left(\frac{4x^2 - 60x + 21}{16x^2}\right) \right) = 0$$

$$\frac{2(9 + a_1)x + 4a_0 + 7a_1}{2x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{63}{4}, a_1 = -9 \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 9x + \frac{63}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \left(x^2 - 9x + \frac{63}{4}\right) \mathrm{e}^{\int \left(\frac{7}{4x} - \frac{1}{2}\right)dx} \\
&= \left(x^2 - 9x + \frac{63}{4}\right) \mathrm{e}^{-\frac{x}{2} + \frac{7\ln(x)}{4}} \\
&= \frac{(4x^2 - 36x + 63)\, x^{7/4} \mathrm{e}^{-\frac{x}{2}}}{4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^2 - x}{2x^2}\,dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{4}} \\
&= z_1 \left(x^{1/4} \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 - 9x^3 + \frac{63}{4}x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2 - x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x + \frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{x + \frac{\ln(x)}{2}}}{\left(x^4 - 9x^3 + \frac{63}{4}x^2\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 - 9x^3 + \frac{63}{4}x^2\right) + c_2 \left(x^4 - 9x^3 + \frac{63}{4}x^2 \left(\int \frac{e^{x+\frac{\ln(x)}{2}}}{\left(x^4 - 9x^3 + \frac{63}{4}x^2\right)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.231.2 Maple step by step solution

Let's solve

$2x^2\left(\frac{d}{dx}y'\right) - x(1+2x)\,y' + 2(4x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x-1)y}{x^2} + \frac{(1+2x)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(1+2x)y'}{2x} + \frac{(4x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1+2x}{2x}, P_3(x) = \frac{4x-1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -\frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -1$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d}{dx}y'\right) - x(1+2x)\,y' + (8x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(k+r-2) - 2a_{k-1}(k-5+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{2, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2(k+r-2)\left(k+r+\frac{1}{2}\right) a_k - 2a_{k-1}(k-5+r) = 0$$

- Shift index using $k->k+1$

  $$2(k+r-1)\left(k+\frac{3}{2}+r\right) a_{k+1} - 2a_k(k+r-4) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{2a_k(k+r-4)}{(k+r-1)(2k+3+2r)}$$

- Recursion relation for $r = 2$; series terminates at $k = 2$

  $$a_{k+1} = \frac{2a_k(k-2)}{(k+1)(2k+7)}$$

- Apply recursion relation for $k = 0$

  $$a_1 = -\frac{4a_0}{7}$$

- Apply recursion relation for $k = 1$

  $$a_2 = -\frac{a_1}{9}$$

- Express in terms of a_0

  $$a_2 = \frac{4a_0}{63}$$

- Terminating series solution of the ODE for $r = 2$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 - \frac{4}{7}x + \frac{4}{63}x^2\right)$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = \frac{2a_k\left(k-\frac{9}{2}\right)}{\left(k-\frac{3}{2}\right)(2k+2)}$
- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{2a_k\left(k-\frac{9}{2}\right)}{\left(k-\frac{3}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 - \frac{4}{7}x + \frac{4}{63}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), b_{k+1} = \frac{2b_k\left(k-\frac{9}{2}\right)}{\left(k-\frac{3}{2}\right)(2k+2)}\right]$

1.231.3 Maple trace

```
Methods for second order ODEs:
```

1.231.4 Maple dsolve solution

Solving time : `0.011` (sec)
Leaf size : 32

```
dsolve(2*x^2*diff(diff(y(x),x),x)-x*(1+2*x)*diff(y(x),x)+2*(4*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1(4x^2 - 36x + 63)\,x^2}{63} + \frac{c_2 \operatorname{hypergeom}\left(\left[-\frac{9}{2}\right], \left[-\frac{3}{2}\right], x\right)}{\sqrt{x}}$$

1.231.5 Mathematica DSolve solution

Solving time : 2.812 (sec)
Leaf size : 89

```
DSolve[{2*x^2*D[y[x],{x,2}]-x*(1+2*x)*D[y[x],x]+2*(4*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1\left(x^4-9x^3+\frac{63x^2}{4}\right)-\frac{4c_2\left(\sqrt{\pi}(-4x^2+36x-63)\,x^{5/2}\text{erfi}\left(\sqrt{x}\right)+2e^x(2x^4-17x^3+24x^2+6x+3)\right)}{945\sqrt{x}}$$

1.232 problem 235

Internal problem ID [8370]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 235
Date solved : `Monday, October 21, 2024 at 05:06:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4x^2y' + (1+2x)\,y = 0$$

1.232.1 Solved as second order ode using Kovacic algorithm

Time used: 0.241 (sec)

Writing the ode as

$$4x^2y'' - 4x^2y' + (1+2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 \\ C &= 1+2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 2x - 1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 2x - 1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 - 2x - 1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 443: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4x^2} - \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} - \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{3}{4x^4} - \frac{5}{4x^5} - \frac{9}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{x-1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 2x - 1}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x^2}{4x^2} \, dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^x}{(y_1)^2}\,dx \\ &= y_1(-\operatorname{Ei}_1(-x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}(-\operatorname{Ei}_1(-x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.232.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) - 4x^2y' + (1+2x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(1+2x)y}{4x^2} + y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - y' + \frac{(1+2x)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -1, P_3(x) = \frac{1+2x}{4x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{4}$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4x^2y' + (1+2x)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$
- Shift index using $k- >k - 1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)^2 - 2a_{k-1}(2k-3+2r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$
- Values of r that satisfy the indicial equation

$r = \frac{1}{2}$

- Each term in the series must be 0, giving the recursion relation

  $a_k (2k+2r-1)^2 + (-4k+6-4r)\, a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+1}(2k+1+2r)^2 + a_k(-4k-4r+2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(2k+2r-1)}{(2k+1+2r)^2}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{4a_k k}{(2k+2)^2}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{4a_k k}{(2k+2)^2}\right]$

1.232.3 Maple trace

```
Methods for second order ODEs:
```

1.232.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 17

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x^2*diff(y(x),x)+(1+2*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = (c_2 \operatorname{Ei}_1(-x) + c_1)\sqrt{x}$$

1.232.5 Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 19

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x^2*D[y[x],x]+(1+2*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{x}(c_2 \operatorname{ExpIntegralEi}(x) + c_1)$$

1.233 problem 236

Internal problem ID [8371]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 236
Date solved : `Monday, October 21, 2024 at 05:06:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(3-2x)\,y' + (1-2x)\,y = 0$$

1.233.1 Solved as second order ode using Kovacic algorithm

Time used: 0.280 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^2+3x\right)y' + (1-2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2+3x \\ C &= 1-2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 - 4x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 445: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = 1 - \frac{1}{4x^2} - \frac{1}{x}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} - \frac{1}{4x^2} - \frac{1}{8x^3} - \frac{3}{32x^4} - \frac{5}{64x^5} - \frac{9}{128x^6} - \frac{17}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-4x - 1}{4x^2}\right) \\ &= 1 + \frac{-4x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 - 4x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\,(1) \\
&= \frac{1}{2x} - 1 \\
&= \frac{1}{2x} - 1
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x} - 1\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - 1\right)^2 - \left(\frac{4x^2 - 4x - 1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - 1\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^2+3x}{x^2} \, dx} \\
&= z_1 e^{x - \frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{x}}{x^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2+3x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2x-3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(-\operatorname{Ei}_1(-2x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}(-\operatorname{Ei}_1(-2x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.233.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + x(3-2x)\,y' + (1-2x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(2x-1)y}{x^2} + \frac{(2x-3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(2x-3)y'}{x} - \frac{(2x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{2x-3}{x}, P_3(x) = -\frac{2x-1}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2\left(\frac{d}{dx}y'\right) - x(2x-3)\,y' + (1-2x)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)^2 - 2a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $r = -1$
- Each term in the series must be 0, giving the recursion relation

  $a_k (k + r + 1)^2 - 2a_{k-1}(k + r) = 0$
- Shift index using $k- >k + 1$

  $a_{k+1}(k + 2 + r)^2 - 2a_k(k + r + 1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r+1)}{(k+2+r)^2}$
- Recursion relation for $r = -1$

  $a_{k+1} = \frac{2a_k k}{(k+1)^2}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{2a_k k}{(k+1)^2}\right]$

1.233.3 Maple trace

```
Methods for second order ODEs:
```

1.233.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(3-2*x)*diff(y(x),x)+(1-2*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \operatorname{Ei}_1(-2x) + c_1}{x}$$

1.233.5 Mathematica DSolve solution

Solving time : 0.084 (sec)
Leaf size : 19

```
DSolve[{x^2*D[y[x],{x,2}]+x*(3-2*x)*D[y[x],x]+(1-2*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 \operatorname{ExpIntegralEi}(2x) + c_1}{x}$$

1.234 problem 237

Internal problem ID [8372]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 237
Date solved : `Monday, October 21, 2024 at 05:06:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(3+x)\,y' + (4-x)\,y = 0$$

1.234.1 Solved as second order ode using Kovacic algorithm

Time used: 0.307 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2 - 3x\right)y' + (4-x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 3x \\ C &= 4 - x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 10x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 10x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 447: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{5}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{5}{2x} - \frac{13}{2x^2} + \frac{65}{2x^3} - \frac{819}{4x^4} + \frac{5785}{4x^5} - \frac{43797}{4x^6} + \frac{347425}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 10x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{10x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{10x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 10. Dividing this by leading coefficient in t which is 4 gives $\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{2}\right) - (0) \\ &= \frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 0\right) &= \frac{5}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 0\right) &= -\frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 10x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{5}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - (\alpha_{c_1}^{+}) \\ &= \frac{5}{2} - \left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \left(\frac{1}{2} \right) \\
&= \frac{1}{2x} + \frac{1}{2} \\
&= \frac{1 + x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r \right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2x} + \frac{1}{2} \right) (2x + a_1) + \left(\left(-\frac{1}{2x^2} \right) + \left(\frac{1}{2x} + \frac{1}{2} \right)^2 - \left(\frac{x^2 + 10x - 1}{4x^2} \right) \right) = 0$$

$$\frac{(-a_1 + 4)\, x - 2a_0 + a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2, a_1 = 4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 4x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + 4x + 2 \right) e^{\int \left(\frac{1}{2x} + \frac{1}{2} \right) dx} \\
&= \left(x^2 + 4x + 2 \right) e^{\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= \left(x^2 + 4x + 2 \right) \sqrt{x}\, e^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{3\ln(x)}{2}} \\
&= z_1 \left(x^{3/2}\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2\mathrm{e}^x\left(x^2+4x+2\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-x}(-x-3)}{4\left(x^2+4x+2\right)} - \frac{\mathrm{Ei}_1\left(x\right)}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\mathrm{e}^x\left(x^2+4x+2\right)\right) + c_2\left(x^2\mathrm{e}^x\left(x^2+4x+2\right)\left(-\frac{\mathrm{e}^{-x}(-x-3)}{4\left(x^2+4x+2\right)} - \frac{\mathrm{Ei}_1\left(x\right)}{4}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.234.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x(3+x)\,y' + (4-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = \frac{(x-4)y}{x^2} + \frac{(3+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{(3+x)y'}{x} - \frac{(x-4)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{3+x}{x}, P_3(x) = -\frac{x-4}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -3$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 4$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right) - x(3+x)\,y' + (4-x)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- ○ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- ○ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 2$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-2)^2 - a_{k-1}(k+r) = 0$$

- Shift index using $k- >k+1$

$$a_{k+1}(k+r-1)^2 - a_k(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r+1)}{(k+r-1)^2}$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k(k+3)}{(k+1)^2}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+3)}{(k+1)^2}\right]$$

1.234.3 Maple trace

```
Methods for second order ODEs:
```

1.234.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 42

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(3+x)*diff(y(x),x)+(4-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \left(c_2 \, \mathrm{e}^x \left(x^2+4x+2\right) \mathrm{Ei}_1\left(x\right)+c_1 \, \mathrm{e}^x \left(x^2+4x+2\right)-c_2(3+x)\right) x^2$$

1.234.5 Mathematica DSolve solution

Solving time : 0.356 (sec)
Leaf size : 52

```
DSolve[{x^2*D[y[x],{x,2}]-x*(3+x)*D[y[x],x]+(4-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}x^2\left(c_2e^x\left(x^2+4x+2\right)\text{ExpIntegralEi}(-x)+4c_1e^x\left(x^2+4x+2\right)+c_2(x+3)\right)$$

1.235 problem 238

Internal problem ID [8373]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 238
Date solved : `Monday, October 21, 2024 at 05:06:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(3-x)\,y' + y = 0$$

1.235.1 Solved as second order ode using Kovacic algorithm

Time used: 0.303 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2+3x\right)y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2+3x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 449: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4x^2} - \frac{3}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2x} - \frac{5}{2x^2} - \frac{15}{2x^3} - \frac{115}{4x^4} - \frac{495}{4x^5} - \frac{2285}{4x^6} - \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-6x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{2x} - \frac{1}{2} \\ &= -\frac{-1+x}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-6x-1}{4x^2}\right) \right) = 0$$

$$\frac{1+a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= (-1+x)\, e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\ &= (-1+x)\, e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\ &= (-1+x)\sqrt{x}\, e^{-\frac{x}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+3x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}-\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{2}}}{x^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{-1+x}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\operatorname{Ei}_1(-x) - \frac{\mathrm{e}^x}{-1+x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{-1+x}{x}\right) + c_2 \left(\frac{-1+x}{x}\left(-\operatorname{Ei}_1(-x) - \frac{\mathrm{e}^x}{-1+x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.235.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+x(3-x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{y}{x^2}+\frac{(x-3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{(x-3)y'}{x}+\frac{y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- Define functions

  $\left[P_2(x)=-\frac{x-3}{x}, P_3(x)=\frac{1}{x^2}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=3$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=1$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right)-x(x-3)\,y'+y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y'$ to series expansion for $m=1..2$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = -1$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+1)^2 - a_{k-1}(k+r-1) = 0$
- Shift index using $k- >k+1$
  $a_{k+1}(k+2+r)^2 - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+r)}{(k+2+r)^2}$
- Recursion relation for $r = -1$; series terminates at $k = 1$
  $a_{k+1} = \frac{a_k(k-1)}{(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second
  $y = a_0 \cdot (1-x)$

1.235.3 Maple trace

```
Methods for second order ODEs:
```

1.235.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 28

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(3-x)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(-1+x)\,c_2\,\mathrm{Ei}_1(-x) + c_2\,\mathrm{e}^x + c_1(-1+x)}{x}$$

1.235.5 Mathematica DSolve solution

Solving time : 0.378 (sec)
Leaf size : 31

```
DSolve[{x^2*D[y[x],{x,2}]+x*(3-x)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{c_2(x-1)\,\text{ExpIntegralEi}(x)+c_1(x-1)-c_2e^x}{x}$$

1.236 problem 239

Internal problem ID [8374]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 239
Date solved : `Monday, October 21, 2024 at 05:06:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2 y'' - \left(2\sqrt{5}-1\right) x y' + \left(\frac{19}{4} - 3x^2\right) y = 0$$

1.236.1 Solved as second order ode using Kovacic algorithm

Time used: 0.172 (sec)

Writing the ode as

$$x^2 y'' + \left(-2x\sqrt{5}+x\right) y' + \left(\frac{19}{4} - 3x^2\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x\sqrt{5}+x \\ C &= \frac{19}{4} - 3x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 3z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 451: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 3$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\sqrt{3}\, x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x\sqrt{5}+x}{x^2}\, dx} \\
&= z_1 e^{\ln(x)\sqrt{5}-\frac{\ln(x)}{2}} \\
&= z_1 \left(x^{\sqrt{5}-\frac{1}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{\sqrt{5}-\frac{1}{2}}\mathrm{e}^{-\sqrt{3}\, x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x\sqrt{5}+x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)\left(2\sqrt{5}-1\right)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{\ln(x)\left(2\sqrt{5}-1\right)} x^{1-2\sqrt{5}}\mathrm{e}^{2\sqrt{3}\, x}\sqrt{3}}{6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{\sqrt{5}-\frac{1}{2}} \mathrm{e}^{-\sqrt{3}\,x}\right) + c_2\left(x^{\sqrt{5}-\frac{1}{2}} \mathrm{e}^{-\sqrt{3}\,x}\left(\frac{\mathrm{e}^{\ln(x)\left(2\sqrt{5}-1\right)} x^{1-2\sqrt{5}} \mathrm{e}^{2\sqrt{3}\,x}\sqrt{3}}{6}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.236.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - \left(2\sqrt{5}-1\right)xy' + \left(\frac{19}{4}-3x^2\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(12x^2-19)y}{4x^2} + \frac{\left(2\sqrt{5}-1\right)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{\left(2\sqrt{5}-1\right)y'}{x} - \frac{(12x^2-19)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2\sqrt{5}-1}{x}, P_3(x) = -\frac{12x^2-19}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1-2\sqrt{5}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{19}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4\left(2\sqrt{5}-1\right)xy' + (-12x^2+19)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$\left(-1+2\sqrt{5}-2r\right)\left(1+2\sqrt{5}-2r\right) a_0 x^r + \left(-3+2\sqrt{5}-2r\right)\left(-1+2\sqrt{5}-2r\right) a_1 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(\left(-\right.\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\left(-1+2\sqrt{5}-2r\right)\left(1+2\sqrt{5}-2r\right) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{2}+\sqrt{5}, \sqrt{5}-\frac{1}{2}\right\}$$

- Each term must be 0

$$\left(-3+2\sqrt{5}-2r\right)\left(-1+2\sqrt{5}-2r\right) a_1 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-8a_k(k+r)\sqrt{5} + \left(4k^2+8kr+4r^2+19\right) a_k - 12a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$-8a_{k+2}(k+2+r)\sqrt{5} + \left(4(k+2)^2+8(k+2)\,r+4r^2+19\right) a_{k+2} - 12a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{12a_k}{-35+8k\sqrt{5}+8\sqrt{5}\,r-4k^2-8kr-4r^2+16\sqrt{5}-16k-16r}$$

- Recursion relation for $r = \frac{1}{2}+\sqrt{5}$

$$a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-16k}$$

- Solution for $r = \frac{1}{2}+\sqrt{5}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}+\sqrt{5}}, a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-16k}, a_1 = 0\right]$$

- Recursion relation for $r = \sqrt{5}-\frac{1}{2}$

$$a_{k+2} = -\frac{12a_k}{-27+8k\sqrt{5}+8\sqrt{5}\left(\sqrt{5}-\frac{1}{2}\right)-4k^2-8k\left(\sqrt{5}-\frac{1}{2}\right)-4\left(\sqrt{5}-\frac{1}{2}\right)^2-16k}$$

- Solution for $r = \sqrt{5}-\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\sqrt{5}-\frac{1}{2}}, a_{k+2} = -\frac{12a_k}{-27+8k\sqrt{5}+8\sqrt{5}\left(\sqrt{5}-\frac{1}{2}\right)-4k^2-8k\left(\sqrt{5}-\frac{1}{2}\right)-4\left(\sqrt{5}-\frac{1}{2}\right)^2-16k}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}+\sqrt{5}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\sqrt{5}-\frac{1}{2}}\right), a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right.}\right.$$

1.236.3 Maple trace

```
Methods for second order ODEs:
```

1.236.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 29

```
dsolve(x^2*diff(diff(y(x),x),x)-(2*5^(1/2)-1)*x*diff(y(x),x)+(19/4-3*x^2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^{\sqrt{5}-\frac{1}{2}}\left(c_1 \sinh\left(\sqrt{3}\,x\right) + c_2 \cosh\left(\sqrt{3}\,x\right)\right)$$

1.236.5 Mathematica DSolve solution

Solving time : `0.14` (sec)
Leaf size : 53

```
DSolve[{x^2*D[y[x],{x,2}]-(2*Sqrt[5]-1)*x*D[y[x],x]+(19/4-3*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{6}e^{-\sqrt{3}x}x^{\sqrt{5}-\frac{1}{2}}\left(\sqrt{3}c_2e^{2\sqrt{3}x} + 6c_1\right)$$

1.237 problem 240

Internal problem ID [8375]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 240
Date solved : `Monday, October 21, 2024 at 05:06:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(x-3)\,y' + (4-x)\,y = 0$$

1.237.1 Solved as second order ode using Kovacic algorithm

Time used: 0.251 (sec)

Writing the ode as

$$x^2y'' + \left(x^2-3x\right)y' + (4-x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2-3x \\ C &= 4-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 2x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 453: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4} - \frac{1}{4x^2} - \frac{1}{2x}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned}
\tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} - \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{3}{4x^4} - \frac{5}{4x^5} - \frac{9}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 2x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{x - 1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 2x - 1}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= \sqrt{x}\, e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 - 3x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} + \frac{3 \ln(x)}{2}} \\
&= z_1 \left(x^{3/2} e^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2 \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-3x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x+3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(-\operatorname{Ei}_1(-x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^2 \mathrm{e}^{-x}\right) + c_2\left(x^2 \mathrm{e}^{-x}(-\operatorname{Ei}_1(-x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.237.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + x(x-3)\,y' + (4-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(x-4)y}{x^2} - \frac{(x-3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(x-3)y'}{x} - \frac{(x-4)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x-3}{x}, P_3(x) = -\frac{x-4}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x(x-3)\,y' + (4-x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2\,x^r + \left(\sum_{k=1}^{\infty}\left(a_k(k+r-2)^2 + a_{k-1}(k+r-2)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

$r = 2$

- Each term in the series must be 0, giving the recursion relation
  $(k+r-2)\left(a_k(k+r-2)+a_{k-1}\right)=0$
- Shift index using $k->k+1$
  $(k+r-1)\left(a_{k+1}(k+r-1)+a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k}{k+r-1}$
- Recursion relation for $r=2$
  $a_{k+1}=-\frac{a_k}{k+1}$
- Solution for $r=2$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=-\frac{a_k}{k+1}\right]$

1.237.3 Maple trace

```
Methods for second order ODEs:
```

1.237.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x-3)*diff(y(x),x)+(4-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^2 \mathrm{e}^{-x}\left(c_1+\mathrm{Ei}_1\left(-x\right) c_2\right)$$

1.237.5 Mathematica DSolve solution

Solving time : 0.06 (sec)
Leaf size : 22

```
DSolve[{x^2*D[y[x],{x,2}]+x*(x-3)*D[y[x],x]+(4-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x} x^2 (c_2 \operatorname{ExpIntegralEi}(x)+c_1)$$

1.238 problem 241

Internal problem ID [8376]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 241
Date solved : `Monday, October 21, 2024 at 05:06:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x^2y' - (2+x)\,y = 0$$

1.238.1 Solved as second order ode using Kovacic algorithm

Time used: 0.222 (sec)

Writing the ode as

$$x^2y'' + x^2y' + (-x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 \\ C &= -x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 455: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{3}{x^4} - \frac{2}{x^5} - \frac{6}{x^6} + \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{4x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned}b &= (1) - (0)\\ &= 1\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2}\\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{\frac{1}{2}} - 0\right) &&= 1\\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{\frac{1}{2}} - 0\right) &&= -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -1$ then

$$\begin{aligned}d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right)\\ &= -1 - (-1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{1}{x} - \frac{1}{2} \\ &= -\frac{2 + x}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{x} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 + 4x + 8}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{x} - \frac{1}{2}\right) dx} \\ &= \frac{\mathrm{e}^{-\frac{x}{2}}}{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^2}{x^2} \, dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1\left(\left(x^2-2x+2\right)\mathrm{e}^{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\mathrm{e}^{-x}}{x}\right) + c_2\left(\frac{\mathrm{e}^{-x}}{x}\left(\left(x^2-2x+2\right)\mathrm{e}^{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.238.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + x^2 y' - (2+x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(2+x)y}{x^2} - y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + y' - \frac{(2+x)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = 1, P_3(x) = -\frac{2+x}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x^2y' + (-x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$

- Shift index using $k- > k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left(a_k(k+r+1)+a_{k-1}\right)=0$
- Shift index using $k->k+1$

  $(k-1+r)\left(a_{k+1}(k+2+r)+a_k\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k}{k+2+r}$
- Recursion relation for $r=-1$

  $a_{k+1}=-\frac{a_k}{k+1}$
- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1}=-\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$

  $a_{k+1}=-\frac{a_k}{k+4}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=-\frac{a_k}{k+4}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=-\frac{a_k}{k+1}, b_{k+1}=-\frac{b_k}{k+4}\right]$

1.238.3 Maple trace

```
Methods for second order ODEs:
```

1.238.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 25

```
dsolve(x^2*diff(diff(y(x),x),x)+x^2*diff(y(x),x)-(2+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1\,\mathrm{e}^{-x}+c_2(x^2-2x+2)}{x}$$

1.238.5 Mathematica DSolve solution

Solving time : 0.057 (sec)
Leaf size : 31

```
DSolve[{x^2*D[y[x],{x,2}]+x^2*D[y[x],x]-(2+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x}(c_2e^x(x^2-2x+2)+c_1)}{x}$$

1.239 problem 242

Internal problem ID [8377]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 242
Date solved : `Monday, October 21, 2024 at 05:06:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right) y = 0$$

1.239.1 Solved as second order ode using Kovacic algorithm

Time used: 0.232 (sec)

Writing the ode as

$$x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 \\ C &= x - \frac{3}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 - 4x + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 457: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{1}{x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} + \frac{1}{4x^2} + \frac{1}{8x^3} + \frac{1}{32x^4} - \frac{1}{64x^5} - \frac{3}{128x^6} - \frac{3}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-4x + 3}{4x^2}\right) \\ &= 1 + \frac{-4x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 - 4x + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - (\alpha_{c_1}^{-}) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (1) \\
&= 1 - \frac{1}{2x} \\
&= 1 - \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(1 - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(1 - \frac{1}{2x}\right)^2 - \left(\frac{4x^2 - 4x + 3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(1 - \frac{1}{2x}\right) dx} \\
&= \frac{\mathrm{e}^x}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^2}{x^2} \, dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(1+2x)\,\mathrm{e}^{-2x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{\sqrt{x}}\right) + c_2 \left(\frac{1}{\sqrt{x}} \left(-\frac{(1+2x)\,\mathrm{e}^{-2x}}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.239.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx}y'\right) + 2x^2 y' + \left(x - \frac{3}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x-3)y}{4x^2} - 2y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + 2y' + \frac{(4x-3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions
  $\left[P_2(x) = 2, P_3(x) = \frac{4x-3}{4x^2}\right]$
- ○ $x \cdot P_2(x)$ is analytic at $x = 0$
  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$$
- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{3}{4}$$
- ○ $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$
- • Multiply by denominators
  $4x^2\left(\frac{d}{dx}y'\right) + 8x^2y' + (4x-3)\,y = 0$
- • Assume series solution for y
  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$
    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - ○ Shift index using $k- > k - m$
    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - ○ Convert $x^2 \cdot y'$ to series expansion
    $$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$$
  - ○ Shift index using $k- > k - 1$
    $$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\, x^{k+r}$$
  - ○ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
    $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0(1+2r)\,(-3+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-3) + 4a_{k-1}(2k-1+2r)\right) x^{k+r}\right) = 0$$
- • $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)(-3+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{-\frac{1}{2},\frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{3}{2}\right)\left(k+r+\frac{1}{2}\right)a_k+8a_{k-1}\left(k-\frac{1}{2}+r\right)=0$$

- Shift index using $k->k+1$

$$4\left(k-\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right)a_{k+1}+8a_k\left(k+r+\frac{1}{2}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{4a_k(2k+2r+1)}{(2k-1+2r)(2k+3+2r)}$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+1}=-\frac{8a_k k}{(2k-2)(2k+2)}$$

- Series not valid for $r=-\frac{1}{2}$, division by 0 in the recursion relation at $k=1$

$$a_{k+1}=-\frac{8a_k k}{(2k-2)(2k+2)}$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+1}=-\frac{4a_k(2k+4)}{(2k+2)(2k+6)}$$

- Solution for $r=\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}},a_{k+1}=-\frac{4a_k(2k+4)}{(2k+2)(2k+6)}\right]$$

1.239.3 Maple trace

```
Methods for second order ODEs:
```

1.239.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 24

```
dsolve(x^2*diff(diff(y(x),x),x)+2*x^2*diff(y(x),x)+(x-3/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{2c_2\,\mathrm{e}^{-2x}x+c_2\,\mathrm{e}^{-2x}+c_1}{\sqrt{x}}$$

1.239.5 Mathematica DSolve solution

Solving time : `0.057` (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]+2*x^2*D[y[x],x]+(x-3/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1 - c_2 e^{-2x}(2x+1)}{4\sqrt{x}}$$

1.240 problem 243

Internal problem ID [8378]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 243
Date solved : `Monday, October 21, 2024 at 05:06:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+x^2y'-2y=0$$

1.240.1 Solved as second order ode using Kovacic algorithm

Time used: 0.214 (sec)

Writing the ode as

$$x^2(1+x)\,y''+x^2y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= x^2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 8x + 8}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 8x + 8 \\ t &= 4\left(x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 8x + 8}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 459: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{1 + x} - \frac{2}{x} + \frac{2}{x^2} - \frac{1}{4\,(1 + x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 8x + 8}{4\,(x^2 + x)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2+8x+8}{4\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2+2x} - \frac{1}{x} + (-)\,(0) \\
&= \frac{1}{2+2x} - \frac{1}{x} \\
&= -\frac{x+2}{2x\,(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2+2x} - \frac{1}{x}\right)(1) + \left(\left(-\frac{1}{2\,(1+x)^2} + \frac{1}{x^2}\right) + \left(\frac{1}{2+2x} - \frac{1}{x}\right)^2 - \left(\frac{-x^2+8x+8}{4\,(x^2+x)^2}\right)\right) = 0
$$

$$
\frac{-2+a_0}{x\,(1+x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, e^{\int \left(\frac{1}{2+2x} - \frac{1}{x}\right) dx} \\
&= (x+2)\, e^{\frac{\ln(1+x)}{2} - \ln(x)} \\
&= \frac{(x+2)\sqrt{1+x}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2}{x^2(1+x)}\,dx} \\
&= z_1 e^{-\frac{\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{1+x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x+2}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\ln(1+x) + \frac{4}{x+2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x+2}{x}\right) + c_2\left(\frac{x+2}{x}\left(\ln(1+x) + \frac{4}{x+2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.240.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right)+x^2y'-2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{2y}{x^2(1+x)}-\frac{y'}{1+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{y'}{1+x}-\frac{2y}{x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{1}{1+x},P_3(x)=-\frac{2}{x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=1$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\left(\frac{d}{dx}y'\right)+x^2y'-2y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(u^2-2u+1)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 u^{-1+r} + \left(a_1(1+r)^2 - 2a_0(r^2+1)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - 2a_k(k^2+2kr+r^2+1) + \right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term must be 0
  $a_1(1+r)^2 - 2a_0(r^2+1) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1)^2 - 2a_k(k^2+1) + a_{k-1}(k-1)^2 = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2)^2 - 2a_{k+1}\left((k+1)^2+1\right) + k^2 a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}, a_1 - 2a_0 = 0\right]$
- Revert the change of variables $u = 1+x$
  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}, a_1 - 2a_0 = 0\right]$

1.240.3 Maple trace

```
Methods for second order ODEs:
```

1.240.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 27

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)+x^2*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2(x+2)\ln(1+x)+c_1x+2c_1+4c_2}{x}$$

1.240.5 Mathematica DSolve solution

Solving time : 0.097 (sec)
Leaf size : 30

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+x^2*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1(x+2)+c_2(x+2)\log(x+1)+4c_2}{x}$$

1.241 problem 244

Internal problem ID [8379]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 244
Date solved : `Monday, October 21, 2024 at 05:06:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x\left(x^2 + 6\right) y' + 6y = 0$$

1.241.1 Solved as second order ode using Kovacic algorithm

Time used: 0.408 (sec)

Writing the ode as

$$x^2y'' + \left(x^3 + 6x\right) y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 + 6x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 14}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 14 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{7}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 461: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{7}{2x} - \frac{49}{4x^3} + \frac{343}{4x^5} - \frac{12005}{16x^7} + \frac{117649}{16x^9} - \frac{2470629}{32x^{11}} + \frac{27176919}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 14}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} + \frac{7}{2}\right) + (0) \\
&= \frac{x^2}{4} + \frac{7}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{7}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{7}{2}\right) - (0) \\
&= \frac{7}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{7}{2}}{\frac{1}{2}} - 1\right) &= 3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{7}{2}}{\frac{1}{2}} - 1\right) &= -4
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{7}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	3	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(6x + 2a_2) + 2\left(\frac{x}{2}\right)\left(3x^2 + 2xa_2 + a_1\right) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{7}{2}\right)\right) = 0$$
$$-a_2 x^2 + (-2a_1 + 6)\, x - 3a_0 + 2a_2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = 3, a_2 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 + 3x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^3 + 3x\right) e^{\int \frac{x}{2} dx} \\ &= \left(x^3 + 3x\right) e^{\frac{x^2}{4}} \\ &= x\left(x^2 + 3\right) e^{\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^3+6x}{x^2} \, dx} \\ &= z_1 e^{-\frac{x^2}{4} - 3\ln(x)} \\ &= z_1 \left(\frac{e^{-\frac{x^2}{4}}}{x^3}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 + 3}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^3+6x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2} - 6\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{e^{-\frac{x^2}{2} - 6\ln(x)} x^4}{\left(x^2 + 3\right)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2+3}{x^2}\right) + c_2\left(\frac{x^2+3}{x^2}\left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}-6\ln(x)}x^4}{(x^2+3)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.241.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + x(x^2+6)\,y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{6y}{x^2} - \frac{(x^2+6)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(x^2+6)y'}{x} + \frac{6y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x^2+6}{x}, P_3(x) = \frac{6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 6$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 6$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x(x^2+6)\,y' + 6y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(2+r)\, x^r + a_1(4+r)\,(3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+3)\,(k+r+2) + a_{k-2}(k-2+r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, -2\}$
- Each term must be 0

  $a_1(4+r)\,(3+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)\,(k+r+2) + a_{k-2}(k-2+r) = 0$
- Shift index using $k->k+2$

  $a_{k+2}(k+5+r)\,(k+4+r) + a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r)}{(k+5+r)(k+4+r)}$
- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{a_k(k-3)}{(k+2)(k+1)}$
- Solution for $r = -3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a_k(k-3)}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = -2$; series terminates at $k = 2$

$a_{k+2} = -\frac{a_k(k-2)}{(k+3)(k+2)}$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k(k-2)}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right), a_{k+2} = -\frac{a_k(k-3)}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k(k-2)}{(k+3)(k+2)}, b_1 = 0\right]$$

1.241.3 Maple trace

```
Methods for second order ODEs:
```

1.241.4 Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 35

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x^2+6)*diff(y(x),x)+6*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \,\mathrm{e}^{-\frac{x^2}{2}} \operatorname{hypergeom}\left([2], \left[\frac{1}{2}\right], \frac{x^2}{2}\right) + c_1(x^2+3)\,x}{x^3}$$

1.241.5 Mathematica DSolve solution

Solving time : `0.715` (sec)
Leaf size : 65

```
DSolve[{x^2*D[y[x],{x,2}]+x*(6+x^2)*D[y[x],x]+6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{2\pi}c_2 x(x^2+3)\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) - 12c_1 x(x^2+3) + 2c_2 e^{-\frac{x^2}{2}}(x^2+2)}{12x^3}$$

1.242 problem 245

Internal problem ID [8380]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 245
Date solved : `Monday, October 21, 2024 at 05:06:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(1-x)\,y' - y = 0$$

1.242.1 Solved as second order ode using Kovacic algorithm

Time used: 0.253 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2+x\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2+x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 2x + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 463: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{2x^2} + \frac{1}{2x^3} + \frac{1}{4x^4} - \frac{1}{4x^5} - \frac{3}{4x^6} - \frac{3}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2x} \\
&= \frac{x - 1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(\frac{1}{2} - \frac{1}{2x}\right)^2 - \left(\frac{x^2 - 2x + 3}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2x}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2 + x}{x^2} \, dx} \\
&= z_1 e^{\frac{x}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(-(1+x)\,x\,e^{x-\ln(x)}e^{-2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^x}{x}\right) + c_2\left(\frac{e^x}{x}\left(-(1+x)\,x\,e^{x-\ln(x)}e^{-2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.242.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + x(1-x)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{y}{x^2} + \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x-1)y'}{x} - \frac{y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

$\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = -\frac{1}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -1$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - x(x-1)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ◦ Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_k(k+r+1) - a_{k-1}\right) = 0$

- Shift index using $k- >k+1$

  $(k+r)\left(a_{k+1}(k+2+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+2+r}$
- Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+3}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

1.242.3 Maple trace

```
Methods for second order ODEs:
```

1.242.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(1-x)*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \, e^{x} + c_1 x + c_1}{x}$$

1.242.5 Mathematica DSolve solution

Solving time : `0.018` (sec)
Leaf size : 23

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1-x)*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 e^x - c_1(x+1)}{x}$$

1.243 problem 246

Internal problem ID [8381]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 246
Date solved : `Monday, October 21, 2024 at 05:06:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(x+3)\,y' + 4y = 0$$

1.243.1 Solved as second order ode using Kovacic algorithm

Time used: 0.291 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2 - 3x\right)y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 6x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 465: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{2x} - \frac{5}{2x^2} + \frac{15}{2x^3} - \frac{115}{4x^4} + \frac{495}{4x^5} - \frac{2285}{4x^6} + \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{6x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 6. Dividing this by leading coefficient in t which is 4 gives $\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{3}{2}\right) - (0) \\ &= \frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2x} + \frac{1}{2} \\
&= \frac{1 + x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 6x - 1}{4x^2}\right)\right) = 0$$

$$\frac{1 - a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1 + x) \, \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{1}{2}\right) dx} \\
&= (1 + x) \, \mathrm{e}^{\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= (1 + x) \sqrt{x} \, \mathrm{e}^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{3\ln(x)}{2}} \\
&= z_1 \left(x^{3/2}\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2\mathrm{e}^{x}(1+x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-x}}{-1-x} - \mathrm{Ei}_1(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2\mathrm{e}^{x}(1+x)\right) + c_2\left(x^2\mathrm{e}^{x}(1+x)\left(-\frac{\mathrm{e}^{-x}}{-1-x} - \mathrm{Ei}_1(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.243.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x(x+3)\,y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = -\frac{4y}{x^2} + \frac{(x+3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{(x+3)y'}{x} + \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = -\frac{x+3}{x}, P_3(x) = \frac{4}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = -3$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 4$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right) - x(x+3)\,y' + 4y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 - a_{k-1}(k+r-1) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+r-1)^2 - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+r-1)^2}$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+2)}{(k+1)^2}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+2)}{(k+1)^2}\right]$

1.243.3 Maple trace

```
Methods for second order ODEs:
```

1.243.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 29

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(x+3)*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = (c_2\, \mathrm{e}^x(1+x)\, \mathrm{Ei}_1(x) + c_1\, \mathrm{e}^x(1+x) - c_2)\, x^2$$

1.243.5 Mathematica DSolve solution

Solving time : `0.092` (sec)
Leaf size : 34

```
DSolve[{x^2*D[y[x],{x,2}]-x*(x+3)*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x^2(c_2e^x(x+1)\,\text{ExpIntegralEi}(-x) + c_1e^x(x+1) + c_2)$$

1.244 problem 247

Internal problem ID [8382]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 247
Date solved : `Monday, October 21, 2024 at 05:07:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x^2y' - 2y = 0$$

1.244.1 Solved as second order ode using Kovacic algorithm

Time used: 0.248 (sec)

Writing the ode as

$$x^2y'' - x^2y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 467: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x^2} - \frac{4}{x^4} + \frac{16}{x^6} - \frac{80}{x^8} + \frac{448}{x^{10}} - \frac{2688}{x^{12}} + \frac{16896}{x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2}{x^2}\right) \\ &= \frac{1}{4} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{x} - \frac{1}{2} \\
&= -\frac{2 + x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{1}{2}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 + 8}{4x^2}\right)\right) &= 0 \\
\frac{-2 + a_0}{x} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2 + x)\, e^{\int \left(-\frac{1}{x} - \frac{1}{2}\right) dx} \\
&= (2 + x)\, e^{-\frac{x}{2} - \ln(x)} \\
&= \frac{(2 + x)\, e^{-\frac{x}{2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2+x}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^x}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(-2+x)\,\mathrm{e}^x}{2+x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{2+x}{x}\right) + c_2\left(\frac{2+x}{x}\left(\frac{(-2+x)\,\mathrm{e}^x}{2+x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.244.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x^2y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{2y}{x^2} + y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - y' - \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -1, P_3(x) = -\frac{2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -2$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - x^2y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

  - Shift index using $k- >k-1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$

  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)(k+r-2) - a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-2) - a_{k-1}(k-1+r) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+2+r)(k-1+r) - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k-1)}{(k+1)(k-2)}$
- Apply recursion relation for $k = 0$

  $a_1 = \frac{a_0}{2}$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 + \frac{x}{2}\right)$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+2)}{(k+4)(k+1)}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+2)}{(k+4)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 + \frac{x}{2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = \frac{b_k(k+2)}{(k+4)(k+1)}\right]$

1.244.3 Maple trace

```
Methods for second order ODEs:
```

1.244.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)-x^2*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$
y = \frac{c_2(-2+x)\,\mathrm{e}^{x}+c_1(2+x)}{x}
$$

1.244.5 Mathematica DSolve solution

Solving time : `0.07` (sec)
Leaf size : 72

```
DSolve[{x^2*D[y[x],{x,2}]-x^2*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to -\frac{2e^{x/2}\left((c_1x+2ic_2)\cosh\left(\frac{x}{2}\right)-(ic_2x+2c_1)\sinh\left(\frac{x}{2}\right)\right)}{\sqrt{\pi}\sqrt{-ix}\sqrt{x}}
$$

1.245 problem 248

Internal problem ID [8383]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 248
Date solved : `Monday, October 21, 2024 at 05:07:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x^2y' - (3x+2)\,y = 0$$

1.245.1 Solved as second order ode using Kovacic algorithm

Time used: 0.281 (sec)

Writing the ode as

$$x^2y'' - x^2y' + (-3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= -3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 12x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 12x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 12x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 469: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} + \frac{3}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{x} - \frac{7}{x^2} + \frac{42}{x^3} - \frac{301}{x^4} + \frac{2394}{x^5} - \frac{20342}{x^6} + \frac{180852}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 12x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{12x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{12x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 12. Dividing this by leading coefficient in t which is 4 gives 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{3}{\frac{1}{2}} - 0\right) &= 3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{3}{\frac{1}{2}} - 0\right) &= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 12x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	3	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= 3 - (2) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{2}{x} + \left(\frac{1}{2}\right) \\
&= \frac{2}{x} + \frac{1}{2} \\
&= \frac{4+x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{2}{x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{2}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2+12x+8}{4x^2}\right)\right) = 0
$$

$$
\frac{4-a_0}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 4\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 4 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (4+x)\, e^{\int \left(\frac{2}{x}+\frac{1}{2}\right) dx} \\
&= (4+x)\, e^{\frac{x}{2}+2\ln(x)} \\
&= (4+x)\, x^2 e^{\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2}{x^2}\, dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x} (4+x)\, x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-x} (x^3+3x^2-2x+2)}{24\,(4+x)\, x^3} + \frac{\mathrm{Ei}_1(x)}{24}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{x} (4+x)\, x^2\right) + c_2 \left(\mathrm{e}^{x} (4+x)\, x^2 \left(-\frac{\mathrm{e}^{-x} (x^3+3x^2-2x+2)}{24\,(4+x)\, x^3} + \frac{\mathrm{Ei}_1(x)}{24}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.245.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x^2y' - (3x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(3x+2)y}{x^2} + y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - y' - \frac{(3x+2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -1, P_3(x) = -\frac{3x+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -2$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - x^2y' + (-3x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$$

- Shift index using $k->k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+2+r) = 0$$

- Shift index using $k->k+1$

$$a_{k+1}(k+2+r)\,(k-1+r) - a_k(k+r+3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r+3)}{(k+2+r)(k-1+r)}$$

- Recursion relation for $r=-1$

$$a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$$

- Series not valid for $r=-1$, division by 0 in the recursion relation at $k=2$

$$a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$$

- Recursion relation for $r=2$

$$a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}$$

- Solution for $r=2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}\right]$$

1.245.3 Maple trace

```
Methods for second order ODEs:
```

1.245.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 48

```
dsolve(x^2*diff(diff(y(x),x),x)-x^2*diff(y(x),x)-(3*x+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-x^3 c_2 \, e^x (4+x) \operatorname{Ei}_1(x) + x^3 c_1 (4+x) \, e^x + c_2(x^3+3x^2-2x+2)}{x}$$

1.245.5 Mathematica DSolve solution

Solving time : 0.264 (sec)
Leaf size : 59

```
DSolve[{x^2*D[y[x],{x,2}]-x^2*D[y[x],x]-(3*x+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{24} c_2 e^x (x+4) x^2 \operatorname{ExpIntegralEi}(-x) + c_1 e^x (x+4) x^2 - \frac{c_2(x^3+3x^2-2x+2)}{24x}$$

1.246 problem 249

Internal problem ID [8384]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 249
Date solved : `Monday, October 21, 2024 at 05:07:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(5 - x)\,y' + 4y = 0$$

1.246.1 Solved as second order ode using Kovacic algorithm

Time used: 0.351 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2 + 5x\right)y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 + 5x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 10x - 1}{4x^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 471: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{5}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{5}{2x} - \frac{13}{2x^2} - \frac{65}{2x^3} - \frac{819}{4x^4} - \frac{5785}{4x^5} - \frac{43797}{4x^6} - \frac{347425}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 10x - 1}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-10x - 1}{4x^2}\right) \\
&= \frac{1}{4} + \frac{-10x - 1}{4x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -10. Dividing this by leading coefficient in t which is 4 gives $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &= -\frac{5}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &= \frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 10x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{5}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{5}{2} - \left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{-1+x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 10x - 1}{4x^2}\right) \right) = 0$$

$$\frac{(a_1 + 4)\, x + 2a_0 + a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2, a_1 = -4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 4x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^2 - 4x + 2)\, e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= (x^2 - 4x + 2)\, e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= (x^2 - 4x + 2)\, \sqrt{x}\, e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+5x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}-\frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{2}}}{x^{5/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2-4x+2}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+5x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x-5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^x (x-3)}{4\,(x^2-4x+2)} - \frac{\mathrm{Ei}_1(-x)}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2-4x+2}{x^2}\right) + c_2 \left(\frac{x^2-4x+2}{x^2}\left(-\frac{\mathrm{e}^x (x-3)}{4\,(x^2-4x+2)} - \frac{\mathrm{Ei}_1(-x)}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.246.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+x(5-x)\,y'+4y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{4y}{x^2}+\frac{(x-5)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{(x-5)y'}{x}+\frac{4y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

 ∘ Define functions

 $\left[P_2(x)=-\frac{x-5}{x},P_3(x)=\frac{4}{x^2}\right]$

 ∘ $x\cdot P_2(x)$ is analytic at $x=0$

 $\left.(x\cdot P_2(x))\right|_{x=0}=5$

 ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

 $\left.(x^2\cdot P_3(x))\right|_{x=0}=4$

 ∘ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right)-x(x-5)\,y'+4y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

 ∘ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

 $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

 ∘ Shift index using $k->k+1-m$

 $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

 ∘ Convert $x^2\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(2+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = -2$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)^2 - a_{k-1}(k+r-1) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+3+r)^2 - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+3+r)^2}$
- Recursion relation for $r = -2$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{(k+1)^2}$
- Apply recursion relation for $k = 0$

  $a_1 = -2a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{2}$
- Terminating series solution of the ODE for $r = -2$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 - 2x + \frac{1}{2}x^2\right)$

1.246.3 Maple trace

```
Methods for second order ODEs:
```

1.246.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 41

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(5-x)*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(x^2 - 4x + 2)\, c_2\ \mathrm{Ei}_1\left(-x\right) + c_2(x - 3)\, \mathrm{e}^x + c_1(x^2 - 4x + 2)}{x^2}$$

1.246.5 Mathematica DSolve solution

Solving time : `0.325` (sec)
Leaf size : 48

```
DSolve[{x^2*D[y[x],{x,2}]+x*(5-x)*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2(x^2 - 4x + 2)\,\mathrm{ExpIntegralEi}(x) + 4c_1(x^2 - 4x + 2) - c_2e^x(x - 3)}{4x^2}$$

1.247 problem 250

Internal problem ID [8385]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 250
Date solved : `Monday, October 21, 2024 at 05:07:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 4x(1-x)\,y' + (2x-9)\,y = 0$$

1.247.1 Solved as second order ode using Kovacic algorithm

Time used: 0.244 (sec)

Writing the ode as

$$4x^2y'' + \left(-4x^2+4x\right)y' + (2x-9)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2+4x \\ C &= 2x-9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 473: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4} + \frac{2}{x^2} - \frac{1}{x}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{x} \\
&= \frac{x - 2}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\
&= \frac{e^{\frac{x}{2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x^2 + 4x}{4x^2} \, dx} \\
&= z_1 e^{\frac{x}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{\frac{x}{2}}}{\sqrt{x}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2+4x}{4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1\left(-(x^2+2x+2)\, x\, e^{x-\ln(x)} e^{-2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^x}{x^{3/2}}\right) + c_2\left(\frac{e^x}{x^{3/2}}\left(-(x^2+2x+2)\, x\, e^{x-\ln(x)} e^{-2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.247.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) + 4x(1-x)\, y' + (2x-9)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(2x-9)y}{4x^2} + \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x-1)y'}{x} + \frac{(2x-9)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = \frac{2x-9}{4x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{9}{4}$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$4x^2\left(\frac{d}{dx}y'\right) - 4x(x-1)\,y' + (2x-9)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- ○ Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- ○ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(3+2r)\,(-3+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+3)\,(2k+2r-3) - 2a_{k-1}(2k+2r-3)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(3+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{3}{2}, \frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{3}{2}\right)\left(\left(k+r+\frac{3}{2}\right)a_k - a_{k-1}\right) = 0$
- Shift index using $k->k+1$

  $4\left(k-\frac{1}{2}+r\right)\left(\left(k+\frac{5}{2}+r\right)a_{k+1} - a_k\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k}{2k+5+2r}$
- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+1} = \frac{2a_k}{2k+2}$
- Solution for $r = -\frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+2}\right]$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = \frac{2a_k}{2k+8}$
- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+8}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{2a_k}{2k+2}, b_{k+1} = \frac{2b_k}{2k+8}\right]$

1.247.3 Maple trace

```
Methods for second order ODEs:
```

1.247.4 Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 23

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*x*(1-x)*diff(y(x),x)+(2*x-9)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \,\mathrm{e}^{x} + c_2(x^2+2x+2)}{x^{3/2}}$$

1.247.5 Mathematica DSolve solution

Solving time : 0.063 (sec)
Leaf size : 30

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x*(1-x)*D[y[x],x]+(2*x-9)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1 e^x - c_2(x^2 + 2x + 2)}{x^{3/2}}$$

1.248 problem 251

Internal problem ID [8386]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 251
Date solved : `Monday, October 21, 2024 at 05:07:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 2x(2+x)\,y' + 2(1+x)\,y = 0$$

1.248.1 Solved as second order ode using Kovacic algorithm

Time used: 0.264 (sec)

Writing the ode as

$$x^2y'' + \left(2x^2 + 4x\right)y' + (2x+2)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 + 4x \\ C &= 2x + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2 + x}{x} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2 + x$$
$$t = x$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2 + x}{x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 475: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{2x^3} - \frac{5}{8x^4} + \frac{7}{8x^5} - \frac{21}{16x^6} + \frac{33}{16x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2 + x}{x} \\ &= Q + \frac{R}{x} \\ &= (1) + \left(\frac{2}{x}\right) \\ &= 1 + \frac{2}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 2. Dividing this by leading coefficient in t which is 1 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{1} - 0\right) &&= 1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{1} - 0\right) &&= -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2 + x}{x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} + (1) \\
&= 1 + \frac{1}{x} \\
&= 1 + \frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(1 + \frac{1}{x}\right)(0) + \left(\left(-\frac{1}{x^2}\right) + \left(1 + \frac{1}{x}\right)^2 - \left(\frac{2+x}{x}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(1+\frac{1}{x}\right) dx} \\ &= x \, \mathrm{e}^{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2x^2+4x}{x^2} \, dx} \\ &= z_1 e^{-x-2\ln(x)} \\ &= z_1 \left(\frac{\mathrm{e}^{-x}}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x-4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2x}}{3x} + \frac{\mathrm{e}^{-2x}}{3} - \frac{2x\,\mathrm{e}^{-2x}}{3} + \frac{4x^2\,\mathrm{Ei}_1(2x)}{3} \right. \\
&\qquad \left. - \frac{4\,\mathrm{Ei}_1(2x)\,x^3 - 2\,\mathrm{e}^{-2x}x^2 + x\,\mathrm{e}^{-2x} - 6x\,\mathrm{Ei}_1(2x) + 2\,\mathrm{e}^{-2x}}{3x} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(-\frac{\mathrm{e}^{-2x}}{3x} + \frac{\mathrm{e}^{-2x}}{3} - \frac{2x\,\mathrm{e}^{-2x}}{3} + \frac{4x^2\,\mathrm{Ei}_1(2x)}{3} \right.\right. \\
&\qquad \left.\left. - \frac{4\,\mathrm{Ei}_1(2x)\,x^3 - 2\,\mathrm{e}^{-2x}x^2 + x\,\mathrm{e}^{-2x} - 6x\,\mathrm{Ei}_1(2x) + 2\,\mathrm{e}^{-2x}}{3x} \right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.248.2 Maple step by step solution

1.248.3 Maple trace

```
Methods for second order ODEs:
```

1.248.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 28

```
dsolve(x^2*diff(diff(y(x),x),x)+2*x*(2+x)*diff(y(x),x)+2*(1+x)*y(x) = 0,
       y(x),singsol=all)
```

$$
y = \frac{2c_2\,\mathrm{Ei}_1(2x)\,x - c_2\,\mathrm{e}^{-2x} + c_1 x}{x^2}
$$

1.248.5 Mathematica DSolve solution

Solving time : `0.119` (sec)
Leaf size : 32

```
DSolve[{x^2*D[y[x],{x,2}]+2*x*(2+x)*D[y[x],x]+2*(1+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-2c_2x\,\text{ExpIntegralEi}(-2x) + c_1x - c_2e^{-2x}}{x^2}$$

1.249 problem 252

Internal problem ID [8387]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 252
Date solved : `Monday, October 21, 2024 at 05:07:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(1-x)\,y' + (1-x)\,y = 0$$

1.249.1 Solved as second order ode using Kovacic algorithm

Time used: 0.265 (sec)

Writing the ode as

$$x^2y'' + \left(x^2 - x\right) y' + (1-x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 - x \\ C &= 1 - x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 2x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 2x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 2x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 476: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2x} - \frac{1}{2x^2} + \frac{1}{2x^3} - \frac{3}{4x^4} + \frac{5}{4x^5} - \frac{9}{4x^6} + \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{2x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 2x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \left(\frac{1}{2} \right) \\
&= \frac{1}{2} + \frac{1}{2x} \\
&= \frac{x + 1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2} + \frac{1}{2x}\right)^2 - \left(\frac{x^2 + 2x - 1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{1}{2x}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 - x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{-\frac{x}{2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x+\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(-\operatorname{Ei}_1(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2(x(-\operatorname{Ei}_1(x))) \end{aligned}$$

Will add steps showing solving for IC soon.

1.249.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x(1-x)\,y' + (1-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(x-1)y}{x^2} - \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(x-1)y'}{x} - \frac{(x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{x-1}{x}, P_3(x) = -\frac{x-1}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x(x-1)\, y' + (1-x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)^2 + a_{k-1}(k-2+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

$r = 1$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)^2 + a_{k-1}(k-2+r) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+r)^2 + a_k(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-1)}{(k+r)^2}$
- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k k}{(k+1)^2}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k k}{(k+1)^2}\right]$

1.249.3 Maple trace

```
Methods for second order ODEs:
```

1.249.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 13

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(1-x)*diff(y(x),x)+(1-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x(c_2 \operatorname{Ei}_1(x) + c_1)$$

1.249.5 Mathematica DSolve solution

Solving time : 0.048 (sec)
Leaf size : 17

```
DSolve[{x^2*D[y[x],{x,2}]-x*(1-x)*D[y[x],x]+(1-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x(c_2 \operatorname{ExpIntegralEi}(-x) + c_1)$$

1.250 problem 253

Internal problem ID [8388]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 253
Date solved : `Monday, October 21, 2024 at 05:07:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 4x(1+2x)\,y' + (4x-1)\,y = 0$$

1.250.1 Solved as second order ode using Kovacic algorithm

Time used: 0.125 (sec)

Writing the ode as

$$4x^2y'' + \left(8x^2+4x\right)y' + (4x-1)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 8x^2+4x \\ C &= 4x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 478: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^2+4x}{4x^2}\,dx}\\ &= z_1 e^{-x-\frac{\ln(x)}{2}}\\ &= z_1\left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-2x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-2x-\ln(x)}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\mathrm{e}^{-2x-\ln(x)}x\,\mathrm{e}^{4x}}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-2x}}{\sqrt{x}}\right) + c_2 \left(\frac{e^{-2x}}{\sqrt{x}} \left(\frac{e^{-2x-\ln(x)} x\, e^{4x}}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.250.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) + 4x(1+2x)\,y' + (4x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x-1)y}{4x^2} - \frac{(1+2x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(1+2x)y'}{x} + \frac{(4x-1)y}{4x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{1+2x}{x}, P_3(x) = \frac{4x-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 4x(1+2x)\,y' + (4x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-1}(2k+2r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(\left(k+r+\frac{1}{2}\right) a_k + 2a_{k-1}\right) = 0$

- Shift index using $k- >k + 1$

  $4\left(k+r+\frac{1}{2}\right)\left(\left(k+\frac{3}{2}+r\right) a_{k+1} + 2a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k}{2k+3+2r}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{4a_k}{2k+2}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{4a_k}{2k+2}\right]$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+1} = -\frac{4a_k}{2k+4}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{4a_k}{2k+4}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{4a_k}{2k+2}, b_{k+1} = -\frac{4b_k}{2k+4}\right]$

1.250.3 Maple trace

```
Methods for second order ODEs:
```

1.250.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 16

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*x*(1+2*x)*diff(y(x),x)+(4*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \, \mathrm{e}^{-2x} + c_1}{\sqrt{x}}$$

1.250.5 Mathematica DSolve solution

Solving time : `0.05` (sec)
Leaf size : 26

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x*(1+2*x)*D[y[x],x]+(4*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-2x} + c_2}{2\sqrt{x}}$$

1.251 problem 254

Internal problem ID [8389]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 254
Date solved : `Monday, October 21, 2024 at 05:07:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(4+x)\,y' + (2+x)\,y = 0$$

1.251.1 Solved as second order ode using Kovacic algorithm

Time used: 0.272 (sec)

Writing the ode as

$$x^2y'' + \left(x^2+4x\right)y' + (2+x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+4x \\ C &= 2+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4 + x}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4 + x$$
$$t = 4x$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4 + x}{4x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 480: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{5}{x^4} + \frac{14}{x^5} - \frac{42}{x^6} + \frac{132}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4+x}{4x} \\ &= Q + \frac{R}{4x} \\ &= \left(\frac{1}{4}\right) + \left(\frac{1}{x}\right) \\ &= \frac{1}{4} + \frac{1}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{1}{\frac{1}{2}} - 0\right) &= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{1}{\frac{1}{2}} - 0\right) &= -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4 + x}{4x}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{x} \\
&= \frac{1}{2} + \frac{1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{1}{x}\right)(0) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{2} + \frac{1}{x}\right)^2 - \left(\frac{4 + x}{4x}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{1}{x}\right) dx} \\
&= x \, \mathrm{e}^{\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 + 4x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - 2\ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{e^{-x}}{3x} + \frac{e^{-x}}{6} - \frac{x\,e^{-x}}{6} + \frac{x^2\,\mathrm{Ei}_1(x)}{6} - \frac{\mathrm{Ei}_1(x)\,x^3 - e^{-x}x^2 - 6x\,\mathrm{Ei}_1(x) + x\,e^{-x} + 4\,e^{-x}}{6x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(-\frac{e^{-x}}{3x} + \frac{e^{-x}}{6} - \frac{x\,e^{-x}}{6} + \frac{x^2\,\mathrm{Ei}_1(x)}{6}\right.\right. \\
&\qquad \left.\left. - \frac{\mathrm{Ei}_1(x)\,x^3 - e^{-x}x^2 - 6x\,\mathrm{Ei}_1(x) + x\,e^{-x} + 4\,e^{-x}}{6x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.251.2 Maple step by step solution

1.251.3 Maple trace

```
Methods for second order ODEs:
```

1.251.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 25

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(4+x)*diff(y(x),x)+(2+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-c_2 \, \mathrm{e}^{-x} + x(c_2 \, \mathrm{Ei}_1(x) + c_1)}{x^2}$$

1.251.5 Mathematica DSolve solution

Solving time : `0.064` (sec)
Leaf size : 32

```
DSolve[{x^2*D[y[x],{x,2}]+x*(4+x)*D[y[x],x]+(2+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-c_2 x \, \text{ExpIntegralEi}(-x) + c_1 x - c_2 e^{-x}}{x^2}$$

1.252 problem 255

Internal problem ID [8390]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 255
Date solved : `Monday, October 21, 2024 at 05:07:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right) y = 0$$

1.252.1 Solved as second order ode using Kovacic algorithm

Time used: 0.293 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{9}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 481: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{x^2} - \frac{i}{2x^4} - \frac{i}{2x^6} - \frac{5i}{8x^8} - \frac{7i}{8x^{10}} - \frac{21i}{16x^{12}} - \frac{33i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{2}{x^2}\right) \\ &= -1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(i) \\
&= -\frac{1}{x} - i \\
&= -\frac{1}{x} - i
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - i\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - i\right)^2 - \left(\frac{-x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{2ia_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -i\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - i
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x - i)\, e^{\int \left(-\frac{1}{x} - i\right) dx} \\
&= (x - i)\, e^{-\ln(x) - ix} \\
&= \frac{(x - i)\, e^{-ix}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x-i)\,\mathrm{e}^{-ix}}{x^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x^{3/2}}\right) + c_2 \left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x^{3/2}} \left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.252.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+xy'+\left(x^2-\frac{9}{4}\right)y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(4x^2-9)y}{4x^2}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{y'}{x}+\frac{(4x^2-9)y}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{4x^2-9}{4x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=-\frac{9}{4}$

  ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$4x^2\left(\frac{d}{dx}y'\right)+4xy'+(4x^2-9)y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x\cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(3+2r)\,(-3+2r)\, x^r + a_1(5+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+3)\,(2k+2r-3) + 4a_{\dots}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{3}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(5+2r)\,(-1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 9) + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 9\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+7}$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+4k-8}$

- Solution for $r = -\frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+4k-8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+28k+40}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+28k+40}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+4k-8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+28k+40}, b_1=0\right]$$

1.252.3 Maple trace

```
Methods for second order ODEs:
```

1.252.4 Maple dsolve solution

Solving time : 0.016 (sec)
Leaf size : 30

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-9/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{-(-x+i)\, c_2\, \mathrm{e}^{-ix}+(x+i)\, \mathrm{e}^{ix} c_1}{x^{3/2}}$$

1.252.5 Mathematica DSolve solution

Solving time : 0.089 (sec)
Leaf size : 44

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-9/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\frac{\sqrt{\frac{2}{\pi}}((c_1x+c_2)\cos(x)+(c_2x-c_1)\sin(x))}{x^{3/2}}$$

1.253 problem 256

Internal problem ID [8391]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 256
Date solved : `Monday, October 21, 2024 at 05:07:08 PM`
CAS classification : `[_Lienard]`

Solve

$$xy'' + 2y' + xy = 0$$

1.253.1 Solved as second order ode using Kovacic algorithm

Time used: 0.145 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 483: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{\cos(x)}{x}\right) + c_2\left(\frac{\cos(x)}{x}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.253.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -y - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion
    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$
  - Shift index using $k- >k-1$
    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$
  - Convert y' to series expansion
    $$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$
  - Shift index using $k- >k+1$
    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
    $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$
  - Shift index using $k- >k+1$
    $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions
  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $$r(1+r) = 0$$
- Values of r that satisfy the indicial equation
  $$r \in \{-1, 0\}$$
- Each term must be 0
  $$a_1(1+r)\,(2+r) = 0$$
- Each term in the series must be 0, giving the recursion relation
  $$a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$$
- Shift index using $k- >k+1$
  $$a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$$
- Recursion relation that defines series solution to ODE
  $$a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$$
- Recursion relation for $r = -1$
  $$a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$$

1.253.3 Maple trace

```
Methods for second order ODEs:
```

1.253.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{x}$$

1.253.5 Mathematica DSolve solution

Solving time : `0.041` (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

1.254 problem 257

Internal problem ID [8392]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 257
Date solved : `Monday, October 21, 2024 at 05:07:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2xy'' + 5(1-2x)\,y' - 5y = 0$$

1.254.1 Solved as second order ode using Kovacic algorithm

Time used: 0.435 (sec)

Writing the ode as

$$2xy'' + (-10x+5)\,y' - 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -10x+5 \\ C &= -5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{100x^2 - 60x + 5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 100x^2 - 60x + 5$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{100x^2 - 60x + 5}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 485: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{25}{4} - \frac{15}{4x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{5}{2} - \frac{3}{4x} - \frac{1}{20x^2} - \frac{3}{200x^3} - \frac{1}{200x^4} - \frac{9}{5000x^5} - \frac{137}{200000x^6} - \frac{543}{2000000x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{5}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{5}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{25}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{100x^2 - 60x + 5}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{25}{4}\right) + \left(\frac{-60x + 5}{16x^2}\right) \\ &= \frac{25}{4} + \frac{-60x + 5}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -60. Dividing this by leading coefficient in t which is 16 gives $-\frac{15}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{15}{4}\right) - (0) \\ &= -\frac{15}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{5}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{15}{4}}{\frac{5}{2}} - 0\right) &&= -\frac{3}{4} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{15}{4}}{\frac{5}{2}} - 0\right) &&= \frac{3}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{100x^2 - 60x + 5}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{5}{2}$	$-\frac{3}{4}$	$\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - (\alpha_{c_1}^{-}) \\ &= \frac{3}{4} - \left(-\frac{1}{4}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4x} + (-)\left(\frac{5}{2}\right) \\
&= -\frac{1}{4x} - \frac{5}{2} \\
&= -\frac{1}{4x} - \frac{5}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{4x} - \frac{5}{2}\right)(1) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x} - \frac{5}{2}\right)^2 - \left(\frac{100x^2 - 60x + 5}{16x^2}\right) \right) &= 0 \\
\frac{-1 + 10a_0}{2x} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{1}{10} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{10}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{10}\right) \mathrm{e}^{\int \left(-\frac{1}{4x} - \frac{5}{2}\right) dx} \\
&= \left(x + \frac{1}{10}\right) \mathrm{e}^{-\frac{5x}{2} - \frac{\ln(x)}{4}} \\
&= \frac{(1 + 10x)\, \mathrm{e}^{-\frac{5x}{2}}}{10x^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-10x+5}{2x} \, dx} \\
&= z_1 e^{\frac{5x}{2} - \frac{5\ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{5x}{2}}}{x^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1 + 10x}{10x^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x+5}{2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{5x - \frac{5\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{100\, \mathrm{e}^{5x - \frac{5\ln(x)}{2}} x^3}{(1 + 10x)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1+10x}{10x^{3/2}}\right) + c_2 \left(\frac{1+10x}{10x^{3/2}} \left(\int \frac{100\, \mathrm{e}^{5x-\frac{5\ln(x)}{2}} x^3}{\left(1+10x\right)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.254.2 Maple step by step solution

Let's solve

$2x\left(\frac{d}{dx}y'\right) + 5(1-2x)\, y' - 5y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{5y}{2x} + \frac{5(2x-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{5(2x-1)y'}{2x} - \frac{5y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{5(2x-1)}{2x}, P_3(x) = -\frac{5}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{5}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x\left(\frac{d}{dx}y'\right) + (-10x+5)\, y' - 5y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
- Shift index using $k->k+1$

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+5+2r) - 5a_k(2k+2r+1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k+1+r)\left(k+\frac{5}{2}+r\right) a_{k+1} - 10\left(k+r+\frac{1}{2}\right) a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{5(2k+2r+1)a_k}{(k+1+r)(2k+5+2r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{5(2k+1)a_k}{(k+1)(2k+5)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{5(2k+1)a_k}{(k+1)(2k+5)}\right]$
- Recursion relation for $r = -\frac{3}{2}$; series terminates at $k = 1$

  $a_{k+1} = \frac{5(2k-2)a_k}{\left(k-\frac{1}{2}\right)(2k+2)}$
- Apply recursion relation for $k = 0$

  $a_1 = 10a_0$
- Terminating series solution of the ODE for $r = -\frac{3}{2}$. Use reduction of order to find the second

  $y = a_0 \cdot (1+10x)$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + b_0 \cdot (1 + 10x), a_{k+1} = \frac{5(2k+1)a_k}{(k+1)(2k+5)}\right]$$

1.254.3 Maple trace

```
Methods for second order ODEs:
```

1.254.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 45

```
dsolve(2*x*diff(diff(y(x),x),x)+5*(1-2*x)*diff(y(x),x)-5*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{10\left(\sqrt{5}\, c_1 \sqrt{\pi}\, \left(x + \frac{1}{10}\right) \operatorname{erfi}\left(\sqrt{5}\, \sqrt{x}\right) - \mathrm{e}^{5x} c_1 \sqrt{x} - \left(x + \frac{1}{10}\right) c_2\right)}{x^{3/2}}$$

1.254.5 Mathematica DSolve solution

Solving time : `0.044` (sec)
Leaf size : 40

```
DSolve[{2*x*D[y[x],{x,2}]+5*(1-2*x)*D[y[x],x]-5*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 L_{-\frac{1}{2}}^{\frac{3}{2}}(5x) + \frac{c_1(10x + 1)}{10\sqrt{5}x^{3/2}}$$

1.255 problem 258

Internal problem ID [8393]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 258
Date solved : `Monday, October 21, 2024 at 05:07:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

1.255.1 Solved as second order ode using Kovacic algorithm

Time used: 0.158 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 487: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.255.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 \left(\frac{d}{dx} y'\right) + 4x y' + (4x^2 - 1) y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$$

1.255.3 Maple trace

```
Methods for second order ODEs:
```

1.255.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{\sqrt{x}}$$

1.255.5 Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

1.256 problem 259

Internal problem ID [8394]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 259
Date solved : `Monday, October 21, 2024 at 05:07:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (x+n)\,y' + (n+1)\,y = 0$$

1.256.1 Solved as second order ode using Kovacic algorithm

Time used: 0.450 (sec)

Writing the ode as

$$xy'' + (x+n)\,y' + (n+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x+n \\ C &= n+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = n^2 - 2xn + x^2 - 2n - 4x$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 489: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{\frac{1}{4}n^2 - \frac{1}{2}n}{x^2} + \frac{-\frac{n}{2} - 1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{1}{4}n^2 - \frac{1}{2}n$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{n}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = 1 - \frac{n}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} - \frac{3n^6}{2x^7} - \frac{3n^5}{2x^6} - \frac{3n^4}{2x^5} - \frac{3n^3}{2x^4} - \frac{3n^2}{2x^3} - \frac{3n}{2x^2} - \frac{77n^5}{2x^7} - \frac{53n^4}{2x^6} - \frac{67n^3}{4x^5} - \frac{37n^2}{4x^4} - \frac{4n}{x^3} - \frac{1075n^4}{4x^7} - \frac{491n^3}{4x^6} - \frac{93n^2}{2x^5} - \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{(-2n-4)x + n^2 - 2n}{4x^2}\right) \\ &= \frac{1}{4} + \frac{(-2n-4)x + n^2 - 2n}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is $-2n-4$. Dividing this by leading coefficient in t which is 4 gives $-\frac{n}{2}-1$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{n}{2}-1\right)-(0) \\ &= -\frac{n}{2}-1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{-\frac{n}{2}-1}{\frac{1}{2}}-0\right) &&= -\frac{n}{2}-1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{-\frac{n}{2}-1}{\frac{1}{2}}-0\right) &&= \frac{n}{2}+1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{n^2-2xn+x^2-2n-4x}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{n}{2}$	$1-\frac{n}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{n}{2}-1$	$\frac{n}{2}+1$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{n}{2}+1$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{n}{2}+1-\left(\frac{n}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{n}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{n}{2x} - \frac{1}{2} \\
&= \frac{n-x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{n}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{n}{2x^2}\right) + \left(\frac{n}{2x} - \frac{1}{2}\right)^2 - \left(\frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2}\right) \right) = 0$$

$$\frac{n + a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -n\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - n$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-n)\, e^{\int \left(\frac{n}{2x} - \frac{1}{2}\right) dx} \\
&= (x-n)\, e^{-\frac{x}{2} + \frac{n\ln(x)}{2}} \\
&= -(n-x)\, x^{\frac{n}{2}} e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x+n}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{n\ln(x)}{2}} \\
&= z_1 \left(x^{-\frac{n}{2}} \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x-n)\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x+n}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-n\ln(x)-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-n\ln(x)-x}\mathrm{e}^{2x}}{(x-n)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x-n)\,\mathrm{e}^{-x}\right) + c_2\left((x-n)\,\mathrm{e}^{-x}\left(\int \frac{\mathrm{e}^{-n\ln(x)-x}\mathrm{e}^{2x}}{(x-n)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.256.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + (x+n)\,y' + (n+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(n+1)y}{x} - \frac{(x+n)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(x+n)y'}{x} + \frac{(n+1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x+n}{x}, P_3(x) = \frac{n+1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = n$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + (x+n)\,y' + (n+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+r+n)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r+n) + a_k(k+r+n+1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+r+n) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, -n+1\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r+n) + a_k(k+r+n+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r+n+1)}{(k+1+r)(k+r+n)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k(k+1+n)}{(k+1)(k+n)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k(k+1+n)}{(k+1)(k+n)}\right]$$

- Recursion relation for $r = -n+1$

$$a_{k+1} = -\frac{a_k(k+2)}{(k+2-n)(k+1)}$$

- Solution for $r = -n+1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-n+1}, a_{k+1} = -\frac{a_k(k+2)}{(k+2-n)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-n+1}\right), a_{k+1} = -\frac{a_k(k+1+n)}{(k+1)(k+n)}, b_{k+1} = -\frac{b_k(k+2)}{(k+2-n)(k+1)}\right]$$

1.256.3 Maple trace

```
Methods for second order ODEs:
```

1.256.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 42

```
dsolve(x*diff(diff(y(x),x),x)+(x+n)*diff(y(x),x)+(n+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{-x}(c_2 x^{-n+1} \operatorname{hypergeom}([-n], [-n+2], x) n + c_1(n-x))}{n}$$

1.256.5 Mathematica DSolve solution

Solving time : 1.863 (sec)
Leaf size : 48

```
DSolve[{x*D[y[x],{x,2}]+(x+n)*D[y[x],x]+(n+1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x}(n-x)\left(c_2 \int_1^x \frac{e^{K[1]}K[1]^{-n}}{(n-K[1])^2} dK[1] + c_1\right)$$

1.257 problem 260

Internal problem ID [8395]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 260
Date solved : `Monday, October 21, 2024 at 05:07:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^4y'' + xy' + y = 0$$

1.257.1 Solved as second order ode using Kovacic algorithm

Time used: 0.350 (sec)

Writing the ode as

$$x^4y'' + xy' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-10x^2 + 1}{4x^6} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -10x^2 + 1 \\ t &= 4x^6 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-10x^2 + 1}{4x^6}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 491: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 2 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^6$. There is a pole at $x = 0$ of order 6. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$
[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}
$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$
\begin{aligned}
\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\
\alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right)
\end{aligned}
$$

The partial fraction decomposition of r is

$$
r = \frac{1}{4x^6} - \frac{5}{2x^4}
$$

There is pole in r at $x = 0$ of order 6, hence $v = 3$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$
[\sqrt{r}]_c \approx \frac{1}{2x^3} - \frac{5}{2x} - \frac{25x}{4} + \dots \tag{2B}
$$

Using eq. (1B), taking the sum up to $v = 3$ the above becomes

$$
[\sqrt{r}]_c = \frac{1}{2x^3} \tag{3B}
$$

The above shows that the coefficient of $\frac{1}{(x-0)^3}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^4}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $-\frac{5}{2}$. Therefore

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^3} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} + 3\right) &&= -1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} + 3\right) &&= 4 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-10x^2 + 1}{4x^6}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	6	$\frac{1}{2x^3}$	-1	4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to

determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = 1$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= 1 - (-1) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x^3} - \frac{1}{x} + (-)\,(0) \\ &= \frac{1}{2x^3} - \frac{1}{x} \\ &= \frac{1}{2x^3} - \frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2x^3} - \frac{1}{x}\right)(2x + a_1) + \left(\left(-\frac{3}{2x^4} + \frac{1}{x^2} \right) + \left(\frac{1}{2x^3} - \frac{1}{x} \right)^2 - \left(\frac{-10x^2 + 1}{4x^6} \right) \right) = 0$$

$$\frac{(2a_0 + 2)\,x + a_1}{x^3} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x^2 - 1)\, e^{\int \left(\frac{1}{2x^3} - \frac{1}{x}\right) dx} \\ &= (x^2 - 1)\, e^{-\frac{1}{4x^2} - \ln(x)} \\ &= \frac{(x^2 - 1)\, e^{-\frac{1}{4x^2}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^4} \, dx} \\ &= z_1 e^{\frac{1}{4x^2}} \\ &= z_1 \left(e^{\frac{1}{4x^2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 - 1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^4} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{1}{2x^2}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{e^{\frac{1}{2x^2}} x^2}{(x^2 - 1)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2-1}{x}\right) + c_2 \left(\frac{x^2-1}{x}\left(\int \frac{e^{\frac{1}{2x^2}} x^2}{(x^2-1)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.257.2 Maple step by step solution

1.257.3 Maple trace

```
Methods for second order ODEs:
```

1.257.4 Maple dsolve solution

Solving time : `0.012` (sec)
Leaf size : 50

```
dsolve(x^4*diff(diff(y(x),x),x)+x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$
y = \frac{c_1 \sqrt{2}\,\sqrt{\pi}\,(x-1)(x+1)\,\mathrm{erfi}\left(\frac{\sqrt{2}}{2x}\right) + c_2\, x^2 + 2\,e^{\frac{1}{2x^2}} c_1 x - c_2}{x}
$$

1.257.5 Mathematica DSolve solution

Solving time : `0.313` (sec)
Leaf size : 61

```
DSolve[{x^4*D[y[x],{x,2}]+x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to -\frac{\sqrt{2\pi} c_2 (x^2-1)\,\mathrm{erfi}\left(\frac{1}{\sqrt{2}x}\right) - 4c_1(x^2-1) + 2c_2 e^{\frac{1}{2x^2}} x}{4x}
$$

1.258 problem 261

Internal problem ID [8396]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 261
Date solved : `Monday, October 21, 2024 at 05:07:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + (2x^2 + x)\, y' - 4y = 0$$

1.258.1 Solved as second order ode using Kovacic algorithm

Time used: 0.275 (sec)

Writing the ode as

$$x^2y'' + (2x^2 + x)\, y' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 + x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 4x + 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 492: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{1}{x} + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} + \frac{7}{4x^2} - \frac{7}{8x^3} - \frac{35}{32x^4} + \frac{133}{64x^5} + \frac{63}{128x^6} - \frac{1239}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 4x + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{4x + 15}{4x^2}\right) \\ &= 1 + \frac{4x + 15}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 + 4x + 15}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{2x} + (-)(1) \\
&= -\frac{3}{2x} - 1 \\
&= -\frac{3}{2x} - 1
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{gathered}
(0) + 2\left(-\frac{3}{2x} - 1\right)(1) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x} - 1\right)^2 - \left(\frac{4x^2 + 4x + 15}{4x^2}\right)\right) = 0 \\
\frac{-3 + 2a_0}{x} = 0
\end{gathered}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{3}{2}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{3}{2}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{3}{2}\right) e^{\int \left(-\frac{3}{2x} - 1\right) dx} \\
&= \left(x + \frac{3}{2}\right) e^{-x - \frac{3\ln(x)}{2}} \\
&= \frac{(3 + 2x)\, e^{-x}}{2x^{3/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2+x}{x^2}\,dx} \\
&= z_1 e^{-x-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{-x}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{-2x}(3+2x)}{2x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(2x^2-4x+3)\,x\,e^{-2x-\ln(x)}e^{4x}}{6+4x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-2x}(3+2x)}{2x^2}\right) + c_2 \left(\frac{e^{-2x}(3+2x)}{2x^2}\left(\frac{(2x^2-4x+3)\,x\,e^{-2x-\ln(x)}e^{4x}}{6+4x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.258.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+\left(2x^2+x\right)y'-4y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{4y}{x^2}-\frac{(2x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(2x+1)y'}{x}-\frac{4y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2x+1}{x},P_3(x)=-\frac{4}{x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-4$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right)+x(2x+1)y'-4y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y'$ to series expansion for $m=1..2$

    $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^2\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(2+r)(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)(k+r-2) + 2a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-2) + 2a_{k-1}(k+r-1) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+3+r)(k+r-1) + 2a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k(k+r)}{(k+3+r)(k+r-1)}$
- Recursion relation for $r = -2$; series terminates at $k = 2$

  $a_{k+1} = -\frac{2a_k(k-2)}{(k+1)(k-3)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{4a_0}{3}$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{2}$
- Express in terms of a_0

  $a_2 = \frac{2a_0}{3}$
- Terminating series solution of the ODE for $r = -2$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 - \frac{4}{3}x + \frac{2}{3}x^2\right)$
- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{2a_k(k+2)}{(k+5)(k+1)}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{2a_k(k+2)}{(k+5)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 - \frac{4}{3}x + \frac{2}{3}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = -\frac{2b_k(k+2)}{(k+5)(k+1)}\right]$

1.258.3 Maple trace

```
Methods for second order ODEs:
```

1.258.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 31

```
dsolve(x^2*diff(diff(y(x),x),x)+(2*x^2+x)*diff(y(x),x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \, e^{-2x}(3+2x) + 2c_1 \left(x^2 - 2x + \frac{3}{2}\right)}{x^2}$$

1.258.5 Mathematica DSolve solution

Solving time : 0.77 (sec)
Leaf size : 44

```
DSolve[{x^2*D[y[x],{x,2}]+(x+2*x^2)*D[y[x],x]-4*y[x]==2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{1}{4}\left(\frac{2c_1 e^{-2x}(2x+3)}{x^2} + \frac{c_2(2x^2-4x+3)}{x^2} - 2\right)$$

1.259 problem 262

Internal problem ID [8397]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 262
Date solved : `Monday, October 21, 2024 at 05:07:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(4x^3-14x^2-2x\right)y''-\left(6x^2-7x+1\right)y'+\left(6x-1\right)y=0$$

1.259.1 Solved as second order ode using Kovacic algorithm

Time used: 0.485 (sec)

Writing the ode as

$$\left(4x^3-14x^2-2x\right)y''+\left(-6x^2+7x-1\right)y'+\left(6x-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^3-14x^2-2x\\B&=-6x^2+7x-1\\C&=6x-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16(2x^3 - 7x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -12x^4 + 156x^3 + 297x^2 - 78x - 3$$
$$t = 16(2x^3 - 7x^2 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16(2x^3 - 7x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 494: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 6 - 4\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^3 - 7x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{7}{4} + \frac{\sqrt{57}}{4}$ of order 2. There is a pole at $x = \frac{7}{4} - \frac{\sqrt{57}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{9}{4x} - \frac{3}{16x^2} + \frac{3}{4\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)^2} + \frac{3}{4\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)^2} + \frac{\frac{9}{8} - \frac{29\sqrt{57}}{152}}{x - \frac{7}{4} - \frac{\sqrt{57}}{4}} + \frac{\frac{9}{8} + \frac{29\sqrt{57}}{152}}{x - \frac{7}{4} + \frac{\sqrt{57}}{4}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4}\end{aligned}$$

For the pole at $x = \frac{7}{4} + \frac{\sqrt{57}}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = \frac{7}{4} - \frac{\sqrt{57}}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16\left(2x^3 - 7x^2 - x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16\left(2x^3 - 7x^2 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$\frac{7}{4} + \frac{\sqrt{57}}{4}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$\frac{7}{4} - \frac{\sqrt{57}}{4}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{1}{4} - \left(-\frac{3}{4}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)} + (-)(0) \\ &= \frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)} \\ &= \frac{-6x^2+7x-1}{8x^3-28x^2-4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)}\right)(1) + \left(\left(-\frac{1}{4x^2} + \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)^2} + \frac{1}{2\left(x-\frac{7}{4}+\right.}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-1)\, \mathrm{e}^{\int \left(\frac{1}{4x} - \frac{1}{2\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)}\right) dx} \\
&= (x-1)\, \mathrm{e}^{\frac{\left(-57+7\sqrt{57}\right)\sqrt{57}\,\ln\left(4x-7+\sqrt{57}\right)}{-798+114\sqrt{57}} - \frac{\left(57+7\sqrt{57}\right)\sqrt{57}\,\ln\left(4x-7-\sqrt{57}\right)}{2\left(399+57\sqrt{57}\right)} + \frac{2\ln(x)}{\left(7+\sqrt{57}\right)\left(-7+\sqrt{57}\right)}} \\
&= \frac{(x-1)\, x^{1/4}}{\sqrt{4x-7+\sqrt{57}}\,\sqrt{4x-7-\sqrt{57}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6x^2+7x-1}{4x^3-14x^2-2x}\, dx} \\
&= z_1 e^{\frac{\ln\left(2x^2-7x-1\right)}{2} - \frac{\ln(x)}{4}} \\
&= z_1 \left(\frac{\sqrt{2x^2-7x-1}}{x^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x-1)\sqrt{2}}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x^2+7x-1}{4x^3-14x^2-2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(2x^2-7x-1)-\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{16x(2x+1)\,\mathrm{e}^{\ln(2x^2-7x-1)-\frac{\ln(x)}{2}}}{(x-1)\,(2x^2-7x-1)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x-1)\sqrt{2}}{4}\right) + c_2 \left(\frac{(x-1)\sqrt{2}}{4}\left(\frac{16x(2x+1)\,\mathrm{e}^{\ln(2x^2-7x-1)-\frac{\ln(x)}{2}}}{(x-1)\,(2x^2-7x-1)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.259.2 Maple step by step solution

Let's solve
$(4x^3-14x^2-2x)\left(\frac{d}{dx}y'\right)-(6x^2-7x+1)\,y'+(6x-1)\,y=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{(6x-1)y}{2x(2x^2-7x-1)} + \frac{(6x^2-7x+1)y'}{2x(2x^2-7x-1)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{(6x^2-7x+1)y'}{2x(2x^2-7x-1)} + \frac{(6x-1)y}{2x(2x^2-7x-1)} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{6x^2-7x+1}{2x(2x^2-7x-1)}, P_3(x) = \frac{6x-1}{2x(2x^2-7x-1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x=0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x(2x^2 - 7x - 1)\left(\frac{d}{dx}y'\right) + (-6x^2 + 7x - 1)\, y' + (6x - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-1 + 2r)\, x^{-1+r} + \left(-a_1(1 + r)\,(1 + 2r) - a_0(14r^2 - 21r + 1)\right) x^r + \left(\sum_{k=1}^{\infty} \left(-a_{k+1}(k + 1 + r\right)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{2}\right\}$

- Each term must be 0

  $-a_1(1+r)(1+2r) - a_0(14r^2 - 21r + 1) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-14a_k + 4a_{k-1} - 2a_{k+1})k^2 + ((-28a_k + 8a_{k-1} - 4a_{k+1})r + 21a_k - 18a_{k-1} - 3a_{k+1})k + (-14a$

- Shift index using $k- >k+1$

  $(-14a_{k+1} + 4a_k - 2a_{k+2})(k+1)^2 + ((-28a_{k+1} + 8a_k - 4a_{k+2})r + 21a_{k+1} - 18a_k - 3a_{k+2})(k +$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} + 8kra_k - 28kra_{k+1} + 4r^2a_k - 14r^2a_{k+1} - 10ka_k - 7ka_{k+1} - 10ra_k - 7ra_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 4kr + 2r^2 + 7k + 7r + 6}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 10ka_k - 7ka_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 10ka_k - 7ka_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}, -a_1 - a_0 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 6ka_k - 21ka_{k+1} + 2a_k - a_{k+1}}{2k^2 + 9k + 10}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 6ka_k - 21ka_{k+1} + 2a_k - a_{k+1}}{2k^2 + 9k + 10}, -3a_1 + 6a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 10ka_k - 7ka_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}, -a_1 - a_0 = 0, b_k\right.$

1.259.3 Maple trace

```
Methods for second order ODEs:
```

1.259.4 Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 21

```
dsolve((4*x^3-14*x^2-2*x)*diff(diff(y(x),x),x)-(6*x^2-7*x+1)*diff(y(x),x)+(6*x-1)*y(x)
       y(x),singsol=all)
```

$$y = c_2\sqrt{x} + c_1(x-1) + 2c_2\, x^{3/2}$$

1.259.5 Mathematica DSolve solution

Solving time : 9.99 (sec)
Leaf size : 26

```
DSolve[{(4*x^3-14*x^2-2*x)*D[y[x],{x,2}]-(6*x^2-7*x+1)*D[y[x],x]+(6*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1(x-1) - 2c_2\sqrt{x}(2x+1)$$

1.260 problem 263

Internal problem ID [8398]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 263
Date solved : `Monday, October 21, 2024 at 05:07:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x^2y' + (x-2)\,y = 0$$

1.260.1 Solved as second order ode using Kovacic algorithm

Time used: 0.236 (sec)

Writing the ode as

$$x^2y'' + x^2y' + (x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 496: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -1$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - (\alpha_{c_1}^{-}) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{x} \\ &= \frac{x-2}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\ &= \frac{\mathrm{e}^{\frac{x}{2}}}{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^2}{x^2} \, dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1\left(-\left(x^2+2x+2\right) \mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(-\left(x^2+2x+2\right) \mathrm{e}^{-x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.260.2 Maple step by step solution

Let's solve
$x^2\left(\frac{d}{dx}y'\right) + x^2 y' + (x-2)\, y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{(x-2)y}{x^2} - y'$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + y' + \frac{(x-2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x)=1, P_3(x)=\frac{x-2}{x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=0$$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-2$$

- ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- • Multiply by denominators

$$x^2\left(\frac{d}{dx}y'\right)+x^2y'+(x-2)\,y=0$$

- • Assume series solution for y

$$y=\sum_{k=0}^{\infty}a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

$$x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$$

  - ◦ Shift index using $k->k-m$

$$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$$

  - ◦ Convert $x^2\cdot y'$ to series expansion

$$x^2\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r+1}$$

  - ◦ Shift index using $k->k-1$

$$x^2\cdot y'=\sum_{k=1}^{\infty}a_{k-1}(k-1+r)\,x^{k+r}$$

  - ◦ Convert $x^2\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k-1+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-2+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+1)\,(k+r-2)+a_{k-1}(k+r)\right)x^{k+r}\right)=0$$

- • $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-2+r)=0$$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-2) + a_{k-1}(k+r) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+2+r)(k-1+r) + a_k(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+1)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k k}{(k+1)(k-2)}$
- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = -\frac{a_k k}{(k+1)(k-2)}$
- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{a_k(k+3)}{(k+4)(k+1)}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k(k+3)}{(k+4)(k+1)}\right]$

1.260.3 Maple trace

```
Methods for second order ODEs:
```

1.260.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 24

```
dsolve(x^2*diff(diff(y(x),x),x)+x^2*diff(y(x),x)+(x-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2(x^2 + 2x + 2) e^{-x} + c_1}{x}$$

1.260.5 Mathematica DSolve solution

Solving time : 0.052 (sec)
Leaf size : 29

```
DSolve[{x^2*D[y[x],{x,2}]+x^2*D[y[x],x]+(x-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1 - c_2 e^{-x}(x^2 + 2x + 2)}{x}$$

1.261 problem 264

Internal problem ID [8399]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 264
Date solved : `Monday, October 21, 2024 at 05:07:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x^2y' + (x-2)\,y = 0$$

1.261.1 Solved as second order ode using Kovacic algorithm

Time used: 0.235 (sec)

Writing the ode as

$$x^2y'' - x^2y' + (x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 498: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{x} \\
&= \frac{x - 2}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2}{x^2} \, dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^x}{(y_1)^2} dx \\ &= y_1 \left(-\left(x^2 + 2x + 2\right) e^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^x}{x}\right) + c_2 \left(\frac{e^x}{x} \left(-\left(x^2 + 2x + 2\right) e^{-x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.261.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) - x^2 y' + (x - 2) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(x-2)y}{x^2} + y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' - y' + \frac{(x-2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -1, P_3(x) = \frac{x-2}{x^2}\right]$

◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$$

◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$$

◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$x^2\left(\frac{d}{dx}y'\right) - x^2y' + (x-2)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

◦ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

◦ Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$$

◦ Shift index using $k- >k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$$

◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+r-2)\right)x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation
  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-2)\,(a_k(k+r+1)-a_{k-1})=0$
- Shift index using $k->k+1$
  $(k-1+r)\,(a_{k+1}(k+2+r)-a_k)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+2+r}$
- Recursion relation for $r=-1$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=-1$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k}{k+4}$
- Solution for $r=2$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=\frac{a_k}{k+4}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+4}\right]$

1.261.3 Maple trace

```
Methods for second order ODEs:
```

1.261.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)-x^2*diff(y(x),x)+(x-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1\,\mathrm{e}^{x}+c_2(x^2+2x+2)}{x}$$

1.261.5 Mathematica DSolve solution

Solving time : 0.052 (sec)
Leaf size : 28

```
DSolve[{x^2*D[y[x],{x,2}]-x^2*D[y[x],x]+(x-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1 e^x - c_2(x^2 + 2x + 2)}{x}$$

1.262 problem 265

Internal problem ID [8400]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 265
Date solved : `Monday, October 21, 2024 at 05:07:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-4x)\,y''-\frac{xy'}{2}-\frac{3xy}{4}=0$$

1.262.1 Solved as second order ode using Kovacic algorithm

Time used: 0.320 (sec)

Writing the ode as

$$\left(-4x^3+x^2\right)y''-\frac{xy'}{2}-\frac{3xy}{4}=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -4x^3+x^2 \\ B &= -\frac{x}{2} \\ C &= -\frac{3x}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-48x^2 - 20x + 5}{16\left(4x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -48x^2 - 20x + 5$$
$$t = 16\left(4x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-48x^2 - 20x + 5}{16\left(4x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 500: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(4x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\left(x - \frac{1}{4}\right)^2} - \frac{5}{4\left(x - \frac{1}{4}\right)} + \frac{5}{4x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

For the pole at $x = \frac{1}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-48x^2 - 20x + 5}{16\left(4x^2 - x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$\begin{aligned}E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\ &= \{1,2,3\}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1,2,5\}$
$\frac{1}{4}$	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{1,2,3\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 2,\ e_2 = 1,\ e_\infty = 3$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\ &= \frac{1}{2}(3-(2+(1)))\\ &= 0\end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{2}{(x-(0))}+\frac{1}{(x-(\frac{1}{4}))}\right)\\ &= \frac{1}{x}+\frac{1}{2x-\frac{1}{2}}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{x} + \frac{1}{2x - \frac{1}{2}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x} + \frac{1}{2x - \frac{1}{2}}\right) w + \frac{144x^2 - 12x - 5}{16x^2 \left(-1 + 4x\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{12x - 2 + 3\sqrt{1 - 4x}}{4x \left(-1 + 4x\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx}\\ &= e^{\int \frac{12x - 2 + 3\sqrt{1-4x}}{4x(-1+4x)} dx}\\ &= \frac{\sqrt{x}\left(-1 + 4x\right)^{1/4} \sqrt{2} \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{3/2}}{4}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}\\ &= z_1 e^{-\int \frac{1}{2} \frac{-\frac{x}{2}}{-4x^3 + x^2} \, dx}\\ &= z_1 e^{\frac{\ln(x)}{4} - \frac{\ln(-1+4x)}{4}}\\ &= z_1 \left(\frac{x^{1/4}}{\left(-1 + 4x\right)^{1/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}\sqrt{2}\left(\sqrt{1-4x}+1\right)\sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{x}{2}}{-4x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}-\frac{\ln(-1+4x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\,\mathrm{e}^{-\frac{\ln(-1+4x)}{2}+\frac{\ln(1-4x)}{2}}\left(-\left(\sqrt{1-4x}+1\right)^2+2\sqrt{1-4x}+2\right)^{3/2}}{3\left(\sqrt{1-4x}+1\right)^3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/4}\sqrt{2}\left(\sqrt{1-4x}+1\right)\sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}}{4}\right) \\
&\quad + c_2\left(\frac{x^{1/4}\sqrt{2}\left(\sqrt{1-4x}+1\right)\sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}}{4}\left(\frac{2\,\mathrm{e}^{-\frac{\ln(-1+4x)}{2}+\frac{\ln(1-4x)}{2}}\left(-\left(\sqrt{1-4x}+1\right)^2+2\sqrt{1-4x}+2\right)^{3/2}}{3\left(\sqrt{1-4x}+1\right)^3}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.262.2 Maple step by step solution

Let's solve

$x^2(1-4x)\left(\frac{d}{dx}y'\right)-\frac{xy'}{2}-\frac{3xy}{4}=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{3y}{4x(-1+4x)}-\frac{y'}{2x(-1+4x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{y'}{2x(-1+4x)}+\frac{3y}{4x(-1+4x)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{1}{2x(-1+4x)},P_3(x)=\frac{3}{4x(-1+4x)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=-\frac{1}{2}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$4x(-1+4x)\left(\frac{d}{dx}y'\right)+2y'+3y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert y' to series expansion

$y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1}$

- ◦ Shift index using $k->k+1$

$y'=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=1..2$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r(-3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (-2a_{k+1}(k+1+r)(2k-1+2r) + a_k(4k+4r-1)(4k+4r-3$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-4\left(k-\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 16a_k\left(k+r-\frac{3}{4}\right)\left(k+r-\frac{1}{4}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(4k+4r-3)(4k+4r-1)}{2(2k-1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k(4k-3)(4k-1)}{2(2k-1)(k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(4k-3)(4k-1)}{2(2k-1)(k+1)}\right]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = \frac{a_k(4k+3)(4k+5)}{2(2k+2)\left(k+\frac{5}{2}\right)}$$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{a_k(4k+3)(4k+5)}{2(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{a_k(4k-3)(4k-1)}{2(2k-1)(k+1)}, b_{k+1} = \frac{b_k(4k+3)(4k+5)}{2(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

1.262.3 Maple trace

```
Methods for second order ODEs:
```

1.262.4 Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 46

```
dsolve(x^2*(1-4*x)*diff(diff(y(x),x),x)-1/2*x*diff(y(x),x)-3/4*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{\left(c_1(x-1)\sqrt{1-4x} - 2c_2\, x^{3/2} + c_1(3x-1)\right)\sqrt{2}}{\left(\sqrt{1-4x}+1\right)^{3/2}}$$

1.262.5 Mathematica DSolve solution

Solving time : 4.239 (sec)
Leaf size : 111

```
DSolve[{x^2*(1-4*x)*D[y[x],{x,2}]+((1-(3/2))*x-(6-4*(3/2))*x^2)*D[y[x],x]+(3/2)*(1-(3/2))*x*
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[4]{x}\sqrt[4]{4x-1}\left(6c_1\left(\sqrt{4x-1}-i\right)^{3/2} + ic_2\left(\sqrt{4x-1}+i\right)^{3/2}\right)}{6\sqrt[4]{1-4x}\sqrt[4]{\sqrt{4x-1}-i}\sqrt[4]{\sqrt{4x-1}+i}}$$

1.263 problem 266

Internal problem ID [8401]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 266
Date solved : `Monday, October 21, 2024 at 05:07:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \left(x^2 + x\right) y' + (x - 9)\, y = 0$$

1.263.1 Solved as second order ode using Kovacic algorithm

Time used: 0.316 (sec)

Writing the ode as

$$x^2y'' + \left(x^2 + x\right) y' + (x - 9)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 + x \\ C &= x - 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 2x + 35$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 35}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 502: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} + \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{17}{2x^2} + \frac{17}{2x^3} - \frac{255}{4x^4} - \frac{833}{4x^5} + \frac{3213}{4x^6} + \frac{21709}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned}\tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 35}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 35}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x + 35}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x + 35}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - (\alpha_{c_1}^{-}) \\ &= -\frac{1}{2} - \left(-\frac{5}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + \left(\frac{1}{2}\right) \\
&= -\frac{5}{2x} + \frac{1}{2} \\
&= \frac{-5 + x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{5}{2x} + \frac{1}{2}\right)(2x + a_1) + \left(\left(\frac{5}{2x^2}\right) + \left(-\frac{5}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 2x + 35}{4x^2}\right)\right) = 0$$

$$\frac{(-a_1 - 8)\, x - 2a_0 - 5a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 20, a_1 = -8\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 8x + 20$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^2 - 8x + 20)\, e^{\int \left(-\frac{5}{2x} + \frac{1}{2}\right) dx} \\
&= (x^2 - 8x + 20)\, e^{\frac{x}{2} - \frac{5 \ln(x)}{2}} \\
&= \frac{(x^2 - 8x + 20)\, e^{\frac{x}{2}}}{x^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2 - 8x + 20}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(x^3 + 9x^2 + 36x + 60)\,x\,\mathrm{e}^{-x-\ln(x)}}{x^2 - 8x + 20}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2 - 8x + 20}{x^3}\right) + c_2 \left(\frac{x^2 - 8x + 20}{x^3}\left(-\frac{(x^3 + 9x^2 + 36x + 60)\,x\,\mathrm{e}^{-x-\ln(x)}}{x^2 - 8x + 20}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.263.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+\left(x^2+x\right)y'+\left(x-9\right)y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(x-9)y}{x^2}-\frac{(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(x+1)y'}{x}+\frac{(x-9)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x+1}{x}, P_3(x)=\frac{x-9}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-9$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right)+x(x+1)y'+\left(x-9\right)y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(-3+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+3)\,(k+r-3) + a_{k-1}(k+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)\,(k+r-3) + a_{k-1}(k+r) = 0$

- Shift index using $k- >k+1$

  $a_{k+1}(k+4+r)\,(k-2+r) + a_k(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+1)}{(k+4+r)(k-2+r)}$

- Recursion relation for $r = -3$; series terminates at $k = 2$

  $a_{k+1} = -\frac{a_k(k-2)}{(k+1)(k-5)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{2a_0}{5}$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{8}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{20}$

- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 - \frac{2}{5}x + \frac{1}{20}x^2\right)$

- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a_k(k+4)}{(k+7)(k+1)}$

- Solution for $r = 3$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1}=-\frac{a_k(k+4)}{(k+7)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y=a_0\cdot\left(1-\frac{2}{5}x+\frac{1}{20}x^2\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), b_{k+1}=-\frac{b_k(k+4)}{(k+7)(k+1)}\right]$$

1.263.3 Maple trace

```
Methods for second order ODEs:
```

1.263.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 38

```
dsolve(x^2*diff(diff(y(x),x),x)+(x^2+x)*diff(y(x),x)+(x-9)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_2(x^3+9x^2+36x+60)\,\mathrm{e}^{-x}+c_1(x^2-8x+20)}{x^3}$$

1.263.5 Mathematica DSolve solution

Solving time : `0.352` (sec)
Leaf size : 42

```
DSolve[{x^2*D[y[x],{x,2}]+(x+x^2)*D[y[x],x]+(x-9)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{c_1((x-8)x+20)-c_2e^{-x}(x^3+9x^2+36x+60)}{x^3}$$

1.264 problem 267

Internal problem ID [8402]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 267
Date solved : `Monday, October 21, 2024 at 05:07:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2 y'' + x(x+1)\, y' + (3x-1)\, y = 0$$

1.264.1 Solved as second order ode using Kovacic algorithm

Time used: 0.307 (sec)

Writing the ode as

$$x^2 y'' + \left(x^2+x\right) y' + (3x-1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 + x \\ C &= 3x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 10x + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 10x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 504: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{5}{2x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{5}{2x} - \frac{11}{2x^2} - \frac{55}{2x^3} - \frac{671}{4x^4} - \frac{4565}{4x^5} - \frac{33231}{4x^6} - \frac{253275}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-10x + 3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-10x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -10. Dividing this by leading coefficient in t which is 4 gives $-\frac{5}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= \frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 10x + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{5}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{3}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{3}{2x} - \frac{1}{2} \\ &= -\frac{-3 + x}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{3}{2x^2}\right) + \left(\frac{3}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 10x + 3}{4x^2}\right)\right) = 0$$

$$\frac{3 + a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -3 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= (-3 + x) \, e^{\int \left(\frac{3}{2x} - \frac{1}{2}\right) dx} \\ &= (-3 + x) \, e^{-\frac{x}{2} + \frac{3 \ln(x)}{2}} \\ &= (-3 + x) \, x^{3/2} e^{-\frac{x}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\,\mathrm{e}^{-x}(-3+x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{x}}{27(-3+x)} - \frac{\mathrm{Ei}_1(-x)}{6} - \frac{7\,\mathrm{e}^{x}}{54x} - \frac{\mathrm{e}^{x}}{18x^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\big(x\,\mathrm{e}^{-x}(-3+x)\big) + c_2\left(x\,\mathrm{e}^{-x}(-3+x)\left(-\frac{\mathrm{e}^{x}}{27(-3+x)} - \frac{\mathrm{Ei}_1(-x)}{6} - \frac{7\,\mathrm{e}^{x}}{54x} - \frac{\mathrm{e}^{x}}{18x^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.264.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+x(x+1)\,y'+(3x-1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(3x-1)y}{x^2}-\frac{(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(x+1)y'}{x}+\frac{(3x-1)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{x+1}{x},P_3(x)=\frac{3x-1}{x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-1$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right)+x(x+1)\,y'+(3x-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^{m}\cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2\cdot\left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+r)\,x^{r}+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+1)\,(k+r-1)+a_{k-1}(k+2+r)\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-1+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{-1,1\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+1)\,(k+r-1)+a_{k-1}(k+2+r)=0$$

- Shift index using $k->k+1$

$$a_{k+1}(k+2+r)\,(k+r)+a_k(k+r+3)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{a_k(k+r+3)}{(k+2+r)(k+r)}$$

- Recursion relation for $r=-1$

$$a_{k+1}=-\frac{a_k(k+2)}{(k+1)(k-1)}$$

- Series not valid for $r=-1$, division by 0 in the recursion relation at $k=1$

$$a_{k+1}=-\frac{a_k(k+2)}{(k+1)(k-1)}$$

- Recursion relation for $r=1$

$$a_{k+1}=-\frac{a_k(k+4)}{(k+3)(k+1)}$$

- Solution for $r=1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1}=-\frac{a_k(k+4)}{(k+3)(k+1)}\right]$$

1.264.3 Maple trace

```
Methods for second order ODEs:
```

1.264.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 48

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x+1)*diff(y(x),x)+(3*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^2 c_2 \,\mathrm{e}^{-x}(-3+x)\,\mathrm{Ei}_1(-x) + x^2 c_1(-3+x)\,\mathrm{e}^{-x} + c_2(x^2-2x-1)}{x}$$

1.264.5 Mathematica DSolve solution

Solving time : 0.263 (sec)
Leaf size : 66

```
DSolve[{x^2*D[y[x],{x,2}]+x*(x+1)*D[y[x],x]+(3*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x}(c_2(x-3)x^2\,\text{ExpIntegralEi}(x) + 6c_1x^3 - x^2(c_2e^x + 18c_1) + 2c_2e^x x + c_2e^x)}{6x}$$

1.265 problem 268

Internal problem ID [8403]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 268
Date solved : `Monday, October 21, 2024 at 05:07:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - (x^2 + 4x)\, y' + 4y = 0$$

1.265.1 Solved as second order ode using Kovacic algorithm

Time used: 0.265 (sec)

Writing the ode as

$$x^2y'' + (-x^2 - 4x)\, y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 8x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 8x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 8x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 506: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} + \frac{2}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x} - \frac{2}{x^2} + \frac{8}{x^3} - \frac{36}{x^4} + \frac{176}{x^5} - \frac{912}{x^6} + \frac{4928}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 8x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{8x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{8x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 8. Dividing this by leading coefficient in t which is 4 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 0\right) &= 2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 0\right) &= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 8x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	2	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= 2 - (2) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{2}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{2}{x} \\
&= \frac{x+4}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{2}{x}\right)(0) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{1}{2} + \frac{2}{x}\right)^2 - \left(\frac{x^2 + 8x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{2}{x}\right) dx} \\
&= x^2 \mathrm{e}^{\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2 - 4x}{x^2} \, dx} \\
&= z_1 e^{\frac{x}{2} + 2\ln(x)} \\
&= z_1 \left(x^2 \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^4 e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-4x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+4\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{e^{-x}}{3x^3} + \frac{e^{-x}}{6x^2} - \frac{e^{-x}}{6x} + \frac{\mathrm{Ei}_1(x)}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^4 e^x\right) + c_2 \left(x^4 e^x \left(-\frac{e^{-x}}{3x^3} + \frac{e^{-x}}{6x^2} - \frac{e^{-x}}{6x} + \frac{\mathrm{Ei}_1(x)}{6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.265.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) - \left(x^2 + 4x\right) y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{4y}{x^2} + \frac{(x+4)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' - \frac{(x+4)y'}{x} + \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x+4}{x}, P_3(x) = \frac{4}{x^2}\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -4$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 4$
- $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $x^2\left(\frac{d}{dx}y'\right) - x(x+4)\,y' + 4y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$
  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k- >k+1-m$
  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$
  Rewrite ODE with series expansions
  $a_0(-1+r)\,(-4+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-4) - a_{k-1}(k+r-1)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)\,(-4+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{1, 4\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_k(k+r-4) - a_{k-1}\right) = 0$
- Shift index using $k- >k+1$

$(k+r)\left(a_{k+1}(k-3+r)-a_k\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k-3+r}$
- Recursion relation for $r=1$

  $a_{k+1}=\frac{a_k}{k-2}$
- Series not valid for $r=1$, division by 0 in the recursion relation at $k=2$

  $a_{k+1}=\frac{a_k}{k-2}$
- Recursion relation for $r=4$

  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=4$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+1}=\frac{a_k}{k+1}\right]$

1.265.3 Maple trace

```
Methods for second order ODEs:
```

1.265.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 35

```
dsolve(x^2*diff(diff(y(x),x),x)-(x^2+4*x)*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y=\left(\mathrm{Ei}_1(x)\,\mathrm{e}^{x} c_2\, x^3+c_1\, x^3\mathrm{e}^{x}-c_2\left(x^2-x+2\right)\right)x$$

1.265.5 Mathematica DSolve solution

Solving time : `0.066` (sec)
Leaf size : 41

```
DSolve[{x^2*D[y[x],{x,2}]-(x^2+4*x)*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_2 e^x x^4-\frac{1}{6}c_1 x\left(e^x x^3\,\text{ExpIntegralEi}(-x)+x^2-x+2\right)$$

1.266 problem 269

Internal problem ID [8404]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 269
Date solved : `Monday, October 21, 2024 at 05:07:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' - (3x+2)\,y' + \frac{(2x-1)\,y}{x} = 0$$

1.266.1 Solved as second order ode using Kovacic algorithm

Time used: 0.486 (sec)

Writing the ode as

$$2x^2y'' + (-3x-2)\,y' + \left(2-\frac{1}{x}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -3x-2 \\ C &= 2-\frac{1}{x} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 36x + 4}{16x^4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 + 36x + 4$$
$$t = 16x^4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 36x + 4}{16x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 508: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_2^v \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{9}{4x^3} + \frac{5}{16x^2} + \frac{1}{4x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^2} + \frac{9}{4x} - \frac{19}{4} + \frac{171x}{8} - \frac{475x^2}{4} + \frac{11799x^3}{16} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $\frac{9}{4}$. Therefore

$$\begin{aligned} b &= \left(\frac{9}{4}\right) - (0) \\ &= \frac{9}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{\frac{9}{4}}{\frac{1}{2}} + 2\right) &= \frac{13}{4} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{\frac{9}{4}}{\frac{1}{2}} + 2\right) &= -\frac{5}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 + 36x + 4}{16x^4}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 + 36x + 4}{16x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{1}{2x^2}$	$\frac{13}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{4} - \left(-\frac{5}{4}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x^2} - \frac{5}{4x} + (-)(0) \\ &= -\frac{1}{2x^2} - \frac{5}{4x} \\ &= \frac{-2 - 5x}{4x^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x^2} - \frac{5}{4x}\right)(1) + \left(\left(\frac{1}{x^3} + \frac{5}{4x^2}\right) + \left(-\frac{1}{2x^2} - \frac{5}{4x}\right)^2 - \left(\frac{5x^2 + 36x + 4}{16x^4}\right)\right) = 0$$

$$\frac{-2 + 5a_0}{2x^2} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{2}{5}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{2}{5}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x + \frac{2}{5}\right) \mathrm{e}^{\int \left(-\frac{1}{2x^2} - \frac{5}{4x}\right) dx} \\ &= \left(x + \frac{2}{5}\right) \mathrm{e}^{-\frac{5\ln(x)}{4} + \frac{1}{2x}} \\ &= \frac{(2 + 5x)\,\mathrm{e}^{\frac{1}{2x}}}{5x^{5/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-3x-2}{2x^2} \, dx} \\ &= z_1 e^{\frac{3\ln(x)}{4} - \frac{1}{2x}} \\ &= z_1 \left(x^{3/4} \mathrm{e}^{-\frac{1}{2x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{2 + 5x}{5\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-2}{2x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{3\ln(x)}{2}-\frac{1}{x}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{25\, \mathrm{e}^{\frac{3\ln(x)}{2}-\frac{1}{x}} x}{(2+5x)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{2+5x}{5\sqrt{x}} \right) + c_2 \left(\frac{2+5x}{5\sqrt{x}} \left(\int \frac{25\, \mathrm{e}^{\frac{3\ln(x)}{2}-\frac{1}{x}} x}{(2+5x)^2} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.266.2 Maple step by step solution

1.266.3 Maple trace

```
Methods for second order ODEs:
```

1.266.4 Maple dsolve solution

Solving time : `0.019` (sec)
Leaf size : 35

```
dsolve(2*x^2*diff(diff(y(x),x),x)-(3*x+2)*diff(y(x),x)+(2*x-1)/x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\, \mathrm{e}^{-\frac{1}{x}} \operatorname{hypergeom}\left([2], \left[-\frac{1}{2}\right], \frac{1}{x}\right) x^{5/2} + 5c_1 x + 2c_1}{\sqrt{x}}$$

1.266.5 Mathematica DSolve solution

Solving time : 0.321 (sec)
Leaf size : 70

```
DSolve[{2*x^2*D[y[x],{x,2}]-(3*x+2)*D[y[x],x]+(2*x-1)/x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{2\sqrt{\pi}c_2(5x+2)\operatorname{erf}\left(\frac{1}{\sqrt{x}}\right)}{3\sqrt{x}} + \frac{2}{3}c_2 e^{-1/x}\left(x^2-4x-2\right) + \frac{c_1(5x+2)}{5\sqrt{x}}$$

1.267 problem 270

Internal problem ID [8405]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 270
Date solved : `Monday, October 21, 2024 at 05:07:22 PM`
CAS classification : `[_Jacobi]`

Solve

$$x(1-x)\,y''+\left(\frac{3}{2}-2x\right)y'-\frac{y}{4}=0$$

1.267.1 Solved as second order ode using Kovacic algorithm

Time used: 0.267 (sec)

Writing the ode as

$$\left(-x^2+x\right)y''+\left(\frac{3}{2}-2x\right)y'-\frac{y}{4}=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+x \\ B &= \frac{3}{2}-2x \\ C &= -\frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x^2 + 4x - 3}{16 (x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4x^2 + 4x - 3$$
$$t = 16(x^2 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^2 + 4x - 3}{16 (x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 509: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{8x} - \frac{3}{16(-1+x)^2} + \frac{1}{-8+8x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4x^2 + 4x - 3}{16(x^2 - x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-4x^2+4x-3}{16\left(x^2-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4x} + \frac{1}{-4 + 4x} + (-)(0) \\
&= \frac{1}{4x} + \frac{1}{-4 + 4x} \\
&= \frac{2x - 1}{4x(-1 + x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} + \frac{1}{-4 + 4x}\right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{1}{4(-1 + x)^2}\right) + \left(\frac{1}{4x} + \frac{1}{-4 + 4x}\right)^2 - \left(\frac{-4x^2 + 4x - 3}{16(x^2 - x)^2}\right)\right) =$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{4x} + \frac{1}{-4 + 4x}\right) dx} \\
&= (x(-1 + x))^{1/4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{3}{2} - 2x}{-x^2 + x} \, dx} \\
&= z_1 e^{-\frac{\ln(-1+x)}{4} - \frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{(-1 + x)^{1/4} x^{3/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x(-1+x)\right)^{1/4}}{\left(-1+x\right)^{1/4} x^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{3}{2}-2x}{-x^2+x}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(-1+x)}{2}-\frac{3\ln(x)}{2}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\ln\left(-\frac{1}{2}+x+\sqrt{x^2-x}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x(-1+x)\right)^{1/4}}{\left(-1+x\right)^{1/4} x^{3/4}}\right) + c_2 \left(\frac{\left(x(-1+x)\right)^{1/4}}{\left(-1+x\right)^{1/4} x^{3/4}} \left(\ln\left(-\frac{1}{2}+x+\sqrt{x^2-x}\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.267.2 Maple step by step solution

Let's solve

$x(1-x)\left(\frac{d}{dx}y'\right)+\left(\frac{3}{2}-2x\right)y'-\frac{y}{4}=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{4x(-1+x)} - \frac{(-3+4x)y'}{2x(-1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(-3+4x)y'}{2x(-1+x)} + \frac{y}{4x(-1+x)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{-3+4x}{2x(-1+x)}, P_3(x) = \frac{1}{4x(-1+x)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{3}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x(-1+x)\left(\frac{d}{dx}y'\right) + (8x-6)\,y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(1+2r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(2k+3+2r) + a_k(2k+2r+1)^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(2k+2r+1)^2 - 4a_{k+1}(k+1+r)\left(k+\frac{3}{2}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(2k+2r+1)^2}{2(k+1+r)(2k+3+2r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k(2k+1)^2}{2(k+1)(2k+3)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(2k+1)^2}{2(k+1)(2k+3)}\right]$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = \frac{2a_k k^2}{\left(k+\frac{1}{2}\right)(2k+2)}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{2a_k k^2}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1} = \frac{a_k(2k+1)^2}{2(k+1)(2k+3)}, b_{k+1} = \frac{2b_k k^2}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

1.267.3 Maple trace

```
Methods for second order ODEs:
```

1.267.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 32

```
dsolve(x*(1-x)*diff(diff(y(x),x),x)+(3/2-2*x)*diff(y(x),x)-1/4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \ln\left(-1+2x+2\sqrt{x(-1+x)}\right) - c_2 \ln(2) + c_1}{\sqrt{x}}$$

1.267.5 Mathematica DSolve solution

Solving time : `0.153` (sec)
Leaf size : 53

```
DSolve[{x*(1-x)*D[y[x],{x,2}]+(3/2-2*x)*D[y[x],x]-1/4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_2\sqrt{x-1}\operatorname{arctanh}\left(\frac{\sqrt{x-1}}{\sqrt{x+1}}\right)}{\sqrt{-((x-1)x)}} + \frac{c_1}{\sqrt{x}}$$

1.268 problem 271

Internal problem ID [8406]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 271
Date solved : `Monday, October 21, 2024 at 05:07:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x(1-x)\,y''+xy'-y=0$$

1.268.1 Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$\left(-2x^2+2x\right)y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-2x^2+2x\\ B&=x\\ C&=-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x + 8}{16x(-1 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x + 8$$
$$t = 16x(-1 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x + 8}{16x(-1 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 511: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x(-1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{2x} - \frac{1}{2(-1+x)} + \frac{5}{16(-1+x)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x+8}{16x(-1+x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-3x+8}{16x\left(-1+x\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{3}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{3}{4}-\left(\frac{3}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{1}{4(-1+x)} + (0) \\
&= \frac{1}{x} - \frac{1}{4(-1+x)} \\
&= \frac{1}{x} - \frac{1}{-4+4x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{x} - \frac{1}{4(-1+x)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{4(-1+x)^2}\right) + \left(\frac{1}{x} - \frac{1}{4(-1+x)}\right)^2 - \left(\frac{-3x+8}{16x(-1+x)^2}\right)\right) = 0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{4(-1+x)}\right) dx} \\
&= \frac{x}{(-1+x)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{-2x^2+2x}\, dx} \\
&= z_1 e^{\frac{\ln(-1+x)}{4}} \\
&= z_1\left((-1+x)^{1/4}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{-2x^2+2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{\ln(-1+x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\sqrt{-1+x}}{x} + \arctan\left(\sqrt{-1+x}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(-\frac{\sqrt{-1+x}}{x} + \arctan\left(\sqrt{-1+x}\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.268.2 Maple step by step solution

1.268.3 Maple trace

```
Methods for second order ODEs:
```

1.268.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 25

```
dsolve(2*x*(1-x)*diff(diff(y(x),x),x)+x*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + \arctan\left(\sqrt{-1+x}\right) x c_2 - \sqrt{-1+x}\, c_2$$

1.268.5 Mathematica DSolve solution

Solving time : 0.139 (sec)
Leaf size : 43

```
DSolve[{2*x*(1-x)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt[4]{2}\left(c_2 x \operatorname{arctanh}\left(\sqrt{1-x}\right) + c_1 x - c_2\sqrt{1-x}\right)$$

1.269 problem 272

Internal problem ID [8407]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 272
Date solved : `Monday, October 21, 2024 at 05:07:24 PM`
CAS classification : `[_Jacobi]`

Solve

$$2x(1-x)\,y''+(1-11x)\,y'-10y=0$$

1.269.1 Solved as second order ode using Kovacic algorithm

Time used: 0.235 (sec)

Writing the ode as

$$\left(-2x^2+2x\right)y''+(1-11x)\,y'-10y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^2+2x \\ B &= 1-11x \\ C &= -10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2 + 66x - 3 \\ t &= 16\left(x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 512: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} - \frac{15}{4(-1+x)} + \frac{15}{4(-1+x)^2} + \frac{15}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 66x - 3}{16(x^2 - x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2+66x-3}{16\left(x^2-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-}\right) \\
&= \frac{1}{4} - \left(-\frac{3}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4x} - \frac{3}{2\left(-1+x\right)} + (-)\left(0\right) \\
&= \frac{3}{4x} - \frac{3}{2\left(-1+x\right)} \\
&= -\frac{3(x+1)}{4x\left(-1+x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4x} - \frac{3}{2\left(-1+x\right)}\right)(1) + \left(\left(-\frac{3}{4x^2} + \frac{3}{2\left(-1+x\right)^2}\right) + \left(\frac{3}{4x} - \frac{3}{2\left(-1+x\right)}\right)^2 - \left(\frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2}\right.\right.
$$

$$
\frac{-3 + 3a_0}{2x\left(-1+x\right)}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+1)\, e^{\int \left(\frac{3}{4x} - \frac{3}{2(-1+x)}\right) dx} \\
&= (x+1)\, e^{\frac{3\ln(x)}{4} - \frac{3\ln(-1+x)}{2}} \\
&= \frac{(x+1)\, x^{3/4}}{\left(-1+x\right)^{3/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-11x}{-2x^2+2x}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{4}-\frac{5\ln(-1+x)}{2}} \\
&= z_1 \left(\frac{1}{x^{1/4}\left(-1+x\right)^{5/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x}\,(x+1)}{\left(-1+x\right)^4}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-11x}{-2x^2+2x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2}-5\ln(-1+x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{2(x^2+6x+1)\left(-1+x\right)^5 \mathrm{e}^{-\frac{\ln(x)}{2}-5\ln(-1+x)}}{x+1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}\,(x+1)}{\left(-1+x\right)^4}\right) + c_2 \left(\frac{\sqrt{x}\,(x+1)}{\left(-1+x\right)^4}\left(\frac{2(x^2+6x+1)\left(-1+x\right)^5 \mathrm{e}^{-\frac{\ln(x)}{2}-5\ln(-1+x)}}{x+1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.269.2 Maple step by step solution

Let's solve

$2x(1-x)\left(\frac{d}{dx}y'\right)+(1-11x)\,y'-10y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{5y}{x(-1+x)}-\frac{(-1+11x)y'}{2x(-1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(-1+11x)y'}{2x(-1+x)}+\frac{5y}{x(-1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{-1+11x}{2x(-1+x)},P_3(x)=\frac{5}{x(-1+x)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x(-1+x)\left(\frac{d}{dx}y'\right)+(-1+11x)\,y'+10y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=1..2$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (-a_{k+1}(k+1+r)(2k+1+2r) + a_k(2k+2r+5)(k+r+2))\, x^{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2(k+1+r)\left(k+\frac{1}{2}+r\right) a_{k+1} + 2(k+r+2)\left(k+r+\frac{5}{2}\right) a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(k+r+2)(2k+2r+5)a_k}{(k+1+r)(2k+1+2r)}$$

- Recursion relation for $r=0$

$$a_{k+1} = \frac{(k+2)(2k+5)a_k}{(k+1)(2k+1)}$$

- Solution for $r=0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(k+2)(2k+5)a_k}{(k+1)(2k+1)}\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1} = \frac{\left(k+\frac{5}{2}\right)(2k+6)a_k}{\left(k+\frac{3}{2}\right)(2k+2)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{\left(k+\frac{5}{2}\right)(2k+6)a_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = \frac{(k+2)(2k+5)a_k}{(k+1)(2k+1)}, b_{k+1} = \frac{\left(k+\frac{5}{2}\right)(2k+6)b_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

1.269.3 Maple trace

```
Methods for second order ODEs:
```

1.269.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 29

```
dsolve(2*x*(1-x)*diff(diff(y(x),x),x)+(1-11*x)*diff(y(x),x)-10*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1(x^2 + 6x + 1) + c_2\sqrt{x}\,(x + 1)}{(-1 + x)^4}$$

1.269.5 Mathematica DSolve solution

Solving time : 0.132 (sec)
Leaf size : 35

```
DSolve[{2*x*(1-x)*D[y[x],{x,2}]+(1-11*x)*D[y[x],x]-10*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1\sqrt{x}(x + 1) - 2c_2(x^2 + 6x + 1)}{(x - 1)^4}$$

1.270 problem 273

Internal problem ID [8408]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 273
Date solved : `Monday, October 21, 2024 at 05:07:25 PM`
CAS classification : `[_Jacobi]`

Solve

$$x(1-x)\, y'' + \frac{(1-2x)\, y'}{3} + \frac{20y}{9} = 0$$

1.270.1 Solved as second order ode using Kovacic algorithm

Time used: 0.256 (sec)

Writing the ode as

$$\left(-x^2+x\right) y'' + \left(-\frac{2x}{3}+\frac{1}{3}\right) y' + \frac{20y}{9} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2 + x \\ B &= -\frac{2x}{3} + \frac{1}{3} \\ C &= \frac{20}{9} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{72x^2 - 72x - 5}{36\,(x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 72x^2 - 72x - 5$$
$$t = 36\left(x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{72x^2 - 72x - 5}{36\,(x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 514: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{5}{36(-1+x)^2} - \frac{5}{36x^2} - \frac{41}{18x} + \frac{41}{18(-1+x)}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{72x^2 - 72x - 5}{36(x^2 - x)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{72x^2-72x-5}{36\left(x^2-x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$
1	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= 2-(1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{6x} + \frac{5}{6(-1+x)} + (0) \\
&= \frac{1}{6x} + \frac{5}{6(-1+x)} \\
&= \frac{-1+6x}{6x(-1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{6x} + \frac{5}{6(-1+x)}\right)(1) + \left(\left(-\frac{1}{6x^2} - \frac{5}{6(-1+x)^2}\right) + \left(\frac{1}{6x} + \frac{5}{6(-1+x)}\right)^2 - \left(\frac{72x^2 - 72x - 5}{36(x^2-x)^2}\right.\right.$$

$$\frac{-1-6a_0}{3x(-1+x)}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{1}{6}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -\frac{1}{6} + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(-\frac{1}{6} + x\right) e^{\int \left(\frac{1}{6x} + \frac{5}{6(-1+x)}\right) dx} \\
&= \left(-\frac{1}{6} + x\right) e^{\frac{5\ln(-1+x)}{6} + \frac{\ln(x)}{6}} \\
&= \left(-\frac{1}{6} + x\right) (-1+x)^{5/6} x^{1/6}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-\frac{2x}{3}+\frac{1}{3}}{-x^2+x}\,dx} \\
&= z_1 e^{-\frac{\ln(x(-1+x))}{6}} \\
&= z_1\left(\frac{1}{\left(x\left(-1+x\right)\right)^{1/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(-1+6x\right)\left(-1+x\right)^{5/6} x^{1/6}}{6\left(x\left(-1+x\right)\right)^{1/6}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{2x}{3}+\frac{1}{3}}{-x^2+x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x(-1+x))}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{54x^{2/3}(-5+6x)}{5\left(-1+6x\right)\left(-1+x\right)^{2/3}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(-1+6x\right)\left(-1+x\right)^{5/6} x^{1/6}}{6\left(x\left(-1+x\right)\right)^{1/6}}\right) + c_2\left(\frac{\left(-1+6x\right)\left(-1+x\right)^{5/6} x^{1/6}}{6\left(x\left(-1+x\right)\right)^{1/6}}\left(-\frac{54x^{2/3}(-5+6x)}{5\left(-1+6x\right)\left(-1+x\right)^{2/3}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.270.2 Maple step by step solution

Let's solve

$x(1-x)\left(\frac{d}{dx}y'\right)+\frac{(1-2x)y'}{3}+\frac{20y}{9}=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{20y}{9x(-1+x)}-\frac{(2x-1)y'}{3x(-1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(2x-1)y'}{3x(-1+x)}-\frac{20y}{9x(-1+x)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{2x-1}{3x(-1+x)}, P_3(x)=-\frac{20}{9x(-1+x)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{1}{3}$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

- ○ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$9x(-1+x)\left(\frac{d}{dx}y'\right)+(6x-3)y'-20y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

$x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=1..2$

$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-3a_0 r(-2+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (-3a_{k+1}(k+1+r)(3k+1+3r) + a_k(3k+3r+4)(3k+3r-5$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-3r(-2+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{2}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-9\left(k+\frac{1}{3}+r\right)(k+1+r)\, a_{k+1} + 9\left(k+r-\frac{5}{3}\right) a_k \left(k+r+\frac{4}{3}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(3k+3r-5)a_k(3k+3r+4)}{3(3k+1+3r)(k+1+r)}$

- Recursion relation for $r=0$

  $a_{k+1} = \frac{(3k-5)a_k(3k+4)}{3(3k+1)(k+1)}$

- Solution for $r=0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(3k-5)a_k(3k+4)}{3(3k+1)(k+1)}\right]$

- Recursion relation for $r=\frac{2}{3}$; series terminates at $k=1$

  $a_{k+1} = \frac{(3k-3)a_k(3k+6)}{3(3k+3)\left(k+\frac{5}{3}\right)}$

- Apply recursion relation for $k=0$

  $a_1 = -\frac{6a_0}{5}$

- Terminating series solution of the ODE for $r=\frac{2}{3}$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(-\frac{6x}{5}+1\right)$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + b_0 \cdot \left(-\frac{6x}{5}+1\right), a_{k+1} = \frac{(3k-5)a_k(3k+4)}{3(3k+1)(k+1)}\right]$

1.270.3 Maple trace

```
Methods for second order ODEs:
```

1.270.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 27

```
dsolve(x*(1-x)*diff(diff(y(x),x),x)+1/3*(1-2*x)*diff(y(x),x)+20/9*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x^{2/3}(-5 + 6x) + c_2(-1 + 6x)(-1 + x)^{2/3}$$

1.270.5 Mathematica DSolve solution

Solving time : 0.081 (sec)
Leaf size : 51

```
DSolve[{x*(1-x)*D[y[x],{x,2}]+1/3*(1-2*x)*D[y[x],x]+20/9*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 \sqrt[3]{-((x-1)x)} Q_1^{\frac{2}{3}}(2x-1) + \frac{c_1 x^{2/3}(6x-5)}{3\,\text{Gamma}\left(\frac{4}{3}\right)}$$

1.271 problem 274

Internal problem ID [8409]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 274
Date solved : `Monday, October 21, 2024 at 05:07:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4y'' + \frac{3(-x^2+2)\,y}{(-x^2+1)^2} = 0$$

1.271.1 Solved as second order ode using Kovacic algorithm

Time used: 0.210 (sec)

Writing the ode as

$$4y'' + \frac{(-3x^2+6)\,y}{(x^2-1)^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 0 \\ C &= \frac{-3x^2+6}{(x^2-1)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 6}{4\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2 - 6 \\ t &= 4\left(x^2 - 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 6}{4\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 516: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{16(x-1)} - \frac{3}{16(x-1)^2} - \frac{3}{16(x+1)^2} - \frac{9}{16(x+1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2-6}{4(x^2-1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 6}{4(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^+ + \alpha_{c_2}^+) \\
&= \frac{3}{2} - \left(\frac{3}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4(x-1)} + \frac{3}{4(x+1)} + (0) \\
&= \frac{3}{4(x-1)} + \frac{3}{4(x+1)} \\
&= \frac{3x}{2x^2-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4(x-1)} + \frac{3}{4(x+1)}\right)(0) + \left(\left(-\frac{3}{4(x-1)^2} - \frac{3}{4(x+1)^2}\right) + \left(\frac{3}{4(x-1)} + \frac{3}{4(x+1)}\right)^2 - \left(\frac{3}{4(\cdots
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{3}{4(x-1)} + \frac{3}{4(x+1)}\right) dx} \\
&= \left(x^2-1\right)^{3/4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \left(x^2-1\right)^{3/4}
\end{aligned}
$$

Which simplifies to

$$y_1 = \left(x^2-1\right)^{3/4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \left(x^2-1\right)^{3/4} \int \frac{1}{\left(x^2-1\right)^{3/2}}\, dx \\ &= \left(x^2-1\right)^{3/4} \left(-\frac{(x-1)(x+1)\, x}{\left(x^2-1\right)^{3/2}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\left(x^2-1\right)^{3/4}\right) + c_2 \left(\left(x^2-1\right)^{3/4} \left(-\frac{(x-1)(x+1)\, x}{\left(x^2-1\right)^{3/2}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.271.2 Maple step by step solution

Let's solve

$4\frac{d}{dx}y' + \frac{3(-x^2+2)y}{(-x^2+1)^2} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{3(x^2-2)y}{4(x^2-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{3(x^2-2)y}{4(x^2-1)^2} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = 0, P_3(x) = -\frac{3(x^2-2)}{4(x^2-1)^2}\right]$
- $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = 0$
- $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = \frac{3}{16}$
- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4(x^2-1)^2 \left(\frac{d}{dx}y'\right) + (-3x^2+6)\, y = 0$
- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^4 - 16u^3 + 16u^2)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-3u^2 + 6u + 3)\, y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
- Shift index using $k->k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 2..4$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+4r)(-3+4r)u^r + (a_1(3+4r)(1+4r) - 2a_0(8r^2-8r-3))u^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(4k+4r-\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$(-1+4r)(-3+4r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$$
- Each term must be 0
$$a_1(3+4r)(1+4r) - 2a_0(8r^2-8r-3) = 0$$
- Solve for the dependent coefficient(s)
$$a_1 = \frac{2a_0(8r^2-8r-3)}{16r^2+16r+3}$$
- Each term in the series must be 0, giving the recursion relation
$$4(4a_k + a_{k-2} - 4a_{k-1})k^2 + 4(2(4a_k + a_{k-2} - 4a_{k-1})r - 4a_k - 5a_{k-2} + 12a_{k-1})k + 4(4a_k + a_{k-2}$$
- Shift index using $k->k+2$
$$4(4a_{k+2} + a_k - 4a_{k+1})(k+2)^2 + 4(2(4a_{k+2} + a_k - 4a_{k+1})r - 4a_{k+2} - 5a_k + 12a_{k+1})(k+2) + 4$$
- Recursion relation that defines series solution to ODE
$$a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 8kra_k - 32kra_{k+1} + 4r^2a_k - 16r^2a_{k+1} - 4ka_k - 16ka_{k+1} - 4ra_k - 16ra_{k+1} - 3a_k + 6a_{k+1}}{16k^2 + 32kr + 16r^2 + 48k + 48r + 35}$$
- Recursion relation for $r = \frac{1}{4}$
$$a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_{k+1}}{16k^2 + 56k + 48}$$
- Solution for $r = \frac{1}{4}$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_{k+1}}{16k^2 + 56k + 48}, a_1 = -\frac{9a_0}{8}\right]$$
- Revert the change of variables $u = x+1$
$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{1}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_{k+1}}{16k^2 + 56k + 48}, a_1 = -\frac{9a_0}{8}\right]$$
- Recursion relation for $r = \frac{3}{4}$
$$a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 2ka_k - 40ka_{k+1} - \frac{15}{4}a_k - 15a_{k+1}}{16k^2 + 72k + 80}$$
- Solution for $r = \frac{3}{4}$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 2ka_k - 40ka_{k+1} - \frac{15}{4}a_k - 15a_{k+1}}{16k^2 + 72k + 80}, a_1 = -\frac{3a_0}{8}\right]$$
- Revert the change of variables $u = x+1$
$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{3}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 2ka_k - 40ka_{k+1} - \frac{15}{4}a_k - 15a_{k+1}}{16k^2 + 72k + 80}, a_1 = -\frac{3a_0}{8}\right]$$
- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+\frac{3}{4}}\right), a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} - 2k a_k - 24k a_{k+1} - \frac{15}{4} a_k + a_{k+1}}{16k^2 + 56k + 48},\right.$$

1.271.3 Maple trace

```
Methods for second order ODEs:
```

1.271.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 24

```
dsolve(4*diff(diff(y(x),x),x)+3*(-x^2+2)/(-x^2+1)^2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \left(x^2 - 1\right)^{3/4} + c_2 x \left(x^2 - 1\right)^{1/4}$$

1.271.5 Mathematica DSolve solution

Solving time : 0.063 (sec)
Leaf size : 51

```
DSolve[{4*D[y[x],{x,2}]+3*(2-x^2)/(1-x^2)^2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{x^2 - 1}\left(c_2 Q_{\frac{1}{2}}^{\frac{1}{2}}(x) + \frac{\sqrt{\frac{2}{\pi}} c_1 x}{\sqrt[4]{1 - x^2}}\right)$$

1.272 problem 275

Internal problem ID [8410]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 275
Date solved : `Monday, October 21, 2024 at 05:07:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' - \frac{2u'}{x} - a^2u = 0$$

1.272.1 Solved as second order ode using Kovacic algorithm

Time used: 0.294 (sec)

Writing the ode as

$$u'' - \frac{2u'}{x} - a^2u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{2}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = a^2x^2 + 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 518: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + a^2$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{2}{x^2}\right) \\ &= \frac{2}{x^2} + a^2 \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{a^2 x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(a) \\
&= -\frac{1}{x} - a \\
&= \frac{-ax - 1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - a\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - a\right)^2 - \left(\frac{a^2x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{2aa_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{1}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{1}{a}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{a}\right) \mathrm{e}^{\int \left(-\frac{1}{x} - a\right) dx} \\
&= \left(x + \frac{1}{a}\right) \mathrm{e}^{-ax - \ln(x)} \\
&= \frac{(ax + 1)\,\mathrm{e}^{-ax}}{ax}
\end{aligned}
$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-\frac{2}{x}}{1}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{(ax+1)\,\mathrm{e}^{-ax}}{a}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{-\frac{2}{x}}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{2\ln(x)}}{(u_1)^2}\,dx \\
&= u_1 \left(\frac{(ax-1)\,\mathrm{e}^{2ax}}{2a\,(ax+1)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a}\right) + c_2 \left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a}\left(\frac{(ax-1)\,\mathrm{e}^{2ax}}{2a\,(ax+1)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.272.2 Maple step by step solution

Let's solve

$\frac{d}{dx}u' - \frac{2u'}{x} - a^2u = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}u'$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = -a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $-a^2ux + \left(\frac{d}{dx}u'\right)x - 2u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k - 1$

    $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert u' to series expansion

    $u' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$
  - Shift index using $k- >k + 1$

    $u' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}u'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\,x^{-1+r} + a_1(1+r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k-2+r) - a^2 a_{k-1}\right) x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term must be 0

  $a_1(1+r)\,(-2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k-2+r) - a^2 a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+r-1) - a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+r-1)}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a^2 a_k}{(k+2)(k-1)}$

- Solution for $r = 0$

  $\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k-1)}, -2a_1 = 0\right]$

- Recursion relation for $r = 3$

  $a_{k+2} = \frac{a^2 a_k}{(k+5)(k+2)}$

- Solution for $r = 3$

  $\left[u = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{a^2 a_k}{(k+5)(k+2)}, 4a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[u = \left(\sum_{k=0}^{\infty} b_k x^k\right) + \left(\sum_{k=0}^{\infty} c_k x^{k+3}\right), b_{k+2} = \frac{a^2 b_k}{(k+2)(k-1)}, -2b_1 = 0, c_{k+2} = \frac{a^2 c_k}{(k+5)(k+2)}, 4c_1 = 0\right]$

1.272.3 Maple trace

```
Methods for second order ODEs:
```

1.272.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 28

```
dsolve(diff(diff(u(x),x),x)-2/x*diff(u(x),x)-a^2*u(x) = 0,
       u(x),singsol=all)
```

$$u = c_1(ax-1)\,\mathrm{e}^{ax} + c_2(ax+1)\,\mathrm{e}^{-ax}$$

1.272.5 Mathematica DSolve solution

Solving time : 0.158 (sec)
Leaf size : 68

```
DSolve[{D[u[x],{x,2}]-2/x*D[u[x],x]-a^2*u[x]==0,{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{\sqrt{\frac{2}{\pi}}\sqrt{x}((iac_2x+c_1)\sinh(ax)-(ac_1x+ic_2)\cosh(ax))}{a\sqrt{-iax}}$$

1.273 problem 276

Internal problem ID [8411]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 276
Date solved : `Monday, October 21, 2024 at 05:07:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' + \frac{2u'}{x} - a^2 u = 0$$

1.273.1 Solved as second order ode using Kovacic algorithm

Time used: 0.144 (sec)

Writing the ode as

$$u'' + \frac{2u'}{x} - a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = a^2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = \left(a^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 520: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{\sqrt{a^2}\, x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{2}{x}}{1} \, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{\mathrm{e}^{\mathrm{csgn}(a) a x}}{x}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{u_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1} \, dx}}{(u_1)^2} \, dx \\
&= u_1 \int \frac{e^{-2\ln(x)}}{(u_1)^2} \, dx \\
&= u_1 \left(-\frac{\mathrm{e}^{-2\,\mathrm{csgn}(a) a x}}{2\,\mathrm{csgn}\,(a)\, a}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1u_1 + c_2u_2 \\
&= c_1\left(\frac{\mathrm{e}^{\operatorname{csgn}(a)ax}}{x}\right) + c_2\left(\frac{\mathrm{e}^{\operatorname{csgn}(a)ax}}{x}\left(-\frac{\mathrm{e}^{-2\operatorname{csgn}(a)ax}}{2\operatorname{csgn}(a)\,a}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.273.2 Maple step by step solution

Let's solve

$\frac{d}{dx}u' + \frac{2u'}{x} - a^2u = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}u'$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - ○ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -a^2\right]$
  - ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$
  - ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $-a^2ux + \left(\frac{d}{dx}u'\right)x + 2u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ○ Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - ○ Shift index using $k- >k-1$

$$x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert u' to series expansion

$$u' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$$

- Shift index using $k- >k+1$

$$u' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1) x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}u'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=0}^{\infty} a_k (k+r) (k+r-1) x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1) (k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r) x^{-1+r} + a_1 (1+r) (2+r) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1) (k+2+r) - a^2 a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 0\}$
- Each term must be 0
  $a_1(1+r)(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+r+1)(k+2+r) - a^2 a_{k-1} = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2+r)(k+3+r) - a^2 a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$
  $a_{k+2} = \frac{a^2 a_k}{(k+1)(k+2)}$
- Solution for $r = -1$
  $\left[u = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{a^2 a_k}{(k+1)(k+2)}, 0 = 0\right]$
- Recursion relation for $r = 0$

$a_{k+2} = \frac{a^2 a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = \frac{a^2 b_k}{(k+1)(k+2)}, 0 = 0, c_{k+2} = \frac{a^2 c_k}{(k+2)(k+3)}, 2c_1 = 0\right]$

1.273.3 Maple trace

```
Methods for second order ODEs:
```

1.273.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 21

```
dsolve(diff(diff(u(x),x),x)+2/x*diff(u(x),x)-a^2*u(x) = 0,
       u(x),singsol=all)
```

$$u = \frac{c_1 \sinh(ax) + c_2 \cosh(ax)}{x}$$

1.273.5 Mathematica DSolve solution

Solving time : `0.043` (sec)
Leaf size : 35

```
DSolve[{D[u[x],{x,2}]+2/x*D[u[x],x]-a^2*u[x]==0,{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{2ac_1 e^{-ax} + c_2 e^{ax}}{2ax}$$

1.274 problem 277

Internal problem ID [8412]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 277
Date solved : `Monday, October 21, 2024 at 05:07:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' + \frac{2u'}{x} + a^2 u = 0$$

1.274.1 Solved as second order ode using Kovacic algorithm

Time used: 0.154 (sec)

Writing the ode as

$$u'' + \frac{2u'}{x} + a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -a^2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = (-a^2)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 522: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{-a^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \frac{\mathrm{e}^{\sqrt{-a^2}\, x}}{x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$u_2 = u_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\,dx}}{(u_1)^2}\,dx$$
$$= u_1 \int \frac{e^{-2\ln(x)}}{(u_1)^2}\,dx$$
$$= u_1 \left(\frac{\sqrt{-a^2}\,\mathrm{e}^{-2\sqrt{-a^2}\,x}}{2a^2}\right)$$

Therefore the solution is

$$u = c_1 u_1 + c_2 u_2$$
$$= c_1 \left(\frac{\mathrm{e}^{\sqrt{-a^2}\,x}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{\sqrt{-a^2}\,x}}{x}\left(\frac{\sqrt{-a^2}\,\mathrm{e}^{-2\sqrt{-a^2}\,x}}{2a^2}\right)\right)$$

Will add steps showing solving for IC soon.

1.274.2 Maple step by step solution

Let's solve

$\frac{d}{dx}u' + \frac{2u'}{x} + a^2u = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}u'$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators
  $a^2ux + \left(\frac{d}{dx}u'\right)x + 2u' = 0$
- Assume series solution for u
  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot u$ to series expansion
    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  ◦ Shift index using $k->k-1$
    $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  ◦ Convert u' to series expansion
    $u' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  ◦ Shift index using $k->k+1$
    $u' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
  ◦ Convert $x \cdot \left(\frac{d}{dx}u'\right)$ to series expansion
    $x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  ◦ Shift index using $k->k+1$
    $x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions
  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a^2 a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 0\}$
- Each term must be 0
  $a_1(1+r)\,(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+r+1)\,(k+2+r) + a^2 a_{k-1} = 0$
- Shift index using $k->k+1$

$$a_{k+2}(k+2+r)(k+3+r) + a^2 a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a^2 a_k}{(k+2+r)(k+3+r)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{a^2 a_k}{(k+1)(k+2)}$$

- Solution for $r = -1$

$$\left[u = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a^2 a_k}{(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+3)}$$

- Solution for $r = 0$

$$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = -\frac{a^2 b_k}{(k+1)(k+2)}, 0 = 0, c_{k+2} = -\frac{a^2 c_k}{(k+2)(k+3)}, 2c_1 = 0\right]$$

1.274.3 Maple trace

```
Methods for second order ODEs:
```

1.274.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 21

```
dsolve(diff(diff(u(x),x),x)+2/x*diff(u(x),x)+a^2*u(x) = 0,
       u(x),singsol=all)
```

$$u = \frac{c_1 \sin(ax) + c_2 \cos(ax)}{x}$$

1.274.5 Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 42

```
DSolve[{D[u[x],{x,2}]+2/x*D[u[x],x]+a^2*u[x]==0,{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{e^{-iax}\left(2c_1 - \frac{ic_2e^{2iax}}{a}\right)}{2x}$$

1.275 problem 278

Internal problem ID [8413]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 278
Date solved : `Monday, October 21, 2024 at 05:07:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' + \frac{4u'}{x} - a^2 u = 0$$

1.275.1 Solved as second order ode using Kovacic algorithm

Time used: 0.294 (sec)

Writing the ode as

$$u'' + \frac{4u'}{x} - a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{4}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 524: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + a^2$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{2}{x^2}\right) \\ &= \frac{2}{x^2} + a^2 \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a^2x^2 + 2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{x} + (-)\,(a) \\ &= -\frac{1}{x} - a \\ &= \frac{-ax - 1}{x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{x} - a\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - a\right)^2 - \left(\frac{a^2x^2 + 2}{x^2}\right)\right) &= 0 \\ \frac{2aa_0 - 2}{x} &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x + \frac{1}{a}\right) e^{\int \left(-\frac{1}{x} - a\right) dx} \\ &= \left(x + \frac{1}{a}\right) e^{-ax - \ln(x)} \\ &= \frac{(ax + 1)\, e^{-ax}}{ax}\end{aligned}$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{4}{x}}{1}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1\left(\frac{1}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{(ax+1)\,\mathrm{e}^{-ax}}{x^3 a}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{4}{x}}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{-4\ln(x)}}{(u_1)^2}\,dx \\
&= u_1\left(\frac{(ax-1)\,\mathrm{e}^{2ax}}{2a\,(ax+1)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1\left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{x^3 a}\right) + c_2\left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{x^3 a}\left(\frac{(ax-1)\,\mathrm{e}^{2ax}}{2a\,(ax+1)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.275.2 Maple step by step solution

Let's solve

$\frac{d}{dx}u' + \frac{4u'}{x} - a^2u = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}u'$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{4}{x}, P_3(x) = -a^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $-a^2ux + \left(\frac{d}{dx}u'\right)x + 4u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot u$ to series expansion

  $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  ◦ Shift index using $k- >k-1$

  $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  ◦ Convert u' to series expansion

  $u' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  ◦ Shift index using $k- >k+1$

  $u' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}u'\right)$ to series expansion

  $$x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(3+r)\,x^{-1+r} + a_1(1+r)\,(4+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+4+r) - a^2 a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r)=0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$
- Each term must be 0

  $a_1(1+r)\,(4+r)=0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+4+r) - a^2 a_{k-1} = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+5+r) - a^2 a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+5+r)}$
- Recursion relation for $r=-3$

  $a_{k+2} = \frac{a^2 a_k}{(k-1)(k+2)}$
- Solution for $r=-3$

  $$\left[u = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = \frac{a^2 a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$$
- Recursion relation for $r=0$

  $a_{k+2} = \frac{a^2 a_k}{(k+2)(k+5)}$
- Solution for $r=0$

  $$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = \frac{a^2 b_k}{(k-1)(k+2)}, -2b_1 = 0, c_{k+2} = \frac{a^2 c_k}{(k+2)(k+5)}, 4c_1 = 0\right]$$

1.275.3 Maple trace

```
Methods for second order ODEs:
```

1.275.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 32

```
dsolve(diff(diff(u(x),x),x)+4/x*diff(u(x),x)-a^2*u(x) = 0,
       u(x),singsol=all)
```

$$u = \frac{c_2(ax+1)\,\mathrm{e}^{-ax} + c_1(ax-1)\,\mathrm{e}^{ax}}{x^3}$$

1.275.5 Mathematica DSolve solution

Solving time : 0.12 (sec)
Leaf size : 68

```
DSolve[{D[u[x],{x,2}]+4/x*D[u[x],x]-a^2*u[x]==0,{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{\sqrt{\frac{2}{\pi}}((iac_2x + c_1)\sinh(ax) - (ac_1x + ic_2)\cosh(ax))}{ax^{5/2}\sqrt{-iax}}$$

1.276 problem 279

Internal problem ID [8414]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 279
Date solved : `Monday, October 21, 2024 at 05:07:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' + \frac{4u'}{x} + a^2 u = 0$$

1.276.1 Solved as second order ode using Kovacic algorithm

Time used: 0.352 (sec)

Writing the ode as

$$u'' + \frac{4u'}{x} + a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{4}{x} \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -a^2x^2 + 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-a^2x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 526: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} - a^2$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx ia - \frac{i}{a\,x^2} - \frac{i}{2a^3x^4} - \frac{i}{2a^5x^6} - \frac{5i}{8a^7x^8} - \frac{7i}{8a^9x^{10}} - \frac{21i}{16a^{11}x^{12}} - \frac{33i}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = ia$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= ia \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-a^2) + \left(\frac{2}{x^2}\right) \\ &= \frac{2}{x^2} - a^2 \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= ia\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{0}{ia}-0\right) &&= 0\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{0}{ia}-0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-a^2x^2+2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	ia	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\
&= 0 - (-1)\\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(ia) \\
&= -\frac{1}{x} - ia \\
&= -\frac{1}{x} - ia
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - ia\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - ia\right)^2 - \left(\frac{-a^2x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{2iaa_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{i}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{i}{a}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{i}{a}\right) e^{\int \left(-\frac{1}{x} - ia\right) dx} \\
&= \left(x - \frac{i}{a}\right) e^{-\ln(x) - iax} \\
&= \frac{(ax - i)\, e^{-iax}}{xa}
\end{aligned}
$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{4}{x}}{1}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1\left(\frac{1}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{(ax-i)\,\mathrm{e}^{-iax}}{x^3 a}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{4}{x}}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{-4\ln(x)}}{(u_1)^2}\,dx \\
&= u_1\left(\frac{(iax-1)\,\mathrm{e}^{2iax}}{2a\,(-ax+i)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1\left(\frac{(ax-i)\,\mathrm{e}^{-iax}}{x^3 a}\right) + c_2\left(\frac{(ax-i)\,\mathrm{e}^{-iax}}{x^3 a}\left(\frac{(iax-1)\,\mathrm{e}^{2iax}}{2a\,(-ax+i)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.276.2 Maple step by step solution

Let's solve

$\frac{d}{dx}u' + \frac{4u'}{x} + a^2u = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}u'$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 4$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $a^2ux + \left(\frac{d}{dx}u'\right)x + 4u' = 0$
- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k - 1$

    $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert u' to series expansion

    $u' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1}$
  - Shift index using $k- >k + 1$

    $u' = \sum_{k=-1}^{\infty} a_{k+1}(k + r + 1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}u'\right)$ to series expansion

  $$x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(3+r)\, x^{-1+r} + a_1(1+r)(4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(k+4+r) + a^2 a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-3, 0\}$$

- Each term must be 0

  $$a_1(1+r)(4+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+r+1)(k+4+r) + a^2 a_{k-1} = 0$$

- Shift index using $k->k+1$

  $$a_{k+2}(k+2+r)(k+5+r) + a^2 a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a^2 a_k}{(k+2+r)(k+5+r)}$$

- Recursion relation for $r = -3$

  $$a_{k+2} = -\frac{a^2 a_k}{(k-1)(k+2)}$$

- Solution for $r = -3$

  $$\left[u = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a^2 a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+5)}$$

- Solution for $r = 0$

  $$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = -\frac{a^2 b_k}{(k-1)(k+2)}, -2b_1 = 0, c_{k+2} = -\frac{a^2 c_k}{(k+2)(k+5)}, 4c_1 = 0\right]$$

1.276.3 Maple trace

```
Methods for second order ODEs:
```

1.276.4 Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 33

```
dsolve(diff(diff(u(x),x),x)+4/x*diff(u(x),x)+a^2*u(x) = 0,
       u(x),singsol=all)
```

$$u = \frac{(c_1 a x + c_2) \cos (a x) + \sin (a x) (c_2 a x - c_1)}{x^3}$$

1.276.5 Mathematica DSolve solution

Solving time : 0.141 (sec)
Leaf size : 57

```
DSolve[{D[u[x],{x,2}]+4/x*D[u[x],x]+a^2*u[x]==0,{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to -\frac{\sqrt{\frac{2}{\pi}}((a c_1 x + c_2) \cos(a x) + (a c_2 x - c_1) \sin(a x))}{x^{3/2}(a x)^{3/2}}$$

1.277 problem 280

Internal problem ID [8415]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 280
Date solved : `Monday, October 21, 2024 at 05:07:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - a^2 y = \frac{6y}{x^2}$$

1.277.1 Solved as second order ode using Kovacic algorithm

Time used: 0.328 (sec)

Writing the ode as

$$y'' + \left(-a^2 - \frac{6}{x^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -a^2 - \frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{a^2 x^2 + 6}{x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= a^2 x^2 + 6 \\
t &= x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{a^2 x^2 + 6}{x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 528: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = a^2 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{3}{a\,x^2} - \frac{9}{2a^3x^4} + \frac{27}{2a^5x^6} - \frac{405}{8a^7x^8} + \frac{1701}{8a^9x^{10}} - \frac{15309}{16a^{11}x^{12}} + \frac{72171}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{6}{x^2}\right) \\ &= a^2 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{a^2 x^2 + 6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x} + (-)(a) \\
&= -\frac{2}{x} - a \\
&= \frac{-ax - 2}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{x} - a\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - a\right)^2 - \left(\frac{a^2x^2 + 6}{x^2}\right)\right) = 0$$

$$\frac{2axa_1 + 4aa_0 - 6x - 4a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{3}{a^2}, a_1 = \frac{3}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + \frac{3x}{a} + \frac{3}{a^2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + \frac{3x}{a} + \frac{3}{a^2}\right) e^{\int \left(-\frac{2}{x} - a\right) dx} \\
&= \left(x^2 + \frac{3x}{a} + \frac{3}{a^2}\right) e^{-ax - 2\ln(x)} \\
&= \frac{(a^2x^2 + 3ax + 3)\, e^{-ax}}{a^2x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax}}{a^2x^2} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax}}{a^2x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax}}{a^2x^2} \int \frac{1}{\frac{(a^2x^2+3ax+3)^2\mathrm{e}^{-2ax}}{a^4x^4}}\,dx \\ &= \frac{(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax}}{a^2x^2} \left(\frac{(a^2x^2 - 3ax + 3)\,\mathrm{e}^{2ax}}{2a\,(a^2x^2 + 3ax + 3)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\left(\frac{(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax}}{a^2x^2}\right) + c_2\left(\frac{(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax}}{a^2x^2}\left(\frac{(a^2x^2 - 3ax + 3)\,\mathrm{e}^{2ax}}{2a\,(a^2x^2 + 3ax + 3)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.277.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - a^2 y = \frac{6y}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{y(a^2x^2+6)}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y(a^2x^2+6)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = -\frac{a^2x^2+6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -6$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d}{dx}y'\right)x^2 + (-a^2x^2 - 6)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$x^2\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(2+r)\,(-3+r)\,x^r+a_1(3+r)\,(-2+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r+2)\,(k+r-3)-a^2a_{k-2}\right)x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-3+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-2,3\}$
- Each term must be 0

  $a_1(3+r)\,(-2+r)=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)\,(k+r-3)-a^2a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)\,(k+r-1)-a^2a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a^2a_k}{(k+4+r)(k+r-1)}$
- Recursion relation for $r=-2$

  $a_{k+2}=\frac{a^2a_k}{(k+2)(k-3)}$
- Solution for $r=-2$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k-2},a_{k+2}=\frac{a^2a_k}{(k+2)(k-3)},a_1=0\right]$
- Recursion relation for $r=3$

  $a_{k+2}=\frac{a^2a_k}{(k+7)(k+2)}$
- Solution for $r=3$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+3},a_{k+2}=\frac{a^2a_k}{(k+7)(k+2)},a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}b_kx^{k-2}\right)+\left(\sum_{k=0}^{\infty}c_kx^{k+3}\right),b_{k+2}=\frac{a^2b_k}{(k+2)(k-3)},b_1=0,c_{k+2}=\frac{a^2c_k}{(k+7)(k+2)},c_1=0\right]$

1.277.3 Maple trace

```
Methods for second order ODEs:
```

1.277.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 48

```
dsolve(diff(diff(y(x),x),x)-a^2*y(x) = 6*y(x)/x^2,
       y(x),singsol=all)
```

$$y = \frac{c_2(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax} + c_1(a^2x^2 - 3ax + 3)\,\mathrm{e}^{ax}}{x^2}$$

1.277.5 Mathematica DSolve solution

Solving time : 0.199 (sec)
Leaf size : 90

```
DSolve[{D[y[x],{x,2}]-a^2*y[x]==6*y[x]/x^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{\frac{2}{\pi}}((a^2c_2x^2 - 3iac_1x + 3c_2)\cosh(ax) + i(c_1(a^2x^2 + 3) + 3iac_2x)\sinh(ax))}{a^2x^{3/2}\sqrt{-iax}}$$

1.278 problem 281

Internal problem ID [8416]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 281
Date solved : `Monday, October 21, 2024 at 05:07:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + n^2 y = \frac{6y}{x^2}$$

1.278.1 Solved as second order ode using Kovacic algorithm

Time used: 0.395 (sec)

Writing the ode as

$$y'' + \left(n^2 - \frac{6}{x^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= n^2 - \frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-n^2x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -n^2x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-n^2x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 530: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -n^2 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx in - \frac{3i}{n\,x^2} - \frac{9i}{2n^3x^4} - \frac{27i}{2n^5x^6} - \frac{405i}{8n^7x^8} - \frac{1701i}{8n^9x^{10}} - \frac{15309i}{16n^{11}x^{12}} - \frac{72171i}{16n^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = in$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= in \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -n^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-n^2x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-n^2) + \left(\frac{6}{x^2}\right) \\ &= -n^2 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= in \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{in} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{in} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-n^2x^2 + 6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	in	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{2}{x} + (-)\,(in) \\ &= -\frac{2}{x} - in \\ &= -\frac{2}{x} - in\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{x} - in\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - in\right)^2 - \left(\frac{-n^2x^2 + 6}{x^2}\right)\right) = 0$$

$$\frac{(2ina_1 - 6)\,x + 4ina_0 - 4a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{3}{n^2}, a_1 = -\frac{3i}{n}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - \frac{3ix}{n} - \frac{3}{n^2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\,dx} \\ &= \left(x^2 - \frac{3ix}{n} - \frac{3}{n^2}\right) \mathrm{e}^{\int \left(-\frac{2}{x} - in\right)dx} \\ &= \left(x^2 - \frac{3ix}{n} - \frac{3}{n^2}\right) \mathrm{e}^{-2\ln(x) - inx} \\ &= \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2 n^2}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} \int \frac{1}{\frac{(n^2x^2-3inx-3)^2\mathrm{e}^{-2inx}}{x^4n^4}}\,dx \\ &= \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} \left(\frac{(in^2x^2 - 3nx - 3i)\,\mathrm{e}^{2inx}}{2n\,(-n^2x^2 + 3inx + 3)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\left(\frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2}\right) + c_2\left(\frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2}\left(\frac{(in^2x^2 - 3nx - 3i)\,\mathrm{e}^{2inx}}{2n\,(-n^2x^2 + 3inx + 3)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.278.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + n^2 y = \frac{6y}{x^2}$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = -\frac{y(n^2x^2-6)}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{y(n^2x^2-6)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{n^2x^2-6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -6$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$\left(\frac{d}{dx}y'\right)x^2 + (n^2x^2 - 6)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

Rewrite ODE with series expansions

$a_0(2+r)(-3+r)x^r + a_1(3+r)(-2+r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)(k+r-3) + n^2 a_{k-2}) x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 3\}$

- Each term must be 0

  $a_1(3+r)(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-3) + n^2 a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+4+r)(k+r-1) + n^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{n^2 a_k}{(k+4+r)(k+r-1)}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{n^2 a_k}{(k+2)(k-3)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{n^2 a_k}{(k+2)(k-3)}, a_1 = 0\right]$

- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{n^2 a_k}{(k+7)(k+2)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{n^2 a_k}{(k+7)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{n^2 a_k}{(k+2)(k-3)}, a_1 = 0, b_{k+2} = -\frac{n^2 b_k}{(k+7)(k+2)}, b_1 = 0\right]$

1.278.3 Maple trace

```
Methods for second order ODEs:
```

1.278.4 Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 53

```
dsolve(diff(diff(y(x),x),x)+n^2*y(x) = 6*y(x)/x^2,
       y(x),singsol=all)
```

$$y = \frac{(c_1 n^2 x^2 + 3c_2 nx - 3c_1) \cos(nx) + \sin(nx)(c_2 n^2 x^2 - 3c_1 nx - 3c_2)}{x^2}$$

1.278.5 Mathematica DSolve solution

Solving time : 0.196 (sec)
Leaf size : 79

```
DSolve[{D[y[x],{x,2}]+n^2*y[x]==6*y[x]/x^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\sqrt{x}((c_2(-n^2)x^2 + 3c_1 nx + 3c_2)\cos(nx) + (c_1(n^2x^2 - 3) + 3c_2 nx)\sin(nx))}{(nx)^{5/2}}$$

1.279 problem 282

Internal problem ID [8417]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 282
Date solved : `Monday, October 21, 2024 at 05:07:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' - \left(x^2 + \frac{1}{4}\right) y = 0$$

1.279.1 Solved as second order ode using Kovacic algorithm

Time used: 0.123 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(-x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 532: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-x}}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-x}}{\sqrt{x}}\right) + c_2 \left(\frac{e^{-x}}{\sqrt{x}} \left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.279.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + xy' - \left(x^2 + \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(4x^2+1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{x} - \frac{(4x^2+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{4x^2+1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 4xy' + (-4x^2 - 1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) - 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) - 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k + 2)^2 + 8(k + 2)\, r + 4r^2 - 1\right) - 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = \frac{4a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = \frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$$

1.279.3 Maple trace

```
Methods for second order ODEs:
```

1.279.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)-(x^2+1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sinh(x) + c_2 \cosh(x)}{\sqrt{x}}$$

1.279.5 Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 32

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]-(x^2+1/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x}(c_2 e^{2x} + 2c_1)}{2\sqrt{x}}$$

1.280 problem 283

Internal problem ID [8418]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 283
Date solved : `Monday, October 21, 2024 at 05:07:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \frac{(-9a^2 + 4x^2)\, y}{4a^2} = 0$$

1.280.1 Solved as second order ode using Kovacic algorithm

Time used: 0.339 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(-\frac{9}{4} + \frac{x^2}{a^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -\frac{9}{4} + \frac{x^2}{a^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2a^2 - x^2}{x^2 a^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2a^2 - x^2$$
$$t = x^2 a^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2a^2 - x^2}{x^2 a^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 534: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2a^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{a^2} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx -\frac{33ia^{13}}{16x^{14}} - \frac{21ia^{11}}{16x^{12}} - \frac{7ia^9}{8x^{10}} - \frac{5ia^7}{8x^8} - \frac{ia^5}{2x^6} - \frac{ia^3}{2x^4} - \frac{ia}{x^2} + \frac{i}{a} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i}{a}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{i}{a}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{1}{a^2}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{2a^2 - x^2}{x^2 a^2} \\ &= Q + \frac{R}{x^2 a^2} \\ &= \left(-\frac{1}{a^2}\right) + \left(\frac{2}{x^2}\right) \\ &= -\frac{1}{a^2} + \frac{2}{x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{i}{a} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{i}{a}} - 0\right) &&= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{i}{a}} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2a^2 - x^2}{x^2 a^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{i}{a}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - (\alpha_{c_1}^{-}) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{i}{a}\right) \\
&= -\frac{1}{x} - \frac{i}{a} \\
&= -\frac{ix+a}{xa}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{i}{a}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{i}{a}\right)^2 - \left(\frac{2a^2-x^2}{x^2a^2}\right)\right) &= 0 \\
\frac{2ia_0 - 2a}{xa} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -ia\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = -ia + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (-ia + x) \, e^{\int \left(-\frac{1}{x} - \frac{i}{a}\right) dx} \\
&= (-ia + x) \, e^{-\ln(x) - \frac{ix}{a}} \\
&= \frac{(-ia + x) \, e^{-\frac{ix}{a}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(-ia + x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(ix + a)\,a(ia + x)\,\mathrm{e}^{\frac{2ix}{a}}}{2\,(ia - x)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(-ia + x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{3/2}}\right) + c_2 \left(\frac{(-ia + x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{3/2}} \left(-\frac{(ix + a)\,a(ia + x)\,\mathrm{e}^{\frac{2ix}{a}}}{2\,(ia - x)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.280.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+xy'+\frac{(-9a^2+4x^2)y}{4a^2}=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{(9a^2-4x^2)y}{4x^2a^2}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{y'}{x}-\frac{(9a^2-4x^2)y}{4x^2a^2}=0$

- Multiply by denominators of the ODE

$4x^2\left(\frac{d}{dx}y'\right)a^2+4xy'a^2-(9a^2-4x^2)\,y=0$

- Make a change of variables

$t=\ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

  $y'=\left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

  $y'=\frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

  $\frac{d}{dx}y'=\left(\frac{d}{dt}\frac{d}{dt}y(t)\right)t'(x)^2+\left(\frac{d}{dx}t'(x)\right)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

  $\frac{d}{dx}y'=\frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$4x^2\left(\frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)a^2+4\left(\frac{d}{dt}y(t)\right)a^2-(9a^2-4x^2)\,y(t)=0$

- Simplify

$4a^2\left(\frac{d}{dt}\frac{d}{dt}y(t)\right)-9y(t)\,a^2+4y(t)\,x^2=0$

- Isolate 2nd derivative

$\frac{d}{dt}\frac{d}{dt}y(t)=\frac{(9a^2-4x^2)y(t)}{4a^2}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d}{dt}\frac{d}{dt}y(t)-\frac{(9a^2-4x^2)y(t)}{4a^2}=0$

- Characteristic polynomial of ODE

$r^2 - \frac{9a^2-4x^2}{4a^2} = 0$

- Factor the characteristic polynomial

  $\frac{4r^2a^2-9a^2+4x^2}{4a^2} = 0$

- Roots of the characteristic polynomial

  $r = \left(\frac{\sqrt{9a^2-4x^2}}{2a}, -\frac{\sqrt{9a^2-4x^2}}{2a}\right)$

- 1st solution of the ODE

  $y_1(t) = e^{\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$

- 2nd solution of the ODE

  $y_2(t) = e^{-\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$

- General solution of the ODE

  $y(t) = C1y_1(t) + C2y_2(t)$

- Substitute in solutions

  $y(t) = C1\,e^{\frac{\sqrt{9a^2-4x^2}\,t}{2a}} + C2\,e^{-\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$

- Change variables back using $t = \ln(x)$

  $y = C1\,e^{\frac{\sqrt{9a^2-4x^2}\,\ln(x)}{2a}} + C2\,e^{-\frac{\sqrt{9a^2-4x^2}\,\ln(x)}{2a}}$

- Simplify

  $y = C1\,x^{\frac{\sqrt{9a^2-4x^2}}{2a}} + C2\,x^{-\frac{\sqrt{9a^2-4x^2}}{2a}}$

1.280.3 Maple trace

```
Methods for second order ODEs:
```

1.280.4 Maple dsolve solution

Solving time : `0.086` (sec)
Leaf size : 37

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+1/4*(-9*a^2+4*x^2)/a^2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2(ix+a)\,e^{-\frac{ix}{a}} + (-ix+a)\,c_1\,e^{\frac{ix}{a}}}{x^{3/2}}$$

1.280.5 Mathematica DSolve solution

Solving time : 0.142 (sec)
Leaf size : 62

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(4*x^2-9*a^2)/(4*a^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\left((ac_2+c_1x)\cos\left(\frac{x}{a}\right)+(c_2x-ac_1)\sin\left(\frac{x}{a}\right)\right)}{x\sqrt{\frac{x}{a}}}$$

1.281 problem 284

Internal problem ID [8419]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 284
Date solved : `Monday, October 21, 2024 at 05:07:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right) y = 0$$

1.281.1 Solved as second order ode using Kovacic algorithm

Time used: 0.316 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{25}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-x^2 + 6}{x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -x^2 + 6 \\
t &= x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-x^2 + 6}{x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 536: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= -1 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 + 6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + (-)\,(i) \\
&= -\frac{2}{x} - i \\
&= -\frac{2}{x} - i
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2 + 6}{x^2}\right)\right) &= 0 \\
\frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -3, a_1 = -3i\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 3ix - 3
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 3ix - 3\right) \mathrm{e}^{\int \left(-\frac{2}{x} - i\right) dx} \\
&= \left(x^2 - 3ix - 3\right) \mathrm{e}^{-2\ln(x) - ix} \\
&= \frac{\left(x^2 - 3ix - 3\right) \mathrm{e}^{-ix}}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}\right) + c_2\left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}\left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.281.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+xy'+\left(x^2-\frac{25}{4}\right)y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(4x^2-25)y}{4x^2}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{y'}{x}+\frac{(4x^2-25)y}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{4x^2-25}{4x^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{25}{4}$

- ○ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

$4x^2\left(\frac{d}{dx}y'\right)+4xy'+(4x^2-25)y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ○ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x\cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(5+2r)\,(-5+2r)\, x^r + a_1(7+2r)\,(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+5)\,(2k+2r-5) + 4a_{\ldots}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+2r)\,(-5+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{5}{2}, \frac{5}{2}\right\}$
- Each term must be 0

  $a_1(7+2r)\,(-3+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r+5)\,(2k+2r-5) + 4a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(2k+9+2r)\,(2k-1+2r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$
- Recursion relation for $r = -\frac{5}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}$
- Solution for $r = -\frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0\right]$$
- Recursion relation for $r = \frac{5}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}$
- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}, a_1 = 0\right]$$
- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(2k+14)(2k+4)}, b_1 = 0\right]$$

1.281.3 Maple trace

```
Methods for second order ODEs:
```

1.281.4 Maple dsolve solution

Solving time : `0.023` (sec)
Leaf size : 43

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-25/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-3c_2\left(ix - \frac{1}{3}x^2 + 1\right)\mathrm{e}^{-ix} + 3c_1\,\mathrm{e}^{ix}\left(ix + \frac{1}{3}x^2 - 1\right)}{x^{5/2}}$$

1.281.5 Mathematica DSolve solution

Solving time : `0.137` (sec)
Leaf size : 59

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-25/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}((-c_2x^2 + 3c_1x + 3c_2)\cos(x) + (c_1(x^2 - 3) + 3c_2x)\sin(x))}{x^{5/2}}$$

1.282 problem 285

Internal problem ID [8420]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 285
Date solved : `Monday, October 21, 2024 at 05:07:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + qy' = \frac{2y}{x^2}$$

1.282.1 Solved as second order ode using Kovacic algorithm

Time used: 0.296 (sec)

Writing the ode as

$$y'' + qy' - \frac{2y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= q \\ C &= -\frac{2}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{q^2x^2 + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = q^2x^2 + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{q^2x^2 + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 538: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{q^2}{4} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{q}{2} + \frac{2}{q\,x^2} - \frac{4}{q^3x^4} + \frac{16}{q^5x^6} - \frac{80}{q^7x^8} + \frac{448}{q^9x^{10}} - \frac{2688}{q^{11}x^{12}} + \frac{16896}{q^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{q}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{q}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{q^2}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{q^2x^2 + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{q^2}{4}\right) + \left(\frac{2}{x^2}\right) \\ &= \frac{q^2}{4} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{q}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{q}{2}} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{q}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{q^2x^2 + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{q}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{q}{2}\right) \\
&= -\frac{1}{x} - \frac{q}{2} \\
&= -\frac{qx + 2}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - \frac{q}{2}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{q}{2}\right)^2 - \left(\frac{q^2x^2 + 8}{4x^2}\right)\right) = 0$$

$$\frac{qa_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{2}{q}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{2}{q}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{2}{q}\right) e^{\int \left(-\frac{1}{x} - \frac{q}{2}\right) dx} \\
&= \left(x + \frac{2}{q}\right) e^{-\frac{qx}{2} - \ln(x)} \\
&= \frac{(qx + 2)\, e^{-\frac{qx}{2}}}{qx}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{q}{1}\,dx} \\
&= z_1 e^{-\frac{qx}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{qx}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-qx}(qx+2)}{qx}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{q}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-qx}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(qx-2)\,\mathrm{e}^{qx}}{q\,(qx+2)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-qx}(qx+2)}{qx}\right) + c_2 \left(\frac{\mathrm{e}^{-qx}(qx+2)}{qx}\left(\frac{(qx-2)\,\mathrm{e}^{qx}}{q\,(qx+2)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.282.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + qy' = \frac{2y}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -qy' + \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + qy' - \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = q, P_3(x) = -\frac{2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $qy'x^2 + \left(\frac{d}{dx}y'\right)x^2 - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

  ◦ Shift index using $k- >k-1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$

  ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k-1+r) x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+r)(-2+r)x^r + \left(\sum_{k=1}^{\infty} (a_k(k+r+1)(k+r-2) + q a_{k-1}(k-1+r)) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-2) + q a_{k-1}(k-1+r) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+2+r)(k-1+r) + q a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{q a_k (k+r)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = -\frac{q a_k (k-1)}{(k+1)(k-2)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{q a_0}{2}$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot \left(-\frac{qx}{2} + 1\right)$
- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{q a_k (k+2)}{(k+4)(k+1)}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{q a_k (k+2)}{(k+4)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(-\frac{qx}{2} + 1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = -\frac{q b_k (k+2)}{(k+4)(k+1)}\right]$

1.282.3 Maple trace

```
Methods for second order ODEs:
```

1.282.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 28

```
dsolve(diff(diff(y(x),x),x)+q*diff(y(x),x) = 2*y(x)/x^2,
       y(x),singsol=all)
```

$$y = \frac{c_2 \,\mathrm{e}^{-q x}(q x+2)+c_1 (q x-2)}{x}$$

1.282.5 Mathematica DSolve solution

Solving time : 0.085 (sec)
Leaf size : 80

```
DSolve[{D[y[x],{x,2}]+q*D[y[x],x]==2*y[x]/x^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{q x^{3/2} e^{-\frac{q x}{2}} \left(2 (i c_2 q x+2 c_1) \sinh \left(\frac{q x}{2}\right)-2 (c_1 q x+2 i c_2) \cosh \left(\frac{q x}{2}\right)\right)}{\sqrt{\pi} (-i q x)^{5/2}}$$

1.283 problem 286

Internal problem ID [8421]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 286
Date solved : `Monday, October 21, 2024 at 05:07:37 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$xy'' + 3y' + 4x^3y = 0$$

1.283.1 Solved as second order ode using Kovacic algorithm

Time used: 0.302 (sec)

Writing the ode as

$$xy'' + 3y' + 4x^3y = 0 \quad (1)$$
$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= 4x^3 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -16x^4 + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 540: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(-4x^2\right) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 2ix \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-16x^4 + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)\left(2ix\right) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\
&= \frac{\mathrm{e}^{-ix^2}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{x} \, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^{-i x^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{i e^{2 i x^2}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-i x^2}}{x^2}\right) + c_2 \left(\frac{e^{-i x^2}}{x^2} \left(-\frac{i e^{2 i x^2}}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.283.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 3y' + 4x^3 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -4x^2 y - \frac{3y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{3y'}{x} + 4x^2 y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + 3y' + 4x^3y = 0$
- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

  $$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$
- Shift index using $k- >k - 3$

  $$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$
- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$$
- Shift index using $k- >k + 1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\, x^{k+r}$$
- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$$
- Shift index using $k- >k + 1$

  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\,(k + r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^r + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 0\}$
- The coefficients of each power of x must be 0
  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$
- Solve for the dependent coefficient(s)
  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$
- Shift index using $k- > k+3$
  $a_{k+4}(k+4+r)\,(k+6+r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$
- Recursion relation for $r = -2$
  $a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$
- Solution for $r = -2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Recursion relation for $r = 0$
  $a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{4b_k}{(k+4)(k+6)}\right.$

1.283.3 Maple trace

```
Methods for second order ODEs:
```

1.283.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 21

```
dsolve(x*diff(diff(y(x),x),x)+3*diff(y(x),x)+4*x^3*y(x) = 0,
       y(x),singsol=all)
```

$$
y = \frac{c_1 \sin\left(x^2\right) + c_2 \cos\left(x^2\right)}{x^2}
$$

1.283.5 Mathematica DSolve solution

Solving time : 0.086 (sec)
Leaf size : 41

```
DSolve[{x*D[y[x],{x,2}]+3*D[y[x],x]+4*x^3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to \frac{4c_1 e^{-ix^2} - ic_2 e^{ix^2}}{4x^2}
$$

1.284 problem 287

Internal problem ID [8422]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 287
Date solved : `Monday, October 21, 2024 at 05:07:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+2x\right)y''-2(x+1)\,y'+2y=0$$

1.284.1 Solved as second order ode using Kovacic algorithm

Time used: 0.234 (sec)

Writing the ode as

$$\left(x^2+2x\right)y''+(-2x-2)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+2x \\ B &= -2x-2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3}{(x^2 + 2x)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3 \\
t &= (x^2 + 2x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3}{(x^2 + 2x)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 542: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x+2)^2} + \frac{3}{4x^2} + \frac{3}{4(x+2)} - \frac{3}{4x}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(x^2+2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x+2)} + \frac{3}{2x} + (-)(0) \\
&= -\frac{1}{2(x+2)} + \frac{3}{2x} \\
&= \frac{x+3}{x(x+2)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)(0) + \left(\left(\frac{1}{2(x+2)^2} - \frac{3}{2x^2}\right) + \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)^2 - \left(\frac{3}{(x^2+2x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right) dx} \\
&= \frac{x^{3/2}}{\sqrt{x+2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x^2+2x} \, dx} \\
&= z_1 e^{\frac{\ln(x(x+2))}{2}} \\
&= z_1 \left(\sqrt{x(x+2)}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x(x+2)}\, x^{3/2}}{\sqrt{x+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x^2+2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x(x+2))}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{1}{x^2} - \frac{1}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x(x+2)}\,x^{3/2}}{\sqrt{x+2}}\right) + c_2 \left(\frac{\sqrt{x(x+2)}\,x^{3/2}}{\sqrt{x+2}} \left(-\frac{1}{x^2} - \frac{1}{x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.284.2 Maple step by step solution

Let's solve

$(x^2+2x)\left(\frac{d}{dx}y'\right) - 2(x+1)\,y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2y}{x(x+2)} + \frac{2(x+1)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(x+1)y'}{x(x+2)} + \frac{2y}{x(x+2)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2(x+1)}{x(x+2)}, P_3(x) = \frac{2}{x(x+2)}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2} = -1$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

$$\left((x+2)^2 \cdot P_3(x)\right)\bigg|_{x=-2} = 0$$

- ○ $x=-2$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -2$

- Multiply by denominators

$x(x+2)\left(\frac{d}{dx}y'\right) + (-2x-2)\,y' + 2y = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(k+r-1) + a_k(k+r-1)\,(k+r-2)\right) u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

$\left((-2k-2r-2)\,a_{k+1} + a_k(k+r-2)\right)(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r-2)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{2(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{4}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - u + \frac{1}{4}u^2\right)$

- Revert the change of variables $u = x + 2$

  $\left[y = \frac{a_0 x^2}{4}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k}{2(k+3)}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$

- Revert the change of variables $u = x + 2$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y = \frac{a_0 x^2}{4} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$

1.284.3 Maple trace

```
Methods for second order ODEs:
```

1.284.4 Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 14

```
dsolve((x^2+2*x)*diff(diff(y(x),x),x)-2*(x+1)*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x^2 + c_2 x + c_2$$

1.284.5 Mathematica DSolve solution

Solving time : 0.054 (sec)
Leaf size : 19

```
DSolve[{(x^2+2*x)*D[y[x],{x,2}]-2*(x+1)*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x^2 - c_2(x+1)$$

1.285 problem 288

Internal problem ID [8423]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 288
Date solved : `Monday, October 21, 2024 at 05:07:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^{2}+2x\right) y''-2(x+1)\, y'+2y=0$$

1.285.1 Solved as second order ode using Kovacic algorithm

Time used: 0.236 (sec)

Writing the ode as

$$\left(x^{2}+2x\right) y''+\left(-2x-2\right) y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}+2x \\ B &= -2x-2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(x^2 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = (x^2 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(x^2 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 544: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2} - \frac{3}{4x} + \frac{3}{4(x+2)} + \frac{3}{4(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x + 2\right)} + \frac{3}{2x} + (-)\left(0\right) \\
&= -\frac{1}{2\left(x + 2\right)} + \frac{3}{2x} \\
&= \frac{x + 3}{x\left(x + 2\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)(0) + \left(\left(\frac{1}{2(x+2)^2} - \frac{3}{2x^2}\right) + \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)^2 - \left(\frac{3}{(x^2+2x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right) dx} \\ &= \frac{x^{3/2}}{\sqrt{x+2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x^2+2x} \, dx} \\ &= z_1 e^{\frac{\ln(x(x+2))}{2}} \\ &= z_1 \left(\sqrt{x(x+2)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x(x+2)}\, x^{3/2}}{\sqrt{x+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2x-2}{x^2+2x} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{\ln(x(x+2))}}{(y_1)^2} dx$$
$$= y_1 \left(-\frac{1}{x^2} - \frac{1}{x}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{\sqrt{x(x+2)}\, x^{3/2}}{\sqrt{x+2}}\right) + c_2 \left(\frac{\sqrt{x(x+2)}\, x^{3/2}}{\sqrt{x+2}} \left(-\frac{1}{x^2} - \frac{1}{x}\right)\right)$$

Will add steps showing solving for IC soon.

1.285.2 Maple step by step solution

Let's solve

$(x^2 + 2x)\left(\frac{d}{dx}y'\right) - 2(x+1)\,y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2y}{x(x+2)} + \frac{2(x+1)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(x+1)y'}{x(x+2)} + \frac{2y}{x(x+2)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(x+1)}{x(x+2)}, P_3(x) = \frac{2}{x(x+2)}\right]$

  - $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

    $\left((x+2) \cdot P_2(x)\right)\Big|_{x=-2} = -1$

  - $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

$$\left((x+2)^2 \cdot P_3(x)\right)\bigg|_{x=-2} = 0$$

- $x = -2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $x(x+2)\left(\frac{d}{dx}y'\right) + (-2x-2)\,y' + 2y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(k+r-1) + a_k(k+r-1)\,(k+r-2)\right)u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left((-2k-2r-2)\,a_{k+1} + a_k(k+r-2)\right) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r-2)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{2(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{4}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - u + \frac{1}{4}u^2\right)$

- Revert the change of variables $u = x + 2$

  $\left[y = \frac{a_0 x^2}{4}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k}{2(k+3)}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$

- Revert the change of variables $u = x + 2$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y = \frac{a_0 x^2}{4} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$

1.285.3 Maple trace

```
Methods for second order ODEs:
```

1.285.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 14

```
dsolve((x^2+2*x)*diff(diff(y(x),x),x)-2*(x+1)*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1\, x^2 + c_2 x + c_2$$

1.285.5 Mathematica DSolve solution

Solving time : `0.05` (sec)
Leaf size : 19

```
DSolve[{(x^2+2*x)*D[y[x],{x,2}]-2*(x+1)*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x^2 - c_2(x+1)$$

1.286 problem 289

Internal problem ID [8424]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 289
Date solved : `Monday, October 21, 2024 at 05:07:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right)y''-2xy'+2y=0$$

1.286.1 Solved as second order ode using Kovacic algorithm

Time used: 0.300 (sec)

Writing the ode as

$$\left(x^2+1\right)y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (x^2+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 546: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{3}{\left(x^{2}+1\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$ then

$$\begin{aligned}
d&=\alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right)\\
&=1-(1)\\
&=0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+\left((+)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^{+}}{x-c_2}\right)+(-)[\sqrt{r}]_\infty\\
&=-\frac{1}{2\left(x-i\right)}+\frac{3}{2\left(x+i\right)}+(-)\left(0\right)\\
&=-\frac{1}{2\left(x-i\right)}+\frac{3}{2\left(x+i\right)}\\
&=\frac{x-2i}{x^{2}+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{3/2}}{(ix+1)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\
&= z_1 e^{\frac{\ln(x^2+1)}{2}} \\
&= z_1 \left(\sqrt{x^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx$$

$$= y_1 \left(-\frac{x}{(x+i)^2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2 \left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)$$

Will add steps showing solving for IC soon.

1.286.2 Maple step by step solution

1.286.3 Maple trace

```
Methods for second order ODEs:
```

1.286.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 16

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2\,x^2 + c_1 x - c_2$$

1.286.5 Mathematica DSolve solution

Solving time : 0.068 (sec)
Leaf size : 21

```
DSolve[{(x^2+1)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x - c_1 (x - i)^2$$

1.287 problem 290

Internal problem ID [8425]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 290
Date solved : `Monday, October 21, 2024 at 05:07:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x^2+1)\,y''-2xy'+2y=0$$

1.287.1 Solved as second order ode using Kovacic algorithm

Time used: 0.293 (sec)

Writing the ode as

$$(x^2+1)\,y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (x^2+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 547: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\ &= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\ &= \frac{x-2i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{3/2}}{(ix+1)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\
&= z_1 e^{\frac{\ln(x^2+1)}{2}} \\
&= z_1\left(\sqrt{x^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx$$
$$= y_1 \left(-\frac{x}{(x+i)^2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2 \left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)$$

Will add steps showing solving for IC soon.

1.287.2 Maple step by step solution

1.287.3 Maple trace

```
Methods for second order ODEs:
```

1.287.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 16

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2\, x^2 + c_1 x - c_2$$

1.287.5 Mathematica DSolve solution

Solving time : 0.056 (sec)
Leaf size : 21

```
DSolve[{(x^2+1)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x - c_1 (x - i)^2$$

1.288 problem 291

Internal problem ID [8426]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 291
Date solved : `Monday, October 21, 2024 at 05:07:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0$$

1.288.1 Solved as second order ode using Kovacic algorithm

Time used: 0.084 (sec)

Writing the ode as

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 548: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\
&= z_1 e^{x^2} \\
&= z_1 \left(\mathrm{e}^{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{x^2}\right) + c_2 \left(\mathrm{e}^{x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.288.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - 4xy' + (4x^2 - 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k- > k + 2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 - 2a_0 + (6a_3 - 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = a_0, a_3 = a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2k a_{k+2} - 2a_k + 5a_{k+2})}{k^2+7k+12}, a_2 = a_0, a_3 = a_1\right]$

1.288.3 Maple trace

```
Methods for second order ODEs:
```

1.288.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(4*x^2-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{x^2}(c_2 x + c_1)$$

1.288.5 Mathematica DSolve solution

Solving time : `0.031` (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x^2}(c_2 x + c_1)$$

1.289 problem 292

Internal problem ID [8427]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 292
Date solved : `Monday, October 21, 2024 at 05:07:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0$$

1.289.1 Solved as second order ode using Kovacic algorithm

Time used: 0.084 (sec)

Writing the ode as

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 550: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4x}{1} \, dx} \\ &= z_1 e^{x^2} \\ &= z_1 \left(\mathrm{e}^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x^2}}{(y_1)^2} \, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{x^2}\right) + c_2\left(\mathrm{e}^{x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.289.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - 4xy' + (4x^2 - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- > k + 2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 - 2a_0 + (6a_3 - 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0
  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$
- Solve for the dependent coefficient(s)
  $\{a_2 = a_0, a_3 = a_1\}$
- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2) a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$
- Shift index using $k- > k + 2$
  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE
  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2ka_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$

1.289.3 Maple trace

```
Methods for second order ODEs:
```

1.289.4 Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(4*x^2-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{x^2}(c_2 x + c_1)$$

1.289.5 Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x^2}(c_2 x + c_1)$$

1.290 problem 293

Internal problem ID [8428]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 293
Date solved : `Monday, October 21, 2024 at 05:07:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2x-3)\,y''-xy'+y=0$$

1.290.1 Solved as second order ode using Kovacic algorithm

Time used: 0.375 (sec)

Writing the ode as

$$(2x-3)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x-3 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 8x + 18}{4(2x-3)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 8x + 18 \\ t &= 4(2x-3)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 8x + 18}{4(2x-3)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 552: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x-3)^2$. There is a pole at $x = \frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{33}{64\left(x - \frac{3}{2}\right)^2} - \frac{5}{16\left(x - \frac{3}{2}\right)}$$

For the pole at $x = \frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{5}{8x} - \frac{11}{16x^2} - \frac{1}{32x^3} + \frac{245}{64x^4} + \frac{2591}{128x^5} + \frac{21117}{256x^6} + \frac{154743}{512x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 8x + 18}{16x^2 - 48x + 36} \\ &= Q + \frac{R}{16x^2 - 48x + 36} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-5x + \frac{63}{4}}{16x^2 - 48x + 36}\right) \\ &= \frac{1}{16} + \frac{-5x + \frac{63}{4}}{16x^2 - 48x + 36}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -5. Dividing this by leading coefficient in t which is 16 gives $-\frac{5}{16}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{16}\right) - (0) \\ &= -\frac{5}{16} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{16}}{\frac{1}{4}} - 0\right) &&= -\frac{5}{8} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{16}}{\frac{1}{4}} - 0\right) &&= \frac{5}{8} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 8x + 18}{4(2x-3)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{3}{2}$	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$-\frac{5}{8}$	$\frac{5}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{5}{8} - \left(-\frac{3}{8}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{8\left(x-\frac{3}{2}\right)} + (-)\left(\frac{1}{4}\right) \\
&= -\frac{3}{8\left(x-\frac{3}{2}\right)} - \frac{1}{4} \\
&= -\frac{x}{4x-6}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{8\left(x-\frac{3}{2}\right)} - \frac{1}{4}\right)(1) + \left(\left(\frac{3}{8\left(x-\frac{3}{2}\right)^2}\right) + \left(-\frac{3}{8\left(x-\frac{3}{2}\right)} - \frac{1}{4}\right)^2 - \left(\frac{x^2-8x+18}{4\left(2x-3\right)^2}\right) \right) = 0$$

$$\frac{a_0}{2x-3} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{3}{8\left(x-\frac{3}{2}\right)} - \frac{1}{4}\right) dx} \\
&= (x)\, \mathrm{e}^{-\frac{x}{4} - \frac{3\ln(2x-3)}{8}} \\
&= \frac{x\, \mathrm{e}^{-\frac{x}{4}}}{(2x-3)^{3/8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{2x-3}\, dx} \\
&= z_1 e^{\frac{x}{4}+\frac{3\ln(2x-3)}{8}} \\
&= z_1\left((2x-3)^{3/8}\, \mathrm{e}^{\frac{x}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{2x-3}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x}{2}+\frac{3\ln(2x-3)}{4}}}{(y_1)^2}\, dx \\
&= y_1\left(\int \frac{\mathrm{e}^{\frac{x}{2}+\frac{3\ln(2x-3)}{4}}}{x^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\int \frac{\mathrm{e}^{\frac{x}{2}+\frac{3\ln(2x-3)}{4}}}{x^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.290.2 Maple step by step solution

Let's solve

$(2x-3)\left(\frac{d}{dx}y'\right)-xy'+y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{y}{2x-3}+\frac{xy'}{2x-3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{xy'}{2x-3}+\frac{y}{2x-3}=0$

☐ Check to see if $x_0=\frac{3}{2}$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{x}{2x-3}, P_3(x)=\frac{1}{2x-3}\right]$

  ◦ $\left(x-\frac{3}{2}\right)\cdot P_2(x)$ is analytic at $x=\frac{3}{2}$

    $\left(\left(x-\frac{3}{2}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{3}{2}}=-\frac{3}{4}$

  ◦ $\left(x-\frac{3}{2}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{3}{2}$

    $\left(\left(x-\frac{3}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{3}{2}}=0$

  ◦ $x=\frac{3}{2}$is a regular singular point

  Check to see if $x_0=\frac{3}{2}$ is a regular singular point

  $x_0=\frac{3}{2}$

- Multiply by denominators

  $(2x-3)\left(\frac{d}{dx}y'\right)-xy'+y=0$

- Change variables using $x=u+\frac{3}{2}$ so that the regular singular point is at $u=0$

  $2u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\left(-u-\frac{3}{2}\right)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

$$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty}a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u \cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)$ to series expansion

$$u \cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -1}$$

- Shift index using $k->k+1$

$$u \cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1}^{\infty}a_{k +1} \left(k +1+r \right) \left(k +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$\frac{a_{0} r \left(-7+4 r \right) u^{-1+r}}{2}+\left(\sum_{k =0}^{\infty}\left(\frac{a_{k +1} \left(k +1+r \right) \left(4 k -3+4 r \right)}{2}-a_{k} \left(k +r -1\right)\right) u^{k +r}\right)=0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$\frac{r \left(-7+4 r \right)}{2}=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{7}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2 \left(k +1+r \right) \left(k -\frac{3}{4}+r \right) a_{k +1}-a_{k} \left(k +r -1\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k +1}=\frac{2 a_{k} \left(k +r -1\right)}{\left(k +1+r \right) \left(4 k -3+4 r \right)}$$

- Recursion relation for $r =0$; series terminates at $k =1$

$$a_{k +1}=\frac{2 a_{k} \left(k -1\right)}{\left(k +1\right) \left(4 k -3\right)}$$

- Apply recursion relation for $k =0$

$$a_{1}=\frac{2 a_{0}}{3}$$

- Terminating series solution of the ODE for $r =0$. Use reduction of order to find the second li

$$y \left(u \right)=a_{0}\cdot \left(1+\frac{2 u}{3}\right)$$

- Revert the change of variables $u =x -\frac{3}{2}$

$$\left[y =\frac{2 a_{0} x}{3}\right]$$

- Recursion relation for $r =\frac{7}{4}$

$$a_{k +1}=\frac{2 a_{k} \left(k +\frac{3}{4}\right)}{\left(k +\frac{11}{4}\right) \left(4 k +4\right)}$$

- Solution for $r =\frac{7}{4}$

$$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k +\frac{7}{4}}, a_{k +1}=\frac{2 a_{k} \left(k +\frac{3}{4}\right)}{\left(k +\frac{11}{4}\right) \left(4 k +4\right)}\right]$$

- Revert the change of variables $u = x - \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{3}{2}\right)^{k+\frac{7}{4}}, a_{k+1} = \frac{2a_k \left(k+\frac{3}{4}\right)}{\left(k+\frac{11}{4}\right)(4k+4)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \frac{2a_0 x}{3} + \left(\sum_{k=0}^{\infty} b_k \left(x - \frac{3}{2}\right)^{k+\frac{7}{4}}\right), b_{k+1} = \frac{2b_k \left(k+\frac{3}{4}\right)}{\left(k+\frac{11}{4}\right)(4k+4)}\right]$$

1.290.3 Maple trace

```
Methods for second order ODEs:
```

1.290.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 29

```
dsolve((2*x-3)*diff(diff(y(x),x),x)-x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = 2\left(x - \frac{3}{2}\right)(2x-3)^{3/4} c_1 \operatorname{KummerM}\left(\frac{3}{4}, \frac{11}{4}, \frac{x}{2} - \frac{3}{4}\right) + c_2 x$$

1.290.5 Mathematica DSolve solution

Solving time : `0.13` (sec)
Leaf size : 63

```
DSolve[{(2*x-3)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 2\ 2^{3/4}(2x-3)\left(c_2(2x-3)^{3/4} L_{-\frac{3}{4}}^{\frac{7}{4}}\left(\frac{x}{2} - \frac{3}{4}\right) + \frac{4\sqrt{2}c_1 x}{2x-3}\right)$$

1.291 problem 294

Internal problem ID [8429]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 294
Date solved : `Monday, October 21, 2024 at 05:07:44 PM`
CAS classification : `[_Hermite]`

Solve

$$y'' - xy' - 3y = 0$$

1.291.1 Solved as second order ode using Kovacic algorithm

Time used: 0.244 (sec)

Writing the ode as

$$y'' - xy' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 10$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 554: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{5}{2x} - \frac{25}{4x^3} + \frac{125}{4x^5} - \frac{3125}{16x^7} + \frac{21875}{16x^9} - \frac{328125}{32x^{11}} + \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} + \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} + \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{2}\right) - (0) \\ &= \frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	2	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{x}{2}\right)(2x + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{5}{2}\right)\right) = 0$$

$$-a_1 x - 2a_0 + 2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2 + 1\right) \mathrm{e}^{\int \frac{x}{2} dx} \\ &= \left(x^2 + 1\right) \mathrm{e}^{\frac{x^2}{4}} \\ &= \left(x^2 + 1\right) \mathrm{e}^{\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x^2}{2}} \left(x^2 + 1\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^2 + 1\right)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x^2}{2}} \left(x^2+1\right) \right) + c_2 \left(\mathrm{e}^{\frac{x^2}{2}} \left(x^2+1\right) \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^2+1\right)^2} dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.291.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(k+3)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} - a_k(k+3) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k(k+3)}{k^2+3k+2}\right]$

1.291.3 Maple trace

```
Methods for second order ODEs:
```

1.291.4 Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-3*y(x) = 0,
       y(x),singsol=all)
```

$$y = (x^2 + 1) \left(c_1 \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) \sqrt{\pi} + c_2 \right) \mathrm{e}^{\frac{x^2}{2}} + \sqrt{2}\, c_1 x$$

1.291.5 Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 35

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 \operatorname{HermiteH}\left(-3, \frac{x}{\sqrt{2}}\right) + c_2 e^{\frac{x^2}{2}} (x^2 + 1)$$

1.292 problem 295

Internal problem ID [8430]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 295
Date solved : `Monday, October 21, 2024 at 05:07:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right)y''-xy'+y=0$$

1.292.1 Solved as second order ode using Kovacic algorithm

Time used: 0.288 (sec)

Writing the ode as

$$\left(x^2+1\right)y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 6}{4(x^2 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 6$$
$$t = 4(x^2 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 6}{4(x^2 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 556: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16(x-i)^2} + \frac{5}{16(x+i)^2} + \frac{7i}{16(x-i)} - \frac{7i}{16(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 6}{4(x^2+1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2-6}{4\left(x^2+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(x - i\right)} - \frac{1}{4\left(x + i\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x - i\right)} - \frac{1}{4\left(x + i\right)} \\
&= -\frac{x}{2x^2 + 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(x - i\right)} - \frac{1}{4\left(x + i\right)}\right)(1) + \left(\left(\frac{1}{4\left(x - i\right)^2} + \frac{1}{4\left(x + i\right)^2}\right) + \left(-\frac{1}{4\left(x - i\right)} - \frac{1}{4\left(x + i\right)}\right)^2 - \left(\frac{-}{4(\cdots}\right.\right.
$$

$$
\frac{(x^2 + 1 \cdots}{(-x + i)^2 (\cdots}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\
&= (x) \frac{1}{\left(\left(-x + i\right)\left(x + i\right)\right)^{1/4}} \\
&= \frac{x}{\left(-x^2 - 1\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2+1}\,dx} \\
&= z_1 e^{\frac{\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\left(x^2+1\right)^{1/4}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(\frac{1}{2} - \frac{i}{2}\right) x\sqrt{2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(i\left(-\frac{\left(x^2+1\right)^{3/2}}{x} + x\sqrt{x^2+1} + \operatorname{arcsinh}(x)\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(\frac{1}{2} - \frac{i}{2}\right) x\sqrt{2}\right) \\
&\quad + c_2\left(\left(\frac{1}{2} - \frac{i}{2}\right) x\sqrt{2}\left(i\left(-\frac{\left(x^2+1\right)^{3/2}}{x} + x\sqrt{x^2+1} + \operatorname{arcsinh}(x)\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.292.2 Maple step by step solution

1.292.3 Maple trace

```
Methods for second order ODEs:
```

1.292.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 23

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = -\sqrt{x^2+1}\, c_2 + x(c_2 \operatorname{arcsinh}(x) + c_1)$$

1.292.5 Mathematica DSolve solution

Solving time : 0.071 (sec)
Leaf size : 39

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x \operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right) - c_2\sqrt{x^2+1} + c_1 x$$

1.293 problem 296

Internal problem ID [8431]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 296
Date solved : `Monday, October 21, 2024 at 05:07:46 PM`
CAS classification : `[_Hermite]`

Solve

$$y'' - xy' + 2y = 0$$

1.293.1 Solved as second order ode using Kovacic algorithm

Time used: 0.273 (sec)

Writing the ode as

$$y'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 10$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 557: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) = 0 \\ a_1 x + 2a_0 + 2 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{-\frac{x^2}{4}} \\
&= \left(x^2 - 1\right) \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2 - 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x^2 - 1)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x^2 - 1) + c_2 \left(x^2 - 1 \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2 - 1)^2} dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.293.2 Maple step by step solution

Let's solve

$\frac{d}{dx} y' - x y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$
  - Convert $\frac{d}{dx} y'$ to series expansion

    $\frac{d}{dx} y' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$
  - Shift index using $k- >k+2$

    $\frac{d}{dx} y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k(k-2) \right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - a_k(k-2) = 0$
- Recursion relation; series terminates at $k = 2$

  $a_{k+2} = \frac{a_k(k-2)}{k^2+3k+2}$
- Apply recursion relation for $k = 0$

$a_2 = -a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

  $y = A_2x^2 + A_1x - a_0$

1.293.3 Maple trace

```
Methods for second order ODEs:
```

1.293.4 Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 39

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -2\,e^{\frac{x^2}{2}} c_1 x + (x-1)(x+1)\left(\operatorname{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{\pi}\,\sqrt{2}\,c_1 + c_2\right)$$

1.293.5 Mathematica DSolve solution

Solving time : `0.142` (sec)
Leaf size : 54

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}c_2\left(\sqrt{2\pi}(x^2-1)\operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) - 2e^{\frac{x^2}{2}}x\right) + c_1(x^2-1)$$

1.294 problem 297

Internal problem ID [8432]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 297
Date solved : `Monday, October 21, 2024 at 05:07:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(-x^2+1\right) y''-y'+y=0$$

1.294.1 Solved as second order ode using Kovacic algorithm

Time used: 0.712 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x - 3}{4\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 - 4x - 3 \\ t &= 4\left(x^2 - 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x - 3}{4\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 559: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{16(x+1)} + \frac{7}{16(x-1)} + \frac{5}{16(x+1)^2} - \frac{3}{16(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\ &= \{1, 2, 3\}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\ &= \{-1, 2, 5\}\end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4x^2 - 4x - 3}{4(x^2-1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 1$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1, 2, 3\}$
-1	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = -1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (-1))) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2} \sum_{c \in \Gamma} \frac{e_c}{x - c} \\ &= \frac{1}{2}\left(\frac{1}{(x - (1))} + \frac{-1}{(x - (-1))}\right) \\ &= \frac{1}{2x - 2} - \frac{1}{2(x+1)} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + (3\theta^2 + 3\theta' - 4r)\, p' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')\, p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{4a_0 - 6}{(x+1)^2 (x-1)} = 0$$

And solving for p gives

$$p = x + \frac{3}{2}$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{x+\frac{3}{2}} + \frac{1}{2x-2} - \frac{1}{2(x+1)}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{3}{2}} + \frac{1}{2x-2} - \frac{1}{2(x+1)}\right) w + \frac{-8x^3 - 4x^2 + 10x + 7}{4(x^2-1)^2 (2x+3)} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{5}\,\sqrt{(x-1)(x+1)}\,x + 2\sqrt{5}\,\sqrt{(x-1)(x+1)} + 2x^2 + 2x + 1}{2(2x+3)(x-1)(x+1)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \,dx}\\ &= e^{\int \frac{2\sqrt{5}\,\sqrt{(x-1)(x+1)}\,x + 2\sqrt{5}\,\sqrt{(x-1)(x+1)} + 2x^2 + 2x + 1}{2(2x+3)(x-1)(x+1)} dx}\\ &= \frac{(x-1)^{1/4}\sqrt{2x+3}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} 5^{1/4}}{(x+1)^{1/4}\sqrt{\frac{5\sqrt{x^2-1}+(2+3x)\sqrt{5}}{\sqrt{x^2-1}\sqrt{-\frac{(2x+3)^2}{x^2-1}}}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{-x^2+1}\,dx} \\
&= z_1 e^{\frac{\operatorname{arctanh}(x)}{2}} \\
&= z_1 \left(\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x+3}\,(5x-5)^{1/4}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{2x+3}}\,(x+1)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\operatorname{arctanh}(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{i\sqrt{x+1}\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{(2x+3)^2\sqrt{5x-5}}dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x+3}\,(5x-5)^{1/4}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{2x+3}}\,(x+1)^{1/4}}\right) \\
&\quad + c_2 \left(\frac{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x+3}\,(5x-5)^{1/4}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{2x+3}}\,(x+1)^{1/4}}\left(\int \frac{i\sqrt{x+1}\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}\left(3\sqrt{5}\,x+5\sqrt{x^{2}\right.}}{(2x+3)^2\sqrt{5x-5}}\right.\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.294.2 Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d}{dx}y'\right) - y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{y}{x^2-1} - \frac{y'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{x^2-1} - \frac{y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x^2-1}, P_3(x) = -\frac{1}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = -\frac{1}{2}$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

$(x^2-1)\left(\frac{d}{dx}y'\right)+y'-y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\frac{d}{du}y(u)-y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $\frac{d}{du}y(u)$ to series expansion

$\frac{d}{du}y(u)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$

  - Shift index using $k->k+1$

$\frac{d}{du}y(u)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

$u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

$u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-a_0r(-3+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(2k-1+2r)+a_k(k^2+2kr+r^2-k-r-1)\right)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(-3+2r)=0$

- Values of r that satisfy the indicial equation

$r\in\left\{0,\frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$-2(k+1+r)\left(k-\frac{1}{2}+r\right)a_{k+1}+a_k(k^2+(2r-1)\,k+r^2-r-1)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{a_k(k^2+2kr+r^2-k-r-1)}{(k+1+r)(2k-1+2r)}$

- Recursion relation for $r=0$

$a_{k+1}=\frac{a_k(k^2-k-1)}{(k+1)(2k-1)}$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k(k^2-k-1)}{(k+1)(2k-1)}\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1}=\frac{a_k(k^2-k-1)}{(k+1)(2k-1)}\right]$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+1}=\frac{a_k(k^2+2k-\frac{1}{4})}{(k+\frac{5}{2})(2k+2)}$$

- Solution for $r=\frac{3}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1}=\frac{a_k(k^2+2k-\frac{1}{4})}{(k+\frac{5}{2})(2k+2)}\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{3}{2}}, a_{k+1}=\frac{a_k(k^2+2k-\frac{1}{4})}{(k+\frac{5}{2})(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k(x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x+1)^{k+\frac{3}{2}}\right), a_{k+1}=\frac{a_k(k^2-k-1)}{(k+1)(2k-1)}, b_{k+1}=\frac{b_k(k^2+2k-\frac{1}{4})}{(k+\frac{5}{2})(2k+2)}\right]$$

1.294.3 Maple trace

```
Methods for second order ODEs:
```

1.294.4 Maple dsolve solution

Solving time : `0.026` (sec)
Leaf size : 66

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y=c_1 \operatorname{hypergeom}\left(\left[-\frac{1}{2}-\frac{\sqrt{5}}{2},-\frac{1}{2}+\frac{\sqrt{5}}{2}\right],\left[-\frac{1}{2}\right],\frac{x}{2}+\frac{1}{2}\right)+2c_2\sqrt{2x+2}\operatorname{hypergeom}\left(\left[1+\frac{\sqrt{5}}{2},1-\frac{\sqrt{5}}{2}\right],\left[\frac{5}{2}\right],\frac{x}{2}+\frac{1}{2}\right)(x+1)$$

1.294.5 Mathematica DSolve solution

Solving time : `131.567` (sec)
Leaf size : 195

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \left(\sqrt{x-1}-\sqrt{x+1}\right)^{-\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(\sqrt{x-1}+\sqrt{x+1}\right)^{\frac{1}{2}\left(\sqrt{5}-1\right)}\left(\sqrt{x-1}-\sqrt{5}\sqrt{x+1}\right)\left(c_2\int_1^x -\frac{2\sqrt{K[1]+1}\left(\sqrt{K[1]-1}-\sqrt{K[1]+1}\right)^{\sqrt{5}}\left(\sqrt{K[1]-1}+\sqrt{K[1]+1}\right)^{-\sqrt{5}}}{\sqrt{1-K[1]}\left(\sqrt{K[1]-1}-\sqrt{5}\sqrt{K[1]+1}\right)^2}dK[1]+c_1\right)$$

1.295 problem 298

Internal problem ID [8433]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 298
Date solved : `Monday, October 21, 2024 at 05:07:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(x+1)^2 y'' + \left(-x^2+1\right) y' + (x-1) y = 0$$

1.295.1 Solved as second order ode using Kovacic algorithm

Time used: 0.200 (sec)

Writing the ode as

$$x(x+1)^2 y'' + \left(-x^2+1\right) y' + (x-1) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x(x+1)^2 \\ B &= -x^2+1 \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 561: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x^2+1}{x(x+1)^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2} + \ln(x+1)} \\ &= z_1 \left(\frac{x+1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x + 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+1}{x(x+1)^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)+2\ln(x+1)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x+1) + c_2(x+1(\ln(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.295.2 Maple step by step solution

Let's solve

$x(x+1)^2 \left(\frac{d}{dx}y'\right) + (-x^2+1)\, y' + (x-1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x-1)y}{x(x+1)^2} + \frac{y'(x-1)}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'(x-1)}{x(x+1)} + \frac{(x-1)y}{x(x+1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x-1}{(x+1)x}, P_3(x) = \frac{x-1}{x(x+1)^2}\right]$

  ◦ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left.((x+1) \cdot P_2(x))\right|_{x=-1} = -2$

  ◦ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left.((x+1)^2 \cdot P_3(x))\right|_{x=-1} = 2$

- $x = -1$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0 = -1$
- Multiply by denominators
  $x(x+1)^2 \left(\frac{d}{dx}y'\right) - (x+1)(x-1)y' + (x-1)y = 0$
- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$
  $(u^3 - u^2)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u^2 + 2u)\left(\frac{d}{du}y(u)\right) + (u-2)y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$
    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k- > k - m$
    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k- > k + 1 - m$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 2..3$
    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k- > k + 2 - m$
    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions
  $-a_0(-1+r)(-2+r)u^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-1)(k+r-2) + a_{k-1}(k+r-2)^2\right)u^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(-1+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation

$r \in \{1, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r-1)(k+r-2)+a_{k-1}(k+r-2)^2=0$
- Shift index using $k->k+1$

  $-a_{k+1}(k+r)(k+r-1)+a_k(k+r-1)^2=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-1)}{k+r}$
- Recursion relation for $r=1$

  $a_{k+1}=\frac{a_k k}{k+1}$
- Solution for $r=1$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+1}=\frac{a_k k}{k+1}\right]$
- Revert the change of variables $u=x+1$

  $\left[y=\sum_{k=0}^{\infty} a_k(x+1)^{k+1}, a_{k+1}=\frac{a_k k}{k+1}\right]$
- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k(k+1)}{k+2}$
- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k(k+1)}{k+2}\right]$
- Revert the change of variables $u=x+1$

  $\left[y=\sum_{k=0}^{\infty} a_k(x+1)^{k+2}, a_{k+1}=\frac{a_k(k+1)}{k+2}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k(x+1)^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k(x+1)^{k+2}\right), a_{k+1}=\frac{a_k k}{k+1}, b_{k+1}=\frac{b_k(k+1)}{k+2}\right]$

1.295.3 Maple trace

```
Methods for second order ODEs:
```

1.295.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(x*(x+1)^2*diff(diff(y(x),x),x)+(-x^2+1)*diff(y(x),x)+(x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = (x + 1) (c_2 \ln (x) + c_1)$$

1.295.5 Mathematica DSolve solution

Solving time : 0.048 (sec)
Leaf size : 17

```
DSolve[{x*(x+1)^2*D[y[x],{x,2}]+(1-x^2)*D[y[x],x]+(x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow (x + 1)(c_2 \log(x) + c_1)$$

1.296 problem 299

Internal problem ID [8434]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 299
Date solved : `Monday, October 21, 2024 at 05:07:49 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$2xy'' - y' + 2y = 0$$

1.296.1 Solved as second order ode using Kovacic algorithm

Time used: 0.248 (sec)

Writing the ode as

$$2xy'' - y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5 - 16x$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 563: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 1 \\
&= 1
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{-1, 2, 5\}
\end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right) \\ &= -\frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = x + \frac{1}{4}$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x+\frac{1}{4}} - \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{1}{4}} - \frac{1}{2x}\right) w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4\,(4x+1)\,x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \,dx} \\ &= e^{\int \frac{16x\sqrt{-x}+4x-1}{4(4x+1)x} dx} \\ &= \frac{\left(2\sqrt{-x}-1\right) e^{2\sqrt{-x}}}{(-x)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{2x}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{4}} \\ &= z_1\left(x^{1/4}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}\left(2\sqrt{-x}-1\right) e^{2\sqrt{-x}}}{(-x)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{2x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{e^{-4\sqrt{-x}}}{8} + \frac{e^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/4} \left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}} \right) + c_2 \left(\frac{x^{1/4} \left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}} \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.296.2 Maple step by step solution

Let's solve

$2x\left(\frac{d}{dx}y'\right) - y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x} + \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'}{2x} + \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x\left(\frac{d}{dx}y'\right) - y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k-\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 2a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k}{(2k-1+2r)(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = -\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}, b_{k+1} = -\frac{2b_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

1.296.3 Maple trace

```
Methods for second order ODEs:
```

1.296.4 Maple dsolve solution

Solving time : `0.015` (sec)
Leaf size : 36

```
dsolve(2*x*diff(diff(y(x),x),x)-diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \left(2c_1\sqrt{x} + c_2\right)\cos\left(2\sqrt{x}\right) - \sin\left(2\sqrt{x}\right)\left(-2c_2\sqrt{x} + c_1\right)$$

1.296.5 Mathematica DSolve solution

Solving time : `0.129` (sec)
Leaf size : 59

```
DSolve[{2*x*D[y[x],{x,2}]-D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{2i\sqrt{x}}\left(2\sqrt{x} + i\right) + \frac{1}{8}c_2 e^{-2i\sqrt{x}}\left(1 + 2i\sqrt{x}\right)$$

1.297 problem 300

Internal problem ID [8435]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 300
Date solved : `Monday, October 21, 2024 at 05:07:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + xy' - 2y = 0$$

1.297.1 Solved as second order ode using Kovacic algorithm

Time used: 0.264 (sec)

Writing the ode as

$$xy'' + xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x+8}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$s = x + 8$$
$$t = 4x$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x+8}{4x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 565: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^{+} &= 1 \\ \alpha_c^{-} &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x} - \frac{4}{x^2} + \frac{16}{x^3} - \frac{80}{x^4} + \frac{448}{x^5} - \frac{2688}{x^6} + \frac{16896}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned}
\tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x+8}{4x} \\
&= Q + \frac{R}{4x} \\
&= \left(\frac{1}{4}\right) + \left(\frac{2}{x}\right) \\
&= \frac{1}{4} + \frac{2}{x}
\end{aligned}
$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 8. Dividing this by leading coefficient in t which is 4 gives 2. Now b can be found.

$$
\begin{aligned}
b &= (2) - (0) \\
&= 2
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 0\right) &&= 2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 0\right) &&= -2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x+8}{4x}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	2	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{x} + \frac{1}{2} \\
&= \frac{1}{x} + \frac{1}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} + \frac{1}{2}\right)^2 - \left(\frac{x+8}{4x}\right)\right) &= 0 \\
\frac{2 - a_0}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 2 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2 + x) \, e^{\int \left(\frac{1}{x} + \frac{1}{2}\right) dx} \\
&= (2 + x) \, e^{\frac{x}{2} + \ln(x)} \\
&= (2 + x) \, x \, e^{\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (2+x)\,x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{-x}}{4x} + \frac{\mathrm{Ei}_1(x)}{2} + \frac{\mathrm{e}^{-x}}{-8-4x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1((2+x)\,x) + c_2\left((2+x)\,x\left(-\frac{\mathrm{e}^{-x}}{4x} + \frac{\mathrm{Ei}_1(x)}{2} + \frac{\mathrm{e}^{-x}}{-8-4x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.297.2 Maple step by step solution

1.297.3 Maple trace

```
Methods for second order ODEs:
```

1.297.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 28

```
dsolve(x*diff(diff(y(x),x),x)+x*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{c_2(x+1)\,\mathrm{e}^{-x}}{2} + \left(c_1 + \frac{c_2\,\mathrm{Ei}_1(x)}{2}\right)x(2+x)$$

1.297.5 Mathematica DSolve solution

Solving time : 0.088 (sec)
Leaf size : 39

```
DSolve[{x*D[y[x],{x,2}]+x*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1x(x+2) - \frac{1}{2}c_2e^{-x}(e^x(x+2)x\,\mathrm{ExpIntegralEi}(-x)+x+1)$$

1.298 problem 301

Internal problem ID [8436]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 301
Date solved : `Monday, October 21, 2024 at 05:07:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(x-1)^2 y'' - 2y = 0$$

1.298.1 Solved as second order ode using Kovacic algorithm

Time used: 0.191 (sec)

Writing the ode as

$$x(x-1)^2 y'' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x(x-1)^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 566: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 0 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x-1)^2} - \frac{2}{x-1} + \frac{2}{x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x\left(x-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{1}{x-1} + (0) \\
&= \frac{1}{x} - \frac{1}{x-1} \\
&= -\frac{1}{x\left(x-1\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{x-1}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{(x-1)^2}\right) + \left(\frac{1}{x} - \frac{1}{x-1}\right)^2 - \left(\frac{2}{x(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{x} - \frac{1}{x-1}\right) dx} \\ &= \frac{x}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{x}{x-1} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{x-1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \frac{x}{x-1} \int \frac{1}{\frac{x^2}{(x-1)^2}}\, dx \\
&= \frac{x}{x-1}\left(x - 2\ln(x) - \frac{1}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{x-1}\right) + c_2\left(\frac{x}{x-1}\left(x - 2\ln(x) - \frac{1}{x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.298.2 Maple step by step solution

Let's solve

$x(x-1)^2\left(\frac{d}{dx}y'\right) - 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{2y}{x(x-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y}{x(x-1)^2} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{2}{x(x-1)^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0 = 0$
- Multiply by denominators
  $x(x-1)^2 \left(\frac{d}{dx}y'\right) - 2y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..3$
    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k- >k+2-m$
    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions
  $a_0 r(-1+r)\, x^{-1+r} + \left(a_1(1+r)\, r - 2a_0(r^2-r+1)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)(k+r) - 2a_k(k^2\right.\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 1\}$
- Each term must be 0
  $a_1(1+r)\, r - 2a_0(r^2-r+1) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(-2a_k + a_{k-1} + a_{k+1})\, k^2 + \left((-4a_k + 2a_{k-1} + 2a_{k+1})\, r + 2a_k - 3a_{k-1} + a_{k+1}\right) k + (-2a_k + a_{k-1}$
- Shift index using $k- >k+1$
  $(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + \left((-4a_{k+1} + 2a_k + 2a_{k+2})\, r + 2a_{k+1} - 3a_k + a_{k+2}\right)(k+1) + (-$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kra_k - 4kra_{k+1} + r^2 a_k - 2r^2 a_{k+1} - ka_k - 2ka_{k+1} - ra_k - 2ra_{k+1} - 2a_{k+1}}{k^2 + 2kr + r^2 + 3k + 3r + 2}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - ka_k - 2ka_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}$
- Solution for $r = 0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{k^2 a_k-2k^2 a_{k+1}-k a_k-2k a_{k+1}-2a_{k+1}}{k^2+3k+2}, -2a_0=0\right]$$

- Recursion relation for $r=1$

  $a_{k+2}=-\frac{k^2 a_k-2k^2 a_{k+1}+k a_k-6k a_{k+1}-6a_{k+1}}{k^2+5k+6}$

- Solution for $r=1$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{k^2 a_k-2k^2 a_{k+1}+k a_k-6k a_{k+1}-6a_{k+1}}{k^2+5k+6}, 2a_1-2a_0=0\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2}=-\frac{k^2 a_k-2k^2 a_{k+1}-k a_k-2k a_{k+1}-2a_{k+1}}{k^2+3k+2}, -2a_0=0, b_{k+2}=-\frac{k^2 b\right.$$

1.298.3 Maple trace

```
Methods for second order ODEs:
```

1.298.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 27

```
dsolve(x*(x-1)^2*diff(diff(y(x),x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{2\ln(x)\, c_2 x-c_2\, x^2+c_1 x+c_2}{x-1}$$

1.298.5 Mathematica DSolve solution

Solving time : `0.055` (sec)
Leaf size : 33

```
DSolve[{x*(x-1)^2*D[y[x],{x,2}]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{-c_2x^2-c_1x+2c_2x\log(x)+c_2}{x-1}$$

1.299 problem 302

Internal problem ID [8437]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 302
Date solved : `Monday, October 21, 2024 at 05:07:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 2xy' + x^2 y = 0$$

1.299.1 Solved as second order ode using Kovacic algorithm

Time used: 0.178 (sec)

Writing the ode as

$$y'' - 2xy' + x^2 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x \\ C &= x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 568: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1}\,dx} \\ &= z_1 e^{\frac{x^2}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x^2}{2}} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x^2}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{x^2}{2}} \cos (x)\right) + c_2\left(\mathrm{e}^{\frac{x^2}{2}} \cos (x) (\tan (x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.299.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - 2xy' + x^2 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - Convert $x^2 \cdot y$ to series expansion

    $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$

  - Shift index using $k- >k-2$

    $x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + (6a_3 - 2a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) - 2a_k k + a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 = 0, 6a_3 - 2a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = 0, a_3 = \frac{a_1}{3}\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - 2a_k k + a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 2a_{k+2}(k+2) + a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2k a_{k+2} - a_k + 4a_{k+2}}{k^2+7k+12}, a_2 = 0, a_3 = \frac{a_1}{3}\right]$

1.299.3 Maple trace

```
Methods for second order ODEs:
```

1.299.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 20

```
dsolve(diff(diff(y(x),x),x)-2*x*diff(y(x),x)+x^2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{\frac{x^2}{2}} (c_1 \cos(x) + c_2 \sin(x))$$

1.299.5 Mathematica DSolve solution

Solving time : `0.049` (sec)
Leaf size : 39

```
DSolve[{D[y[x],{x,2}]-2*x*D[y[x],x]+x^2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{\frac{1}{2}x(x-2i)} \left(2c_1 - ic_2 e^{2ix}\right)$$

1.300 problem 303

Internal problem ID [8438]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 303
Date solved : `Monday, October 21, 2024 at 05:07:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x\left(-x^2+2\right)y''-\left(x^2+4x+2\right)\left((1-x)\,y'+y\right)=0$$

1.300.1 Solved as second order ode using Kovacic algorithm

Time used: 0.550 (sec)

Writing the ode as

$$\left(-x^3+2x\right)y''+\left(x^3+3x^2-2x-2\right)y'+\left(-x^2-4x-2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^3+2x\\B&=x^3+3x^2-2x-2\\C&=-x^2-4x-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4\left(x^3 - 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12$$
$$t = 4\left(x^3 - 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4\left(x^3 - 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 570: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\left(x - \sqrt{2}\right)^2} + \frac{3}{4\left(x + \sqrt{2}\right)^2} + \frac{-\frac{5\sqrt{2}}{8} - \frac{1}{2}}{x - \sqrt{2}} + \frac{\frac{5\sqrt{2}}{8} - \frac{1}{2}}{x + \sqrt{2}} + \frac{3}{2x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - \sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{2} + \frac{1}{2x} - \frac{1}{2x^2} - \frac{3}{2x^3} + \frac{21}{4x^4} - \frac{43}{4x^5} + \frac{135}{4x^6} - \frac{147}{4x^7} + \dots \qquad (9)
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4x^6 - 16x^4 + 16x^2} \\
&= Q + \frac{R}{4x^6 - 16x^4 + 16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{2x^5 - x^4 - 16x^3 + 20x^2 + 24x + 12}{4x^6 - 16x^4 + 16x^2}\right) \\
&= \frac{1}{4} + \frac{2x^5 - x^4 - 16x^3 + 20x^2 + 24x + 12}{4x^6 - 16x^4 + 16x^2}
\end{aligned}$$

Since the degree of t is 6, then we see that the coefficient of the term x^5 in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{1}{2}\right) - (0) \\
&= \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4\left(x^3 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \left(\frac{1}{2}\right) \\ &= \frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2} \\ &= \frac{x^3 + x^2 - 2x - 6}{2x^3 - 4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{3}{2x}-\frac{1}{2\left(x-\sqrt{2}\right)}-\frac{1}{2\left(x+\sqrt{2}\right)}+\frac{1}{2}\right)(0)+\left(\left(-\frac{3}{2x^{2}}+\frac{1}{2\left(x-\sqrt{2}\right)^{2}}+\frac{1}{2\left(x+\sqrt{2}\right)^{2}}\right)+\left(\frac{3}{2x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{2x}-\frac{1}{2\left(x-\sqrt{2}\right)}-\frac{1}{2\left(x+\sqrt{2}\right)}+\frac{1}{2}\right) dx} \\
&= \frac{x^{3/2}\mathrm{e}^{\frac{x}{2}}}{\sqrt{x+\sqrt{2}}\,\sqrt{x-\sqrt{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3+3x^2-2x-2}{-x^3+2x} \, dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{2}+\frac{\ln\left(x^2-2\right)}{2}} \\
&= z_1\left(\sqrt{x}\,\sqrt{x^2-2}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2 \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^3+3x^2-2x-2}{-x^3+2x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+\ln(x)+\ln(x^2-2)}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{(x-1)\,\mathrm{e}^{x+\ln(x)+\ln(x^2-2)}\,\mathrm{e}^{-2x}}{x^3\,(x^2-2)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^2 \mathrm{e}^x\right) + c_2 \left(x^2 \mathrm{e}^x \left(-\frac{(x-1)\,\mathrm{e}^{x+\ln(x)+\ln(x^2-2)}\,\mathrm{e}^{-2x}}{x^3\,(x^2-2)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.300.2 Maple step by step solution

Let's solve

$x(-x^2+2)\left(\frac{d}{dx}y'\right) - (x^2+4x+2)\left((1-x)\,y' + y\right) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+4x+2)y}{x(x^2-2)} + \frac{(x^2+4x+2)(x-1)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x^2+4x+2)(x-1)y'}{x(x^2-2)} + \frac{(x^2+4x+2)y}{x(x^2-2)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{(x^2+4x+2)(x-1)}{x(x^2-2)}, P_3(x) = \frac{x^2+4x+2}{x(x^2-2)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x^2-2)\left(\frac{d}{dx}y'\right) - (x^2+4x+2)(x-1)\,y' + (x^2+4x+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- > k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k- > k + 2 - m$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

    Rewrite ODE with series expansions

    $-2a_0 r(-2+r)\,x^{-1+r} + \left(-2a_1(1+r)(-1+r) + 2a_0(1+r)\right)x^r + \left(-2a_2(2+r)\,r + 2a_1(2+r) + \right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 2\}$

- The coefficients of each power of x must be 0

  $\left[-2a_1(1+r)(-1+r)+2a_0(1+r)=0, -2a_2(2+r)r+2a_1(2+r)+a_0(-2+r)^2=0\right]$

- Solve for the dependent coefficient(s)

  $\left\{a_1=\frac{a_0}{-1+r}, a_2=\frac{a_0(r^2-5r+10)}{2(r^2+r-2)}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k-1}(k-3+r)^2-2a_{k+1}(k+r+1)(k+r-1)+(2a_k-a_{k-2})k+(2a_k-a_{k-2})r+2a_k+3a_{k-}$

- Shift index using $k->k+2$

  $a_{k+1}(k+r-1)^2-2a_{k+3}(k+3+r)(k+r+1)+(2a_{k+2}-a_k)(k+2)+(2a_{k+2}-a_k)r+2a_{k+2}$

- Recursion relation that defines series solution to ODE

  $a_{k+3}=\frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}-a_kr-2ra_{k+1}+2ra_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3+r)(k+r+1)}$

- Recursion relation for $r=0$

  $a_{k+3}=\frac{k^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty}a_kx^k, a_{k+3}=\frac{k^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}, a_1=-a_0, a_2=-\frac{5a_0}{2}\right]$

- Recursion relation for $r=2$

  $a_{k+3}=\frac{k^2a_{k+1}-a_kk+2ka_{k+1}+2ka_{k+2}-a_k+a_{k+1}+10a_{k+2}}{2(k+5)(k+3)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+2}, a_{k+3}=\frac{k^2a_{k+1}-a_kk+2ka_{k+1}+2ka_{k+2}-a_k+a_{k+1}+10a_{k+2}}{2(k+5)(k+3)}, a_1=a_0, a_2=\frac{a_0}{2}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+2}\right), a_{k+3}=\frac{k^2a_{k+1}-ka_k-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}, a_1=-a_0, a_2=\right.$

1.300.3 Maple trace

```
Methods for second order ODEs:
```

1.300.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 17

```
dsolve(x*(-x^2+2)*diff(diff(y(x),x),x)-(x^2+4*x+2)*((1-x)*diff(y(x),x)+y(x)) = 0,
       y(x),singsol=all)
```

$$y = c_1(x - 1) + c_2\, x^2 \mathrm{e}^{x}$$

1.300.5 Mathematica DSolve solution

Solving time : 0.312 (sec)
Leaf size : 21

```
DSolve[{x*(2-x^2)*D[y[x],{x,2}]-(x^2+4*x+2)*((1-x)*D[y[x],x]+y[x])==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x x^2 + c_2(x - 1)$$

1.301 problem 304

Internal problem ID [8439]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 304
Date solved : `Monday, October 21, 2024 at 05:07:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y'' - (1+2x)\,(xy' - y) = 0$$

1.301.1 Solved as second order ode using Kovacic algorithm

Time used: 0.294 (sec)

Writing the ode as

$$x^2(1+x)\,y'' + \left(-2x^2 - x\right)y' + (1+2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= -2x^2 - x \\ C &= 1+2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x - 1}{4(x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4x - 1$$
$$t = 4(x^2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x - 1}{4(x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 572: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 1 \\
&= 3
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2} + \frac{1}{2 + 2x} - \frac{1}{2x} + \frac{3}{4(1 + x)^2}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $3 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4x-1}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 0 - (0) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(1+x\right)} + \frac{1}{2x} + (0) \\ &= -\frac{1}{2\left(1+x\right)} + \frac{1}{2x} \\ &= \frac{1}{2x\left(1+x\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(1+x)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{2(1+x)} + \frac{1}{2x}\right)^2 - \left(\frac{-4x-1}{4(x^2+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(1+x)} + \frac{1}{2x}\right) dx} \\ &= \frac{\sqrt{x}}{\sqrt{1+x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-x}{x^2(1+x)}\, dx} \\ &= z_1 e^{\frac{\ln(x(1+x))}{2}} \\ &= z_1 \left(\sqrt{x(1+x)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x(1+x)}\,\sqrt{x}}{\sqrt{1+x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2x^2-x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{\ln(x(1+x))}}{(y_1)^2}\,dx$$
$$= y_1(x + \ln(x))$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{\sqrt{x(1+x)}\,\sqrt{x}}{\sqrt{1+x}}\right) + c_2\left(\frac{\sqrt{x(1+x)}\,\sqrt{x}}{\sqrt{1+x}}(x+\ln(x))\right)$$

Will add steps showing solving for IC soon.

1.301.2 Maple step by step solution

Let's solve
$x^2(1+x)\left(\frac{d}{dx}y'\right) - (1+2x)(xy'-y) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{(1+2x)y}{x^2(1+x)} + \frac{(1+2x)y'}{x(1+x)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{(1+2x)y'}{x(1+x)} + \frac{(1+2x)y}{x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{1+2x}{x(1+x)}, P_3(x) = \frac{1+2x}{x^2(1+x)}\right]$
  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$
    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = -1$
  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x = -1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -1$

- Multiply by denominators

$x^2(1+x)\left(\frac{d}{dx}y'\right) - x(1+2x)\,y' + (1+2x)\,y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

$(u^3 - 2u^2 + u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-2u^2 + 3u - 1)\left(\frac{d}{du}y(u)\right) + (-1 + 2u)\,y(u) = 0$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\,u^{-1+r} + \left(a_1(1+r)\,(-1+r) - a_0(2r^2 - 5r + 1)\right)u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r -\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term must be 0
  $a_1(1+r)(-1+r) - a_0(2r^2 - 5r + 1) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(-2a_k + a_{k-1} + a_{k+1})k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1})r + 5a_k - 5a_{k-1})k + (-2a_k + a_{k-1} + a_{k+1})$
- Shift index using $k- >k+1$
  $(-2a_{k+1} + a_k + a_{k+2})(k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2})r + 5a_{k+1} - 5a_k)(k+1) + (-2a_{k+1} +$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2kra_k - 4kra_{k+1} + r^2a_k - 2r^2a_{k+1} - 3ka_k + ka_{k+1} - 3ra_k + ra_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2kr + r^2 + 2k + 2r}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 3ka_k + ka_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 3ka_k + ka_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2k}$
- Recursion relation for $r = 2$
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2 + 6k + 8}$
- Solution for $r = 2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2 + 6k + 8}, 3a_1 + a_0 = 0\right]$
- Revert the change of variables $u = 1 + x$
  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+2}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2 + 6k + 8}, 3a_1 + a_0 = 0\right]$

1.301.3 Maple trace

```
Methods for second order ODEs:
```

1.301.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 15

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)-(1+2*x)*(x*diff(y(x),x)-y(x)) = 0,
       y(x),singsol=all)
```

$$y = x(c_2 \ln(x) + c_2 x + c_1)$$

1.301.5 Mathematica DSolve solution

Solving time : 0.28 (sec)
Leaf size : 132

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-(1+2*x)*(x*D[y[x],x]+y[x])==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x^{1+\sqrt{2}}\,\text{Hypergeometric2F1}\left(-\frac{1}{2}+\sqrt{2}-\frac{\sqrt{17}}{2},-\frac{1}{2}+\sqrt{2}+\frac{\sqrt{17}}{2},1+2\sqrt{2},-x\right) + c_1 x^{1-\sqrt{2}}\,\text{Hypergeometric2F1}\left(\frac{1}{2}\left(-1-2\sqrt{2}-\sqrt{17}\right),\frac{1}{2}\left(-1-2\sqrt{2}+\sqrt{17}\right),1-2\sqrt{2},-x\right)$$

1.302 problem 305

Internal problem ID [8440]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 305
Date solved : `Monday, October 21, 2024 at 05:07:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2(2-x)\,x^2y''-(4-x)\,xy'+(3-x)\,y=0$$

1.302.1 Solved as second order ode using Kovacic algorithm

Time used: 0.187 (sec)

Writing the ode as

$$\left(-2x^3+4x^2\right)y''+\left(x^2-4x\right)y'+(3-x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+4x^2 \\ B &= x^2-4x \\ C &= 3-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{16(-2+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = 16(-2+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{16(-2+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 574: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(-2+x)^2$. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(-2+x)^2}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(-2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16(-2+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{16\left(-2+x\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
2	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{-8+4x} + (-)\left(0\right) \\ &= \frac{1}{-8+4x} \\ &= \frac{1}{-8+4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{-8+4x}\right)(0) + \left(\left(-\frac{1}{4\left(-2+x\right)^2}\right) + \left(\frac{1}{-8+4x}\right)^2 - \left(-\frac{3}{16\left(-2+x\right)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{-8+4x} dx} \\
&= (-2+x)^{1/4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2-4x}{-2x^3+4x^2}\, dx} \\
&= z_1 e^{\frac{\ln(x)}{2} - \frac{\ln(-2+x)}{4}} \\
&= z_1 \left(\frac{\sqrt{x}}{(-2+x)^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{x^2-4x}{-2x^3+4x^2}\, dx}}{(y_1)^2}\, dx$$
$$= y_1 \int \frac{e^{\ln(x)-\frac{\ln(-2+x)}{2}}}{(y_1)^2}\, dx$$
$$= y_1 \left(\frac{2\,\mathrm{e}^{\ln(x)-\frac{\ln(-2+x)}{2}}(-2+x)}{x}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\sqrt{x}\right) + c_2 \left(\sqrt{x}\left(\frac{2\,\mathrm{e}^{\ln(x)-\frac{\ln(-2+x)}{2}}(-2+x)}{x}\right)\right)$$

Will add steps showing solving for IC soon.

1.302.2 Maple step by step solution

Let's solve

$2(2-x)\,x^2\left(\frac{d}{dx}y'\right) - (4-x)\,xy' + (3-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(-3+x)y}{2(-2+x)x^2} + \frac{(x-4)y'}{2(-2+x)x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x-4)y'}{2(-2+x)x} + \frac{(-3+x)y}{2(-2+x)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x-4}{2x(-2+x)}, P_3(x) = \frac{-3+x}{2(-2+x)x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{3}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2(-2+x)\,x^2\left(\frac{d}{dx}y'\right) - x(x-4)\,y' + (-3+x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k- >k + 2 - m$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(-1+2r)\,(-3+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(2k+2r-1)\,(2k+2r-3) + a_{k-1}(2k+2r-3)\,(k-2\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-1+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-4\left(\left(-\frac{k}{2}-\frac{r}{2}+1\right)a_{k-1}+a_k\left(k+r-\frac{1}{2}\right)\right)\left(k+r-\frac{3}{2}\right)=0$

- Shift index using $k->k+1$

  $-4\left(\left(-\frac{k}{2}+\frac{1}{2}-\frac{r}{2}\right)a_k+a_{k+1}\left(k+\frac{1}{2}+r\right)\right)\left(k+r-\frac{1}{2}\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{(k+r-1)a_k}{2k+1+2r}$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=\frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+1}=\frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}\right]$

- Recursion relation for $r=\frac{3}{2}$

  $a_{k+1}=\frac{\left(k+\frac{1}{2}\right)a_k}{2k+4}$

- Solution for $r=\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}}, a_{k+1}=\frac{\left(k+\frac{1}{2}\right)a_k}{2k+4}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+1}=\frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}, b_{k+1}=\frac{\left(k+\frac{1}{2}\right)b_k}{2k+4}\right]$

1.302.3 Maple trace

```
Methods for second order ODEs:
```

1.302.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 19

```
dsolve(2*(2-x)*x^2*diff(diff(y(x),x),x)-(4-x)*x*diff(y(x),x)+(3-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=c_1\sqrt{x}+c_2\sqrt{(-2+x)\,x}$$

1.302.5 Mathematica DSolve solution

Solving time : 0.097 (sec)
Leaf size : 41

```
DSolve[{2*(2-x)*x^2*D[y[x],{x,2}]-(4-x)*x*D[y[x],x]+(3-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[4]{x-2}\sqrt{x}\left(2c_2\sqrt{x-2}+c_1\right)}{\sqrt[4]{2-x}}$$

1.303 problem 306

Internal problem ID [8441]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 306
Date solved : `Monday, October 21, 2024 at 05:07:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-x)\,x^2y''+(5x-4)\,xy'+(6-9x)\,y=0$$

1.303.1 Solved as second order ode using Kovacic algorithm

Time used: 0.205 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(5x^2-4x\right)y'+(6-9x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^3+x^2\\B&=5x^2-4x\\C&=6-9x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x+4}{4x(-1+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x + 4$$
$$t = 4x(-1+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x+4}{4x(-1+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 576: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(-1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(-1+x)^2} - \frac{1}{-1+x} + \frac{1}{x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+4}{4x(-1+x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x+4}{4x\left(-1+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{1}{2(-1+x)} + (-)(0) \\
&= \frac{1}{x} - \frac{1}{2(-1+x)} \\
&= \frac{-2+x}{2(-1+x)x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{2(-1+x)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{2(-1+x)^2}\right) + \left(\frac{1}{x} - \frac{1}{2(-1+x)}\right)^2 - \left(\frac{-x+4}{4x(-1+x)^2}\right)\right) = 0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{2(-1+x)}\right) dx} \\
&= \frac{x}{\sqrt{-1+x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x^2-4x}{-x^3+x^2}\, dx} \\
&= z_1 e^{2\ln(x) + \frac{\ln(-1+x)}{2}} \\
&= z_1 \left(x^2\sqrt{-1+x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2-4x}{-x^3+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{4\ln(x)+\ln(-1+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{1}{x} + \ln(x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^3) + c_2 \left(x^3 \left(\frac{1}{x} + \ln(x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.303.2 Maple step by step solution

1.303.3 Maple trace

```
Methods for second order ODEs:
```

1.303.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 18

```
dsolve((1-x)*x^2*diff(diff(y(x),x),x)+(5*x-4)*x*diff(y(x),x)+(6-9*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^2 (c_2 x \ln(x) + c_1 x + c_2)$$

1.303.5 Mathematica DSolve solution

Solving time : 0.063 (sec)
Leaf size : 24

```
DSolve[{(1-x)*x^2*D[y[x],{x,2}]+(5*x-4)*x*D[y[x],x]+(6-9*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x^2(c_1x - c_2(x\log(x) + 1))$$

1.304 problem 307

Internal problem ID [8442]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 307
Date solved : `Monday, October 21, 2024 at 05:07:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (4x^2 + 1)\, y' + 4x(x^2 + 1)\, y = 0$$

1.304.1 Solved as second order ode using Kovacic algorithm

Time used: 0.186 (sec)

Writing the ode as

$$xy'' + (4x^2 + 1)\, y' + (4x^3 + 4x)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 4x^2 + 1 \\ C &= 4x^3 + 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 577: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x^2+1}{x} \, dx} \\ &= z_1 e^{-x^2 - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^{-x^2}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^2+1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(\ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.304.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + (4x^2+1)\,y' + 4x(x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = (-4x^2-4)\,y - \frac{(4x^2+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(4x^2+1)y'}{x} + (4x^2+4)\,y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{4x^2+1}{x}, P_3(x) = 4x^2+4\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + (4x^2+1)\, y' + 4x(x^2+1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 1..3$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k + 1$

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + a_1(1+r)^2 x^r + \left(a_2(2+r)^2 + 4a_0(1+r)\right) x^{1+r} + \left(a_3(3+r)^2 + 4a_1(2+r)\right) x^{2+r} + \Big($

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- The coefficients of each power of x must be 0

  $\left[a_1(1+r)^2 = 0, a_2(2+r)^2 + 4a_0(1+r) = 0, a_3(3+r)^2 + 4a_1(2+r) = 0\right]$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = -\frac{4a_0(1+r)}{r^2+4r+4}, a_3 = 0\right\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + 4a_{k-1}k + 4a_{k-3} = 0$
- Shift index using $k- > k+3$

  $a_{k+4}(k+4)^2 + 4a_{k+2}(k+3) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}$
- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}, a_1 = 0, a_2 = -a_0, a_3 = 0\right]$

1.304.3 Maple trace

```
Methods for second order ODEs:
```

1.304.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+(4*x^2+1)*diff(y(x),x)+4*x*(x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{-x^2}(c_1 + \ln(x)\, c_2)$$

1.304.5 Mathematica DSolve solution

Solving time : `0.048` (sec)
Leaf size : 21

```
DSolve[{x*D[y[x],{x,2}]+(4*x^2+1)*D[y[x],x]+4*x*(x^2+1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow e^{-x^2}(c_2 \log(x) + c_1)$$

1.305 problem 309

Internal problem ID [8443]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 309
Date solved : `Monday, October 21, 2024 at 05:07:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 2xy' + 8y = 0$$

1.305.1 Solved as second order ode using Kovacic algorithm

Time used: 0.241 (sec)

Writing the ode as

$$y'' - 2xy' + 8y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 9$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 - 9)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 579: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{9}{2x} - \frac{81}{8x^3} - \frac{729}{16x^5} - \frac{32805}{128x^7} - \frac{413343}{256x^9} - \frac{11160261}{1024x^{11}} - \frac{157837977}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2-9}{1} \\
&= Q + \frac{R}{1} \\
&= (x^2-9) + (0) \\
&= x^2 - 9
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -9. Now b can be found.

$$\begin{aligned}
b &= (-9) - (0) \\
&= -9
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= x \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-9}{1} - 1\right) &&= -5 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-9}{1} - 1\right) &&= 4
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 - 9$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	-5	4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 4$, and since there are no poles then

$$\begin{aligned}
d &= \alpha_\infty^{-} \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)(x) \\ &= -x \\ &= -x\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}(12x^2 + 6xa_3 + 2a_2) + 2(-x)(4x^3 + 3x^2a_3 + 2xa_2 + a_1) + ((-1) + (-x)^2 - (x^2 - 9)) = 0 \\ 2a_3x^3 + 4(3 + a_2)\, x^2 + 6(a_1 + a_3)\, x + 8a_0 + 2a_2 = 0\end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{3}{4}, a_1 = 0, a_2 = -3, a_3 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 3x^2 + \frac{3}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\, dx} \\ &= \left(x^4 - 3x^2 + \frac{3}{4}\right) e^{\int -x dx} \\ &= \left(x^4 - 3x^2 + \frac{3}{4}\right) e^{-\frac{x^2}{2}} \\ &= \frac{(4x^4 - 12x^2 + 3)\, e^{-\frac{x^2}{2}}}{4}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1}\,dx} \\
&= z_1 e^{\frac{x^2}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 - 3x^2 + \frac{3}{4}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{x^2}}{\left(x^4 - 3x^2 + \frac{3}{4}\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 - 3x^2 + \frac{3}{4}\right) + c_2 \left(x^4 - 3x^2 + \frac{3}{4}\left(\int \frac{\mathrm{e}^{x^2}}{\left(x^4 - 3x^2 + \frac{3}{4}\right)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.305.2 Maple trace

```
Methods for second order ODEs:
```

1.305.3 Maple dsolve solution

Solving time : `0.012` (sec)
Leaf size : 44

```
dsolve(diff(diff(y(x),x),x)-2*x*diff(y(x),x)+8*y(x) = 0,
       y(x),singsol=all)
```

$$y = 2c_1(2x^3 - 5x)\,e^{x^2} - 4(\sqrt{\pi}\ \mathrm{erfi}\,(x)\,c_1 - c_2)\left(x^4 - 3x^2 + \frac{3}{4}\right)$$

1.305.4 Mathematica DSolve solution

Solving time : `1.632` (sec)
Leaf size : 63

```
DSolve[{D[y[x],{x,2}]-2*x*D[y[x],x]+8*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) - \frac{1}{12}c_2\left(\sqrt{\pi}(-4x^4 + 12x^2 - 3)\,\mathrm{erfi}(x) + 2e^{x^2}x(2x^2 - 5)\right)$$

1.306 problem 310

Internal problem ID [8444]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 310
Date solved : `Monday, October 21, 2024 at 05:07:59 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right)y''-2xy'+12y=0$$

1.306.1 Solved as second order ode using Kovacic algorithm

Time used: 0.338 (sec)

Writing the ode as

$$\left(-x^2+1\right)y''-2xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= -x^2+1\\ B &= -2x\\ C &= 12\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{12x^2 - 13}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 12x^2 - 13$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{12x^2 - 13}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 580: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{25}{4(x-1)} - \frac{25}{4(x+1)} - \frac{1}{4(x+1)^2} - \frac{1}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{12x^2 - 13}{(x^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=12$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = 4\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{12x^2-13}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	4	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=4$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= 4-(1)\\
&= 3
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(6x+2a_2) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(3x^2 + 2xa_2 + a_1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(x+1)^2}\right) + \left(\frac{1}{2x-2} + \frac{}{2x} \right.\right.$$

$$\frac{-6a_2x^2 + (-10a_1}{x^{}}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = -\frac{3}{5}, a_2 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 - \frac{3}{5}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x^3 - \frac{3}{5}x\right) e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= \left(x^3 - \frac{3}{5}x\right) \sqrt{(x-1)(x+1)} \\
&= \frac{(5x^3 - 3x)\sqrt{x^2-1}}{5}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(5x^3-3x)\sqrt{x^2-1}}{5\sqrt{x-1}\,\sqrt{x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{125x}{36\left(x^2-\frac{3}{5}\right)}+\frac{25\ln(x-1)}{8}+\frac{25}{9x}-\frac{25\ln(x+1)}{8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(5x^3-3x)\sqrt{x^2-1}}{5\sqrt{x-1}\,\sqrt{x+1}}\right) \\
&\quad + c_2\left(\frac{(5x^3-3x)\sqrt{x^2-1}}{5\sqrt{x-1}\,\sqrt{x+1}}\left(\frac{125x}{36\left(x^2-\frac{3}{5}\right)}+\frac{25\ln(x-1)}{8}+\frac{25}{9x}-\frac{25\ln(x+1)}{8}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.306.2 Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d}{dx}y'\right) - 2xy' + 12y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = \frac{12y}{x^2-1} - \frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{2xy'}{x^2-1} - \frac{12y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{12}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = 1$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -1$

- Multiply by denominators

$(x^2-1)\left(\frac{d}{dx}y'\right) + 2xy' - 12y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 12y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+4)\,(k+r-3)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2a_{k+1}(k+1)^2 + a_k(k+4)\,(k-3) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+4)(k-3)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 3$
  $a_{k+1} = \frac{a_k(k+4)(k-3)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -6a_0$
- Apply recursion relation for $k = 1$
  $a_2 = -\frac{5a_1}{4}$
- Express in terms of a_0
  $a_2 = \frac{15a_0}{2}$
- Apply recursion relation for $k = 2$
  $a_3 = -\frac{a_2}{3}$
- Express in terms of a_0
  $a_3 = -\frac{5a_0}{2}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y(u) = a_0 \cdot \left(1 - 6u + \frac{15}{2}u^2 - \frac{5}{2}u^3\right)$
- Revert the change of variables $u = x + 1$

$$\left[y = a_0\left(\frac{3}{2}x - \frac{5}{2}x^3\right)\right]$$

1.306.3 Maple trace

```
Methods for second order ODEs:
```

1.306.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 55

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+12*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(5x^3 - 3x)\, c_2 \ln(x-1)}{24} + \frac{(-5x^3 + 3x)\, c_2 \ln(x+1)}{24} - \frac{5c_1\, x^3}{3} + \frac{5c_2\, x^2}{12} + c_1 x - \frac{c_2}{9}$$

1.306.5 Mathematica DSolve solution

Solving time : `0.037` (sec)
Leaf size : 59

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}c_1 x(5x^2 - 3) + c_2\left(-\frac{5x^2}{2} - \frac{1}{4}(5x^2 - 3)\, x(\log(1-x) - \log(x+1)) + \frac{2}{3}\right)$$

1.307 problem 311

Internal problem ID [8445]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 311
Date solved : `Monday, October 21, 2024 at 05:08:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(x+2)\,y''+2(x+1)\,y'-2y=0$$

1.307.1 Solved as second order ode using Kovacic algorithm

Time used: 0.254 (sec)

Writing the ode as

$$\left(x^2+2x\right)y''+(2x+2)\,y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+2x\\B&=2x+2\\C&=-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{2x^2 + 4x - 1}{(x^2 + 2x)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 2x^2 + 4x - 1 \\
t &= (x^2 + 2x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{2x^2 + 4x - 1}{(x^2 + 2x)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 582: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\&= 4 - 2\\&= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{4x} - \frac{5}{4(x+2)} - \frac{1}{4(x+2)^2} - \frac{1}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 + 4x - 1}{(x^2 + 2x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{2x^2+4x-1}{(x^2+2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{+}\right) \\
&= 2-(1) \\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x+4} + \frac{1}{2x} + (0) \\
&= \frac{1}{2x+4} + \frac{1}{2x} \\
&= \frac{x+1}{x\,(x+2)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x+4} + \frac{1}{2x}\right)(1) + \left(\left(-\frac{1}{2\,(x+2)^2} - \frac{1}{2x^2}\right) + \left(\frac{1}{2x+4} + \frac{1}{2x}\right)^2 - \left(\frac{2x^2+4x-1}{(x^2+2x)^2}\right)\right) &= 0 \\
\frac{2-2a_0}{x\,(x+2)} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= (x+1)\,\mathrm{e}^{\int \left(\frac{1}{2x+4} + \frac{1}{2x}\right) dx} \\
&= (x+1)\,\sqrt{x\,(x+2)} \\
&= (x+1)\,\sqrt{x\,(x+2)}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x+2}{x^2+2x}\,dx} \\
&= z_1 e^{-\frac{\ln(x(x+2))}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x\,(x+2)}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x + 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x+2}{x^2+2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x(x+2))}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\ln(x+2)}{2} + \frac{1}{x+1} + \frac{\ln(x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x+1) + c_2\left(x+1\left(-\frac{\ln(x+2)}{2} + \frac{1}{x+1} + \frac{\ln(x)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.307.2 Maple step by step solution

Let's solve

$x(x+2)\left(\frac{d}{dx}y'\right)+2(x+1)\,y'-2y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{2y}{x(x+2)}-\frac{2(x+1)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{2(x+1)y'}{x(x+2)}-\frac{2y}{x(x+2)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{2(x+1)}{x(x+2)},P_3(x)=-\frac{2}{x(x+2)}\right]$

- ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

$\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=1$

- ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

$\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

- ◦ $x=-2$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-2$

- Multiply by denominators

$x(x+2)\left(\frac{d}{dx}y'\right)+(2x+2)\,y'-2y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+2)\,(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2a_{k+1}(k+1)^2 + a_k(k+2)\,(k-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y(u) = a_0 \cdot (-u+1)$
- Revert the change of variables $u = x+2$
  $[y = a_0(-x-1)]$

1.307.3 Maple trace

```
Methods for second order ODEs:
```

1.307.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 28

```
dsolve(x*(x+2)*diff(diff(y(x),x),x)+2*(x+1)*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{c_2(x+1)\ln(x+2)}{2} + \frac{c_2(x+1)\ln(x)}{2} + c_1x + c_1 + c_2$$

1.307.5 Mathematica DSolve solution

Solving time : 0.036 (sec)
Leaf size : 37

```
DSolve[{x*(x+2)*D[y[x],{x,2}]+2*(x+1)*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1(x+1) - \frac{1}{2}c_2((x+1)\log(-x) - (x+1)\log(x+2) + 2)$$

1.308 problem 313

Internal problem ID [8446]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 313
Date solved : `Monday, October 21, 2024 at 05:08:00 PM`
CAS classification :
`[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]`

Solve

$$x(x+2)\, y''+(x+1)\, y'-4y=0$$

1.308.1 Solved as second order ode using Kovacic algorithm

Time used: 0.233 (sec)

Writing the ode as

$$\left(x^{2}+2x\right) y''+(x+1)\, y'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}+2x \\ B &= x+1 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15x^2 + 30x - 3$$
$$t = 4\left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 584: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{33}{16\,(x+2)} - \frac{3}{16\,(x+2)^2} + \frac{33}{16x} - \frac{3}{16x^2}
$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{15x^2 + 30x - 3}{4\,(x^2 + 2x)^2}
$$

Since the gcd$(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{15x^2+30x-3}{4\left(x^2+2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{+}\right) \\
&= \frac{5}{2}-\left(\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4\left(x+2\right)} + \frac{3}{4x} + (0) \\
&= \frac{3}{4\left(x+2\right)} + \frac{3}{4x} \\
&= \frac{\frac{3x}{2}+\frac{3}{2}}{x\left(x+2\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4\left(x+2\right)} + \frac{3}{4x}\right)(1) + \left(\left(-\frac{3}{4\left(x+2\right)^2} - \frac{3}{4x^2}\right) + \left(\frac{3}{4\left(x+2\right)} + \frac{3}{4x}\right)^2 - \left(\frac{15x^2+30x-3}{4\left(x^2+2x\right)^2}\right)\right) = \frac{3-3a_0}{x\left(x+2\right)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+1)\, e^{\int \left(\frac{3}{4(x+2)} + \frac{3}{4x}\right) dx} \\
&= (x+1)\left(x(x+2)\right)^{3/4} \\
&= (x+1)\left(x(x+2)\right)^{3/4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x+1}{x^2+2x}\,dx} \\
&= z_1 e^{-\frac{\ln(x(x+2))}{4}} \\
&= z_1 \left(\frac{1}{\left(x\left(x+2\right)\right)^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x\left(x+2\right)}\left(x+1\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x+1}{x^2+2x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x(x+2))}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{2x^2+4x+1}{\sqrt{x\left(x+2\right)}\,\left(x+1\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x\left(x+2\right)}\left(x+1\right)\right) + c_2\left(\sqrt{x\left(x+2\right)}\left(x+1\right)\left(-\frac{2x^2+4x+1}{\sqrt{x\left(x+2\right)}\,\left(x+1\right)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.308.2 Maple step by step solution

Let's solve

$x(x+2)\left(\frac{d}{dx}y'\right)+(x+1)\,y'-4y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{4y}{x(x+2)}-\frac{(x+1)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(x+1)y'}{x(x+2)}-\frac{4y}{x(x+2)}=0$

□ Check to see if x_0 is a regular singular point

- ∘ Define functions

$\left[P_2(x)=\frac{x+1}{x(x+2)}, P_3(x)=-\frac{4}{x(x+2)}\right]$

- ∘ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

$\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=\frac{1}{2}$

- ∘ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

$\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

- ∘ $x=-2$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-2$

- Multiply by denominators

$x(x+2)\left(\frac{d}{dx}y'\right)+(x+1)\,y'-4y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)-4y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ∘ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ∘ Shift index using $k->k+1-m$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

$$u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-a_{0}r(-1+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(2k+1+2r)+a_{k}(k+r+2)\,(k+r-2)\right)u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-1+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(k+\frac{1}{2}+r\right)(k+1+r)\,a_{k+1}+a_{k}(k+r+2)\,(k+r-2)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_{k}(k+r+2)(k+r-2)}{(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r=0$; series terminates at $k=2$

$$a_{k+1}=\frac{a_{k}(k+2)(k-2)}{(2k+1)(k+1)}$$

- Apply recursion relation for $k=0$

$$a_{1}=-4a_{0}$$

- Apply recursion relation for $k=1$

$$a_{2}=-\frac{a_{1}}{2}$$

- Express in terms of a_0

$$a_{2}=2a_{0}$$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

$$y(u)=a_{0}\cdot\left(2u^{2}-4u+1\right)$$

- Revert the change of variables $u=x+2$

$$\left[y=a_{0}\left(2x^{2}+4x+1\right)\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1}=\frac{a_{k}\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1}=\frac{a_k(k+\frac{5}{2})(k-\frac{3}{2})}{(2k+2)(k+\frac{3}{2})}\right]$$

- Revert the change of variables $u=x+2$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+2)^{k+\frac{1}{2}}, a_{k+1}=\frac{a_k(k+\frac{5}{2})(k-\frac{3}{2})}{(2k+2)(k+\frac{3}{2})}\right]$$

- Combine solutions and rename parameters

$$\left[y=a_0(2x^2+4x+1)+\left(\sum_{k=0}^{\infty} b_k(x+2)^{k+\frac{1}{2}}\right), b_{k+1}=\frac{b_k(k+\frac{5}{2})(k-\frac{3}{2})}{(2k+2)(k+\frac{3}{2})}\right]$$

1.308.3 Maple trace

```
Methods for second order ODEs:
```

1.308.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 28

```
dsolve(x*(x+2)*diff(diff(y(x),x),x)+(x+1)*diff(y(x),x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y=c_2\sqrt{x(x+2)}\,(x+1)+2c_1\left(x^2+2x+\frac{1}{2}\right)$$

1.308.5 Mathematica DSolve solution

Solving time : 2.196 (sec)
Leaf size : 73

```
DSolve[{x*(x+2)*D[y[x],{x,2}]+(x+1)*D[y[x],x]-4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_1\cosh\left(8\operatorname{arctanh}\left(\frac{\sqrt{x}-1}{\sqrt{3}-\sqrt{x+2}}\right)\right)-ic_2\sinh\left(8\operatorname{arctanh}\left(\frac{\sqrt{x}-1}{\sqrt{3}-\sqrt{x+2}}\right)\right)$$

1.309 problem 314

Internal problem ID [8447]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 314
Date solved : `Monday, October 21, 2024 at 05:08:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\,y''-xy'+y=0$$

1.309.1 Solved as second order ode using Kovacic algorithm

Time used: 0.255 (sec)

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 6 \\ t &= 4(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 586: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\,(x-1)^2} - \frac{1}{2\,(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4(x-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - (\alpha_{c_1}^{-}) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2(x-1)} + \frac{1}{2} \\ &= \frac{x-2}{2x-2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\ &= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1}\, dx} \\ &= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\ &= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{x\, e^{x+\ln(x-1)} e^{-2x}}{x-1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2\left(e^x\left(-\frac{x\, e^{x+\ln(x-1)} e^{-2x}}{x-1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.309.2 Maple step by step solution

Let's solve

$(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$
- $(x-1)\cdot P_2(x)$ is analytic at $x=1$
  $$\left((x-1)\cdot P_2(x)\right)\bigg|_{x=1} = -1$$
- $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$
  $$\left((x-1)^2\cdot P_3(x)\right)\bigg|_{x=1} = 0$$
- $x=1$is a regular singular point
  Check to see if $x_0 = 1$ is a regular singular point
  $x_0 = 1$

- Multiply by denominators
  $(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$
- Change variables using $x = u+1$ so that the regular singular point is at $u=0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions
- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$
- Shift index using $k->k+1$
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

  Rewrite ODE with series expansions
  $$a_0r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$
- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

1.309.3 Maple trace

```
Methods for second order ODEs:
```

1.309.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2 \,\mathrm{e}^{x}$$

1.309.5 Mathematica DSolve solution

Solving time : 0.052 (sec)
Leaf size : 17

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2 x$$

1.310 problem 315

Internal problem ID [8448]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 315
Date solved : `Monday, October 21, 2024 at 05:08:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-2xy'+2y=0$$

1.310.1 Solved as second order ode using Kovacic algorithm

Time used: 0.300 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= (x^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 588: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{3/2}}{(ix+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1\left(\sqrt{x^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx$$
$$= y_1 \left(-\frac{x}{(x+i)^2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2 \left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)$$

Will add steps showing solving for IC soon.

1.310.2 Maple step by step solution

1.310.3 Maple trace

```
Methods for second order ODEs:
```

1.310.4 Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 16

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2\, x^2 + c_1 x - c_2$$

1.310.5 Mathematica DSolve solution

Solving time : 0.07 (sec)
Leaf size : 21

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x - c_1 (x - i)^2$$

1.311 problem 316

Internal problem ID [8449]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 316
Date solved : `Monday, October 21, 2024 at 05:08:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-2x+10\right) y''+xy'-4y=0$$

1.311.1 Solved as second order ode using Kovacic algorithm

Time used: 0.437 (sec)

Writing the ode as

$$\left(x^2-2x+10\right) y''+xy'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-2x+10 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2 - 32x + 180 \\ t &= 4\left(x^2 - 2x + 10\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 589: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x + 10)^2$. There is a pole at $x = 1 + 3i$ of order 2. There is a pole at $x = 1 - 3i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{7}{36} + \frac{i}{24}}{(x - 1 - 3i)^2} + \frac{-\frac{7}{36} - \frac{i}{24}}{(x - 1 + 3i)^2} - \frac{149i}{216\,(x - 1 - 3i)} + \frac{149i}{216\,(x - 1 + 3i)}$$

For the pole at $x = 1 + 3i$ let b be the coefficient of $\frac{1}{(x-1-3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} + \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} + \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} - \frac{i}{12} \end{aligned}$$

For the pole at $x = 1 - 3i$ let b be the coefficient of $\frac{1}{(x-1+3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} - \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} - \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} + \frac{i}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$1+3i$	2	0	$\frac{3}{4} + \frac{i}{12}$	$\frac{1}{4} - \frac{i}{12}$
$1-3i$	2	0	$\frac{3}{4} - \frac{i}{12}$	$\frac{1}{4} + \frac{i}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i} + (0) \\
&= \frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i} \\
&= \frac{3x-4}{2x^2-4x+20}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right)(1) + \left(\left(\frac{-\frac{3}{4}-\frac{i}{12}}{(x-1-3i)^2} + \frac{-\frac{3}{4}+\frac{i}{12}}{(x-1+3i)^2}\right) + \left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i \ldots}\right.\right.
$$

$$
-\frac{3\ldots}{(-\ldots}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{4}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{4}{3}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{\frac{3\ln\left(x^2-2x+10\right)}{4} - \frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}} \\
&= \frac{(3x-4)\left(x^2-2x+10\right)^{3/4} e^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}}}{3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2-2x+10}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2-2x+10\right)}{4}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}}}{\left(x^2-2x+10\right)^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2-2x+10}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{9(3x^2-4x+15)\,\mathrm{e}^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}\mathrm{e}^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}\right) \\
&\quad + c_2 \left(\frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}\left(-\frac{9(3x^2-4x+15)\,\mathrm{e}^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}\mathrm{e}^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.311.2 Maple step by step solution

1.311.3 Maple trace

```
Methods for second order ODEs:
```

1.311.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 31

```
dsolve((x^2-2*x+10)*diff(diff(y(x),x),x)+x*diff(y(x),x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y = 3(x-1+3i)^{\frac{1}{2}-\frac{i}{6}} c_2 \left(x-\frac{4}{3}\right) (x-1-3i)^{\frac{1}{2}+\frac{i}{6}} + c_1 \left(x^2-\frac{4}{3}x+5\right)$$

1.311.5 Mathematica DSolve solution

Solving time : 1.253 (sec)
Leaf size : 92

```
DSolve[{(x^2-2*x+10)*D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3}(3x - 4)\sqrt{x^2-2x+10}\,e^{-\frac{1}{3}\arctan\left(\frac{x-1}{3}\right)} \left(c_2 \int_1^x \frac{9e^{\frac{1}{3}\arctan\left(\frac{1}{3}(K[1]-1)\right)}}{(4-3K[1])^2 \left(K[1]^2-2K[1]+10\right)^{3/2}} dK[1] + c_1\right)$$

1.312 problem 317

Internal problem ID [8450]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 317
Date solved : `Monday, October 21, 2024 at 05:08:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-2x+10\right)y''+xy'-4y=0$$

1.312.1 Solved as second order ode using Kovacic algorithm

Time used: 0.385 (sec)

Writing the ode as

$$\left(x^2-2x+10\right)y''+xy'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-2x+10 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 15x^2 - 32x + 180 \\
t &= 4\left(x^2 - 2x + 10\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 590: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x + 10)^2$. There is a pole at $x = 1 + 3i$ of order 2. There is a pole at $x = 1 - 3i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{7}{36} + \frac{i}{24}}{(x - 1 - 3i)^2} + \frac{-\frac{7}{36} - \frac{i}{24}}{(x - 1 + 3i)^2} - \frac{149i}{216\,(x - 1 - 3i)} + \frac{149i}{216\,(x - 1 + 3i)}$$

For the pole at $x = 1 + 3i$ let b be the coefficient of $\frac{1}{(x-1-3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} + \frac{i}{24}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} + \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} - \frac{i}{12}\end{aligned}$$

For the pole at $x = 1 - 3i$ let b be the coefficient of $\frac{1}{(x-1+3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} - \frac{i}{24}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} - \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} + \frac{i}{12}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$1+3i$	2	0	$\frac{3}{4} + \frac{i}{12}$	$\frac{1}{4} - \frac{i}{12}$
$1-3i$	2	0	$\frac{3}{4} - \frac{i}{12}$	$\frac{1}{4} + \frac{i}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i} + (0) \\
&= \frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i} \\
&= \frac{3x - 4}{2x^2 - 4x + 20}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i}\right)(1) + \left(\left(\frac{-\frac{3}{4} - \frac{i}{12}}{(x - 1 - 3i)^2} + \frac{-\frac{3}{4} + \frac{i}{12}}{(x - 1 + 3i)^2}\right) + \left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3 \dots}\right.\right.
$$

$$
- \frac{3 \dots}{(- \dots}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{4}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{4}{3}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{\frac{3 \ln(x^2 - 2x + 10)}{4} - \frac{\arctan\left(\frac{x}{3} - \frac{1}{3}\right)}{6}} \\
&= \frac{(3x - 4)\,(x^2 - 2x + 10)^{3/4}\, e^{-\frac{\arctan\left(\frac{x}{3} - \frac{1}{3}\right)}{6}}}{3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2-2x+10}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2-2x+10\right)}{4}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}}}{\left(x^2-2x+10\right)^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2-2x+10}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{9(3x^2-4x+15)\,\mathrm{e}^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}\mathrm{e}^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}\right) \\
&\quad + c_2 \left(\frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}\left(-\frac{9(3x^2-4x+15)\,\mathrm{e}^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}\mathrm{e}^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.312.2 Maple step by step solution

1.312.3 Maple trace

```
Methods for second order ODEs:
```

1.312.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 31

```
dsolve((x^2-2*x+10)*diff(diff(y(x),x),x)+x*diff(y(x),x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y = 3\left(x-\frac{4}{3}\right)(x-1+3i)^{\frac{1}{2}-\frac{i}{6}}\, c_2 (x-1-3i)^{\frac{1}{2}+\frac{i}{6}} + c_1\left(x^2-\frac{4}{3}x+5\right)$$

1.312.5 Mathematica DSolve solution

Solving time : 0.849 (sec)
Leaf size : 92

```
DSolve[{(x^2-2*x+10)*D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3}(3x - 4)\sqrt{x^2-2x+10}\, e^{-\frac{1}{3}\arctan\left(\frac{x-1}{3}\right)}\left(c_2\int_1^x \frac{9e^{\frac{1}{3}\arctan\left(\frac{1}{3}(K[1]-1)\right)}}{(4-3K[1])^2\left(K[1]^2-2K[1]+10\right)^{3/2}}dK[1] + c_1\right)$$

1.313 problem 318

Internal problem ID [8451]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 318
Date solved : `Monday, October 21, 2024 at 05:08:05 PM`
CAS classification : `[_Hermite]`

Solve

$$y'' - xy' + 2y = 0$$

1.313.1 Solved as second order ode using Kovacic algorithm

Time used: 0.260 (sec)

Writing the ode as

$$y'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 10}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 10 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 591: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) = 0$$

$$a_1 x + 2a_0 + 2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= (x^2 - 1) \, e^{\int -\frac{x}{2} dx} \\
&= (x^2 - 1) \, e^{-\frac{x^2}{4}} \\
&= (x^2 - 1) \, e^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(e^{\frac{x^2}{4}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2 - 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2 - 1)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1(x^2-1) + c_2\left(x^2-1\left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right)\right)$$

Will add steps showing solving for IC soon.

1.313.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) - a_k(k-2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} - a_k(k-2) = 0$

- Recursion relation; series terminates at $k = 2$

  $a_{k+2} = \frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

$a_2 = -a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

  $y = A_2x^2 + A_1x - a_0$

1.313.3 Maple trace

```
Methods for second order ODEs:
```

1.313.4 Maple dsolve solution

Solving time : 0.011 (sec)
Leaf size : 42

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = 2\,\mathrm{e}^{\frac{x^2}{2}} c_1 x - (x-1)(x+1)\left(\sqrt{2}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{\pi}\,c_1 - c_2\right)$$

1.313.5 Mathematica DSolve solution

Solving time : 0.137 (sec)
Leaf size : 54

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}c_2\left(\sqrt{2\pi}(x^2-1)\,\mathrm{erfi}\left(\frac{x}{\sqrt{2}}\right) - 2e^{\frac{x^2}{2}}x\right) + c_1(x^2-1)$$

1.314 problem 319

Internal problem ID [8452]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 319
Date solved : `Monday, October 21, 2024 at 05:08:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x+2)\,y''+xy'-y=0$$

1.314.1 Solved as second order ode using Kovacic algorithm

Time used: 0.278 (sec)

Writing the ode as

$$(x+2)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x+2 \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 12}{4\left(x + 2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x + 12 \\ t &= 4(x + 2)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 12}{4\left(x + 2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 593: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+2)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x^2} - \frac{8}{x^3} + \frac{20}{x^4} - \frac{32}{x^5} + \frac{16}{x^6} + \frac{64}{x^7} - \frac{80}{x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 12}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{8}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{8}{4x^2 + 16x + 16} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$
\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 4x + 12}{4(x+2)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x+2} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{x+2} - \frac{1}{2} \\
&= -\frac{4+x}{2\left(x+2\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x+2} - \frac{1}{2}\right)(1) + \left(\left(\frac{1}{\left(x+2\right)^2}\right) + \left(-\frac{1}{x+2} - \frac{1}{2}\right)^2 - \left(\frac{x^2+4x+12}{4\left(x+2\right)^2}\right)\right) = 0$$

$$\frac{a_0-4}{x+2} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 4 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (4+x)\, \mathrm{e}^{\int \left(-\frac{1}{x+2} - \frac{1}{2}\right) dx} \\
&= (4+x)\, \mathrm{e}^{-\frac{x}{2} - \ln(x+2)} \\
&= \frac{(4+x)\, \mathrm{e}^{-\frac{x}{2}}}{x+2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x+2}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\ln(x+2)} \\
&= z_1\left((x+2)\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}(4+x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x+2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+2\ln(x+2)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x\,\mathrm{e}^{-x+2\ln(x+2)}\mathrm{e}^{2x}}{(4+x)\,(x+2)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}(4+x)\right) + c_2\left(\mathrm{e}^{-x}(4+x)\left(\frac{x\,\mathrm{e}^{-x+2\ln(x+2)}\mathrm{e}^{2x}}{(4+x)\,(x+2)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.314.2 Maple step by step solution

Let's solve

$(x+2)\left(\frac{d}{dx}y'\right)+xy'-y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{y}{x+2}-\frac{xy'}{x+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{xy'}{x+2}-\frac{y}{x+2}=0$

□ Check to see if $x_0=-2$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x}{x+2}, P_3(x)=-\frac{1}{x+2}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=-2$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$

- Multiply by denominators

$(x+2)\left(\frac{d}{dx}y'\right)+xy'-y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)-y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k->k+1$

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r-1)}{(k+1+r)(k-2+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$$a_{k+1} = -\frac{a_k(k-1)}{(k+1)(k-2)}$$

- Apply recursion relation for $k = 0$

$$a_1 = -\frac{a_0}{2}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot \left(1 - \frac{u}{2}\right)$$

- Revert the change of variables $u = x + 2$

$$\left[y = -\frac{a_0 x}{2}\right]$$

- Recursion relation for $r = 3$

$$a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}$$

- Solution for $r = 3$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+3}, a_{k+1} = -\frac{a_k (k+2)}{(k+4)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = -\frac{a_0 x}{2} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+3}\right), b_{k+1} = -\frac{b_k (k+2)}{(k+4)(k+1)}\right]$$

1.314.3 Maple trace

```
Methods for second order ODEs:
```

1.314.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 17

```
dsolve((x+2)*diff(diff(y(x),x),x)+x*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2 \,\mathrm{e}^{-x} (4 + x)$$

1.314.5 Mathematica DSolve solution

Solving time : 0.157 (sec)
Leaf size : 72

```
DSolve[{(x+2)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{2\sqrt{\frac{2}{\pi}} e^{-x-2} \sqrt{x+2} \left(c_1 \left(e^{x+2} x + x + 4\right) - i c_2 \left(\left(e^{x+2} - 1\right) x - 4\right)\right)}{\sqrt{-i(x+2)}}$$

1.315 problem 320

Internal problem ID [8453]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 320
Date solved : `Monday, October 21, 2024 at 05:08:07 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$(x^2+1)\,y''-6y=0$$

1.315.1 Solved as second order ode using Kovacic algorithm

Time used: 0.252 (sec)

Writing the ode as

$$(x^2+1)\,y''-6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+1\\B&=0\\C&=-6\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{x^2 + 1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 6$$
$$t = x^2 + 1$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{6}{x^2 + 1}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 595: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2 + 1$. There is a pole at $x = i$ of order 1. There is a pole at $x = -i$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = i$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6}{x^2 + 1}$$

Since the $\gcd(s, t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6}{x^2 + 1}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= 3 - (1) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x-i} + (0) \\ &= \frac{1}{x-i} \\ &= \frac{1}{x-i} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2)+2\left(\frac{1}{x-i}\right)(2x+a_1)+\left(\left(-\frac{1}{(x-i)^2}\right)+\left(\frac{1}{x-i}\right)^2-\left(\frac{6}{x^2+1}\right)\right)=0$$

$$2+\frac{-4x-2a_1}{-x+i}+\frac{-6x^2-6a_1x-6a_0}{x^2+1}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=0, a_1=i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x^2+ix$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \left(x^2+ix\right) e^{\int \frac{1}{x-i}dx} \\
&= \left(x^2+ix\right) e^{\frac{\ln\left(x^2+1\right)}{2}+i\arctan(x)} \\
&= x(x+i)(ix+1)
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= x(x+i)(ix+1)
\end{aligned}$$

Which simplifies to

$$y_1=ix^3+ix$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= ix^3 + ix \int \frac{1}{(ix^3 + ix)^2}\, dx \\
&= ix^3 + ix \left(\frac{x}{2x^2+2} + \frac{3\arctan(x)}{2} + \frac{1}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (ix^3 + ix) + c_2 \left(ix^3 + ix \left(\frac{x}{2x^2+2} + \frac{3\arctan(x)}{2} + \frac{1}{x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.315.2 Maple step by step solution

1.315.3 Maple trace

```
Methods for second order ODEs:
```

1.315.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 31

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-6*y(x) = 0,
       y(x),singsol=all)
```

$$
y = \frac{3xc_2(x^2+1)\arctan(x)}{2} + c_1 x^3 + \frac{3c_2 x^2}{2} + c_1 x + c_2
$$

1.315.5 Mathematica DSolve solution

Solving time : 0.081 (sec)
Leaf size : 36

```
DSolve[{(x^2+1)*D[y[x],{x,2}]-6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1(x^3 + x) - \frac{1}{2}c_2(3(x^3 + x)\arctan(x) + 3x^2 + 2)$$

1.316 problem 321

Internal problem ID [8454]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 321
Date solved : `Monday, October 21, 2024 at 05:08:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+2\right) y''+3xy'-y=0$$

1.316.1 Solved as second order ode using Kovacic algorithm

Time used: 0.381 (sec)

Writing the ode as

$$\left(x^2+2\right) y''+3xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2+2\\ B&=3x\\ C&=-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 20}{4(x^2 + 2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 7x^2 + 20$$
$$t = 4(x^2 + 2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 20}{4(x^2 + 2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 596: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+2)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\left(x - i\sqrt{2}\right)^2} - \frac{3}{16\left(x + i\sqrt{2}\right)^2} - \frac{17i\sqrt{2}}{32\left(x - i\sqrt{2}\right)} + \frac{17i\sqrt{2}}{32\left(x + i\sqrt{2}\right)}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7x^2 + 20}{4\left(x^2 + 2\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{7}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
$i\sqrt{2}$	2	$\{1,2,3\}$
$-i\sqrt{2}$	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2-(1+(1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(i\sqrt{2}))} + \frac{1}{(x-(-i\sqrt{2}))}\right) \\ &= \frac{1}{2x-2i\sqrt{2}} + \frac{1}{2x+2i\sqrt{2}} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x - 2i\sqrt{2}} + \frac{1}{2x + 2i\sqrt{2}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2x - 2i\sqrt{2}} + \frac{1}{2x + 2i\sqrt{2}}\right) w + \frac{7x^2 + 16}{4\left(\sqrt{2} + ix\right)^2 \left(x + i\sqrt{2}\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{x + 2\sqrt{2x^2 + 4}}{2x^2 + 4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{x + 2\sqrt{2x^2+4}}{2x^2+4} dx} \\ &= \left(x^2 + 2\right)^{1/4} \mathrm{e}^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2+2} \, dx} \\ &= z_1 e^{-\frac{3\ln\left(x^2+2\right)}{4}} \\ &= z_1 \left(\frac{1}{\left(x^2+2\right)^{3/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2+2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+2\right)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{e^{-2\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} \right) + c_2 \left(\frac{e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} \left(\int \frac{e^{-2\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.316.2 Maple step by step solution

1.316.3 Maple trace

```
Methods for second order ODEs:
```

1.316.4 Maple dsolve solution

Solving time : 0.024 (sec)
Leaf size : 45

```
dsolve((x^2+2)*diff(diff(y(x),x),x)+3*x*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\left(\sqrt{x^2+2}+x\right)^{-\sqrt{2}} + c_1\left(\sqrt{x^2+2}+x\right)^{\sqrt{2}}}{\sqrt{x^2+2}}$$

1.316.5 Mathematica DSolve solution

Solving time : 0.14 (sec)
Leaf size : 92

```
DSolve[{(x^2+2)*D[y[x],{x,2}]+3*x*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2^{3/4} c_1 \cos\left(2\sqrt{2}\arcsin\left(\frac{1}{2}\sqrt{2-i\sqrt{2}x}\right)\right)}{\sqrt{\pi}\sqrt{x^2+2}} + \frac{c_2 Q_{-\frac{1}{2}+\sqrt{2}}^{\frac{1}{2}}\left(\frac{ix}{\sqrt{2}}\right)}{\sqrt[4]{x^2+2}}$$

1.317 problem 322

Internal problem ID [8455]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 322
Date solved : `Monday, October 21, 2024 at 05:08:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\,y''-xy'+y=0$$

1.317.1 Solved as second order ode using Kovacic algorithm

Time used: 0.251 (sec)

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x-1\\B&=-x\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 4x + 6}{4 (x - 1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 4x + 6 \\
t &= 4(x - 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 - 4x + 6}{4 (x - 1)^2} \right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 597: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{2}\right) - (0)\\&= -\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2}\\\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2}\\\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - (\alpha_{c_1}^-)\\&= -\frac{1}{2} - \left(-\frac{1}{2}\right)\\&= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-\frac{x\,\mathrm{e}^{x+\ln(x-1)}\mathrm{e}^{-2x}}{x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.317.2 Maple step by step solution

Let's solve

$(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$
- $(x-1)\cdot P_2(x)$ is analytic at $x=1$
  $$\left((x-1)\cdot P_2(x)\right)\bigg|_{x=1} = -1$$
- $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$
  $$\left((x-1)^2\cdot P_3(x)\right)\bigg|_{x=1} = 0$$
- $x=1$is a regular singular point
  Check to see if $x_0=1$ is a regular singular point
  $x_0=1$

- Multiply by denominators
  $(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$
- Change variables using $x=u+1$ so that the regular singular point is at $u=0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions
- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$
- Shift index using $k->k+1$
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$
- Revert the change of variables $u = x - 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$
- Solution for $r = 2$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$
- Revert the change of variables $u = x - 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

1.317.3 Maple trace

```
Methods for second order ODEs:
```

1.317.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2 \, \mathrm{e}^{x}$$

1.317.5 Mathematica DSolve solution

Solving time : 0.05 (sec)
Leaf size : 17

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2 x$$

1.318 problem 325

Internal problem ID [8456]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 325
Date solved : `Monday, October 21, 2024 at 05:08:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0$$

1.318.1 Solved as second order ode using Kovacic algorithm

Time used: 0.388 (sec)

Writing the ode as

$$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= \frac{5}{3}x + x^2 \\ C &= -\frac{1}{3} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9x^2 + 30x + 7}{36x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9x^2 + 30x + 7 \\
t &= 36x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{9x^2 + 30x + 7}{36x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 599: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{7}{36x^2} + \frac{5}{6x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{5}{6x} - \frac{1}{2x^2} + \frac{5}{6x^3} - \frac{59}{36x^4} + \frac{385}{108x^5} - \frac{2681}{324x^6} + \frac{19525}{972x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^2 + 30x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{30x + 7}{36x^2}\right) \\ &= \frac{1}{4} + \frac{30x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 30. Dividing this by leading coefficient in t which is 36 gives $\frac{5}{6}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{6}\right) - (0) \\ &= \frac{5}{6} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{6}}{\frac{1}{2}} - 0\right) &&= \frac{5}{6} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{6}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^2 + 30x + 7}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{5}{6}$	$-\frac{5}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{5}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{5}{6} - \left(-\frac{1}{6}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{6x} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{6x} + \frac{1}{2} \\
&= -\frac{1}{6x} + \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{6x} + \frac{1}{2}\right)(1) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} + \frac{1}{2}\right)^2 - \left(\frac{9x^2 + 30x + 7}{36x^2}\right) \right) &= 0 \\
\frac{-1 - 3a_0}{3x} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = -\frac{1}{3} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{1}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{1}{3}\right) e^{\int \left(-\frac{1}{6x} + \frac{1}{2}\right) dx} \\
&= \left(x - \frac{1}{3}\right) e^{\frac{x}{2} - \frac{\ln(x)}{6}} \\
&= \frac{(-1 + 3x) e^{\frac{x}{2}}}{3x^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{5}{3}x + x^2}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \frac{5\ln(x)}{6}} \\
&= z_1 \left(\frac{e^{-\frac{x}{2}}}{x^{5/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{-1 + 3x}{3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{5}{3}x + x^2}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x - \frac{5\ln(x)}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{9 e^{-x - \frac{5\ln(x)}{3}} x^2}{(-1 + 3x)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{-1+3x}{3x}\right) + c_2 \left(\frac{-1+3x}{3x} \left(\int \frac{9\,\mathrm{e}^{-x-\frac{5\ln(x)}{3}} x^2}{(-1+3x)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.318.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{y}{3x^2} - \frac{(3x+5)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3x+5)y'}{3x} - \frac{y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3x+5}{3x}, P_3(x) = -\frac{1}{3x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{3}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $3x^2\left(\frac{d}{dx}y'\right) + x(3x+5)y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(3k+3r-1) + 3a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-1, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3(k+r+1)\left(k+r-\frac{1}{3}\right) a_k + 3a_{k-1}(k+r-1) = 0$$

- Shift index using $k- > k+1$

$$3(k+2+r)\left(k+\frac{2}{3}+r\right) a_{k+1} + 3a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{3a_k(k+r)}{(k+2+r)(3k+2+3r)}$$

- Recursion relation for $r = -1$; series terminates at $k = 1$

$$a_{k+1} = -\frac{3a_k(k-1)}{(k+1)(3k-1)}$$

- Apply recursion relation for $k = 0$

$$a_1 = -3a_0$$

- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

$$y = a_0 \cdot (-3x+1)$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = -\frac{3a_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{3a_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot (-3x+1) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), b_{k+1} = -\frac{3b_k \left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}\right]$$

1.318.3 Maple trace

```
Methods for second order ODEs:
```

1.318.4 Maple dsolve solution

Solving time : 0.025 (sec)
Leaf size : 29

```
dsolve(x^2*diff(diff(y(x),x),x)+(5/3*x+x^2)*diff(y(x),x)-1/3*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 x^{4/3} \operatorname{hypergeom}\left([2], \left[\frac{7}{3}\right], x\right) e^{-x} - 3c_2 x + c_2}{x}$$

1.318.5 Mathematica DSolve solution

Solving time : 0.858 (sec)
Leaf size : 47

```
DSolve[{x^2*D[y[x],{x,2}]+(5/3*x+x^2)*D[y[x],x]-1/3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{-3c_1 x + 3c_2 e^{-x} \sqrt[3]{x} + c_2(1-3x)\Gamma\left(\frac{1}{3}, x\right) + c_1}{3x}$$

1.319 problem 326

Internal problem ID [8457]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 326
Date solved : `Monday, October 21, 2024 at 05:08:11 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$2xy'' - y' + 2y = 0$$

1.319.1 Solved as second order ode using Kovacic algorithm

Time used: 0.227 (sec)

Writing the ode as

$$2xy'' - y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{5 - 16x}{16x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 5 - 16x \\
t &= 16x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 601: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -1, \ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right) \\ &= -\frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = x + \frac{1}{4}$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x+\frac{1}{4}} - \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{1}{4}} - \frac{1}{2x}\right) w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4(4x+1)x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(4x+1)x} dx} \\ &= \frac{\left(2\sqrt{-x} - 1\right) \mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{2x} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{4}} \\ &= z_1\left(x^{1/4}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}\left(2\sqrt{-x} - 1\right) \mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{2x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x} - 4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/4}\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{\left(-x\right)^{1/4}}\right) + c_2 \left(\frac{x^{1/4}\left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{\left(-x\right)^{1/4}} \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.319.2 Maple step by step solution

Let's solve

$2x\left(\frac{d}{dx}y'\right) - y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x} + \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'}{2x} + \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x\left(\frac{d}{dx}y'\right) - y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k+1+r)\left(k-\frac{1}{2}+r\right) a_{k+1} + 2a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k}{(k+1+r)(2k-1+2r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}\right]$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = -\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}$
- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}, b_{k+1} = -\frac{2b_k}{(k+\frac{5}{2})(2k+2)}\right]$$

1.319.3 Maple trace

```
Methods for second order ODEs:
```

1.319.4 Maple dsolve solution

Solving time : 0.016 (sec)
Leaf size : 36

```
dsolve(2*x*diff(diff(y(x),x),x)-diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = (2c_1\sqrt{x} + c_2)\cos(2\sqrt{x}) - \sin(2\sqrt{x})(-2c_2\sqrt{x} + c_1)$$

1.319.5 Mathematica DSolve solution

Solving time : 0.121 (sec)
Leaf size : 59

```
DSolve[{2*x*D[y[x],{x,2}]-D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{2i\sqrt{x}}(2\sqrt{x} + i) + \frac{1}{8}c_2 e^{-2i\sqrt{x}}(1 + 2i\sqrt{x})$$

1.320 problem 327

Internal problem ID [8458]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 327
Date solved : `Monday, October 21, 2024 at 05:08:12 PM`
CAS classification : `[_Laguerre]`

Solve

$$2xy'' - (3+2x)\,y' + y = 0$$

1.320.1 Solved as second order ode using Kovacic algorithm

Time used: 0.338 (sec)

Writing the ode as

$$2xy'' + (-3-2x)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -3-2x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 4x + 21 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x + 21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 603: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{4x} + \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4x} + \frac{5}{4x^2} - \frac{5}{8x^3} - \frac{5}{4x^4} + \frac{35}{16x^5} + \frac{105}{64x^6} - \frac{1005}{128x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{4x^2 + 4x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x + 21}{16x^2}\right) \\ &= \frac{1}{4} + \frac{4x + 21}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 4x + 21}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= \frac{1}{4} - \left(-\frac{3}{4}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{3}{4x} + \left(\frac{1}{2}\right) \\
&= -\frac{3}{4x} + \frac{1}{2} \\
&= -\frac{3}{4x} + \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{3}{4x} + \frac{1}{2}\right)(1) + \left(\left(\frac{3}{4x^2}\right) + \left(-\frac{3}{4x} + \frac{1}{2}\right)^2 - \left(\frac{4x^2 + 4x + 21}{16x^2}\right) \right) &= 0 \\
\frac{-3 - 2a_0}{2x} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{3}{2}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{3}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{3}{2}\right) \mathrm{e}^{\int \left(-\frac{3}{4x} + \frac{1}{2}\right) dx} \\
&= \left(x - \frac{3}{2}\right) \mathrm{e}^{\frac{x}{2} - \frac{3 \ln(x)}{4}} \\
&= \frac{(-3 + 2x) \, \mathrm{e}^{\frac{x}{2}}}{2x^{3/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3-2x}{2x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{3 \ln(x)}{4}} \\
&= z_1 \left(x^{3/4} \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{x}(-3 + 2x)}{2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3-2x}{2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x + \frac{3 \ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{4 \, \mathrm{e}^{x + \frac{3 \ln(x)}{2}} \mathrm{e}^{-2x}}{(-3 + 2x)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{x}(-3+2x)}{2}\right) + c_2\left(\frac{e^{x}(-3+2x)}{2}\left(\int \frac{4\,e^{x+\frac{3\ln(x)}{2}}e^{-2x}}{(-3+2x)^{2}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.320.2 Maple step by step solution

Let's solve

$2x\left(\frac{d}{dx}y'\right) - (3+2x)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{2x} + \frac{(3+2x)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(3+2x)y'}{2x} + \frac{y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{3+2x}{2x}, P_3(x) = \frac{1}{2x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -\frac{3}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x\left(\frac{d}{dx}y'\right) + (-3-2x)\,y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-5+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-3+2r) - a_k(2k+2r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-5+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{5}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+1+r)\left(k - \frac{3}{2} + r\right) a_{k+1} - 2a_k\left(k + r - \frac{1}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(2k+2r-1)}{(k+1+r)(2k-3+2r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k(2k-1)}{(k+1)(2k-3)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(2k-1)}{(k+1)(2k-3)}\right]$$

- Recursion relation for $r = \frac{5}{2}$

$$a_{k+1} = \frac{a_k(2k+4)}{\left(k+\frac{7}{2}\right)(2k+2)}$$

- Solution for $r = \frac{5}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+1} = \frac{a_k(2k+4)}{\left(k+\frac{7}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+1} = \frac{a_k(2k-1)}{(k+1)(2k-3)}, b_{k+1} = \frac{b_k(2k+4)}{(k+\frac{7}{2})(2k+2)}\right]$$

1.320.3 Maple trace

```
Methods for second order ODEs:
```

1.320.4 Maple dsolve solution

Solving time : `0.015` (sec)
Leaf size : 24

```
dsolve(2*x*diff(diff(y(x),x),x)-(3+2*x)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \,\mathrm{hypergeom}\left([2], \left[\frac{7}{2}\right], x\right) x^{5/2} - \frac{2\,\mathrm{e}^{x} c_2 \left(x - \frac{3}{2}\right)}{3}$$

1.320.5 Mathematica DSolve solution

Solving time : `1.059` (sec)
Leaf size : 54

```
DSolve[{2*x*D[y[x],{x,2}]-(3+2*x)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}\left(-\sqrt{\pi} c_2 e^x (2x-3)\mathrm{erf}\left(\sqrt{x}\right) + 2c_1 e^x (2x-3) - 6c_2\sqrt{x}\right)$$

1.321 problem 328

Internal problem ID [8459]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 328
Date solved : `Monday, October 21, 2024 at 05:08:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + 3xy' + (2x - 1)\, y = 0$$

1.321.1 Solved as second order ode using Kovacic algorithm

Time used: 0.229 (sec)

Writing the ode as

$$2x^2y'' + 3xy' + (2x - 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 3x \\ C &= 2x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 - 16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 605: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right) \\ &= -\frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d=1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = x + \frac{1}{4}$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x+\frac{1}{4}} - \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{1}{4}} - \frac{1}{2x}\right) w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4\,(4x+1)\,x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(4x+1)x}dx} \\ &= \frac{\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{\left(-x\right)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{2x^2}\,dx} \\ &= z_1 e^{-\frac{3\ln(x)}{4}} \\ &= z_1\left(\frac{1}{x^{3/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{x^{3/4}\left(-x\right)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{3x}{2x^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx$$
$$= y_1 \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{x^{3/4}\left(-x\right)^{1/4}}\right) + c_2 \left(\frac{\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{x^{3/4}\left(-x\right)^{1/4}}\left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)\right)$$

Will add steps showing solving for IC soon.

1.321.2 Maple step by step solution

Let's solve

$2x^2\left(\frac{d}{dx}y'\right) + 3xy' + (2x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(2x-1)y}{2x^2} - \frac{3y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{3y'}{2x} + \frac{(2x-1)y}{2x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{3}{2x}, P_3(x) = \frac{2x-1}{2x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{3}{2}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{2}$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d}{dx}y'\right) + 3xy' + (2x - 1)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ○ Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ○ Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

  - ○ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

    Rewrite ODE with series expansions

    $$a_0(1 + r)\,(-1 + 2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k + r + 1)\,(2k + 2r - 1) + 2a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-1, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k + r - \frac{1}{2}\right)(k + r + 1)\, a_k + 2a_{k-1} = 0$

- Shift index using $k- >k + 1$

  $2\left(k + \frac{1}{2} + r\right)(k + 2 + r)\, a_{k+1} + 2a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{2a_k}{(2k+1+2r)(k+2+r)}$

- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}, b_{k+1} = -\frac{2b_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

1.321.3 Maple trace

```
Methods for second order ODEs:
```

1.321.4 Maple dsolve solution

Solving time : `0.088` (sec)
Leaf size : 73

```
dsolve(2*x^2*diff(diff(y(x),x),x)+3*x*diff(y(x),x)+(2*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\sqrt{\frac{(2\sqrt{x}-i)(4x+1)}{2\sqrt{x}+i}}\, \mathrm{e}^{-2i\sqrt{x}} + c_1\sqrt{\frac{(2\sqrt{x}+i)(4x+1)}{2\sqrt{x}-i}}\, \mathrm{e}^{2i\sqrt{x}}}{x}$$

1.321.5 Mathematica DSolve solution

Solving time : `0.135` (sec)
Leaf size : 64

```
DSolve[{2*x^2*D[y[x],{x,2}]+3*x*D[y[x],x]+(2*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-2i\sqrt{x}}\left(8c_1e^{4i\sqrt{x}}\left(2\sqrt{x}+i\right)+c_2\left(1+2i\sqrt{x}\right)\right)}{8x}$$

1.322 problem 329

Internal problem ID [8460]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 329
Date solved : `Monday, October 21, 2024 at 05:08:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + 2y' - xy = 0$$

1.322.1 Solved as second order ode using Kovacic algorithm

Time used: 0.086 (sec)

Writing the ode as

$$xy'' + 2y' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 607: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{-x}}{x}\right) + c_2\left(\frac{e^{-x}}{x}\left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.322.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 2y' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = y - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2y'}{x} - y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -1\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + 2y' - xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- > k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - Shift index using $k- > k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- > k+1$

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) - a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) - a_{k-1} = 0$
- Shift index using $k- > k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) - a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = \frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = \frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

1.322.3 Maple trace

```
Methods for second order ODEs:
```

1.322.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sinh(x) + c_2 \cosh(x)}{x}$$

1.322.5 Mathematica DSolve solution

Solving time : 0.038 (sec)
Leaf size : 28

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-x} + c_2 e^x}{2x}$$

1.323 problem 330

Internal problem ID [8461]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 330
Date solved : `Monday, October 21, 2024 at 05:08:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

1.323.1 Solved as second order ode using Kovacic algorithm

Time used: 0.164 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 609: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(x)}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.323.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 4xy' + (4x^2 - 1)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0
  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$
  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$$

1.323.3 Maple trace

```
Methods for second order ODEs:
```

1.323.4 Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{\sqrt{x}}$$

1.323.5 Mathematica DSolve solution

Solving time : `0.052` (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

1.324 problem 331

Internal problem ID [8462]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 331
Date solved : `Monday, October 21, 2024 at 05:08:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (x-6)\,y' - 3y = 0$$

1.324.1 Solved as second order ode using Kovacic algorithm

Time used: 0.280 (sec)

Writing the ode as

$$xy'' + (x-6)\,y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x-6 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 48}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 48$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 48}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 611: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{12}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{12}{x^2} - \frac{144}{x^4} + \frac{3456}{x^6} - \frac{103680}{x^8} + \frac{3483648}{x^{10}} - \frac{125411328}{x^{12}} + \frac{4729798656}{x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 48}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{12}{x^2}\right) \\ &= \frac{1}{4} + \frac{12}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 48}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= 0 - (-3) \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{x} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{3}{x} - \frac{1}{2} \\
&= -\frac{6+x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(6x + 2a_2) + 2\left(-\frac{3}{x} - \frac{1}{2}\right)(3x^2 + 2a_2 x + a_1) + \left(\left(\frac{3}{x^2}\right) + \left(-\frac{3}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2+48}{4x^2}\right)\right) = 0
$$

$$
\frac{(a_2 - 12)\, x^2 + 2(a_1 - 5a_2)\, x + 3a_0 - 6a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 120, a_1 = 60, a_2 = 12\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^3 + 12x^2 + 60x + 120
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^3 + 12x^2 + 60x + 120)\, e^{\int \left(-\frac{3}{x} - \frac{1}{2}\right) dx} \\
&= (x^3 + 12x^2 + 60x + 120)\, e^{-\frac{x}{2} - 3\ln(x)} \\
&= \frac{(x^3 + 12x^2 + 60x + 120)\, e^{-\frac{x}{2}}}{x^3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x-6}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}+3\ln(x)} \\
&= z_1 \left(x^3 \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}\left(x^3+12x^2+60x+120\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x-6}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+6\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(x^3-12x^2+60x-120)\,\mathrm{e}^{-x+6\ln(x)}\mathrm{e}^{2x}}{(x^3+12x^2+60x+120)\,x^6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\left(x^3+12x^2+60x+120\right)\right) \\
&\quad + c_2\left(\mathrm{e}^{-x}\left(x^3+12x^2+60x+120\right)\left(\frac{(x^3-12x^2+60x-120)\,\mathrm{e}^{-x+6\ln(x)}\mathrm{e}^{2x}}{(x^3+12x^2+60x+120)\,x^6}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.324.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right)+(x-6)\,y'-3y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{3y}{x}-\frac{(x-6)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(x-6)y'}{x}-\frac{3y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x-6}{x}, P_3(x)=-\frac{3}{x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-6$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

$x\left(\frac{d}{dx}y'\right)+(x-6)\,y'-3y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-7+r) x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k-6+r) + a_k(k+r-3)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-7+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 7\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-6+r) + a_k(k+r-3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-3)}{(k+1+r)(k-6+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 3$

  $a_{k+1} = -\frac{a_k(k-3)}{(k+1)(k-6)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{a_0}{2}$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{5}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{10}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2}{12}$
- Express in terms of a_0

  $a_3 = -\frac{a_0}{120}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right)$
- Recursion relation for $r = 7$

  $a_{k+1} = -\frac{a_k(k+4)}{(k+8)(k+1)}$
- Solution for $r = 7$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+7}, a_{k+1} = -\frac{a_k(k+4)}{(k+8)(k+1)}\right]$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot \left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+7}\right), b_{k+1} = -\frac{b_k(k+4)}{(k+8)(k+1)}\right]$$

1.324.3 Maple trace

```
Methods for second order ODEs:
```

1.324.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 39

```
dsolve(x*diff(diff(y(x),x),x)+(x-6)*diff(y(x),x)-3*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1(x^3 - 12x^2 + 60x - 120) + c_2 \,\mathrm{e}^{-x}(x^3 + 12x^2 + 60x + 120)$$

1.324.5 Mathematica DSolve solution

Solving time : `0.094` (sec)
Leaf size : 98

```
DSolve[{x*D[y[x],{x,2}]+(x-6)*D[y[x],x]-3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2e^{-x/2}\sqrt{x}\left((c_1x^3 + 12ic_2x^2 + 60c_1x + 120ic_2)\cosh\left(\frac{x}{2}\right) - (12c_1(x^2 + 10) + ic_2x(x^2 + 60))\sinh\left(\frac{x}{2}\right)\right)}{\sqrt{\pi}\sqrt{-ix}}$$

1.325 problem 332

Internal problem ID [8463]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 332
Date solved : `Monday, October 21, 2024 at 05:08:17 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$x^4 y'' + \lambda y = 0$$

1.325.1 Solved as second order ode using Kovacic algorithm

Time used: 0.252 (sec)

Writing the ode as

$$x^4 y'' + \lambda y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= 0 \\ C &= \lambda \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\lambda}{x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\lambda \\ t &= x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{\lambda}{x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 613: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{\lambda}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i\sqrt{\lambda}}{x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{i\sqrt{\lambda}}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = i\sqrt{\lambda}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{i\sqrt{\lambda}}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{0}{i\sqrt{\lambda}} + 2\right) &= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{0}{i\sqrt{\lambda}} + 2\right) &= 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{\lambda}{x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{i\sqrt{\lambda}}{x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x} + (-)(0) \\
&= -\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x} \\
&= \frac{-i\sqrt{\lambda} + x}{x^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{2i\sqrt{\lambda}}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right)^2 - \left(-\frac{\lambda}{x^4}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right) dx} \\
&= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \int \frac{1}{x^2 \mathrm{e}^{\frac{2i\sqrt{\lambda}}{x}}}\,dx \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \left(-\frac{i\mathrm{e}^{-\frac{2i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}\right) + c_2\left(x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}\left(-\frac{i\mathrm{e}^{-\frac{2i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.325.2 Maple step by step solution

1.325.3 Maple trace

```
Methods for second order ODEs:
```

1.325.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 31

```
dsolve(x^4*diff(diff(y(x),x),x)+lambda*y(x) = 0,
       y(x),singsol=all)
```

$$y = x\left(c_1 \sinh\left(\frac{\sqrt{-\lambda}}{x}\right) + c_2 \cosh\left(\frac{\sqrt{-\lambda}}{x}\right)\right)$$

1.325.5 Mathematica DSolve solution

Solving time : 0.177 (sec)
Leaf size : 52

```
DSolve[{x^4*D[y[x],{x,2}]+\[Lambda]*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x e^{\frac{i\sqrt{\lambda}}{x}} - \frac{i c_2 x e^{-\frac{i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}$$

1.326 problem 333

Internal problem ID [8464]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 333
Date solved : `Monday, October 21, 2024 at 05:08:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 4xy' + \left(4x^2 - 25\right) y = 0$$

1.326.1 Solved as second order ode using Kovacic algorithm

Time used: 0.312 (sec)

Writing the ode as

$$4x^2y'' + 4xy' + \left(4x^2 - 25\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 4x \\ C &= 4x^2 - 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 614: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= -1 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{0}{i}-0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{0}{i}-0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2+6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + (-)\,(i) \\
&= -\frac{2}{x} - i \\
&= -\frac{2}{x} - i
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2 + 6}{x^2}\right)\right) = 0
$$

$$
\frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -3, a_1 = -3i\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 3ix - 3
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 3ix - 3\right) e^{\int \left(-\frac{2}{x} - i\right) dx} \\
&= \left(x^2 - 3ix - 3\right) e^{-2\ln(x) - ix} \\
&= \frac{\left(x^2 - 3ix - 3\right) e^{-ix}}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{4x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}\right) + c_2 \left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}} \left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.326.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right)+4xy'+\left(4x^2-25\right)y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(4x^2-25)y}{4x^2}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{y'}{x}+\frac{(4x^2-25)y}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{4x^2-25}{4x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-\frac{25}{4}$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right)+4xy'+\left(4x^2-25\right)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x\cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(5+2r)\,(-5+2r)\, x^r + a_1(7+2r)\,(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+5)\,(2k+2r-5) + 4a_{\cdot}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+2r)\,(-5+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{5}{2}, \frac{5}{2}\right\}$
- Each term must be 0

  $a_1(7+2r)\,(-3+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r+5)\,(2k+2r-5) + 4a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(2k+9+2r)\,(2k-1+2r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$
- Recursion relation for $r = -\frac{5}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}$
- Solution for $r = -\frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0\right]$$
- Recursion relation for $r = \frac{5}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}$
- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}, a_1 = 0\right]$$
- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(2k+14)(2k+4)}, b_1 = 0\right]$$

1.326.3 Maple trace

```
Methods for second order ODEs:
```

1.326.4 Maple dsolve solution

Solving time : `0.016` (sec)
Leaf size : 43

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(4*x^2-25)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-3c_2\left(ix - \frac{1}{3}x^2 + 1\right) e^{-ix} + 3c_1\left(ix + \frac{1}{3}x^2 - 1\right) e^{ix}}{x^{5/2}}$$

1.326.5 Mathematica DSolve solution

Solving time : `0.139` (sec)
Leaf size : 59

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2-25)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\left(\left(-c_2x^2 + 3c_1x + 3c_2\right)\cos(x) + \left(c_1\left(x^2 - 3\right) + 3c_2x\right)\sin(x)\right)}{x^{5/2}}$$

1.327 problem 334

Internal problem ID [8465]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 334
Date solved : `Monday, October 21, 2024 at 05:08:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0$$

1.327.1 Solved as second order ode using Kovacic algorithm

Time used: 0.175 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 36x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-36}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -36 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -36z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 616: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -36$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(6x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(6x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(6x)}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(6x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(6x)}{\sqrt{x}} \left(\frac{\tan(6x)}{6}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.327.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(144x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{x} + \frac{(144x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{144x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 4xy' + (144x^2 - 1)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (1 + 2r)\, (-1 + 2r)\, x^r + a_1 (3 + 2r)\, (1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k (2k + 2r + 1)\, (2k + 2r - 1) + 144a$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$(1 + 2r)\, (-1 + 2r) = 0$
- Values of r that satisfy the indicial equation
$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0
$a_1 (3 + 2r)\, (1 + 2r) = 0$
- Solve for the dependent coefficient(s)
$a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
$a_k (4k^2 + 8kr + 4r^2 - 1) + 144a_{k-2} = 0$
- Shift index using $k- >k + 2$
$a_{k+2} \left(4(k + 2)^2 + 8(k + 2)\, r + 4r^2 - 1\right) + 144a_k = 0$
- Recursion relation that defines series solution to ODE
$a_{k+2} = -\frac{144a_k}{4k^2 + 8kr + 4r^2 + 16k + 16r + 15}$
- Recursion relation for $r = -\frac{1}{2}$
$a_{k+2} = -\frac{144a_k}{4k^2 + 12k + 8}$
- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{144a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{144a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{144a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{144a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{144b_k}{4k^2+20k+24}, b_1 = 0\right]$$

1.327.3 Maple trace

```
Methods for second order ODEs:
```

1.327.4 Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(36*x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(6x) + c_2 \cos(6x)}{\sqrt{x}}$$

1.327.5 Mathematica DSolve solution

Solving time : `0.061` (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(36*x^2-1/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-6ix}(12c_1 - ic_2 e^{12ix})}{12\sqrt{x}}$$

1.328 problem 335

Internal problem ID [8466]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 335
Date solved : `Monday, October 21, 2024 at 05:08:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \left(x^2 - 2\right) y = 0$$

1.328.1 Solved as second order ode using Kovacic algorithm

Time used: 0.283 (sec)

Writing the ode as

$$x^2y'' + \left(x^2 - 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 618: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{x^2} - \frac{i}{2x^4} - \frac{i}{2x^6} - \frac{5i}{8x^8} - \frac{7i}{8x^{10}} - \frac{21i}{16x^{12}} - \frac{33i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{2}{x^2}\right) \\ &= -1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)(i) \\
&= -\frac{1}{x} - i \\
&= -\frac{1}{x} - i
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - i\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - i\right)^2 - \left(\frac{-x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{2ia_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -i\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - i
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x - i) \, \mathrm{e}^{\int \left(-\frac{1}{x} - i\right) dx} \\
&= (x - i) \, \mathrm{e}^{-\ln(x) - ix} \\
&= \frac{(x - i) \, \mathrm{e}^{-ix}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \frac{(x - i)\,\mathrm{e}^{-ix}}{x}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x - i)\,\mathrm{e}^{-ix}}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \frac{(x - i)\,\mathrm{e}^{-ix}}{x} \int \frac{1}{\frac{(x-i)^2\mathrm{e}^{-2ix}}{x^2}}\,dx \\
&= \frac{(x - i)\,\mathrm{e}^{-ix}}{x} \left(\frac{(ix - 1)\,\mathrm{e}^{2ix}}{-2x + 2i}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x - i)\,\mathrm{e}^{-ix}}{x}\right) + c_2 \left(\frac{(x - i)\,\mathrm{e}^{-ix}}{x} \left(\frac{(ix - 1)\,\mathrm{e}^{2ix}}{-2x + 2i}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.328.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+\left(x^2-2\right)y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(x^2-2)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(x^2-2)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=0, P_3(x)=\frac{x^2-2}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-2$

  ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right)+\left(x^2-2\right)y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+r)(-2+r)\, x^r + a_1(2+r)(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)(k+r-2) + a_{k-2}\right) x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 2\}$
- Each term must be 0
  $a_1(2+r)(-1+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+1)(k+r-2) + a_{k-2} = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(k+3+r)(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{(k+3+r)(k+r)}$
- Recursion relation for $r = -1$
  $a_{k+2} = -\frac{a_k}{(k+2)(k-1)}$
- Solution for $r = -1$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+2)(k-1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$
  $a_{k+2} = -\frac{a_k}{(k+5)(k+2)}$
- Solution for $r = 2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+5)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k-1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+5)(k+2)}, b_1 = 0\right]$

1.328.3 Maple trace

```
Methods for second order ODEs:
```

1.328.4 Maple dsolve solution

Solving time : `0.074` (sec)
Leaf size : 27

```
dsolve(x^2*diff(diff(y(x),x),x)+(x^2-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(c_1x + c_2)\cos(x) + \sin(x)(c_2x - c_1)}{x}$$

1.328.5 Mathematica DSolve solution

Solving time : `0.03` (sec)
Leaf size : 21

```
DSolve[{x^2*D[y[x],{x,2}]+(x^2-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x(c_1 j_1(x) - c_2 y_1(x))$$

1.329 problem 336

Internal problem ID [8467]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 336
Date solved : `Monday, October 21, 2024 at 05:08:20 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$xy'' + 3y' + x^3y = 0$$

1.329.1 Solved as second order ode using Kovacic algorithm

Time used: 0.286 (sec)

Writing the ode as

$$xy'' + 3y' + x^3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4x^4 + 3}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4x^4 + 3 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-4x^4 + 3}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 620: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 4\\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx ix - \frac{3i}{8x^3} - \frac{9i}{128x^7} - \frac{27i}{1024x^{11}} - \frac{405i}{32768x^{15}} - \frac{1701i}{262144x^{19}} - \frac{15309i}{4194304x^{23}} - \frac{72171i}{33554432x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-4x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-x^2) + \left(\frac{3}{4x^2}\right) \\ &= -x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= ix \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-4x^4 + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	ix	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{2x} + (-)\,(ix) \\ &= -\frac{1}{2x} - ix \\ &= -\frac{1}{2x} - ix\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{2x} - ix\right)(0) + \left(\left(\frac{1}{2x^2} - i\right) + \left(-\frac{1}{2x} - ix\right)^2 - \left(\frac{-4x^4 + 3}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2x} - ix\right) dx} \\ &= \frac{\mathrm{e}^{-\frac{i x^2}{2}}}{\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{x}\, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{x^{3/2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{i x^2}{2}}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{i e^{i x^2}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-\frac{i x^2}{2}}}{x^2}\right) + c_2 \left(\frac{e^{-\frac{i x^2}{2}}}{x^2}\left(-\frac{i e^{i x^2}}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.329.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 3y' + x^3 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -x^2 y - \frac{3y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{3y'}{x} + x^2 y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3}{x}, P_3(x) = x^2\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + 3y' + x^3y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

  $$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$

- Shift index using $k- >k - 3$

  $$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r)\,x^{-1+r}+a_1(1+r)\,(3+r)\,x^{r}+a_2(2+r)\,(4+r)\,x^{1+r}+a_3(3+r)\,(5+r)\,x^{2+r}+\left(\sum_{k=3}^{\infty}\right($$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(2+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{-2,0\}$
- The coefficients of each power of x must be 0
  $[a_1(1+r)\,(3+r)=0, a_2(2+r)\,(4+r)=0, a_3(3+r)\,(5+r)=0]$
- Solve for the dependent coefficient(s)
  $\{a_1=0, a_2=0, a_3=0\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k+r+3)+a_{k-3}=0$
- Shift index using $k->k+3$
  $a_{k+4}(k+4+r)\,(k+6+r)+a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+4}=-\frac{a_k}{(k+4+r)(k+6+r)}$
- Recursion relation for $r=-2$
  $a_{k+4}=-\frac{a_k}{(k+2)(k+4)}$
- Solution for $r=-2$
  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-2}, a_{k+4}=-\frac{a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0\right]$
- Recursion relation for $r=0$
  $a_{k+4}=-\frac{a_k}{(k+4)(k+6)}$
- Solution for $r=0$
  $\left[y=\sum_{k=0}^{\infty}a_k x^{k}, a_{k+4}=-\frac{a_k}{(k+4)(k+6)}, a_1=0, a_2=0, a_3=0\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k}\right), a_{k+4}=-\frac{a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{b_k}{(k+4)(k+6)}\right.$

1.329.3 Maple trace

```
Methods for second order ODEs:
```

1.329.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 25

```
dsolve(x*diff(diff(y(x),x),x)+3*diff(y(x),x)+x^3*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(\frac{x^2}{2}\right) + c_2 \cos\left(\frac{x^2}{2}\right)}{x^2}$$

1.329.5 Mathematica DSolve solution

Solving time : 0.089 (sec)
Leaf size : 43

```
DSolve[{x*D[y[x],{x,2}]+3*D[y[x],x]+x^3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{ix^2}{2}} \left(2c_1 - ic_2 e^{ix^2}\right)}{2x^2}$$

1.330 problem 337

Internal problem ID [8468]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 337
Date solved : `Monday, October 21, 2024 at 05:08:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 4xy' + \left(x^2+2\right)y = 0$$

1.330.1 Solved as second order ode using Kovacic algorithm

Time used: 0.149 (sec)

Writing the ode as

$$x^2y'' + 4xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 622: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\,dx} \\ &= z_1 e^{-2\ln(x)} \\ &= z_1\left(\frac{1}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x^2}\right) + c_2 \left(\frac{\cos(x)}{x^2}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.330.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + 4xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+2)y}{x^2} - \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{4y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + 4xy' + (x^2+2)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (2+r)\,(1+r)\, x^r + a_1 (3+r)\,(2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k+r+2)\,(k+r+1) + a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)\,(1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-2, -1\}$

- Each term must be 0

$a_1 (3+r)\,(2+r) = 0$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$a_k (k+r+2)\,(k+r+1) + a_{k-2} = 0$

- Shift index using $k- > k + 2$

$a_{k+2} (k+4+r)\,(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k}{(k+4+r)(k+3+r)}$

- Recursion relation for $r = -2$

$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = -2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$$

- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$

- Solution for $r=-1$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$$

1.330.3 Maple trace

```
Methods for second order ODEs:
```

1.330.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1 \sin(x)+c_2 \cos(x)}{x^2}$$

1.330.5 Mathematica DSolve solution

Solving time : `0.045` (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{2c_1 e^{-ix}-ic_2 e^{ix}}{2x^2}$$

1.331 problem 338

Internal problem ID [8469]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 338
Date solved : `Monday, October 21, 2024 at 05:08:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$16x^2y'' + 32xy' + \left(x^4 - 12\right)y = 0$$

1.331.1 Solved as second order ode using Kovacic algorithm

Time used: 0.288 (sec)

Writing the ode as

$$16x^2y'' + 32xy' + \left(x^4 - 12\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^2 \\ B &= 32x \\ C &= x^4 - 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 + 12}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^4 + 12$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 + 12}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 624: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 4\\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{x^2}{16} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{ix}{4} - \frac{3i}{2x^3} - \frac{9i}{2x^7} - \frac{27i}{x^{11}} - \frac{405i}{2x^{15}} - \frac{1701i}{x^{19}} - \frac{15309i}{x^{23}} - \frac{144342i}{x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{ix}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{-x^4 + 12}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(-\frac{x^2}{16}\right) + \left(\frac{3}{4x^2}\right) \\ &= -\frac{x^2}{16} + \frac{3}{4x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned}b &= (0) - (0) \\ &= 0\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{ix}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{i}{4}} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{i}{4}} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^4 + 12}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{ix}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{2x} + (-)\left(\frac{ix}{4}\right) \\ &= -\frac{1}{2x} - \frac{ix}{4} \\ &= -\frac{1}{2x} - \frac{ix}{4}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{2x} - \frac{ix}{4}\right)(0) + \left(\left(\frac{1}{2x^2} - \frac{i}{4}\right) + \left(-\frac{1}{2x} - \frac{ix}{4}\right)^2 - \left(\frac{-x^4 + 12}{16x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{ix}{4}\right) dx} \\ &= \frac{\mathrm{e}^{-\frac{ix^2}{8}}}{\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{32x}{16x^2}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{i x^2}{8}}}{x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{32x}{16x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-2i e^{\frac{i x^2}{4}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-\frac{i x^2}{8}}}{x^{3/2}}\right) + c_2 \left(\frac{e^{-\frac{i x^2}{8}}}{x^{3/2}} \left(-2i e^{\frac{i x^2}{4}}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.331.2 Maple step by step solution

Let's solve

$16x^2 \left(\frac{d}{dx} y'\right) + 32 x y' + (x^4 - 12)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(x^4-12)y}{16x^2} - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{2y'}{x} + \frac{(x^4-12)y}{16x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = \frac{x^4-12}{16x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{3}{4}$$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2\left(\frac{d}{dx}y'\right) + 32xy' + (x^4 - 12)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..4$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ◦ Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r}$$

- ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$4a_0(3 + 2r)\,(-1 + 2r)\, x^r + 4a_1(5 + 2r)\,(1 + 2r)\, x^{1+r} + 4a_2(7 + 2r)\,(3 + 2r)\, x^{2+r} + 4a_3(9 + 2r)\,($$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4(3 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{3}{2}, \frac{1}{2}\right\}$
- The coefficients of each power of x must be 0

  $[4a_1(5+2r)(1+2r)=0, 4a_2(7+2r)(3+2r)=0, 4a_3(9+2r)(5+2r)=0]$
- Solve for the dependent coefficient(s)

  $\{a_1=0, a_2=0, a_3=0\}$
- Each term in the series must be 0, giving the recursion relation

  $16\left(k+r-\frac{1}{2}\right)\left(k+r+\frac{3}{2}\right)a_k+a_{k-4}=0$
- Shift index using $k->k+4$

  $16\left(k+\frac{7}{2}+r\right)\left(k+\frac{11}{2}+r\right)a_{k+4}+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+4}=-\frac{a_k}{4(2k+7+2r)(2k+11+2r)}$
- Recursion relation for $r=-\frac{3}{2}$

  $a_{k+4}=-\frac{a_k}{4(2k+4)(2k+8)}$
- Solution for $r=-\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+4}=-\frac{a_k}{4(2k+4)(2k+8)}, a_1=0, a_2=0, a_3=0\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+4}=-\frac{a_k}{4(2k+8)(2k+12)}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+4}=-\frac{a_k}{4(2k+8)(2k+12)}, a_1=0, a_2=0, a_3=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+4}=-\frac{a_k}{4(2k+4)(2k+8)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{}{4(2k\text{-}}\right.$

1.331.3 Maple trace

```
Methods for second order ODEs:
```

1.331.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 25

```
dsolve(16*x^2*diff(diff(y(x),x),x)+32*x*diff(y(x),x)+(x^4-12)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(\frac{x^2}{8}\right) + c_2 \cos\left(\frac{x^2}{8}\right)}{x^{3/2}}$$

1.331.5 Mathematica DSolve solution

Solving time : 0.101 (sec)
Leaf size : 42

```
DSolve[{16*x^2*D[y[x],{x,2}]+32*x*D[y[x],x]+(x^4-12)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{ix^2}{8}} \left(c_1 - 2ic_2 e^{\frac{ix^2}{4}}\right)}{x^{3/2}}$$

1.332 problem 339

Internal problem ID [8470]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 339
Date solved : `Monday, October 21, 2024 at 05:08:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2 y' + xy = 0$$

1.332.1 Solved as second order ode using Kovacic algorithm

Time used: 0.377 (sec)

Writing the ode as

$$y'' - x^2 y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3 - 8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x(x^3 - 8)$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3 - 8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 626: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} - \frac{2}{x} - \frac{4}{x^4} - \frac{16}{x^7} - \frac{80}{x^{10}} - \frac{448}{x^{13}} - \frac{2688}{x^{16}} - \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x(x^3-8)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^4 - 2x\right) + (0) \\
&= \frac{1}{4}x^4 - 2x
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -2. Now b can be found.

$$\begin{aligned}
b &= (-2) - (0) \\
&= -2
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^2}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 2\right) &= -3 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 2\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3-8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	-3	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^2}{2}\right) \\ &= -\frac{x^2}{2} \\ &= -\frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x^2}{2}\right)(1) + \left((-x) + \left(-\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3-8)}{4}\right)\right) = 0$$

$$xa_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x)\, \mathrm{e}^{\int -\frac{x^2}{2} dx} \\ &= (x)\, \mathrm{e}^{-\frac{x^3}{6}} \\ &= x\, \mathrm{e}^{-\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{1} \, dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{3^{2/3}(-1)^{1/3}\left(-\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}} + \frac{3\, 3^{1/3}(-1)^{2/3}\mathrm{e}^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{2/3}}\right)}{9} \right) \end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1(x) + c_2\left(x\left(\frac{3^{2/3}(-1)^{1/3}\left(-\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}} + \frac{3\,3^{1/3}(-1)^{2/3}\mathrm{e}^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{2/3}}\right)}{9}\right)\right)$$

Will add steps showing solving for IC soon.

1.332.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - x^2y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  - Shift index using $k- >k-1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

$\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_{k-1}(k-2)\right)x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\,a_{k+2} - a_{k-1}(k-2) = 0$

- Shift index using $k- > k+1$

  $\left((k+1)^2+3k+5\right)a_{k+3} - a_k(k-1) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k(k-1)}{k^2+5k+6}, 2a_2 = 0\right]$

1.332.3 Maple trace

```
Methods for second order ODEs:
```

1.332.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x^2*diff(y(x),x)+x*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2\left(-x^3\right)^{1/3} 3^{2/3}\Gamma\left(\frac{2}{3}\right) - c_2\left(-x^3\right)^{1/3} 3^{2/3}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) + 3\,\mathrm{e}^{\frac{x^3}{3}} c_2 + c_1 x$$

1.332.5 Mathematica DSolve solution

Solving time : 0.082 (sec)
Leaf size : 41

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]+x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x - \frac{c_2 \sqrt[3]{-x^3}\Gamma\left(-\frac{1}{3}, -\frac{x^3}{3}\right)}{3\sqrt[3]{3}}$$

1.333 problem 340

Internal problem ID [8471]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 340
Date solved : `Monday, October 21, 2024 at 05:08:24 PM`
CAS classification : `[_Laguerre]`

Solve

$$xy'' - (x+2)\,y' + 2y = 0$$

1.333.1 Solved as second order ode using Kovacic algorithm

Time used: 0.233 (sec)

Writing the ode as

$$xy'' + (-x-2)\,y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x-2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 4x + 8 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 628: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{x} \\ &= \frac{x - 2}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\ &= \frac{\mathrm{e}^{\frac{x}{2}}}{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x-2}{x} \, dx} \\ &= z_1 e^{\frac{x}{2} + \ln(x)} \\ &= z_1 \left(x \, \mathrm{e}^{\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-2}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(x^2+2x+2)\, \mathrm{e}^{x+2\ln(x)} \mathrm{e}^{-2x}}{x^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2 \left(\mathrm{e}^x \left(-\frac{(x^2+2x+2)\, \mathrm{e}^{x+2\ln(x)} \mathrm{e}^{-2x}}{x^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.333.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) - (x+2)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2y}{x} + \frac{(x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x+2)y'}{x} + \frac{2y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x+2}{x}, P_3(x) = \frac{2}{x}\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -2$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
- $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $x\left(\frac{d}{dx}y'\right) + (-x-2)\,y' + 2y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$
  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
- Shift index using $k- >k+1-m$
  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$
- Shift index using $k- >k+1$
  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions
  $a_0 r(-3+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-2) - a_k(k+r-2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation
  $(k+r-2)\left(a_{k+1}(k+1+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=3$
  $a_{k+1}=\frac{a_k}{k+4}$
- Solution for $r=3$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1}=\frac{a_k}{k+4}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+4}\right]$

1.333.3 Maple trace

```
Methods for second order ODEs:
```

1.333.4 Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 19

```
dsolve(x*diff(diff(y(x),x),x)-(x+2)*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y=c_1\,\mathrm{e}^{x}+c_2\left(x^2+2x+2\right)$$

1.333.5 Mathematica DSolve solution

Solving time : 0.05 (sec)
Leaf size : 24

```
DSolve[{x*D[y[x],{x,2}]-(x+2)*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2 \left(x^2 + 2x + 2\right)$$

1.334 problem 341

Internal problem ID [8472]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 341
Date solved : `Monday, October 21, 2024 at 05:08:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

1.334.1 Solved as second order ode using Kovacic algorithm

Time used: 0.241 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 630: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) = 0$$
$$a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, \mathrm{e}^{\int -\frac{x}{2} dx} \\ &= (x) \, \mathrm{e}^{-\frac{x^2}{4}} \\ &= x \, \mathrm{e}^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{2}} x\left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.334.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

1.334.3 Maple trace

```
Methods for second order ODEs:
```

1.334.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -x\left(c_2\pi\ \mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)-c_1\right)\mathrm{e}^{-\frac{x^2}{2}}+i\sqrt{\pi}\,\sqrt{2}\,c_2$$

1.334.5 Mathematica DSolve solution

Solving time : 0.09 (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\sqrt{\frac{\pi}{2}}c_2e^{-\frac{x^2}{2}}\sqrt{x^2}\mathrm{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right)+\sqrt{2}c_1e^{-\frac{x^2}{2}}x+c_2$$

1.335 problem 342

Internal problem ID [8473]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 342
Date solved : `Monday, October 21, 2024 at 05:08:26 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-2xy'+2y=0$$

1.335.1 Solved as second order ode using Kovacic algorithm

Time used: 0.274 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 3}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 - 3$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 3}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 632: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\&= 4 - 2\\&= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x+1)^2} + \frac{5}{4(x-1)} - \frac{1}{4(x-1)^2} - \frac{5}{4(x+1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 3}{(x^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^2-3}{(x^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= 2-(1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(x+1)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)^2 - \left(\frac{2x^2-3}{(x^2-1)^2}\right.\right. \\
-\frac{2a_0}{x^2-1} =
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= (x)\sqrt{(x-1)(x+1)} \\
&= x\sqrt{x^2-1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{1}{x}+\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}\right) + c_2 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}\left(\frac{1}{x}+\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.335.2 Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d}{dx}y'\right)-2xy'+2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{2y}{x^2-1}-\frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{2xy'}{x^2-1}-\frac{2y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{2}{x^2-1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=1$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d}{dx}y'\right)+2xy'-2y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+2)\,(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-2a_{k+1}(k+1)^2 + a_k(k+2)\,(k-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$$a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$$

- Apply recursion relation for $k = 0$

$$a_1 = -a_0$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot (-u+1)$$

- Revert the change of variables $u = x + 1$

$$[y = -a_0x]$$

1.335.3 Maple trace

```
Methods for second order ODEs:
```

1.335.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 25

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\ln(x-1)\, c_2 x}{2} - \frac{\ln(x+1)\, c_2 x}{2} + c_1 x + c_2$$

1.335.5 Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 33

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

1.336 problem 343

Internal problem ID [8474]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 343
Date solved : `Monday, October 21, 2024 at 05:08:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0$$

1.336.1 Solved as second order ode using Kovacic algorithm

Time used: 0.084 (sec)

Writing the ode as

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 634: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\ &= z_1 e^{x^2} \\ &= z_1 \left(\mathrm{e}^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{x^2}\right) + c_2\left(e^{x^2}(x)\right)\end{aligned}$$

Will add steps showing solving for IC soon.

1.336.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - 4xy' + (4x^2 - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k + 2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 - 2a_0 + (6a_3 - 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = a_0, a_3 = a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2ka_{k+2} - 2a_k + 5a_{k+2})}{k^2+7k+12}, a_2 = a_0, a_3 = a_1\right]$

1.336.3 Maple trace

```
Methods for second order ODEs:
```

1.336.4 Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(4*x^2-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{x^2}(c_2 x + c_1)$$

1.336.5 Mathematica DSolve solution

Solving time : `0.031` (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x^2}(c_2 x + c_1)$$

1.337 problem 344

Internal problem ID [8475]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 344
Date solved : `Monday, October 21, 2024 at 05:08:27 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-2 x y'+30 y=0$$

1.337.1 Solved as second order ode using Kovacic algorithm

Time used: 0.323 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-2 x y'+30 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -2x \\ C &= 30 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{30x^2 - 31}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 30x^2 - 31$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{30x^2 - 31}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 636: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{61}{4(x-1)} - \frac{1}{4(x+1)^2} - \frac{1}{4(x-1)^2} - \frac{61}{4(x+1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{30x^2 - 31}{(x^2 - 1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=30$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=6 \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-5
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{30x^2-31}{(x^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	6	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=6$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= 6-(1) \\
&= 5
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(20x^3 + 12x^2 a_4 + 6x a_3 + 2a_2) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(5x^4 + 4x^3 a_4 + 3x^2 a_3 + 2x a_2 + a_1) + \left(\left(-\frac{1}{2(x -
$$

$$
-10a_4 x^4 + (-18a_3 - 20
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = 0, a_1 = \frac{5}{21}, a_2 = 0, a_3 = -\frac{10}{9}, a_4 = 0\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right) e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= \left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right) \sqrt{(x-1)(x+1)} \\
&= \frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(63x^5-70x^3+15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{3969\ln(x+1)}{128}+\frac{441}{25x}-\frac{3087\left(-23x^3+\frac{935}{63}x\right)}{1600\left(x^4-\frac{10}{9}x^2+\frac{5}{21}\right)}+\frac{3969\ln(x-1)}{128}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left(\frac{(63x^5-70x^3+15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{x+1}}\right) \\
&\quad + c_2\left(\frac{(63x^5-70x^3+15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{x+1}}\left(-\frac{3969\ln(x+1)}{128}+\frac{441}{25x}\right.\right. \\
&\qquad \left.\left.-\frac{3087\left(-23x^3+\frac{935}{63}x\right)}{1600\left(x^4-\frac{10}{9}x^2+\frac{5}{21}\right)}+\frac{3969\ln(x-1)}{128}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.337.2 Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d}{dx}y'\right) - 2xy' + 30y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = \frac{30y}{x^2-1} - \frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{2xy'}{x^2-1} - \frac{30y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

◦ Define functions

$\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{30}{x^2-1}\right]$

◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = 1$

◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$

◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -1$

- Multiply by denominators

$(x^2-1)\left(\frac{d}{dx}y'\right) + 2xy' - 30y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 30y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

◦ Shift index using $k->k+1-m$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

$$u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_{0}r^{2}u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^{2}+a_{k}(k+r+6)\,(k+r-5)\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r^{2}=0$

- Values of r that satisfy the indicial equation

$r=0$

- Each term in the series must be 0, giving the recursion relation

$-2a_{k+1}(k+1)^{2}+a_{k}(k+6)\,(k-5)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{a_{k}(k+6)(k-5)}{2(k+1)^{2}}$

- Recursion relation for $r=0$; series terminates at $k=5$

$a_{k+1}=\frac{a_{k}(k+6)(k-5)}{2(k+1)^{2}}$

- Apply recursion relation for $k=0$

$a_{1}=-15a_{0}$

- Apply recursion relation for $k=1$

$a_{2}=-\frac{7a_{1}}{2}$

- Express in terms of a_0

$a_{2}=\frac{105a_{0}}{2}$

- Apply recursion relation for $k=2$

$a_{3}=-\frac{4a_{2}}{3}$

- Express in terms of a_0

$a_{3}=-70a_{0}$

- Apply recursion relation for $k=3$

$a_{4}=-\frac{9a_{3}}{16}$

- Express in terms of a_0

$a_{4}=\frac{315a_{0}}{8}$

- Apply recursion relation for $k = 4$

  $a_5 = -\frac{a_4}{5}$
- Express in terms of a_0

  $a_5 = -\frac{63a_0}{8}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 15u + \frac{105}{2}u^2 - 70u^3 + \frac{315}{8}u^4 - \frac{63}{8}u^5\right)$
- Revert the change of variables $u = x + 1$

  $\left[y = a_0\left(-\frac{15}{8}x + \frac{35}{4}x^3 - \frac{63}{8}x^5\right)\right]$

1.337.3 Maple trace

```
Methods for second order ODEs:
```

1.337.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 71

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+30*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{21c_2\left(x^4 - \frac{10}{9}x^2 + \frac{5}{21}\right)x\ln(x-1)}{640} - \frac{21c_2\left(x^4 - \frac{10}{9}x^2 + \frac{5}{21}\right)x\ln(x+1)}{640} + \frac{21c_1x^5}{5} + \frac{21c_2x^4}{320} - \frac{14c_1x^3}{3} - \frac{49c_2x^2}{960} + c_1x + \frac{c_2}{225}$$

1.337.5 Mathematica DSolve solution

Solving time : `0.039` (sec)
Leaf size : 76

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+30*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{8}c_1x\left(63x^4 - 70x^2 + 15\right) + c_2\left(-\frac{63x^4}{8} + \frac{49x^2}{8} - \frac{1}{16}\left(63x^4 - 70x^2 + 15\right)x(\log(1-x) - \log(x+1)) - \frac{8}{15}\right)$$

1.338 problem 345

Internal problem ID [8476]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 345
Date solved : `Monday, October 21, 2024 at 05:08:28 PM`
CAS classification : `[_Lienard]`

Solve

$$xy'' + 2y' + xy = 0$$

1.338.1 Solved as second order ode using Kovacic algorithm

Time used: 0.142 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 638: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx}\\ &= z_1 e^{-\ln(x)}\\ &= z_1\left(\frac{1}{x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx\\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x}\right) + c_2 \left(\frac{\cos(x)}{x} (\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.338.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -y - \frac{2y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + \frac{2y'}{x} + y = 0$
- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators
  $x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion
    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k->k-1$
    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert y' to series expansion
    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - Shift index using $k->k+1$
    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k->k+1$
    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 0\}$
- Each term must be 0
  $a_1(1+r)\,(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$
- Shift index using $k->k+1$
  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$
  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

1.338.3 Maple trace

```
Methods for second order ODEs:
```

1.338.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{x}$$

1.338.5 Mathematica DSolve solution

Solving time : `0.039` (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

1.339 problem 346

Internal problem ID [8477]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 346
Date solved : `Monday, October 21, 2024 at 05:08:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (2x+1)\,y' + (x+1)\,y = 0$$

1.339.1 Solved as second order ode using Kovacic algorithm

Time used: 0.177 (sec)

Writing the ode as

$$xy'' + (2x+1)\,y' + (x+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2x+1 \\ C &= x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 640: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2x+1}{x} \, dx} \\ &= z_1 e^{-x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x+1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.339.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + (2x+1)\,y' + (x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x+1)y}{x} - \frac{(2x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(2x+1)y'}{x} + \frac{(x+1)y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x+1}{x}, P_3(x) = \frac{x+1}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x\left(\frac{d}{dx}y'\right) + (2x+1)\,y' + (x+1)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

  - Shift index using $k->k+1$

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 + a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k(2k+2r+1) + a_{k-1}\right) x^{k}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

$a_1(1+r)^2 + a_0(1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 + 2a_k k + a_k + a_{k-1} = 0$$

- Shift index using $k- > k+1$

$$a_{k+2}(k+2)^2 + 2a_{k+1}(k+1) + a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}, a_1 + a_0 = 0\right]$$

1.339.3 Maple trace

```
Methods for second order ODEs:
```

1.339.4 Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 15

```
dsolve(x*diff(diff(y(x),x),x)+(2*x+1)*diff(y(x),x)+(x+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-x}(c_1 + \ln(x)\, c_2)$$

1.339.5 Mathematica DSolve solution

Solving time : `0.043` (sec)
Leaf size : 19

```
DSolve[{x*D[y[x],{x,2}]+(2*x+1)*D[y[x],x]+(x+1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x}(c_2 \log(x) + c_1)$$

1.340 problem 347

Internal problem ID [8478]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 347
Date solved : `Monday, October 21, 2024 at 05:08:30 PM`
CAS classification : `[_Jacobi]`

Solve

$$2x(x-1)\,y''-(x+1)\,y'+y=0$$

1.340.1 Solved as second order ode using Kovacic algorithm

Time used: 0.211 (sec)

Writing the ode as

$$(2x^2-2x)\,y''+(-x-1)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^2-2x\\B&=-x-1\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 + 18x - 3}{16(x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 + 18x - 3$$
$$t = 16(x^2 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 + 18x - 3}{16(x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 642: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-1)^2} + \frac{3}{4x} - \frac{3}{16x^2} - \frac{3}{4(x-1)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 18x - 3}{16(x^2 - x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-3x^2+18x-3}{16\left(x^2-x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{4}-\left(\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4x} - \frac{1}{2\left(x-1\right)} + (-)\left(0\right) \\
&= \frac{3}{4x} - \frac{1}{2\left(x-1\right)} \\
&= \frac{x-3}{4x\left(x-1\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{3}{4x} - \frac{1}{2\left(x-1\right)}\right)(0) + \left(\left(-\frac{3}{4x^2} + \frac{1}{2\left(x-1\right)^2}\right) + \left(\frac{3}{4x} - \frac{1}{2\left(x-1\right)}\right)^2 - \left(\frac{-3x^2+18x-3}{16\left(x^2-x\right)^2}\right)\right) &= \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{4x} - \frac{1}{2(x-1)}\right)dx} \\
&= \frac{x^{3/4}}{\sqrt{x-1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x-1}{2x^2-2x}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{4} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\frac{\sqrt{x-1}}{x^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-1}{2x^2-2x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x)}{2}+\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{2(x+1)\,\mathrm{e}^{-\frac{\ln(x)}{2}+\ln(x-1)}}{x-1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\left(\frac{2(x+1)\,\mathrm{e}^{-\frac{\ln(x)}{2}+\ln(x-1)}}{x-1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.340.2 Maple step by step solution

Let's solve

$2x(x-1)\left(\frac{d}{dx}y'\right) - (x+1)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = -\frac{y}{2x(x-1)} + \frac{(x+1)y'}{2x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{(x+1)y'}{2x(x-1)} + \frac{y}{2x(x-1)} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{x+1}{2x(x-1)}, P_3(x) = \frac{1}{2x(x-1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x(x-1)\left(\frac{d}{dx}y'\right) + (-x-1)\, y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (-a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(2k+2r-1)\,(k+r-1))\, x^{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-1+2r) = 0$

- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $-2(k+1+r)\left(k+\frac{1}{2}+r\right) a_{k+1}+2\left(k+r-\frac{1}{2}\right) a_{k}(k+r-1)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{(2k+2r-1)a_{k}(k+r-1)}{(k+1+r)(2k+1+2r)}$
- Recursion relation for $r=0$; series terminates at $k=1$
  $a_{k+1}=\frac{(2k-1)a_{k}(k-1)}{(k+1)(2k+1)}$
- Apply recursion relation for $k=0$
  $a_{1}=a_{0}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li
  $y=a_{0}\cdot(x+1)$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1}=\frac{2ka_{k}\left(k-\frac{1}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$
- Solution for $r=\frac{1}{2}$
  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k+\frac{1}{2}}, a_{k+1}=\frac{2ka_{k}\left(k-\frac{1}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y=a_{0}\cdot(x+1)+\left(\sum_{k=0}^{\infty} b_{k} x^{k+\frac{1}{2}}\right), b_{k+1}=\frac{2kb_{k}\left(k-\frac{1}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

1.340.3 Maple trace

```
Methods for second order ODEs:
```

1.340.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(2*x*(x-1)*diff(diff(y(x),x),x)-(x+1)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y=c_{2}\sqrt{x}+c_{1}x+c_{1}$$

1.340.5 Mathematica DSolve solution

Solving time : 0.083 (sec)
Leaf size : 21

```
DSolve[{2*x*(x-1)*D[y[x],{x,2}]-(x+1)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1\sqrt{x} - 2c_2(x+1)$$

1.341 problem 348

Internal problem ID [8479]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 348
Date solved : `Monday, October 21, 2024 at 05:08:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + 2y' + 4xy = 0$$

1.341.1 Solved as second order ode using Kovacic algorithm

Time used: 0.155 (sec)

Writing the ode as

$$xy'' + 2y' + 4xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 644: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(2x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(2x)}{x}\right) + c_2 \left(\frac{\cos(2x)}{x} \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.341.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 2y' + 4xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -4y - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2y'}{x} + 4y = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 4\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + 2y' + 4xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x \cdot y$ to series expansion

$$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- ◦ Shift index using $k->k-1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- ◦ Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$$

- ◦ Shift index using $k->k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1) x^{k+r}$$

- ◦ Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- ◦ Shift index using $k->k+1$

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r) x^{-1+r} + a_1(1+r)(2+r) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(k+2+r) + 4a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r) + 4a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+3+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{4a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{4a_k}{(k+1)(k+2)}, 0=0\right]$$

- Recursion relation for $r=0$

  $$a_{k+2}=-\frac{4a_k}{(k+2)(k+3)}$$

- Solution for $r=0$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k}, a_{k+2}=-\frac{4a_k}{(k+2)(k+3)}, 2a_1=0\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+2}=-\frac{4a_k}{(k+1)(k+2)}, 0=0, b_{k+2}=-\frac{4b_k}{(k+2)(k+3)}, 2b_1=0\right]$$

1.341.3 Maple trace

```
Methods for second order ODEs:
```

1.341.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 21

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+4*x*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1\sin(2x)+c_2\cos(2x)}{x}$$

1.341.5 Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+4*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{4c_1e^{-2ix}-ic_2e^{2ix}}{4x}$$

1.342 problem 349

Internal problem ID [8480]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 349
Date solved : `Monday, October 21, 2024 at 05:08:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (2-2x)\,y' + (x-2)\,y = 0$$

1.342.1 Solved as second order ode using Kovacic algorithm

Time used: 0.092 (sec)

Writing the ode as

$$xy'' + (2-2x)\,y' + (x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2-2x \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 646: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2-2x}{x}\,dx} \\
&= z_1 e^{x-\ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^x}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2-2x}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{x}\right) + c_2 \left(\frac{e^x}{x}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.342.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + (2 - 2x)\, y' + (x - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x-2)y}{x} + \frac{2(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(x-1)y'}{x} + \frac{(x-2)y}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2(x-1)}{x}, P_3(x) = \frac{x-2}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + (2 - 2x)\, y' + (x - 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r)\, x^{-1+r} + (a_1(1+r)\,(2+r) - 2a_0(1+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k+2+r) - 2a_k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- Each term must be 0

$$a_1(1+r)\,(2+r) - 2a_0(1+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(k+2+r) - 2a_k k - 2a_k r - 2a_k + a_{k-1} = 0$$

- Shift index using $k- >k + 1$

$$a_{k+2}(k+2+r)\,(k+3+r) - 2a_{k+1}(k+1) - 2r a_{k+1} - 2a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2k a_{k+1} + 2r a_{k+1} - a_k + 4a_{k+1}}{(k+2+r)(k+3+r)}$$

- Recursion relation for $r = -1$

$a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}, 2a_1 - 2a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0, b_{k+2} = \frac{2kb_{k+1}-b_k+4b_{k+1}}{(k+2)(k+3)}, 2b_1 - 2b\right.$

1.342.3 Maple trace

```
Methods for second order ODEs:
```

1.342.4 Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 15

```
dsolve(x*diff(diff(y(x),x),x)+(2-2*x)*diff(y(x),x)+(x-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^x(c_1 x + c_2)}{x}$$

1.342.5 Mathematica DSolve solution

Solving time : 0.041 (sec)
Leaf size : 19

```
DSolve[{x*D[y[x],{x,2}]+(2-2*x)*D[y[x],x]+(x-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^x(c_2 x + c_1)}{x}$$

1.343 problem 350

Internal problem ID [8481]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 350
Date solved : `Monday, October 21, 2024 at 05:08:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 6xy' + \left(4x^2 + 6\right) y = 0$$

1.343.1 Solved as second order ode using Kovacic algorithm

Time used: 0.160 (sec)

Writing the ode as

$$x^2y'' + 6xy' + \left(4x^2 + 6\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 6x \\ C &= 4x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 648: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6x}{x^2}\, dx} \\
&= z_1 e^{-3\ln(x)} \\
&= z_1 \left(\frac{1}{x^3}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(2x)}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-6\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(2x)}{x^3}\right) + c_2 \left(\frac{\cos(2x)}{x^3} \left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.343.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + 6xy' + (4x^2 + 6) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{2(2x^2+3)y}{x^2} - \frac{6y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{6y'}{x} + \frac{2(2x^2+3)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{6}{x}, P_3(x) = \frac{2(2x^2+3)}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 6$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 6$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2 \left(\frac{d}{dx} y'\right) + 6xy' + (4x^2 + 6) y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(2+r)\, x^r + a_1(4+r)\,(3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+3)\,(k+r+2) + 4a_{k-2}\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, -2\}$

- Each term must be 0

  $a_1(4+r)\,(3+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)\,(k+r+2) + 4a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(k+5+r)\,(k+4+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(k+5+r)(k+4+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}$

- Solution for $r = -3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}, a_1 = 0\right]$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{4a_k}{(k+3)(k+2)}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{4a_k}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right), a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(k+3)(k+2)}, b_1 = 0\right]$$

1.343.3 Maple trace

```
Methods for second order ODEs:
```

1.343.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+6*x*diff(y(x),x)+(4*x^2+6)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(2x) + c_2 \cos(2x)}{x^3}$$

1.343.5 Mathematica DSolve solution

Solving time : `0.047` (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]+6*x*D[y[x],x]+(4*x^2+6)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1 e^{-2ix} - ic_2 e^{2ix}}{4x^3}$$

1.344 problem 351

Internal problem ID [8482]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 351
Date solved : `Monday, October 21, 2024 at 05:08:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (1 - 2x)\, y' + (x - 1)\, y = 0$$

1.344.1 Solved as second order ode using Kovacic algorithm

Time used: 0.177 (sec)

Writing the ode as

$$xy'' + (1 - 2x)\, y' + (x - 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1 - 2x \\ C &= x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 650: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\, dx}\\ &= e^{\int \frac{1}{2x} dx}\\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{1-2x}{x}\, dx}\\ &= z_1 e^{x - \frac{\ln(x)}{2}}\\ &= z_1\left(\frac{e^x}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{1-2x}{x}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx$$
$$= y_1(\ln(x))$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\ln(x)))$$

Will add steps showing solving for IC soon.

1.344.2 Maple step by step solution

Let's solve
$x\left(\frac{d}{dx}y'\right) + (1-2x)\,y' + (x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{(x-1)y}{x} + \frac{(2x-1)y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x\left(\frac{d}{dx}y'\right) + (1-2x)\,y' + (x-1)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 - a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k+2r+1) + a_{k-1}\right) x^{k}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

$a_1(1+r)^2 - a_0(1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2+(-2k-1)\,a_k+a_{k-1}=0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2)^2+(-2k-3)\,a_{k+1}+a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=\frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r=0$

$$a_{k+2}=\frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+2}=\frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2}, a_1-a_0=0\right]$$

1.344.3 Maple trace

```
Methods for second order ODEs:
```

1.344.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 13

```
dsolve(x*diff(diff(y(x),x),x)+(1-2*x)*diff(y(x),x)+(x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\mathrm{e}^{x}(c_1+\ln(x)\,c_2)$$

1.344.5 Mathematica DSolve solution

Solving time : `0.037` (sec)
Leaf size : 17

```
DSolve[{x*D[y[x],{x,2}]+(1-2*x)*D[y[x],x]+(x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to e^x(c_2\log(x)+c_1)$$

1.345 problem 352

Internal problem ID [8483]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 352
Date solved : `Monday, October 21, 2024 at 05:08:34 PM`
CAS classification : `[_Jacobi]`

Solve

$$x(1-x)\,y''+\left(\frac{1}{2}+2x\right)y'-2y=0$$

1.345.1 Solved as second order ode using Kovacic algorithm

Time used: 0.278 (sec)

Writing the ode as

$$\left(-x^2+x\right)y''+\left(\frac{1}{2}+2x\right)y'-2y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+x \\ B &= \frac{1}{2}+2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{48x - 3}{16\left(x^2 - x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 48x - 3 \\
t &= 16\left(x^2 - x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{48x - 3}{16\left(x^2 - x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 652: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{21}{8x} + \frac{45}{16(-1+x)^2} - \frac{21}{8(-1+x)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{48x - 3}{16\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} - \frac{5}{4(-1 + x)} + (0) \\ &= \frac{1}{4x} - \frac{5}{4(-1 + x)} \\ &= -\frac{4x + 1}{4x(-1 + x)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} - \frac{5}{4\,(-1+x)}\right)(1) + \left(\left(-\frac{1}{4x^2} + \frac{5}{4\,(-1+x)^2}\right) + \left(\frac{1}{4x} - \frac{5}{4\,(-1+x)}\right)^2 - \left(\frac{48x-3}{16\,(x^2-x)^2}\right)\right) \frac{-1+4a_0}{2x\,(-1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{4}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{4}\right) e^{\int \left(\frac{1}{4x} - \frac{5}{4(-1+x)}\right) dx} \\
&= \left(x + \frac{1}{4}\right) e^{-\frac{5\ln(-1+x)}{4} + \frac{\ln(x)}{4}} \\
&= \frac{\left(x + \frac{1}{4}\right) x^{1/4}}{(-1+x)^{5/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{1}{2}+2x}{-x^2+x} \, dx} \\
&= z_1 e^{\frac{5\ln(-1+x)}{4} - \frac{\ln(x)}{4}} \\
&= z_1 \left(\frac{(-1+x)^{5/4}}{x^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x + \frac{1}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{1}{2}+2x}{-x^2+x} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{\frac{5\ln(-1+x)}{2} - \frac{\ln(x)}{2}}}{(y_1)^2} dx \\
&= y_1 \left(-\frac{\sqrt{-1+x}\,\sqrt{x}\left(12\ln\left(-\frac{1}{2}+x+\sqrt{x(-1+x)}\right)x - 4\sqrt{x(-1+x)}\,x + 3\ln\left(-\frac{1}{2}+x+\sqrt{x(-1\right.\right.}{\sqrt{x(-1+x)}\,(4x+1)}
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x+\frac{1}{4}\right) + c_2\left(x\right. \\
&\quad + \frac{1}{4}\left(-\frac{\sqrt{-1+x}\,\sqrt{x}\left(12\ln\left(-\frac{1}{2}+x+\sqrt{x(-1+x)}\right)x - 4\sqrt{x(-1+x)}\,x + 3\ln\left(-\frac{1}{2}+x+\sqrt{x(-1\right.\right.}{\sqrt{x(-1+x)}\,(4x+1)}
\end{aligned}$$

Will add steps showing solving for IC soon.

1.345.2 Maple step by step solution

Let's solve

$x(1-x)\left(\frac{d}{dx}y'\right)+\left(\frac{1}{2}+2x\right)y'-2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{2y}{x(-1+x)}+\frac{(4x+1)y'}{2x(-1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{(4x+1)y'}{2x(-1+x)}+\frac{2y}{x(-1+x)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{4x+1}{2x(-1+x)}, P_3(x)=\frac{2}{x(-1+x)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x(-1+x)\left(\frac{d}{dx}y'\right)+(-4x-1)y'+4y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=1..2$

$$x^{m} \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\left(k+r-1\right) x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^{m} \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\left(k+1-m+r\right) x^{k+r}$$

Rewrite ODE with series expansions

$$-a_{0}r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\left(2k+1+2r\right) + 2a_{k}(k+r-1)\left(k+r-2\right)\right) x^{k+r}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2(k+1+r)\left(k+\frac{1}{2}+r\right) a_{k+1} + 2a_{k}(k+r-1)\left(k+r-2\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_{k}(k+r-1)(k+r-2)}{(k+1+r)(2k+1+2r)}$$

- Recursion relation for $r=0$; series terminates at $k=1$

$$a_{k+1} = \frac{2a_{k}(k-1)(k-2)}{(k+1)(2k+1)}$$

- Apply recursion relation for $k=0$

$$a_{1} = 4a_{0}$$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

$$y = a_{0} \cdot (4x+1)$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1} = \frac{2a_{k}\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_{k}x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_{k}\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_{0} \cdot (4x+1) + \left(\sum_{k=0}^{\infty} b_{k}x^{k+\frac{1}{2}}\right), b_{k+1} = \frac{2b_{k}\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

1.345.3 Maple trace

```
Methods for second order ODEs:
```

1.345.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 53

```
dsolve(x*(1-x)*diff(diff(y(x),x),x)+(1/2+2*x)*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$
\begin{aligned}
y = &(-12x-3)\,c_2 \ln\left(-1+2x+2\sqrt{x\,(-1+x)}\right) \\
&+(4x+26)\,c_2\sqrt{x\,(-1+x)}+4\left(x+\frac{1}{4}\right)(3\ln(2)\,c_2+c_1)
\end{aligned}
$$

1.345.5 Mathematica DSolve solution

Solving time : 0.306 (sec)
Leaf size : 64

```
DSolve[{x*(1-x)*D[y[x],{x,2}]+(1/2+2*x)*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x)\to \frac{1}{2}c_2\left(\sqrt{-((x-1)x)}(2x+13)-6(4x+1)\arctan\left(\frac{\sqrt{1-x}}{\sqrt{x}+1}\right)\right)+c_1\left(x+\frac{1}{4}\right)
$$

1.346 problem 353

Internal problem ID [8484]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 353
Date solved : `Monday, October 21, 2024 at 05:08:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4\left(t^2-3t+2\right)y''-2y'+y=0$$

1.346.1 Solved as second order ode using Kovacic algorithm

Time used: 0.271 (sec)

Writing the ode as

$$\left(4t^2-12t+8\right)y''-2y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4t^2-12t+8 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4t^2 + 20t - 19}{16\left(t^2 - 3t + 2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4t^2 + 20t - 19$$
$$t = 16\left(t^2 - 3t + 2\right)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{-4t^2 + 20t - 19}{16\left(t^2 - 3t + 2\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 654: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(t^2 - 3t + 2)^2$. There is a pole at $t = 2$ of order 2. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(t-1)^2} + \frac{3}{8\,(t-1)} - \frac{3}{8\,(t-2)} + \frac{5}{16\,(t-2)^2}$$

For the pole at $t = 2$ let b be the coefficient of $\frac{1}{(t-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4t^2 + 20t - 19}{16\,(t^2 - 3t + 2)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-4t^2+20t-19}{16\left(t^2-3t+2\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{t-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4(t-2)} + \frac{3}{4(t-1)} + (-)(0) \\
&= -\frac{1}{4(t-2)} + \frac{3}{4(t-1)} \\
&= \frac{2t-5}{4(t-1)(t-2)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right)(0) + \left(\left(\frac{1}{4(t-2)^2} - \frac{3}{4(t-1)^2}\right) + \left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right)^2 - \left(\frac{-4t}{16(
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right) dt} \\
&= \frac{(t-1)^{3/4}}{(t-2)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{4t^2-12t+8}\, dt} \\
&= z_1 e^{\frac{\ln(t-2)}{4} - \frac{\ln(t-1)}{4}} \\
&= z_1 \left(\frac{(t-2)^{1/4}}{(t-1)^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \sqrt{t-1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{4t^2-12t+8}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{\frac{\ln(t-2)}{2}-\frac{\ln(t-1)}{2}}}{(y_1)^2}\,dt \\
&= y_1 \left(-\frac{2\sqrt{t-2}}{\sqrt{t-1}} + \frac{\ln\left(-\frac{3}{2}+t+\sqrt{t^2-3t+2}\right)\sqrt{(t-1)(t-2)}}{\sqrt{t-2}\,\sqrt{t-1}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{t-1}\right) + c_2\left(\sqrt{t-1}\left(-\frac{2\sqrt{t-2}}{\sqrt{t-1}} + \frac{\ln\left(-\frac{3}{2}+t+\sqrt{t^2-3t+2}\right)\sqrt{(t-1)(t-2)}}{\sqrt{t-2}\,\sqrt{t-1}}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.346.2 Maple step by step solution

Let's solve

$4(t^2-3t+2)\left(\frac{d}{dt}y'\right) - 2y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{y}{4(t^2-3t+2)} + \frac{y'}{2(t^2-3t+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y' - \frac{y'}{2(t^2-3t+2)} + \frac{y}{4(t^2-3t+2)} = 0$

☐ Check to see if t_0 is a regular singular point

- Define functions

  $\left[P_2(t) = -\frac{1}{2(t^2-3t+2)}, P_3(t) = \frac{1}{4(t^2-3t+2)}\right]$

- $(t-1) \cdot P_2(t)$ is analytic at $t = 1$

  $\left((t-1) \cdot P_2(t)\right)\Big|_{t=1} = \frac{1}{2}$

- $(t-1)^2 \cdot P_3(t)$ is analytic at $t = 1$

  $\left((t-1)^2 \cdot P_3(t)\right)\Big|_{t=1} = 0$

- $t = 1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = 1$

- Multiply by denominators

  $(4t^2 - 12t + 8)\left(\frac{d}{dt}y'\right) - 2y' + y = 0$

- Change variables using $t = u + 1$ so that the regular singular point is at $u = 0$

  $(4u^2 - 4u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) - 2\frac{d}{du}y(u) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$-2a_0r(-1+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(2k+1+2r)+a_k(2k+2r-1)^2\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r(-1+2r)=0$
- Values of r that satisfy the indicial equation
  $r\in\left\{0,\frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $a_k(2k+2r-1)^2-4(k+1+r)\left(k+\frac{1}{2}+r\right)a_{k+1}=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k(2k+2r-1)^2}{2(k+1+r)(2k+1+2r)}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k(2k-1)^2}{2(k+1)(2k+1)}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+1}=\frac{a_k(2k-1)^2}{2(k+1)(2k+1)}\right]$
- Revert the change of variables $u=t-1$
  $\left[y=\sum_{k=0}^{\infty}a_k(t-1)^k,a_{k+1}=\frac{a_k(2k-1)^2}{2(k+1)(2k+1)}\right]$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1}=\frac{2a_kk^2}{\left(k+\frac{3}{2}\right)(2k+2)}$
- Solution for $r=\frac{1}{2}$
  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+\frac{1}{2}},a_{k+1}=\frac{2a_kk^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Revert the change of variables $u=t-1$
  $\left[y=\sum_{k=0}^{\infty}a_k(t-1)^{k+\frac{1}{2}},a_{k+1}=\frac{2a_kk^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty}a_k(t-1)^k\right)+\left(\sum_{k=0}^{\infty}b_k(t-1)^{k+\frac{1}{2}}\right),a_{k+1}=\frac{a_k(2k-1)^2}{2(k+1)(2k+1)},b_{k+1}=\frac{2b_kk^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

1.346.3 Maple trace

```
Methods for second order ODEs:
```

1.346.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 56

```
dsolve(4*(t^2-3*t+2)*diff(diff(y(t),t),t)-2*diff(y(t),t)+y(t) = 0,
       y(t),singsol=all)
```

$$y = c_1\sqrt{t-1} + \frac{c_2\left(-\frac{\sqrt{t^2-3t+2}\left(-\ln(2)+\ln\left(-3+2t+2\sqrt{(t-1)(t-2)}\right)\right)}{2} + t - 2\right)}{\sqrt{t-2}}$$

1.346.5 Mathematica DSolve solution

Solving time : 0.285 (sec)
Leaf size : 53

```
DSolve[{4*(t^2-3*t+2)*D[y[t],{t,2}]-2*D[y[t],t]+y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \sqrt{1-t}\left(-2c_2\text{arctanh}\left(\frac{1}{\sqrt{\frac{t-1}{t-2}}}\right) + \frac{2c_2}{\sqrt{\frac{t-1}{t-2}}} + c_1\right)$$

1.347 problem 354

Internal problem ID [8485]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 354
Date solved : `Monday, October 21, 2024 at 05:08:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2\left(t^{2}-5t+6\right)y''+\left(2t-3\right)y'-8y=0$$

1.347.1 Solved as second order ode using Kovacic algorithm

Time used: 0.257 (sec)

Writing the ode as

$$\left(2t^{2}-10t+12\right)y''+\left(2t-3\right)y'-8y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2t^{2}-10t+12\\B&=2t-3\\C&=-8\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{60t^2 - 308t + 381}{16\left(t^2 - 5t + 6\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 60t^2 - 308t + 381$$
$$t = 16\left(t^2 - 5t + 6\right)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{60t^2 - 308t + 381}{16\left(t^2 - 5t + 6\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 656: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(t^2 - 5t + 6)^2$. There is a pole at $t = 3$ of order 2. There is a pole at $t = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{29}{8(t-3)} - \frac{3}{16(t-3)^2} - \frac{29}{8(t-2)} + \frac{5}{16(t-2)^2}$$

For the pole at $t = 3$ let b be the coefficient of $\frac{1}{(t-3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $t = 2$ let b be the coefficient of $\frac{1}{(t-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{60t^2 - 308t + 381}{16(t^2 - 5t + 6)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{60t^2-308t+381}{16\left(t^2-5t+6\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
3	2	0	$\frac{3}{4}$	$\frac{1}{4}$
2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^+\right) \\
&= \frac{5}{2}-\left(\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{t-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4t-12} + \frac{5}{4(t-2)} + (0) \\
&= \frac{1}{4t-12} + \frac{5}{4(t-2)} \\
&= \frac{6t-17}{4(t-2)(t-3)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = t + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right)(1) + \left(\left(-\frac{1}{4(t-3)^2} - \frac{5}{4(t-2)^2}\right) + \left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right)^2 - \left(\frac{60t^2 - \cdots}{16(t^2 \cdots}\right.\right.
$$

$$
\frac{-6\cdots}{2t^2 - \cdots}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{17}{6}\right\}
$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$
p(t) = t - \frac{17}{6}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t - \frac{17}{6}\right) e^{\int \left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right) dt} \\
&= \left(t - \frac{17}{6}\right) e^{\frac{\ln(t-3)}{4} + \frac{5\ln(t-2)}{4}} \\
&= \left(t - \frac{17}{6}\right) (t-3)^{1/4} (t-2)^{5/4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2t-3}{2t^2-10t+12}\, dt} \\
&= z_1 e^{-\frac{3\ln(t-3)}{4}+\frac{\ln(t-2)}{4}} \\
&= z_1 \left(\frac{(t-2)^{1/4}}{(t-3)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(t-2)^{3/2}\,(6t-17)}{6\sqrt{t-3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2t-3}{2t^2-10t+12}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-\frac{3\ln(t-3)}{2}+\frac{\ln(t-2)}{2}}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{24(t-3)^2\,(24t^2-104t+111)\,\mathrm{e}^{-\frac{3\ln(t-3)}{2}+\frac{\ln(t-2)}{2}}}{5\,(6t-17)\,(t-2)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(t-2)^{3/2}\,(6t-17)}{6\sqrt{t-3}}\right) \\
&\quad + c_2 \left(\frac{(t-2)^{3/2}\,(6t-17)}{6\sqrt{t-3}} \left(\frac{24(t-3)^2\,(24t^2-104t+111)\,\mathrm{e}^{-\frac{3\ln(t-3)}{2}+\frac{\ln(t-2)}{2}}}{5\,(6t-17)\,(t-2)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.347.2 Maple step by step solution

Let's solve

$2(t^2-5t+6)\left(\frac{d}{dt}y'\right)+(2t-3)\,y'-8y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y'=\frac{4y}{t^2-5t+6}-\frac{(2t-3)y'}{2(t^2-5t+6)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y'+\frac{(2t-3)y'}{2(t^2-5t+6)}-\frac{4y}{t^2-5t+6}=0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(t)=\frac{2t-3}{2(t^2-5t+6)},P_3(t)=-\frac{4}{t^2-5t+6}\right]$

  ◦ $(t-2)\cdot P_2(t)$ is analytic at $t=2$

  $\left((t-2)\cdot P_2(t)\right)\Big|_{t=2}=-\frac{1}{2}$

  ◦ $(t-2)^2\cdot P_3(t)$ is analytic at $t=2$

  $\left((t-2)^2\cdot P_3(t)\right)\Big|_{t=2}=0$

  ◦ $t=2$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0=2$

- Multiply by denominators

  $(2t^2-10t+12)\left(\frac{d}{dt}y'\right)+(2t-3)\,y'-8y=0$

- Change variables using $t=u+2$ so that the regular singular point is at $u=0$

  $(2u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2u+1)\left(\frac{d}{du}y(u)\right)-8y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-3+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k(k+r+2)\,(k+r-2)\right) u^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(k-\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 2a_k(k+r+2)\,(k+r-2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k(k+r+2)(k+r-2)}{(2k-1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 2$

$$a_{k+1} = \frac{2a_k(k+2)(k-2)}{(2k-1)(k+1)}$$

- Apply recursion relation for $k = 0$

$$a_1 = 8a_0$$

- Apply recursion relation for $k = 1$

$$a_2 = -3a_1$$

- Express in terms of a_0

$$a_2 = -24a_0$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot (-24u^2 + 8u + 1)$$

- Revert the change of variables $u = t - 2$

$$[y = a_0(-24t^2 + 104t - 111)]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = \frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{(2k+2)\left(k+\frac{5}{2}\right)}$$

- Solution for $r = \frac{3}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1}=\frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

- Revert the change of variables $u=t-2$

$$\left[y=\sum_{k=0}^{\infty} a_k(t-2)^{k+\frac{3}{2}}, a_{k+1}=\frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=a_0(-24t^2+104t-111)+\left(\sum_{k=0}^{\infty} b_k(t-2)^{k+\frac{3}{2}}\right), b_{k+1}=\frac{2b_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

1.347.3 Maple trace

```
Methods for second order ODEs:
```

1.347.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 35

```
dsolve(2*(t^2-5*t+6)*diff(diff(y(t),t),t)+(2*t-3)*diff(y(t),t)-8*y(t) = 0,
       y(t),singsol=all)
```

$$y=\frac{c_1(24t^2-104t+111)}{24}+\frac{c_2(t-2)^{3/2}(6t-17)}{\sqrt{t-3}}$$

1.347.5 Mathematica DSolve solution

Solving time : `0.731` (sec)
Leaf size : 140

```
DSolve[{2*(t^2-5*t+6)*D[y[t],{t,2}]+(2*t-3)*D[y[t],t]-8*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{\sqrt[4]{2-t}\sqrt[4]{t-3}(t-2)^{5/4}\left(5c_1(6t-17)-\frac{24c_2(\sqrt{t-2}-1)\sqrt{t-3}(-t^2+(4\sqrt{t-2}-2)t-4\sqrt{t-2}+7)(24t^2-104t+111)}{(-t+\sqrt{t-2}+2)^3(-t+2\sqrt{t-2}+1)}\right)}{30(3-t)^{3/4}}$$

1.348 problem 355

Internal problem ID [8486]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 355
Date solved : `Monday, October 21, 2024 at 05:08:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3t(1+t)\,y''+ty'-y=0$$

1.348.1 Solved as second order ode using Kovacic algorithm

Time used: 0.308 (sec)

Writing the ode as

$$\left(3t^2+3t\right)y''+ty'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3t^2+3t \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7t + 12}{36t\,(1 + t)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 7t + 12$$
$$t = 36t(1 + t)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{7t + 12}{36t\,(1 + t)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 658: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36t(1+t)^2$. There is a pole at $t = 0$ of order 1. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $t = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36\,(1+t)^2} + \frac{1}{3t} - \frac{1}{3\,(1+t)}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(1+t)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7t + 12}{36t\,(1+t)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{7}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{7}{6} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{7t+12}{36t\,(1+t)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
-1	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{7}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{7}{6} - \left(\frac{7}{6}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{t-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{t} + \frac{1}{6+6t} + (0) \\
&= \frac{1}{t} + \frac{1}{6+6t} \\
&= \frac{1}{t} + \frac{1}{6+6t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{t} + \frac{1}{6+6t}\right)(0) + \left(\left(-\frac{1}{t^2} - \frac{1}{6\left(1+t\right)^2}\right) + \left(\frac{1}{t} + \frac{1}{6+6t}\right)^2 - \left(\frac{7t+12}{36t\left(1+t\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{t} + \frac{1}{6+6t}\right) dt} \\
&= \left(1+t\right)^{1/6} t
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t}{3t^2+3t} \, dt} \\
&= z_1 e^{-\frac{\ln(1+t)}{6}} \\
&= z_1 \left(\frac{1}{\left(1+t\right)^{1/6}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{3t^2+3t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-\frac{\ln(1+t)}{3}}}{(y_1)^2}\, dt \\
&= y_1 \left(-\frac{1}{3\left((1+t)^{1/3}-1\right)} - \frac{\ln\left((1+t)^{1/3}-1\right)}{3} + \frac{-2(1+t)^{1/3}-1}{3\,(1+t)^{2/3}+3\,(1+t)^{1/3}+3} \right. \\
&\quad \left. + \frac{\ln\left((1+t)^{2/3}+(1+t)^{1/3}+1\right)}{6} - \frac{\sqrt{3}\,\arctan\left(\frac{\left(1+2(1+t)^{1/3}\right)\sqrt{3}}{3}\right)}{3} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(t) + c_2 \left(t \left(-\frac{1}{3\left((1+t)^{1/3}-1\right)} - \frac{\ln\left((1+t)^{1/3}-1\right)}{3} \right.\right. \\
&\quad + \frac{-2(1+t)^{1/3}-1}{3\,(1+t)^{2/3}+3\,(1+t)^{1/3}+3} + \frac{\ln\left((1+t)^{2/3}+(1+t)^{1/3}+1\right)}{6} \\
&\quad \left.\left. - \frac{\sqrt{3}\,\arctan\left(\frac{\left(1+2(1+t)^{1/3}\right)\sqrt{3}}{3}\right)}{3} \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.348.2 Maple step by step solution

Let's solve

$3t(1+t)\left(\frac{d}{dt}y'\right)+ty'-y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y'=\frac{y}{3t(1+t)}-\frac{y'}{3(1+t)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y'+\frac{y'}{3(1+t)}-\frac{y}{3t(1+t)}=0$

☐ Check to see if t_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(t)=\frac{1}{3(1+t)},P_3(t)=-\frac{1}{3t(1+t)}\right]$

  ◦ $(1+t)\cdot P_2(t)$ is analytic at $t=-1$

    $\left((1+t)\cdot P_2(t)\right)\Big|_{t=-1}=\frac{1}{3}$

  ◦ $(1+t)^2\cdot P_3(t)$ is analytic at $t=-1$

    $\left((1+t)^2\cdot P_3(t)\right)\Big|_{t=-1}=0$

  ◦ $t=-1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0=-1$

- Multiply by denominators

  $3t(1+t)\left(\frac{d}{dt}y'\right)+ty'-y=0$

- Change variables using $t=u-1$ so that the regular singular point is at $u=0$

  $(3u^2-3u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)-y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- ○ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- ○ Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(-2+3r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(3k+3r+1) + a_k(3k+3r+1)\,(k+r-1)\right) u^{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-2+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{2}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3((-k-r-1)\, a_{k+1} + a_k(k+r-1))\left(k+r+\frac{1}{3}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r-1)}{k+1+r}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$$a_{k+1} = \frac{a_k(k-1)}{k+1}$$

- Apply recursion relation for $k = 0$

$$a_1 = -a_0$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot (-u+1)$$

- Revert the change of variables $u = 1+t$

$$[y = -a_0t]$$

- Recursion relation for $r = \frac{2}{3}$

$$a_{k+1} = \frac{a_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}$$

- Solution for $r = \frac{2}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{2}{3}}, a_{k+1} = \frac{a_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$$

- Revert the change of variables $u = 1+t$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+t)^{k+\frac{2}{3}}, a_{k+1} = \frac{a_k \left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$$

- Combine solutions and rename parameters

$$\left[y = -a_0 t + \left(\sum_{k=0}^{\infty} b_k (1+t)^{k+\frac{2}{3}}\right), b_{k+1} = \frac{b_k \left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$$

1.348.3 Maple trace

```
Methods for second order ODEs:
```

1.348.4 Maple dsolve solution

Solving time : `0.012` (sec)
Leaf size : 67

```
dsolve(3*t*(1+t)*diff(diff(y(t),t),t)+t*diff(y(t),t)-y(t) = 0,
       y(t),singsol=all)
```

$$y = c_1 t + 2\sqrt{3}\,\arctan\left(\frac{\left(1+2(1+t)^{1/3}\right)\sqrt{3}}{3}\right) t c_2 + 2\ln\left((1+t)^{1/3}-1\right) t c_2 - \ln\left((1+t)^{2/3}+(1+t)^{1/3}+1\right) t c_2 + 6(1+t)^{2/3} c_2$$

1.348.5 Mathematica DSolve solution

Solving time : `0.315` (sec)
Leaf size : 93

```
DSolve[{3*t*(1+t)*D[y[t],{t,2}]+t*D[y[t],t]-y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{6c_1 t - c_2\left(2\sqrt{3}t\arctan\left(\frac{2\sqrt[3]{t+1}+1}{\sqrt{3}}\right) + 6(t+1)^{2/3} + 2t\log\left(\sqrt[3]{t+1}-1\right) - t\log\left((t+1)^{2/3}+\sqrt[3]{t+1}+1\right.\right.}{6\sqrt[6]{3}}$$

1.349 problem 356

Internal problem ID [8487]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 356
Date solved : `Monday, October 21, 2024 at 05:08:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \frac{\left(x + \frac{3}{4}\right)y}{4} = 0$$

1.349.1 Solved as second order ode using Kovacic algorithm

Time used: 0.217 (sec)

Writing the ode as

$$x^2y'' + \left(\frac{x}{4} + \frac{3}{16}\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= \frac{x}{4} + \frac{3}{16} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4x - 3$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 660: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} - \frac{1}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{1}{2}(1-(1)) \\
&= 0
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\
&= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\
&= \frac{1}{2x}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}
$$

Since $d=0$, then letting

$$
p = 1 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$
0 = 0
$$

And solving for p gives

$$
p = 1
$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta + \frac{p'}{p} \\
&= \frac{1}{2x}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0
$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1+4x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+2\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{-x}}{4x} dx} \\ &= x^{1/4} \mathrm{e}^{\sqrt{-x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{1/4} \mathrm{e}^{\sqrt{-x}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{1/4} \mathrm{e}^{\sqrt{-x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= x^{1/4} \mathrm{e}^{\sqrt{-x}} \int \frac{1}{\sqrt{x}\, \mathrm{e}^{2\sqrt{-x}}} \, dx \\ &= x^{1/4} \mathrm{e}^{\sqrt{-x}} \left(-\frac{\sqrt{-x}\left(1 - \mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{1/4} \mathrm{e}^{\sqrt{-x}}\right) + c_2 \left(x^{1/4} \mathrm{e}^{\sqrt{-x}} \left(-\frac{\sqrt{-x}\left(1-\mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.349.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + \frac{\left(x+\frac{3}{4}\right)y}{4} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$
- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y(4x+3)}{16x^2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y(4x+3)}{16x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{4x+3}{16x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{16}$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2\left(\frac{d}{dx}y'\right) + (4x+3)\,y = 0$
- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ◦ Shift index using $k- > k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+4r)(-3+4r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(4k+4r-1)(4k+4r-3) + 4a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+4r)(-3+4r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$16\left(k+r-\frac{1}{4}\right)\left(k+r-\frac{3}{4}\right) a_k + 4a_{k-1} = 0$$

- Shift index using $k- > k + 1$

  $$16\left(k+\frac{3}{4}+r\right)\left(k+\frac{1}{4}+r\right) a_{k+1} + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{4a_k}{(4k+3+4r)(4k+1+4r)}$$

- Recursion relation for $r = \frac{1}{4}$

  $$a_{k+1} = -\frac{4a_k}{(4k+4)(4k+2)}$$

- Solution for $r = \frac{1}{4}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = -\frac{4a_k}{(4k+4)(4k+2)}\right]$$

- Recursion relation for $r = \frac{3}{4}$

  $$a_{k+1} = -\frac{4a_k}{(4k+6)(4k+4)}$$

- Solution for $r = \frac{3}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = -\frac{4a_k}{(4k+6)(4k+4)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1} = -\frac{4a_k}{(4k+4)(4k+2)}, b_{k+1} = -\frac{4b_k}{(4k+6)(4k+4)}\right]$$

1.349.3 Maple trace

```
Methods for second order ODEs:
```

1.349.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+1/4*(x+3/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^{1/4}\left(c_1 \sin\left(\sqrt{x}\right) + c_2 \cos\left(\sqrt{x}\right)\right)$$

1.349.5 Mathematica DSolve solution

Solving time : 0.088 (sec)
Leaf size : 43

```
DSolve[{x^2*D[y[x],{x,2}]+1/4*(x+3/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow e^{-i\sqrt{x}} \sqrt[4]{x}\left(c_1 e^{2i\sqrt{x}} + ic_2\right)$$

1.350 problem 357

Internal problem ID [8488]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 357
Date solved : `Monday, October 21, 2024 at 05:08:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \frac{(x^2-1)\,y}{4} = 0$$

1.350.1 Solved as second order ode using Kovacic algorithm

Time used: 0.171 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(\frac{x^2}{4} - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= \frac{x^2}{4} - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 662: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(2\tan\left(\frac{x}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}} \left(2\tan\left(\frac{x}{2}\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.350.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + x y' + \frac{(x^2-1) y}{4} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(x^2-1) y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{y'}{x} + \frac{(x^2-1) y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 \left(\frac{d}{dx} y'\right) + 4x y' + (x^2 - 1) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + a_{k-2}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{a_k}{4k^2+12k+8}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{a_k}{4k^2+20k+24}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{a_k}{4k^2+20k+24}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{b_k}{4k^2+20k+24}, b_1=0\right]$$

1.350.3 Maple trace

```
Methods for second order ODEs:
```

1.350.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+1/4*(x^2-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1 \sin\left(\frac{x}{2}\right)+c_2 \cos\left(\frac{x}{2}\right)}{\sqrt{x}}$$

1.350.5 Mathematica DSolve solution

Solving time : 0.068 (sec)
Leaf size : 36

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+1/4*(x^2-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{e^{-\frac{ix}{2}}\left(c_1-ic_2e^{ix}\right)}{\sqrt{x}}$$

1.351 problem 358

Internal problem ID [8489]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 358
Date solved : `Monday, October 21, 2024 at 05:08:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (1 - 2x)\,y' + (x - 1)\,y = 0$$

1.351.1 Solved as second order ode using Kovacic algorithm

Time used: 0.177 (sec)

Writing the ode as

$$xy'' + (1 - 2x)\,y' + (x - 1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1 - 2x \\ C &= x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 664: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx}\\ &= e^{\int \frac{1}{2x} dx}\\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{1-2x}{x} \, dx}\\ &= z_1 e^{x - \frac{\ln(x)}{2}}\\ &= z_1 \left(\frac{e^x}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-2x}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(\ln(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.351.2 Maple step by step solution

Let's solve
$x\left(\frac{d}{dx}y'\right) + (1-2x)\,y' + (x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{(x-1)y}{x} + \frac{(2x-1)y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x\left(\frac{d}{dx}y'\right) + (1-2x)\,y' + (x-1)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  ◦ Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  ◦ Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  ◦ Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$

  ◦ Shift index using $k->k+1$

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 - a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k+2r+1) + a_{k-1}\right) x^{k}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

$a_1(1+r)^2 - a_0(1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 + (-2k-1)\,a_k + a_{k-1} = 0$$

- Shift index using $k- > k+1$

$$a_{k+2}(k+2)^2 + (-2k-3)\,a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$$

1.351.3 Maple trace

```
Methods for second order ODEs:
```

1.351.4 Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 13

```
dsolve(x*diff(diff(y(x),x),x)+(1-2*x)*diff(y(x),x)+(x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{x}(c_1 + \ln(x)\,c_2)$$

1.351.5 Mathematica DSolve solution

Solving time : `0.037` (sec)
Leaf size : 17

```
DSolve[{x*D[y[x],{x,2}]+(1-2*x)*D[y[x],x]+(x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^x(c_2 \log(x) + c_1)$$

1.352 problem 359

Internal problem ID [8490]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 359
Date solved : `Monday, October 21, 2024 at 05:08:40 PM`
CAS classification : `[_Laguerre]`

Solve

$$xy'' - (x+1)\,y' + y = 0$$

1.352.1 Solved as second order ode using Kovacic algorithm

Time used: 0.236 (sec)

Writing the ode as

$$xy'' + (-x-1)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 2x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 666: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4} + \frac{3}{4x^2} - \frac{1}{2x}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{2x^2} + \frac{1}{2x^3} + \frac{1}{4x^4} - \frac{1}{4x^5} - \frac{3}{4x^6} - \frac{3}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - (\alpha_{c_1}^{-}) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2x} \\
&= \frac{x-1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(\frac{1}{2} - \frac{1}{2x}\right)^2 - \left(\frac{x^2-2x+3}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2} - \frac{1}{2x}\right) dx} \\
&= \frac{e^{\frac{x}{2}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x-1}{x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-1}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(x+1)\,\mathrm{e}^{x+\ln(x)}\mathrm{e}^{-2x}}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-\frac{(x+1)\,\mathrm{e}^{x+\ln(x)}\mathrm{e}^{-2x}}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.352.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) - (x+1)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x} + \frac{(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x+1)y'}{x} + \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x+1}{x}, P_3(x) = \frac{1}{x}\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$
  $(x \cdot P_2(x))\Big|_{x=0} = -1$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$
- $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $x\left(\frac{d}{dx}y'\right) + (-x-1)\,y' + y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions
- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$
  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k->k+1-m$
  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
- Shift index using $k->k+1$
  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions
  $a_0 r(-2+r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) - a_k(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)(a_{k+1}(k+1+r)-a_k)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

1.352.3 Maple trace

```
Methods for second order ODEs:
```

1.352.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 13

```
dsolve(x*diff(diff(y(x),x),x)-(x+1)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y=c_2\,\mathrm{e}^{x}+c_1x+c_1$$

1.352.5 Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 19

```
DSolve[{x*D[y[x],{x,2}]-(x+1)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2(x+1)$$

1.353 problem 360

Internal problem ID [8491]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 360
Date solved : `Monday, October 21, 2024 at 05:08:41 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$xy'' + 3y' + 4x^3y = 0$$

1.353.1 Solved as second order ode using Kovacic algorithm

Time used: 0.284 (sec)

Writing the ode as

$$xy'' + 3y' + 4x^3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= 4x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 668: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(-4x^2\right) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 2ix \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\,(2ix) \\ &= -\frac{1}{2x} - 2ix \\ &= -\frac{1}{2x} - 2ix\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4 + 3}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2x} - 2ix\right)dx} \\ &= \frac{\mathrm{e}^{-ix^2}}{\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{x}\,dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2}} \\ &= z_1\left(\frac{1}{x^{3/2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-i x^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{i e^{2 i x^2}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-i x^2}}{x^2}\right) + c_2 \left(\frac{e^{-i x^2}}{x^2}\left(-\frac{i e^{2 i x^2}}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.353.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 3y' + 4x^3 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -4x^2 y - \frac{3y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{3y'}{x} + 4x^2 y = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions
    $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$
    $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$
  - ◦ $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- • Multiply by denominators
  $x\left(\frac{d}{dx}y'\right) + 3y' + 4x^3y = 0$
- • Assume series solution for y
  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^3 \cdot y$ to series expansion
    $$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$
  - ◦ Shift index using $k- > k - 3$
    $$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$
  - ◦ Convert y' to series expansion
    $$y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1}$$
  - ◦ Shift index using $k- > k + 1$
    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\, x^{k+r}$$
  - ◦ Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
    $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r-1}$$
  - ◦ Shift index using $k- > k + 1$
    $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\,(k + r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r)\,x^{-1+r} + a_1(1+r)\,(3+r)\,x^r + a_2(2+r)\,(4+r)\,x^{1+r} + a_3(3+r)\,(5+r)\,x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 0\}$
- The coefficients of each power of x must be 0
  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$
- Solve for the dependent coefficient(s)
  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$
- Shift index using $k- >k+3$
  $a_{k+4}(k+4+r)\,(k+6+r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$
- Recursion relation for $r = -2$
  $a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$
- Solution for $r = -2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Recursion relation for $r = 0$
  $a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{4b_k}{(k+4)(k+6)}\right.$

1.353.3 Maple trace

```
Methods for second order ODEs:
```

1.353.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 21

```
dsolve(x*diff(diff(y(x),x),x)+3*diff(y(x),x)+4*x^3*y(x) = 0,
       y(x),singsol=all)
```

$$
y = \frac{c_1 \sin\left(x^2\right) + c_2 \cos\left(x^2\right)}{x^2}
$$

1.353.5 Mathematica DSolve solution

Solving time : 0.075 (sec)
Leaf size : 41

```
DSolve[{x*D[y[x],{x,2}]+3*D[y[x],x]+4*x^3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to \frac{4c_1 e^{-ix^2} - ic_2 e^{ix^2}}{4x^2}
$$

1.354 problem 361

Internal problem ID [8492]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 361
Date solved : `Monday, October 21, 2024 at 05:08:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-x^2+1\right)y''+2x\left(-x^2+1\right)y'-2y=0$$

1.354.1 Solved as second order ode using Kovacic algorithm

Time used: 0.225 (sec)

Writing the ode as

$$\left(-x^4+x^2\right)y''+\left(-2x^3+2x\right)y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^4+x^2\\B&=-2x^3+2x\\C&=-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{x^2(x^2-1)} \tag{6}$$

Comparing the above to (5) shows that

$$s = -2$$
$$t = x^2(x^2-1)$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{x^2(x^2-1)}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 670: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2(x^2 - 1)$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 1. There is a pole at $x = -1$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 1$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x-1} + \frac{2}{x^2} + \frac{1}{x+1}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{x^2 (x^2 - 1)}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	1	0	0	1
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= 1 - (0) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{x - 1} - \frac{1}{x} + (-)(0) \\ &= \frac{1}{x - 1} - \frac{1}{x} \\ &= \frac{1}{x^2 - x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x-1} - \frac{1}{x}\right)(1) + \left(\left(-\frac{1}{(x-1)^2} + \frac{1}{x^2}\right) + \left(\frac{1}{x-1} - \frac{1}{x}\right)^2 - \left(-\frac{2}{x^2\,(x^2-1)}\right)\right) = 0$$

$$\frac{-2a_0 + 2}{x^3 - x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x+1)\, \mathrm{e}^{\int \left(\frac{1}{x-1} - \frac{1}{x}\right) dx} \\
&= (x+1)\, \mathrm{e}^{\ln(x-1) - \ln(x)} \\
&= \frac{x^2 - 1}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^3+2x}{-x^4+x^2}\, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 - 1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+2x}{-x^4+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{1}{4(x+1)} - \frac{\ln(x+1)}{4} - \frac{1}{4(x-1)} + \frac{\ln(x-1)}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2-1}{x^2}\right) + c_2 \left(\frac{x^2-1}{x^2}\left(-\frac{1}{4(x+1)} - \frac{\ln(x+1)}{4} - \frac{1}{4(x-1)} + \frac{\ln(x-1)}{4}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.354.2 Maple step by step solution

1.354.3 Maple trace

```
Methods for second order ODEs:
```

1.354.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 47

```
dsolve(x^2*(-x^2+1)*diff(diff(y(x),x),x)+2*x*(-x^2+1)*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2(x^2-1)\ln(x-1)+(-x^2+1)\,c_2\ln(x+1)+2c_1\,x^2-2c_2x-2c_1}{2x^2}$$

1.354.5 Mathematica DSolve solution

Solving time : 0.095 (sec)
Leaf size : 56

```
DSolve[{x^2*(1-x^2)*D[y[x],{x,2}]+2*x*(1-x^2)*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-4c_1x^2-c_2(x^2-1)\log(1-x)+c_2(x^2-1)\log(x+1)+2c_2x+4c_1}{4x^2}$$

1.355 problem 362

Internal problem ID [8493]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 362
Date solved : `Monday, October 21, 2024 at 05:08:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2xy'' + (x-2)\,y' - y = 0$$

1.355.1 Solved as second order ode using Kovacic algorithm

Time used: 0.243 (sec)

Writing the ode as

$$2xy'' + (x-2)\,y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= x-2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 12}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x + 12 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 12}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 671: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{1}{4x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{1}{2x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{2}{x^4} + \frac{4}{x^5} - \frac{24}{x^6} + \frac{48}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 12}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{4x + 12}{16x^2}\right) \\ &= \frac{1}{16} + \frac{4x + 12}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{4}} - 0\right) &= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{4}} - 0\right) &= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 12}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\left(\frac{1}{4}\right) \\
&= -\frac{1}{2x} - \frac{1}{4} \\
&= -\frac{x+2}{4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - \frac{1}{4}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x} - \frac{1}{4}\right)^2 - \left(\frac{x^2+4x+12}{16x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{1}{4}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{4}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x-2}{2x}\, dx} \\
&= z_1 e^{-\frac{x}{4} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{-\frac{x}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x-2}{2x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x}{2}+\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{2(x-2)\,\mathrm{e}^{-\frac{x}{2}+\ln(x)}\mathrm{e}^{x}}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}}\left(\frac{2(x-2)\,\mathrm{e}^{-\frac{x}{2}+\ln(x)}\mathrm{e}^{x}}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.355.2 Maple step by step solution

Let's solve

$2x\left(\frac{d}{dx}y'\right) + (x-2)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{y}{2x} - \frac{(x-2)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(x-2)y'}{2x} - \frac{y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{x-2}{2x}, P_3(x) = -\frac{1}{2x}\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x\left(\frac{d}{dx}y'\right) + (x-2)\,y' - y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
- Shift index using $k- >k+1$

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(-2+r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)\,(k+r-1) + a_k(k+r-1)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r-1)\left(a_{k+1}(k+1+r)+\frac{a_k}{2}\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k}{2(k+1+r)}$
- Recursion relation for $r=0$

  $a_{k+1}=-\frac{a_k}{2(k+1)}$
- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=-\frac{a_k}{2(k+1)}\right]$
- Recursion relation for $r=2$

  $a_{k+1}=-\frac{a_k}{2(k+3)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=-\frac{a_k}{2(k+3)}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=-\frac{a_k}{2(k+1)}, b_{k+1}=-\frac{b_k}{2(k+3)}\right]$

1.355.3 Maple trace

```
Methods for second order ODEs:
```

1.355.4 Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 16

```
dsolve(2*x*diff(diff(y(x),x),x)+(x-2)*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y=c_1(x-2)+c_2\,\mathrm{e}^{-\frac{x}{2}}$$

1.355.5 Mathematica DSolve solution

Solving time : 0.056 (sec)
Leaf size : 23

```
DSolve[{2*x*D[y[x],{x,2}]+(x-2)*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-x/2} + 2c_2(x-2)$$

1.356 problem 363

Internal problem ID [8494]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 363
Date solved : `Monday, October 21, 2024 at 05:08:44 PM`
CAS classification : `[_Lienard]`

Solve

$$xy'' + 2y' + xy = 0$$

1.356.1 Solved as second order ode using Kovacic algorithm

Time used: 0.144 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 673: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos (x)}{x}\right) + c_2 \left(\frac{\cos (x)}{x} (\tan (x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.356.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -y - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2y'}{x} + y = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k->k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$$

1.356.3 Maple trace

```
Methods for second order ODEs:
```

1.356.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{x}$$

1.356.5 Mathematica DSolve solution

Solving time : `0.043` (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

1.357 problem 364

Internal problem ID [8495]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 364
Date solved : `Monday, October 21, 2024 at 05:08:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 2x^2 y' + \left(x^4 + 2x - 1\right) y = 0$$

1.357.1 Solved as second order ode using Kovacic algorithm

Time used: 0.122 (sec)

Writing the ode as

$$y'' + 2x^2 y' + \left(x^4 + 2x - 1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x^2 \\ C &= x^4 + 2x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 675: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2}{1}\,dx}\\ &= z_1 e^{-\frac{x^3}{3}}\\ &= z_1\left(\mathrm{e}^{-\frac{x^3}{3}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{2x^2}{1}\, dx}}{(y_1)^2}\, dx$$

$$= y_1 \int \frac{e^{-\frac{2x^3}{3}}}{(y_1)^2}\, dx$$

$$= y_1 \left(\frac{\mathrm{e}^{-\frac{2x^3}{3}}\, \mathrm{e}^{\frac{2x(x^2+3)}{3}}}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\mathrm{e}^{-\frac{x(x^2+3)}{3}}\right) + c_2 \left(\mathrm{e}^{-\frac{x(x^2+3)}{3}} \left(\frac{\mathrm{e}^{-\frac{2x^3}{3}}\, \mathrm{e}^{\frac{2x(x^2+3)}{3}}}{2}\right)\right)$$

Will add steps showing solving for IC soon.

1.357.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 2x^2 y' + (x^4 + 2x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..4$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ◦ Convert $x^2 \cdot y'$ to series expansion

$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

- Shift index using $k- >k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

- Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 - a_0 + (6a_3 - a_1 + 2a_0)\, x + (12a_4 - a_2 + 4a_1)\, x^2 + (20a_5 - a_3 + 6a_2)\, x^3 + \left(\sum_{k=4}^{\infty} (a_{k+2}(k+2$

- The coefficients of each power of x must be 0

  $[2a_2 - a_0 = 0, 6a_3 - a_1 + 2a_0 = 0, 12a_4 - a_2 + 4a_1 = 0, 20a_5 - a_3 + 6a_2 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6} - \frac{a_0}{3}, a_4 = \frac{a_0}{24} - \frac{a_1}{3}, a_5 = \frac{a_1}{120} - \frac{a_0}{6}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + 2a_{k-1}k - a_k + a_{k-4} = 0$

- Shift index using $k- >k+4$

  $\left((k+4)^2 + 3k + 14\right) a_{k+6} + 2a_{k+3}(k+4) - a_{k+4} + a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{2ka_{k+3} + a_k + 8a_{k+3} - a_{k+4}}{k^2 + 11k + 30}, a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6} - \frac{a_0}{3}, a_4 = \frac{a_0}{24} - \frac{a_1}{3}, a_5 = \frac{a_1}{120} - \frac{a_0}{6}\right]$

1.357.3 Maple trace

```
Methods for second order ODEs:
```

1.357.4 Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)+2*x^2*diff(y(x),x)+(x^4+2*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \,\mathrm{e}^{-\frac{x\left(x^2-3\right)}{3}} + c_2 \,\mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}}$$

1.357.5 Mathematica DSolve solution

Solving time : 0.053 (sec)
Leaf size : 34

```
DSolve[{D[y[x],{x,2}]+2*x^2*D[y[x],x]+(x^4+2*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-\frac{1}{3}x(x^2+3)} \left(c_2 e^{2x} + 2c_1\right)$$

1.358 problem 365

Internal problem ID [8496]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 365
Date solved : `Monday, October 21, 2024 at 05:08:45 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$u'' + \frac{u}{x^2} = 0$$

1.358.1 Solved as second order ode using Kovacic algorithm

Time used: 0.243 (sec)

Writing the ode as

$$u'' + \frac{u}{x^2} = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{1}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 677: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -1$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -1$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \frac{i\sqrt{3}}{2} - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} + (-)(0) \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} \\
&= \frac{1 - i\sqrt{3}}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x^2}\right) + \left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)^2 - \left(-\frac{1}{x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} dx} \\ &= x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \end{aligned}$$

The first solution to the original ode in u is found from

$$u_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} u_1 &= z_1 \\ &= x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \end{aligned}$$

Which simplifies to

$$u_1 = x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{u_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
u_2 &= u_1 \int \frac{1}{u_1^2} \,dx \\
&= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \int \frac{1}{x^{1-i\sqrt{3}}} \,dx \\
&= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{ix\sqrt{3}\, x^{i\sqrt{3}-1}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right) + c_2 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{ix\sqrt{3}\, x^{i\sqrt{3}-1}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.358.2 Maple step by step solution

Let's solve

$\frac{d}{dx}u' + \frac{u}{x^2} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}u'$

- Multiply by denominators of the ODE

  $\left(\frac{d}{dx}u'\right) x^2 + u = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of u with respect to x , using the chain rule

    $u' = \left(\frac{d}{dt}u(t)\right) t'(x)$

  - Compute derivative

    $u' = \frac{\frac{d}{dt}u(t)}{x}$

  - Calculate the 2nd derivative of u with respect to x , using the chain rule

    $\frac{d}{dx}u' = \left(\frac{d}{dt}\frac{d}{dt}u(t)\right) t'(x)^2 + \left(\frac{d}{dx}t'(x)\right)\left(\frac{d}{dt}u(t)\right)$

- Compute derivative

  $\frac{d}{dx}u' = \frac{\frac{d}{dt}\frac{d}{dt}u(t)}{x^2} - \frac{\frac{d}{dt}u(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d}{dt}\frac{d}{dt}u(t)}{x^2} - \frac{\frac{d}{dt}u(t)}{x^2}\right)x^2 + u(t) = 0$

- Simplify

  $\frac{d}{dt}\frac{d}{dt}u(t) - \frac{d}{dt}u(t) + u(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1 \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the ODE

  $u_1(t) = \mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)$

- 2nd solution of the ODE

  $u_2(t) = \mathrm{e}^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- General solution of the ODE

  $u(t) = C1 u_1(t) + C2 u_2(t)$

- Substitute in solutions

  $u(t) = C1\,\mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + C2\,\mathrm{e}^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- Change variables back using $t = \ln(x)$

  $u = C1\sqrt{x}\,\cos\left(\frac{\ln(x)\sqrt{3}}{2}\right) + C2\sqrt{x}\,\sin\left(\frac{\ln(x)\sqrt{3}}{2}\right)$

- Simplify

  $u = \sqrt{x}\left(C1\cos\left(\frac{\ln(x)\sqrt{3}}{2}\right) + C2\sin\left(\frac{\ln(x)\sqrt{3}}{2}\right)\right)$

1.358.3 Maple trace

```
Methods for second order ODEs:
```

1.358.4 Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 29

```
dsolve(diff(diff(u(x),x),x)+1/x^2*u(x) = 0,
       u(x),singsol=all)
```

$$u = \sqrt{x}\left(c_1 \sin\left(\frac{\ln\left(x\right)\sqrt{3}}{2}\right) + c_2 \cos\left(\frac{\ln\left(x\right)\sqrt{3}}{2}\right)\right)$$

1.358.5 Mathematica DSolve solution

Solving time : `0.04` (sec)
Leaf size : 42

```
DSolve[{D[u[x],{x,2}]+1/x^2*u[x]==0,{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \sqrt{x}\left(c_1 \cos\left(\frac{1}{2}\sqrt{3}\log(x)\right) + c_2 \sin\left(\frac{1}{2}\sqrt{3}\log(x)\right)\right)$$

1.359 problem 366

Internal problem ID [8497]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 366
Date solved : `Monday, October 21, 2024 at 05:08:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' - (2x+1)\,u' + \left(x^2+x-1\right)u = 0$$

1.359.1 Solved as second order ode using Kovacic algorithm

Time used: 0.102 (sec)

Writing the ode as

$$u'' + (-2x-1)\,u' + \left(x^2+x-1\right)u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x-1 \\ C &= x^2+x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 679: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-1}{1}\,dx} \\ &= z_1 e^{\frac{1}{2}x^2 + \frac{1}{2}x} \\ &= z_1 \left(e^{\frac{x(x+1)}{2}}\right) \end{aligned}$$

Which simplifies to

$$u_1 = e^{\frac{x^2}{2}}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{-2x-1}{1}\,dx}}{(u_1)^2}\,dx \\ &= u_1 \int \frac{e^{x^2+x}}{(u_1)^2}\,dx \\ &= u_1 \left(e^{x^2+x} e^{-x^2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1\left(\mathrm{e}^{\frac{x^2}{2}}\right) + c_2\left(\mathrm{e}^{\frac{x^2}{2}}\left(\mathrm{e}^{x^2+x}\mathrm{e}^{-x^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.359.2 Maple step by step solution

Let's solve

$\frac{d}{dx}u' - (2x+1)\,u' + (x^2+x-1)\,u = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}u'$

- Isolate 2nd derivative

  $\frac{d}{dx}u' = (-x^2-x+1)\,u + (2x+1)\,u'$

- Group terms with u on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}u' + (-2x-1)\,u' + (x^2+x-1)\,u = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot u$ to series expansion for $m = 0..2$

  $x^m \cdot u = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k-m$

  $x^m \cdot u = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x^m \cdot u'$ to series expansion for $m = 0..1$

  $x^m \cdot u' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

- ○ Shift index using $k- >k+1-m$

  $x^m \cdot u' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

- ○ Convert $\frac{d}{dx}u'$ to series expansion

$$\frac{d}{dx}u' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

$$\frac{d}{dx}u' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 - a_1 - a_0 + (6a_3 - 2a_2 - 3a_1 + a_0)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - a_{k+1}(k+1) - a_k(2k+1$$

- The coefficients of each power of x must be 0

$$[2a_2 - a_1 - a_0 = 0, 6a_3 - 2a_2 - 3a_1 + a_0 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = \frac{a_1}{2} + \frac{a_0}{2}, a_3 = \frac{2a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+2} + (-2a_k - a_{k+1} + 3a_{k+2})\, k - a_k + a_{k-2} + a_{k-1} - a_{k+1} + 2a_{k+2} = 0$$

- Shift index using $k->k+2$

$$(k+2)^2\, a_{k+4} + (-2a_{k+2} - a_{k+3} + 3a_{k+4})\,(k+2) - a_{k+2} + a_k + a_{k+1} - a_{k+3} + 2a_{k+4} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2k a_{k+2} + k a_{k+3} - a_k - a_{k+1} + 5a_{k+2} + 3a_{k+3}}{k^2+7k+12}, a_2 = \frac{a_1}{2} + \frac{a_0}{2}, a_3 = \frac{2a_1}{3}\right]$$

1.359.3 Maple trace

```
Methods for second order ODEs:
```

1.359.4 Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 22

```
dsolve(diff(diff(u(x),x),x)-(2*x+1)*diff(u(x),x)+(x^2+x-1)*u(x) = 0,
       u(x),singsol=all)
```

$$u = c_1\, \mathrm{e}^{\frac{x^2}{2}} + c_2\, \mathrm{e}^{\frac{x(x+2)}{2}}$$

1.359.5 Mathematica DSolve solution

Solving time : `0.039` (sec)
Leaf size : 24

```
DSolve[{D[u[x],{x,2}]-(2*x+1)*D[u[x],x]+(x^2+x-1)*u[x]==0,{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to e^{\frac{x^2}{2}}(c_2 e^x + c_1)$$

1.360 problem 367

Internal problem ID [8498]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 367
Date solved : `Monday, October 21, 2024 at 05:08:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 2y' + \left(1 + \frac{2}{(1+3x)^2}\right) y = 0$$

1.360.1 Solved as second order ode using Kovacic algorithm

Time used: 0.153 (sec)

Writing the ode as

$$y'' + 2y' + \left(1 + \frac{2}{(1+3x)^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 + \frac{2}{(1+3x)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{(1+3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -2$$
$$t = (1+3x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{(1+3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 681: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+3x)^2$. There is a pole at $x = -\frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{2}{9\left(x+\frac{1}{3}\right)^2}
$$

For the pole at $x = -\frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{2}{(1+3x)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{2}{9}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{(1+3x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{3}$	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{1+3x} + (-)(0) \\
&= \frac{1}{1+3x} \\
&= \frac{1}{1+3x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{1+3x}\right)(0) + \left(\left(-\frac{1}{3\left(x+\frac{1}{3}\right)^2}\right) + \left(\frac{1}{1+3x}\right)^2 - \left(-\frac{2}{(1+3x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{1+3x} dx} \\ &= (1+3x)^{1/3} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\, dx} \\ &= z_1 e^{-x} \\ &= z_1 \left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}(1+3x)^{1/3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1 \left((1+3x)^{1/3}\, \mathrm{e}^{-2x} \mathrm{e}^{2x} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x} (1+3x)^{1/3} \right) + c_2 \left(\mathrm{e}^{-x} (1+3x)^{1/3} \left((1+3x)^{1/3}\, \mathrm{e}^{-2x} \mathrm{e}^{2x} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.360.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 2y' + \left(1 + \frac{2}{(1+3x)^2}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{3(3x^2+2x+1)y}{(1+3x)^2} - 2y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + 2y' + \frac{3(3x^2+2x+1)y}{(1+3x)^2} = 0$

☐ Check to see if $x_0 = -\frac{1}{3}$ is a regular singular point

- ∘ Define functions

  $\left[P_2(x) = 2, P_3(x) = \frac{3(3x^2+2x+1)}{(1+3x)^2}\right]$

- ∘ $\left(x+\frac{1}{3}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{3}$

  $\left(\left(x+\frac{1}{3}\right) \cdot P_2(x)\right)\Bigg|_{x=-\frac{1}{3}} = 0$

- ∘ $\left(x+\frac{1}{3}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{3}$

$$\left(\left(x+\frac{1}{3}\right)^{2}\cdot P_{3}(x)\right)\Bigg|_{x=-\frac{1}{3}}=\frac{2}{9}$$

- $x=-\frac{1}{3}$is a regular singular point

  Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

  $x_0=-\frac{1}{3}$

- Multiply by denominators

  $(1+3x)^{2}\left(\frac{d}{dx}y'\right)+2(1+3x)^{2}y'+(9x^{2}+6x+3)y=0$

- Change variables using $x=u-\frac{1}{3}$ so that the regular singular point is at $u=0$

  $9u^{2}\left(\frac{d}{du}\frac{d}{du}y(u)\right)+18u^{2}\left(\frac{d}{du}y(u)\right)+(9u^{2}+2)y(u)=0$

- Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^{m}\cdot y(u)$ to series expansion for $m=0..2$

  $$u^{m}\cdot y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^{m}\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$$

- Convert $u^{2}\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion

  $$u^{2}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)u^{k+r+1}$$

- Shift index using $k->k-1$

  $$u^{2}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=1}^{\infty}a_{k-1}(k-1+r)u^{k+r}$$

- Convert $u^{2}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $$u^{2}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)(k-1+r)u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}(-1+3r)(-2+3r)u^{r}+(a_{1}(2+3r)(1+3r)+18a_{0}r)u^{1+r}+\left(\sum_{k=2}^{\infty}(a_{k}(3k+3r-1)(3k+3r\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)(-2+3r)=0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$

- Each term must be 0

  $a_1(2+3r)(1+3r)+18a_0r=0$
- Solve for the dependent coefficient(s)

  $a_1=-\frac{18a_0r}{9r^2+9r+2}$
- Each term in the series must be 0, giving the recursion relation

  $9\left(k+r-\frac{2}{3}\right)\left(k+r-\frac{1}{3}\right)a_k+18a_{k-1}k+18a_{k-1}r+9a_{k-2}-18a_{k-1}=0$
- Shift index using $k->k+2$

  $9\left(k+\frac{4}{3}+r\right)\left(k+\frac{5}{3}+r\right)a_{k+2}+18a_{k+1}(k+2)+18a_{k+1}r+9a_k-18a_{k+1}=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{9(2ka_{k+1}+2a_{k+1}r+a_k+2a_{k+1})}{(3k+4+3r)(3k+5+3r)}$
- Recursion relation for $r=\frac{1}{3}$

  $a_{k+2}=-\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}$
- Solution for $r=\frac{1}{3}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{1}{3}}, a_{k+2}=-\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1=-a_0\right]$
- Revert the change of variables $u=x+\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{3}\right)^{k+\frac{1}{3}}, a_{k+2}=-\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1=-a_0\right]$
- Recursion relation for $r=\frac{2}{3}$

  $a_{k+2}=-\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}$
- Solution for $r=\frac{2}{3}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{2}{3}}, a_{k+2}=-\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}, a_1=-a_0\right]$
- Revert the change of variables $u=x+\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{3}\right)^{k+\frac{2}{3}}, a_{k+2}=-\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}, a_1=-a_0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{3}\right)^{k+\frac{1}{3}}\right)+\left(\sum_{k=0}^{\infty}b_k\left(x+\frac{1}{3}\right)^{k+\frac{2}{3}}\right), a_{k+2}=-\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1=-a_0, b_{k+2}=\right.$

1.360.3 Maple trace

```
Methods for second order ODEs:
```

1.360.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)+2*diff(y(x),x)+(1+2/(1+3*x)^2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{-x}(1+3x)^{1/3}\left(c_2(1+3x)^{1/3}+c_1\right)$$

1.360.5 Mathematica DSolve solution

Solving time : 0.077 (sec)
Leaf size : 35

```
DSolve[{D[y[x],{x,2}]+2*D[y[x],x]+(1+2/(1+3*x)^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x}\sqrt[3]{3x+1}\left(c_2\sqrt[3]{3x+1}+c_1\right)$$

1.361 problem 368

Internal problem ID [8499]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 368
Date solved : `Monday, October 21, 2024 at 05:08:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + (x^2+2)\,y = 0$$

1.361.1 Solved as second order ode using Kovacic algorithm

Time used: 0.145 (sec)

Writing the ode as

$$x^2y'' - 2xy' + (x^2+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 683: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.361.2 Maple step by step solution

Let's solve
$x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions
    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2+2)\,y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term must be 0

  $a_1 r(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) + a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+1+r)\,(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$

$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

1.361.3 Maple trace

```
Methods for second order ODEs:
```

1.361.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+(x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x(c_1 \sin(x) + c_2 \cos(x))$$

1.361.5 Mathematica DSolve solution

Solving time : `0.043` (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

1.362 problem 369

Internal problem ID [8500]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 369
Date solved : `Monday, October 21, 2024 at 05:08:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0$$

1.362.1 Solved as second order ode using Kovacic algorithm

Time used: 0.147 (sec)

Writing the ode as

$$y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= -\frac{2}{(1+x)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{2}{(1+x)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 2 \\
t &= (1+x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{2}{(1+x)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 685: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+x)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{(1+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{(1+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{1+x} + (-)(0) \\ &= -\frac{1}{1+x} \\ &= -\frac{1}{1+x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{1+x}\right)(0) + \left(\left(\frac{1}{(1+x)^2}\right) + \left(-\frac{1}{1+x}\right)^2 - \left(\frac{2}{(1+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{1+x} dx} \\
&= \frac{1}{1+x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1} \, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{(1+x)^3}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2+x}\right) + c_2 \left(\frac{1}{x^2+x}\left(\frac{(1+x)^3}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.362.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -\frac{2}{(1+x)^2}\right]$
  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = 0$
  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = -2$
  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

$x_0 = -1$

- Multiply by denominators

  $x(1+x)^2 \left(\frac{d}{dx}y'\right) + 2(1+x)^2 y' - 2yx = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^3 - u^2)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + 2u^2\left(\frac{d}{du}y(u)\right) + (-2u+2)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- >k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^2 \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

    $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r+1}$

  - Shift index using $k- >k-1$

    $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 2..3$

    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,u^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(1+r)\,(-2+r)\,u^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r+1)\,(k+r-2)\right) u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$

- Each term in the series must be 0, giving the recursion relation

$-(k+r+1)(k+r-2)(a_k - a_{k-1}) = 0$

- Shift index using $k- >k+1$

  $-(k+r+2)(k-1+r)(a_{k+1} - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = a_k$

- Recursion relation for $r = -1$

  $a_{k+1} = a_k$

- Solution for $r = -1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+1} = a_k\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-1}, a_{k+1} = a_k\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = a_k$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = a_k\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+2}, a_{k+1} = a_k\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+2}\right), a_{k+1} = a_k, b_{k+1} = b_k\right]$

1.362.3 Maple trace

```
Methods for second order ODEs:
```

1.362.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 29

```
dsolve(diff(diff(y(x),x),x)+2/x*diff(y(x),x)-2/(1+x)^2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(x^3 + 3x^2 + 3x)\, c_2 + c_1}{x\,(1 + x)}$$

1.362.5 Mathematica DSolve solution

Solving time : 0.049 (sec)
Leaf size : 34

```
DSolve[{D[y[x],{x,2}]+2/x*D[y[x],x]-2/(1+x)^2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x(x^2 + 3x + 3) + 3c_1}{3x(x + 1)}$$

1.363 problem 370

Internal problem ID [8501]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 370
Date solved : `Monday, October 21, 2024 at 05:08:49 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$y'' + \frac{y}{2x^4} = 0$$

1.363.1 Solved as second order ode using Kovacic algorithm

Time used: 0.251 (sec)

Writing the ode as

$$y'' + \frac{y}{2x^4} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{1}{2x^4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{2x^4} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 2x^4$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{2x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 687: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 2x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{1}{2x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i\sqrt{2}}{2x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{i\sqrt{2}}{2x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = \frac{i\sqrt{2}}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{i\sqrt{2}}{2x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{0}{\frac{i\sqrt{2}}{2}} + 2\right) &= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{i\sqrt{2}}{2}} + 2\right) &= 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{2x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{i\sqrt{2}}{2x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to

determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{i\sqrt{2}}{2x^2} + \frac{1}{x} + (-)\,(0) \\
&= -\frac{i\sqrt{2}}{2x^2} + \frac{1}{x} \\
&= \frac{-i\sqrt{2}+2x}{2x^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{i\sqrt{2}}{2x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{i\sqrt{2}}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{i\sqrt{2}}{2x^2} + \frac{1}{x}\right)^2 - \left(-\frac{1}{2x^4}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{i\sqrt{2}}{2x^2}+\frac{1}{x}\right) dx} \\ &= x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} \int \frac{1}{x^2 \mathrm{e}^{\frac{i\sqrt{2}}{x}}}\, dx \\ &= x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} \left(-\frac{i\sqrt{2}\,\mathrm{e}^{-\frac{i\sqrt{2}}{x}}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}}\right) + c_2 \left(x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} \left(-\frac{i\sqrt{2}\,\mathrm{e}^{-\frac{i\sqrt{2}}{x}}}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.363.2 Maple step by step solution

1.363.3 Maple trace

```
Methods for second order ODEs:
```

1.363.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 29

```
dsolve(diff(diff(y(x),x),x)+1/2/x^4*y(x) = 0,
       y(x),singsol=all)
```

$$y = x\left(c_1 \sin\left(\frac{\sqrt{2}}{2x}\right) + c_2 \cos\left(\frac{\sqrt{2}}{2x}\right)\right)$$

1.363.5 Mathematica DSolve solution

Solving time : `0.142` (sec)
Leaf size : 50

```
DSolve[{D[y[x],{x,2}]+1/(2*x^4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{\frac{i}{\sqrt{2}x}} x - \frac{i c_2 e^{-\frac{i}{\sqrt{2}x}} x}{\sqrt{2}}$$

1.364 problem 371

Internal problem ID [8502]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 371
Date solved : `Monday, October 21, 2024 at 05:08:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.364.1 Solved as second order ode using Kovacic algorithm

Time used: 0.250 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 688: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) = 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2 + x) \, \mathrm{e}^{\int (-1-\frac{x}{2}) dx} \\
&= (2 + x) \, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (2 + x) \, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, \mathrm{e}^{-2}\sqrt{2} \, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.364.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

1.364.3 Maple trace

```
Methods for second order ODEs:
```

1.364.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x} \pi c_2 (2+x) \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i\mathrm{e}^{\frac{x(2+x)}{2}} \sqrt{\pi}\,\sqrt{2}\,c_2 + c_1(2+x)\,\mathrm{e}^{-x}$$

1.364.5 Mathematica DSolve solution

Solving time : 0.223 (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-x}\left(-\sqrt{2\pi} c_2 \sqrt{(x+2)^2} \operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2} c_1 (x+2) + 2 c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

1.365 problem 372

Internal problem ID [8503]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 372
Date solved : `Monday, October 21, 2024 at 05:08:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.365.1 Solved as second order ode using Kovacic algorithm

Time used: 0.248 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 690: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0 \\ -2 + a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x) \, \mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2 + x) \, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, \mathrm{e}^{-2}\sqrt{2} \, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.365.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ∘ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

- ∘ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

- ∘ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ∘ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ∘ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - a_k k - a_{k-1} = 0$
- Shift index using $k- > k + 1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

1.365.3 Maple trace

```
Methods for second order ODEs:
```

1.365.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x}\pi c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i\mathrm{e}^{\frac{x(2+x)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1(2+x)\,\mathrm{e}^{-x}$$

1.365.5 Mathematica DSolve solution

Solving time : `0.164` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

1.366 problem 373

Internal problem ID [8504]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 373
Date solved : `Monday, October 21, 2024 at 05:08:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.366.1 Solved as second order ode using Kovacic algorithm

Time used: 0.248 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 692: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2 + x) \, e^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (2 + x) \, e^{-x-\frac{1}{4}x^2} \\
&= (2 + x) \, e^{-\frac{x(4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(e^{\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, e^{-2}\sqrt{2} \, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.366.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$\left(k^2 + 3k + 2\right) a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

1.366.3 Maple trace

```
Methods for second order ODEs:
```

1.366.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x}\pi c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i\mathrm{e}^{\frac{x(2+x)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1(2+x)\,\mathrm{e}^{-x}$$

1.366.5 Mathematica DSolve solution

Solving time : `0.166` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

1.367 problem 374

Internal problem ID [8505]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 374
Date solved : `Monday, October 21, 2024 at 05:08:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.367.1 Solved as second order ode using Kovacic algorithm

Time used: 0.251 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 694: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x) \, \mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2 + x) \, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, \mathrm{e}^{-2} \sqrt{2} \, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\big((2+x)\,e^{-x}\big) + c_2\left((2+x)\,e^{-x}\left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.367.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

- ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

- ◦ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

- ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ◦ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0
  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$
- Shift index using $k- > k+1$
  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$
- Recursion relation that defines the series solution to the ODE
  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

1.367.3 Maple trace

```
Methods for second order ODEs:
```

1.367.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x} \pi c_2 (2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i\mathrm{e}^{\frac{x(2+x)}{2}} \sqrt{\pi}\,\sqrt{2}\,c_2 + c_1 (2+x)\,\mathrm{e}^{-x}$$

1.367.5 Mathematica DSolve solution

Solving time : `0.168` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-x} \left(-\sqrt{2\pi} c_2 \sqrt{(x+2)^2} \text{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2} c_1 (x+2) + 2 c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

1.368 problem 375

Internal problem ID [8506]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 375
Date solved : `Monday, October 21, 2024 at 05:08:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.368.1 Solved as second order ode using Kovacic algorithm

Time used: 0.252 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 696: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2 + x) \, e^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (2 + x) \, e^{-x-\frac{1}{4}x^2} \\
&= (2 + x) \, e^{-\frac{x(4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(e^{\frac{x^2}{4}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i \sqrt{\pi} \, e^{-2} \sqrt{2} \, \operatorname{erf}\left(\frac{i \sqrt{2}\,(2+x)}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.368.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- > k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- > k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

1.368.3 Maple trace

```
Methods for second order ODEs:
```

1.368.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \pi \,\mathrm{e}^{-2-x} c_2 (2+x) \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{\frac{x(2+x)}{2}} c_2 + c_1 (2+x)\,\mathrm{e}^{-x}$$

1.368.5 Mathematica DSolve solution

Solving time : `0.166` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-x} \left(-\sqrt{2\pi} c_2 \sqrt{(x+2)^2} \operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2} c_1 (x+2) + 2 c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

1.369 problem 376

Internal problem ID [8507]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 376
Date solved : `Monday, October 21, 2024 at 05:08:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.369.1 Solved as second order ode using Kovacic algorithm

Time used: 0.257 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 698: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 2\\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i\\ &= \frac{x}{2} + 1 \qquad (10)\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, e^{\int \left(-1 - \frac{x}{2}\right) dx} \\ &= (2 + x) \, e^{-x - \frac{1}{4}x^2} \\ &= (2 + x) \, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{-2 + \frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, e^{-2} \sqrt{2} \, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2} \right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.369.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2) a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- > k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

1.369.3 Maple trace

```
Methods for second order ODEs:
```

1.369.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x} \pi c_2 (2+x) \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i \mathrm{e}^{\frac{x(2+x)}{2}} \sqrt{\pi}\,\sqrt{2}\, c_2 + c_1 (2+x)\, \mathrm{e}^{-x}$$

1.369.5 Mathematica DSolve solution

Solving time : `0.168` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
      y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-x} \left(-\sqrt{2\pi} c_2 \sqrt{(x+2)^2} \operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2} c_1 (x+2) + 2 c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

1.370 problem 377

Internal problem ID [8508]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 377
Date solved : `Monday, October 21, 2024 at 05:08:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.370.1 Solved as second order ode using Kovacic algorithm

Time used: 0.250 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 700: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0$$

$$-2 + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x) \, \mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2 + x) \, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, \mathrm{e}^{-2} \sqrt{2} \, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.370.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- > k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- > k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- > k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$

1.370.3 Maple trace

```
Methods for second order ODEs:
```

1.370.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x}\pi c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i\mathrm{e}^{\frac{x(2+x)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1(2+x)\,\mathrm{e}^{-x}$$

1.370.5 Mathematica DSolve solution

Solving time : `0.166` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

1.371 problem 378

Internal problem ID [8509]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 378
Date solved : `Monday, October 21, 2024 at 05:08:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.371.1 Solved as second order ode using Kovacic algorithm

Time used: 0.253 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 702: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, \mathrm{e}^{\int \left(-1 - \frac{x}{2}\right) dx} \\ &= (2 + x) \, \mathrm{e}^{-x - \frac{1}{4}x^2} \\ &= (2 + x) \, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2 + \frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, \mathrm{e}^{-2} \sqrt{2} \, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\big((2+x)\,\mathrm{e}^{-x}\big) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.371.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k->k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$
- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$

1.371.3 Maple trace

```
Methods for second order ODEs:
```

1.371.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x}\pi c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i\mathrm{e}^{\frac{x(2+x)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1(2+x)\,\mathrm{e}^{-x}$$

1.371.5 Mathematica DSolve solution

Solving time : `0.169` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

1.372 problem 379

Internal problem ID [8510]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 379
Date solved : `Monday, October 21, 2024 at 05:08:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.372.1 Solved as second order ode using Kovacic algorithm

Time used: 0.250 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 704: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} + 1 \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, e^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x) \, e^{-x-\frac{1}{4}x^2} \\ &= (2 + x) \, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, e^{-2}\sqrt{2} \, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.372.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k->k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2) a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- > k + 1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

1.372.3 Maple trace

```
Methods for second order ODEs:
```

1.372.4 Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x} \pi c_2 (2+x) \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i\mathrm{e}^{\frac{x(2+x)}{2}} \sqrt{\pi}\,\sqrt{2}\,c_2 + c_1 (2+x)\,\mathrm{e}^{-x}$$

1.372.5 Mathematica DSolve solution

Solving time : `0.169` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-x} \left(-\sqrt{2\pi} c_2 \sqrt{(x+2)^2} \operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2} c_1 (x+2) + 2 c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

1.373 problem 380

Internal problem ID [8511]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 380
Date solved : `Monday, October 21, 2024 at 05:08:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.373.1 Solved as second order ode using Kovacic algorithm

Time used: 0.252 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 706: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2 + x)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (2 + x)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (2 + x)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (2 + x)\, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi}\, \mathrm{e}^{-2}\sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.373.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

$$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2 + 3k + 2\right) a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k + 1$

$$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

1.373.3 Maple trace

```
Methods for second order ODEs:
```

1.373.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x} \pi c_2 (2 + x) \operatorname{erf}\left(\frac{i\sqrt{2}\,(2 + x)}{2}\right) - i\mathrm{e}^{\frac{x(2+x)}{2}} \sqrt{\pi}\,\sqrt{2}\,c_2 + c_1 (2 + x)\,\mathrm{e}^{-x}$$

1.373.5 Mathematica DSolve solution

Solving time : `0.169` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{1}{2} e^{-x} \left(-\sqrt{2\pi} c_2 \sqrt{(x+2)^2} \operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2} c_1 (x+2) + 2 c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

1.374 problem 381

Internal problem ID [8512]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 381
Date solved : `Monday, October 21, 2024 at 05:08:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.374.1 Solved as second order ode using Kovacic algorithm

Time used: 0.259 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 708: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} {[\sqrt{r}]_\infty} &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} {[\sqrt{r}]_\infty} &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2 + x)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (2 + x)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (2 + x)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (2 + x)\, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi}\, \mathrm{e}^{-2} \sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.374.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2+\left(\sum_{k=1}^{\infty}\left(a_{k+2}(k+2)\left(k+1\right)-a_k k-a_{k-1}\right)x^k\right)=0$$

- Each term must be 0

  $2a_2=0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\,a_{k+2}-a_k k-a_{k-1}=0$
- Shift index using $k->k+1$

  $\left((k+1)^2+3k+5\right)a_{k+3}-a_{k+1}(k+1)-a_k=0$
- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+3}=\frac{k a_{k+1}+a_k+a_{k+1}}{k^2+5k+6}, 2a_2=0\right]$

1.374.3 Maple trace

```
Methods for second order ODEs:
```

1.374.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y=\pi\,\mathrm{e}^{-2-x}c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{\frac{x(2+x)}{2}}c_2+c_1(2+x)\,\mathrm{e}^{-x}$$

1.374.5 Mathematica DSolve solution

Solving time : `0.168` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right)+2\sqrt{2}c_1(x+2)+2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

1.375 problem 382

Internal problem ID [8513]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 382
Date solved : `Monday, October 21, 2024 at 05:08:59 PM`
CAS classification : `[_Lienard]`

Solve

$$xy'' + 2y' + xy = 0$$

1.375.1 Solved as second order ode using Kovacic algorithm

Time used: 0.145 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 710: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(x)}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x}\right) + c_2 \left(\frac{\cos(x)}{x} (\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.375.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -y - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k->k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$$

1.375.3 Maple trace

```
Methods for second order ODEs:
```

1.375.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{x}$$

1.375.5 Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

1.376 problem 383

Internal problem ID [8514]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 383
Date solved : `Monday, October 21, 2024 at 05:09:00 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$2x^2y'' + 3xy' - xy = 0$$

1.376.1 Solved as second order ode using Kovacic algorithm

Time used: 0.199 (sec)

Writing the ode as

$$2x^2y'' + 3xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 3x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 8x - 3$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 712: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\
&= \frac{1}{2}(1-(1))\\
&= 0
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\
&= \frac{1}{2}\left(\frac{1}{(x-(0))}\right)\\
&= \frac{1}{2x}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}
$$

Since $d=0$, then letting

$$
p = 1 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$
0 = 0
$$

And solving for p gives

$$
p = 1
$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta + \frac{p'}{p}\\
&= \frac{1}{2x}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0
$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1-8x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+2\sqrt{2}\,\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= e^{\int \frac{1+2\sqrt{2}\,\sqrt{x}}{4x} dx} \\ &= x^{1/4} e^{\sqrt{2}\,\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{2x^2} \, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{4}} \\ &= z_1 \left(\frac{1}{x^{3/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{2x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{-2\sqrt{2}\,\sqrt{x}}\sqrt{2}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}\right) + c_2\left(\frac{e^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}\left(-\frac{e^{-2\sqrt{2}\,\sqrt{x}}\sqrt{2}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.376.2 Maple step by step solution

Let's solve

$2x^2\left(\frac{d}{dx}y'\right) + 3xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{y}{2x} - \frac{3y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{3y'}{2x} - \frac{y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3}{2x}, P_3(x) = -\frac{1}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{3}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2\left(\frac{d}{dx}y'\right)x + 3y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+3+2r) - a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+1+r)\left(k+\frac{3}{2}+r\right) a_{k+1} - a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{(k+1+r)(2k+3+2r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{(k+1)(2k+3)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{(k+1)(2k+3)}\right]$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = \frac{a_k}{\left(k+\frac{1}{2}\right)(2k+2)}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1}=\frac{a_k}{(k+1)(2k+3)}, b_{k+1}=\frac{b_k}{(k+\frac{1}{2})(2k+2)}\right]$$

1.376.3 Maple trace

```
Methods for second order ODEs:
```

1.376.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 29

```
dsolve(2*x^2*diff(diff(y(x),x),x)+3*x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1 \sinh\left(\sqrt{2}\,\sqrt{x}\right)+c_2 \cosh\left(\sqrt{2}\,\sqrt{x}\right)}{\sqrt{x}}$$

1.376.5 Mathematica DSolve solution

Solving time : `0.119` (sec)
Leaf size : 56

```
DSolve[{2*x^2*D[y[x],{x,2}]+3*x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{e^{-\sqrt{2}\sqrt{x}}\left(2c_1 e^{2\sqrt{2}\sqrt{x}}-\sqrt{2}c_2\right)}{2\sqrt{x}}$$

1.377 problem 384

Internal problem ID [8515]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 384
Date solved : `Monday, October 21, 2024 at 05:09:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \left(3x^2 + 2x\right) y' - 2y = 0$$

1.377.1 Solved as second order ode using Kovacic algorithm

Time used: 0.237 (sec)

Writing the ode as

$$x^2y'' + \left(3x^2 + 2x\right) y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x^2 + 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 + 12x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^2 + 12x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2 + 12x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 714: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{2}{x^2} + \frac{3}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} + \frac{1}{x} + \frac{1}{3x^2} - \frac{2}{9x^3} + \frac{1}{9x^4} - \frac{2}{81x^5} - \frac{2}{81x^6} + \frac{28}{729x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^2 + 12x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{9}{4}\right) + \left(\frac{12x + 8}{4x^2}\right) \\ &= \frac{9}{4} + \frac{12x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 12. Dividing this by leading coefficient in t which is 4 gives 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{3}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{3}{\frac{3}{2}} - 0\right) = 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{3}{\frac{3}{2}} - 0\right) = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^2 + 12x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{3}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{3}{2}\right) \\
&= -\frac{1}{x} - \frac{3}{2} \\
&= -\frac{1}{x} - \frac{3}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{3}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{3}{2}\right)^2 - \left(\frac{9x^2 + 12x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{x} - \frac{3}{2}\right) dx} \\
&= \frac{e^{-\frac{3x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x^2 + 2x}{x^2} \, dx} \\
&= z_1 e^{-\frac{3x}{2} - \ln(x)} \\
&= z_1 \left(\frac{e^{-\frac{3x}{2}}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^{-3x}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2+2x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3x-2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{(9x^2-6x+2)\, x^2 e^{-3x-2\ln(x)} e^{6x}}{27}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-3x}}{x^2}\right) + c_2 \left(\frac{e^{-3x}}{x^2} \left(\frac{(9x^2-6x+2)\, x^2 e^{-3x-2\ln(x)} e^{6x}}{27}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.377.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + (3x^2+2x)\, y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{2y}{x^2} - \frac{(3x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{(3x+2)y'}{x} - \frac{2y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions
  $\left[P_2(x)=\frac{3x+2}{x}, P_3(x)=-\frac{2}{x^2}\right]$
- $x\cdot P_2(x)$ is analytic at $x=0$
  $$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=2$$
- $x^2\cdot P_3(x)$ is analytic at $x=0$
  $$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-2$$
- $x=0$is a regular singular point
  Check to see if $x_0=0$ is a regular singular point
  $x_0=0$

- Multiply by denominators
  $x^2\left(\frac{d}{dx}y'\right)+x(3x+2)\,y'-2y=0$
- Assume series solution for y
  $$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions
- Convert $x^m\cdot y'$ to series expansion for $m=1..2$
  $$x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$
- Convert $x^2\cdot\left(\frac{d}{dx}y'\right)$ to series expansion
  $$x^2\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0(2+r)\,(-1+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+2)\,(k+r-1)+3a_{k-1}(k+r-1)\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(2+r)\,(-1+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{-2,1\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_k(k+r+2)+3a_{k-1}\right)=0$
- Shift index using $k->k+1$

$(k+r)\left(a_{k+1}(k+3+r)+3a_k\right)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=-\frac{3a_k}{k+3+r}$

- Recursion relation for $r=-2$

$a_{k+1}=-\frac{3a_k}{k+1}$

- Solution for $r=-2$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1}=-\frac{3a_k}{k+1}\right]$

- Recursion relation for $r=1$

$a_{k+1}=-\frac{3a_k}{k+4}$

- Solution for $r=1$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1}=-\frac{3a_k}{k+4}\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1}=-\frac{3a_k}{k+1}, b_{k+1}=-\frac{3b_k}{k+4}\right]$

1.377.3 Maple trace

```
Methods for second order ODEs:
```

1.377.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 27

```
dsolve(x^2*diff(diff(y(x),x),x)+(3*x^2+2*x)*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1\,\mathrm{e}^{-3x}+c_2(9x^2-6x+2)}{x^2}$$

1.377.5 Mathematica DSolve solution

Solving time : 0.044 (sec)
Leaf size : 35

```
DSolve[{x^2*D[y[x],{x,2}]+(2*x+3*x^2)*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1(9x^2 - 6x + 2) + 27c_2e^{-3x}}{27x^2}$$

1.378 problem 385

Internal problem ID [8516]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 385
Date solved : `Monday, October 21, 2024 at 05:09:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+x+1\right)y''+x\left(11x^2+11x+9\right)y'+\left(7x^2+10x+6\right)y=0$$

1.378.1 Solved as second order ode using Kovacic algorithm

Time used: 1.037 (sec)

Writing the ode as

$$\left(2x^4+2x^3+2x^2\right)y''+\left(11x^3+11x^2+9x\right)y'+\left(7x^2+10x+6\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^4+2x^3+2x^2\\B&=11x^3+11x^2+9x\\C&=7x^2+10x+6\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int\frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 21x^4 + 18x^3 + 27x^2 - 2x - 3$$
$$t = 16\left(x^3 + x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 716: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{5}{24} + \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{24} - \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{1}{8} - \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{1}{8} + \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} - \frac{3}{16x^2} + \frac{1}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} + \frac{i\sqrt{3}}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{6 + 6i\sqrt{3}}}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{6 + 6i\sqrt{3}}}{12} \end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} - \frac{i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6+6i\sqrt{3}}}{12}$
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\ &= \frac{7}{4} - \left(\frac{7}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x - c_3} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} + (0) \\ &= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \\ &= \frac{7x^2 + 3x + 1}{4x\left(x^2 + x + 1\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} \right) + \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{4x}+\frac{\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}}+\frac{\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)dx} \\
&= 2\left(x^2+x+1\right)^{3/4}\sqrt{2}\, x^{1/4} \mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2}\, dx} \\
&= z_1 e^{-\frac{9\ln(x)}{4}-\frac{\ln\left(x^2+x+1\right)}{4}-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{x^{9/4}\left(x^2+x+1\right)^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{9\ln(x)}{2}-\frac{\ln\left(x^2+x+1\right)}{2}-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{(y_1)^2}\, dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{9\ln(x)}{2}-\frac{\ln\left(x^2+x+1\right)}{2}-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} x^4 \mathrm{e}^{\frac{2\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2+8x+8} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\right) \\
&\quad + c_2 \left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\left(\int \frac{\mathrm{e}^{-\frac{9\ln(x)}{2}-\frac{\ln\left(x^2+x+1\right)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}x^4\mathrm{e}^{\frac{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2+8x+8}\right.\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.378.2 Maple step by step solution

Let's solve

$2x^2(x^2+x+1)\left(\frac{d}{dx}y'\right)+x(11x^2+11x+9)\,y'+(7x^2+10x+6)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(7x^2+10x+6)y}{2x^2(x^2+x+1)}-\frac{(11x^2+11x+9)y'}{2x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(11x^2+11x+9)y'}{2x(x^2+x+1)}+\frac{(7x^2+10x+6)y}{2x^2(x^2+x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{11x^2+11x+9}{2x(x^2+x+1)}, P_3(x)=\frac{7x^2+10x+6}{2x^2(x^2+x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{9}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=3$

  ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

  $2x^2(x^2+x+1)\left(\frac{d}{dx}y'\right)+x(11x^2+11x+9)\,y'+(7x^2+10x+6)\,y=0$
- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..2$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m=1..3$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m=2..4$

    $x^m \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(3+2r)\,x^r+(a_1(3+r)\,(5+2r)+a_0(5+2r)\,(2+r))\,x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(2k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(3+2r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{-2,-\frac{3}{2}\right\}$
- Each term must be 0

  $a_1(3+r)\,(5+2r)+a_0(5+2r)\,(2+r)=0$
- Solve for the dependent coefficient(s)

  $a_1=-\frac{(2+r)a_0}{3+r}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{3}{2}\right)\left(\left(a_k+a_{k-2}+a_{k-1}\right)k+\left(a_k+a_{k-2}+a_{k-1}\right)r+2a_k-a_{k-2}+a_{k-1}\right)=0$
- Shift index using $k->k+2$

  $2\left(k+\frac{7}{2}+r\right)\left(\left(a_{k+2}+a_k+a_{k+1}\right)(k+2)+\left(a_{k+2}+a_k+a_{k+1}\right)r+2a_{k+2}-a_k+a_{k+1}\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{ka_k+ka_{k+1}+ra_k+ra_{k+1}+a_k+3a_{k+1}}{k+4+r}$
- Recursion relation for $r=-2$

  $a_{k+2}=-\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2}$
- Solution for $r=-2$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-2}, a_{k+2}=-\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2}, a_1=0\right]$
- Recursion relation for $r=-\frac{3}{2}$

  $a_{k+2}=-\frac{ka_k+ka_{k+1}-\frac{1}{2}a_k+\frac{3}{2}a_{k+1}}{k+\frac{5}{2}}$
- Solution for $r=-\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{3}{2}}, a_{k+2}=-\frac{ka_k+ka_{k+1}-\frac{1}{2}a_k+\frac{3}{2}a_{k+1}}{k+\frac{5}{2}}, a_1=-\frac{a_0}{3}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k-\frac{3}{2}}\right), a_{k+2}=-\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2}, a_1=0, b_{k+2}=-\frac{kb_k+kb_{k+1}-\frac{1}{2}b_k+}{k+\frac{5}{2}}\right.$

1.378.3 Maple trace

```
Methods for second order ODEs:
```

1.378.4 Maple dsolve solution

Solving time : `0.119` (sec)
Leaf size : 231

```
dsolve(2*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+x*(11*x^2+11*x+9)*diff(y(x),x)+(7*x^2+10*x
       y(x),singsol=all)
```

$$y=\frac{\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}\left(2x+1+i\sqrt{3}\right)^{\frac{5\sqrt{3}+3i}{6\sqrt{3}+6i}}\left(i\sqrt{3}-2x-1\right)^{\frac{64i\sqrt{3}+2368}{\left(\sqrt{3}+i\right)^3\left(i-\sqrt{3}\right)^4\left(13\sqrt{3}+9i\right)}}\left(\mathrm{HeunG}\left(\frac{\sqrt{3}+i}{i-\sqrt{3}},0,0,\frac{5}{2},\frac{1}{2},\right.\right.}{x^{5/2}\left(x^2+x+1\right)}$$

1.378.5 Mathematica DSolve solution

Solving time : 1.07 (sec)
Leaf size : 93

```
DSolve[{2*x^2*(1+x+x^2)*D[y[x],{x,2}] + x*(9+11*x+11*x^2)*D[y[x],x] + (6+10*x+7*x^2)*y[x] ==
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x^2+x+1}e^{-\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}}\left(c_2\int_1^x \frac{e^{\frac{\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}{\sqrt{3}}}}{\sqrt{K[1]}(K[1]^2+K[1]+1)^{3/2}}dK[1]+c_1\right)}{x^2}$$

1.379 problem 388

Internal problem ID [8517]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 388
Date solved : `Monday, October 21, 2024 at 05:09:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (1 + x)\, y' + 2y = 0$$

1.379.1 Solved as second order ode using Kovacic algorithm

Time used: 0.286 (sec)

Writing the ode as

$$xy'' + (1 + x)\, y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1 + x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 718: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4x^2} - \frac{3}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2x} - \frac{5}{2x^2} - \frac{15}{2x^3} - \frac{115}{4x^4} - \frac{495}{4x^5} - \frac{2285}{4x^6} - \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-6x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{-1 + x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 6x - 1}{4x^2}\right) \right) = 0$$

$$\frac{1 + a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (-1 + x)\, e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= (-1 + x)\, e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= (-1 + x)\sqrt{x}\, e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+x}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}(-1+x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+x}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\operatorname{Ei}_1(-x) - \frac{\mathrm{e}^{x}}{-1+x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}(-1+x)\right) + c_2\left(\mathrm{e}^{-x}(-1+x)\left(-\operatorname{Ei}_1(-x) - \frac{\mathrm{e}^{x}}{-1+x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.379.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right)+(1+x)\,y'+2y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{2y}{x}-\frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(1+x)y'}{x}+\frac{2y}{x}=0$

☐ Check to see if $x_0=0$ is a regular singular point

 ◦ Define functions

 $\left[P_2(x)=\frac{1+x}{x},P_3(x)=\frac{2}{x}\right]$

 ◦ $x\cdot P_2(x)$ is analytic at $x=0$

 $\left.(x\cdot P_2(x))\right|_{x=0}=1$

 ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

 $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

 ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$x\left(\frac{d}{dx}y'\right)+(1+x)\,y'+2y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

 ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

 $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

 ◦ Shift index using $k->k+1-m$

 $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

 ◦ Convert $x\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$$

- ○ Shift index using $k->k+1$

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k(k+r+2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 + a_k(k+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}\right]$$

1.379.3 Maple trace

```
Methods for second order ODEs:
```

1.379.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 29

```
dsolve(x*diff(diff(y(x),x),x)+(1+x)*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2\, e^{-x}(-1+x)\, \mathrm{Ei}_1(-x) + c_1\, e^{-x}(-1+x) + c_2$$

1.379.5 Mathematica DSolve solution

Solving time : `0.094` (sec)
Leaf size : 33

```
DSolve[{x*D[y[x],{x,2}] +(1+x)*D[y[x],x]+2*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x}(c_2(x-1)\,\text{ExpIntegralEi}(x) + c_1(x-1) - c_2e^x)$$

1.380 problem 389

Internal problem ID [8518]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 389
Date solved : `Monday, October 21, 2024 at 05:09:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2-2x+1)\,y''-x(3+x)\,y'+(4+x)\,y=0$$

1.380.1 Solved as second order ode using Kovacic algorithm

Time used: 0.321 (sec)

Writing the ode as

$$x^2(x-1)^2\,y''+\left(-x^2-3x\right)y'+(4+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(x-1)^2\\ B&=-x^2-3x\\ C&=4+x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 10x - 1}{4x^2 (x-1)^4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 7x^2 + 10x - 1$$
$$t = 4x^2(x-1)^4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 10x - 1}{4x^2 (x-1)^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 720: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2(x-1)^4$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{2(x-1)} + \frac{7}{4(x-1)^2} - \frac{1}{4x^2} + \frac{3}{2x} + \frac{4}{(x-1)^4} - \frac{2}{(x-1)^3}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{3}{2(x-1)} + \frac{7}{4(x-1)^2} - \frac{1}{4x^2} + \frac{3}{2x} + \frac{4}{(x-1)^4} - \frac{2}{(x-1)^3}$$

There is pole in r at $x = 1$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 1$ gives

$$[\sqrt{r}]_c \approx \frac{2}{(x-1)^2} - \frac{1}{2(x-1)} + \frac{21}{32} - \frac{9x}{32} + \frac{53(x-1)^2}{256} - \frac{149(x-1)^3}{1024} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{2}{(x-1)^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-1)^2}$ is

$$a = 2$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 1$. This term becomes $\frac{1}{(x-1)^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -2. Therefore

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{2}{(x-1)^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{-2}{2} + 2\right) &&= \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{-2}{2} + 2\right) &&= \frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2 + 10x - 1}{4x^2 (x - 1)^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	4	$\frac{2}{(x-1)^2}$	$\frac{1}{2}$	$\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{2}{(x - 1)^2} + \frac{1}{2x - 2} + (-)(0) \\
&= \frac{1}{2x} + \frac{2}{(x - 1)^2} + \frac{1}{2x - 2} \\
&= \frac{2x^2 + x + 1}{2x (x - 1)^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{4}{(x-1)^3} - \frac{1}{2(x-1)^2}\right) + \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right) dx} \\
&= \sqrt{x}\,\sqrt{x-1}\,\mathrm{e}^{-\frac{2}{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2-3x}{x^2(x-1)^2} \, dx} \\
&= z_1 e^{\frac{3\ln(x)}{2} - \frac{2}{x-1} - \frac{3\ln(x-1)}{2}} \\
&= z_1 \left(\frac{x^{3/2}\mathrm{e}^{-\frac{2}{x-1}}}{(x-1)^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}}{(x-1)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2(x-1)^2} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{3\ln(x)-\frac{4}{x-1}-3\ln(x-1)}}{(y_1)^2} dx$$
$$= y_1 \left(\mathrm{e}^{-4} \mathrm{Ei}_1 \left(-\frac{4}{x-1} - 4\right)\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x^{3/2} \mathrm{e}^{-\frac{4}{x-1}} \sqrt{x(x-1)}}{(x-1)^{3/2}}\right) + c_2 \left(\frac{x^{3/2} \mathrm{e}^{-\frac{4}{x-1}} \sqrt{x(x-1)}}{(x-1)^{3/2}} \left(\mathrm{e}^{-4} \mathrm{Ei}_1 \left(-\frac{4}{x-1} - 4\right)\right)\right)$$

Will add steps showing solving for IC soon.

1.380.2 Maple step by step solution

Let's solve

$x^2(x^2-2x+1)\left(\frac{d}{dx}y'\right) - x(3+x)\, y' + (4+x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4+x)y}{x^2(x^2-2x+1)} + \frac{(3+x)y'}{x(x^2-2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(3+x)y'}{x(x^2-2x+1)} + \frac{(4+x)y}{x^2(x^2-2x+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{3+x}{x(x^2-2x+1)}, P_3(x) = \frac{4+x}{x^2(x^2-2x+1)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -3$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2-2x+1)\left(\frac{d}{dx}y'\right) - x(3+x)\,y' + (4+x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2\,x^r + \left(a_1(-1+r)^2 - a_0(1+2r)\,(-1+r)\right)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(2k -\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term must be 0

  $a_1(-1+r)^2 - a_0(1+2r)(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0(1+2r)}{-1+r}$
- Each term in the series must be 0, giving the recursion relation

  $((a_k + a_{k-2} - 2a_{k-1})k + (a_k + a_{k-2} - 2a_{k-1})r - 2a_k - 3a_{k-2} + a_{k-1})(k+r-2) = 0$
- Shift index using $k->k+2$

  $((a_{k+2} + a_k - 2a_{k+1})(k+2) + (a_{k+2} + a_k - 2a_{k+1})r - 2a_{k+2} - 3a_k + a_{k+1})(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{ka_k - 2ka_{k+1} + ra_k - 2ra_{k+1} - a_k - 3a_{k+1}}{k+r}$
- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{ka_k - 2ka_{k+1} + a_k - 7a_{k+1}}{k+2}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{ka_k - 2ka_{k+1} + a_k - 7a_{k+1}}{k+2}, a_1 = 5a_0\right]$

1.380.3 Maple trace

```
Methods for second order ODEs:
```

1.380.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 45

```
dsolve(x^2*(x^2-2*x+1)*diff(diff(y(x),x),x)-x*(3+x)*diff(y(x),x)+(4+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^2\left(c_2\,e^{-\frac{4x}{x-1}}\operatorname{Ei}_1\left(-\frac{4x}{x-1}\right) + e^{-\frac{4}{x-1}}c_1\right)}{x-1}$$

1.380.5 Mathematica DSolve solution

Solving time : 0.296 (sec)
Leaf size : 54

```
DSolve[{x^2*(1-2*x+x^2)*D[y[x],{x,2}] -x*(3+x)*D[y[x],x]+(4+x)*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{4x}{x-1}}\sqrt{1-x}x^2\left(c_2 \operatorname{ExpIntegralEi}\left(\frac{4x}{x-1}\right)+e^4c_1\right)}{(x-1)^{3/2}}$$

1.381 problem 390

Internal problem ID [8519]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 390
Date solved : `Monday, October 21, 2024 at 05:09:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''+5x^2y'+(1+x)\,y=0$$

1.381.1 Solved as second order ode using Kovacic algorithm

Time used: 0.222 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y''+5x^2y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2 \\ B &= 5x^2 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 - 24x - 16$$
$$t = 16\left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 722: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16 + 8x} - \frac{1}{4x^2} + \frac{5}{16\left(2 + x\right)^2} - \frac{1}{8x}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 24x - 16}{16\,(x^2 + 2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(2+x\right)} + \frac{1}{2x} + (-)\left(0\right) \\
&= -\frac{1}{4\left(2+x\right)} + \frac{1}{2x} \\
&= \frac{x+4}{4x\left(2+x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(2+x\right)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{4\left(2+x\right)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{4\left(2+x\right)} + \frac{1}{2x}\right)^2 - \left(\frac{-3x^2-24x-16}{16\left(x^2+2x\right)^2}\right)\right.
$$

$$
0 =
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{4(2+x)} + \frac{1}{2x}\right) dx} \\
&= \frac{\sqrt{x}}{\left(2+x\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x^2}{2x^3+4x^2} \, dx} \\
&= z_1 e^{-\frac{5\ln(2+x)}{4}} \\
&= z_1 \left(\frac{1}{\left(2+x\right)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(2+x)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2}{2x^3+4x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(2+x)}{2}}}{(y_1)^2} dx \\
&= y_1 \left(2\sqrt{2+x} - 2\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{(2+x)^{3/2}}\right) + c_2 \left(\frac{\sqrt{x}}{(2+x)^{3/2}} \left(2\sqrt{2+x} - 2\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.381.2 Maple step by step solution

Let's solve

$2x^2(2+x)\left(\frac{d}{dx}y'\right) + 5x^2y' + (1+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(1+x)y}{2x^2(2+x)} - \frac{5y'}{2(2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{5y'}{2(2+x)} + \frac{(1+x)y}{2x^2(2+x)} = 0$

- [ ] Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{5}{2(2+x)}, P_3(x) = \frac{1+x}{2x^2(2+x)}\right]$
  - $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((2+x)\cdot P_2(x)\right)\bigg|_{x=-2} = \frac{5}{2}$
  - $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((2+x)^2\cdot P_3(x)\right)\bigg|_{x=-2} = 0$
  - $x=-2$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -2$
- Multiply by denominators

  $2x^2(2+x)\left(\frac{d}{dx}y'\right) + 5x^2y' + (1+x)\,y = 0$
- Change variables using $x = u-2$ so that the regular singular point is at $u=0$

  $(2u^3 - 8u^2 + 8u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (5u^2 - 20u + 20)\left(\frac{d}{du}y(u)\right) + (-1+u)\,y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- [ ] Rewrite ODE with series expansions
  - Convert $u^m\cdot y(u)$ to series expansion for $m = 0..1$

    $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k->k-m$

    $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $4a_0r(3+2r)\, u^{-1+r} + (4a_1(1+r)(5+2r) - a_0(8r^2+12r+1))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)(2k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r(3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{3}{2}\right\}$
- Each term must be 0

  $4a_1(1+r)(5+2r) - a_0(8r^2+12r+1) = 0$
- Each term in the series must be 0, giving the recursion relation

  $2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\, r - 12a_k - a_{k-1} + 28a_{k+1})\, k + 2(-4a_k +$
- Shift index using $k->k+1$

  $2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\, r - 12a_{k+1} - a_k + 28a_{k+2})\,(k+1)$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} + 3ka_k - 28ka_{k+1} + 3ra_k - 28ra_{k+1} + a_k - 21a_{k+1}}{4(2k^2+4kr+2r^2+11k+11r+14)}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$
- Revert the change of variables $u = 2 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(2+x)^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$
- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 3ka_k - 4ka_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}$
- Solution for $r = -\frac{3}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 3ka_k - 4ka_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$

- Revert the change of variables $u = 2 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (2+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (2+x)^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a\right.$$

1.381.3 Maple trace

```
Methods for second order ODEs:
```

1.381.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 39

```
dsolve(2*x^2*(2+x)*diff(diff(y(x),x),x)+5*x^2*diff(y(x),x)+(1+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\left(\sqrt{2}\,\sqrt{2+x}\,c_2 - 2\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)c_2 + c_1\right)\sqrt{x}}{(2+x)^{3/2}}$$

1.381.5 Mathematica DSolve solution

Solving time : `0.105` (sec)
Leaf size : 55

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}] +5*x^2*D[y[x],x]+(1+x)*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}\left(-2\sqrt{2}c_2\text{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right) + 2c_2\sqrt{x+2} + c_1\right)}{(x+2)^{3/2}}$$

1.382 problem 391

Internal problem ID [8520]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 391
Date solved : `Monday, October 21, 2024 at 05:09:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 4xy' + \left(x^2 + 2\right) y = 0$$

1.382.1 Solved as second order ode using Kovacic algorithm

Time used: 0.155 (sec)

Writing the ode as

$$x^2y'' + 4xy' + \left(x^2 + 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 724: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\,dx} \\ &= z_1 e^{-2\ln(x)} \\ &= z_1\left(\frac{1}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{x^2}\right) + c_2\left(\frac{\cos(x)}{x^2}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.382.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + 4xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+2)y}{x^2} - \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{4y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + 4xy' + (x^2+2)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (2+r)\,(1+r)\, x^r + a_1 (3+r)\,(2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k+r+2)\,(k+r+1) + a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$
- Each term must be 0

  $a_1 (3+r)\,(2+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k (k+r+2)\,(k+r+1) + a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2} (k+4+r)\,(k+3+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+4+r)(k+3+r)}$
- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$$

1.382.3 Maple trace

```
Methods for second order ODEs:
```

1.382.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{x^2}$$

1.382.5 Mathematica DSolve solution

Solving time : `0.048` (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x^2}$$

1.383 problem 392

Internal problem ID [8521]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 392
Date solved : `Monday, October 21, 2024 at 05:09:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

1.383.1 Solved as second order ode using Kovacic algorithm

Time used: 0.161 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 726: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.383.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 4xy' + (4x^2 - 1)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

$$a_1(3+2r)\,(1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$$

- Shift index using $k- >k + 2$

$$a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4a_k}{4k^2 + 8kr + 4r^2 + 16k + 16r + 15}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+2} = -\frac{4a_k}{4k^2 + 12k + 8}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

1.383.3 Maple trace

```
Methods for second order ODEs:
```

1.383.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1 \sin(x)+c_2 \cos(x)}{\sqrt{x}}$$

1.383.5 Mathematica DSolve solution

Solving time : 0.049 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{e^{-ix}(2c_1-ic_2e^{2ix})}{2\sqrt{x}}$$

1.384 problem 394

Internal problem ID [8522]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 394
Date solved : `Monday, October 21, 2024 at 05:09:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - xy' - \left(x^2 + \frac{5}{4}\right) y = 0$$

1.384.1 Solved as second order ode using Kovacic algorithm

Time used: 0.261 (sec)

Writing the ode as

$$x^2y'' - xy' + \left(-x^2 - \frac{5}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= -x^2 - \frac{5}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 + 2}{x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 2 \\
t &= x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 + 2}{x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 728: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{x^2} - \frac{1}{2x^4} + \frac{1}{2x^6} - \frac{5}{8x^8} + \frac{7}{8x^{10}} - \frac{21}{16x^{12}} + \frac{33}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (1) + \left(\frac{2}{x^2}\right) \\ &= 1 + \frac{2}{x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned}b &= (0) - (0) \\ &= 0\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{1} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{1} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(1) \\
&= -\frac{1}{x} - 1 \\
&= -\frac{1+x}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - 1\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - 1\right)^2 - \left(\frac{x^2+2}{x^2}\right) \right) &= 0 \\
\frac{-2 + 2a_0}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 1 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1 + x)\, e^{\int \left(-\frac{1}{x} - 1\right) dx} \\
&= (1 + x)\, e^{-x - \ln(x)} \\
&= \frac{(1 + x)\, e^{-x}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(-1+x)\,\mathrm{e}^{2x}}{2+2x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}}\right) + c_2\left(\frac{(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}}\left(\frac{(-1+x)\,\mathrm{e}^{2x}}{2+2x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.384.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - xy' - \left(x^2 + \frac{5}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = \frac{(4x^2+5)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{y'}{x} - \frac{(4x^2+5)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{4x^2+5}{4x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\bigg|_{x=0} = -1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{5}{4}$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$4x^2\left(\frac{d}{dx}y'\right) - 4xy' + (-4x^2 - 5)y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-5+2r)\, x^r + a_1(3+2r)\,(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-5) - 4a\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-5+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{5}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(-3+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r+\frac{1}{2}\right)\left(k+r-\frac{5}{2}\right) a_k - 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $4\left(k+\frac{5}{2}+r\right)\left(k-\frac{1}{2}+r\right) a_{k+2} - 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4a_k}{(2k+5+2r)(2k-1+2r)}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4a_k}{(2k+4)(2k-2)}$

- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4a_k}{(2k+4)(2k-2)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{5}{2}$

  $a_{k+2} = \frac{4a_k}{(2k+10)(2k+4)}$

- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = \frac{4a_k}{(2k+10)(2k+4)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{5}{2}}\right), a_{k+2}=\frac{4a_k}{(2k+4)(2k-2)}, a_1=0, b_{k+2}=\frac{4b_k}{(2k+10)(2k+4)}, b_1=0\right]$$

1.384.3 Maple trace

```
Methods for second order ODEs:
```

1.384.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 25

```
dsolve(x^2*diff(diff(y(x),x),x)-x*diff(y(x),x)-(x^2+5/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{(1+x)\,c_2\,\mathrm{e}^{-x}+c_1\,\mathrm{e}^{x}(-1+x)}{\sqrt{x}}$$

1.384.5 Mathematica DSolve solution

Solving time : 0.107 (sec)
Leaf size : 53

```
DSolve[{x^2*D[y[x],{x,2}]-x*D[y[x],x]-(x^2+5/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{\sqrt{\frac{2}{\pi}}((ic_2x+c_1)\sinh(x)-(c_1x+ic_2)\cosh(x))}{\sqrt{-ix}}$$

1.385 problem 395

Internal problem ID [8523]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 395
Date solved : `Monday, October 21, 2024 at 05:09:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

1.385.1 Solved as second order ode using Kovacic algorithm

Time used: 0.159 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 730: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.385.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 4xy' + (4x^2 - 1)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r}$$

- ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1 + 2r)\,(-1 + 2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

$$a_1(3 + 2r)\,(1 + 2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$$

- Shift index using $k- >k + 2$

$$a_{k+2}\left(4(k + 2)^2 + 8(k + 2)\, r + 4r^2 - 1\right) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

1.385.3 Maple trace

```
Methods for second order ODEs:
```

1.385.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1\sin(x)+c_2\cos(x)}{\sqrt{x}}$$

1.385.5 Mathematica DSolve solution

Solving time : 0.043 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{e^{-ix}(2c_1-ic_2e^{2ix})}{2\sqrt{x}}$$

1.386 problem 396

Internal problem ID [8524]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 396
Date solved : `Monday, October 21, 2024 at 05:09:11 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$x^2y'' + 3xy' + 4x^4y = 0$$

1.386.1 Solved as second order ode using Kovacic algorithm

Time used: 0.288 (sec)

Writing the ode as

$$x^2y'' + 3xy' + 4x^4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x \\ C &= 4x^4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-16x^4 + 3}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -16x^4 + 3 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 732: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(-4x^2\right) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 2ix \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-16x^4 + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{2x} + (-)(2ix) \\ &= -\frac{1}{2x} - 2ix \\ &= -\frac{1}{2x} - 2ix\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\ &= \frac{e^{-ix^2}}{\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2}\, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2}} \\ &= z_1\left(\frac{1}{x^{3/2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-i x^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{i e^{2 i x^2}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-i x^2}}{x^2}\right) + c_2 \left(\frac{e^{-i x^2}}{x^2} \left(-\frac{i e^{2 i x^2}}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.386.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + 3 x y' + 4 x^4 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -4 x^2 y - \frac{3 y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{3 y'}{x} + 4 x^2 y = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $4yx^3 + \left(\frac{d}{dx}y'\right)x + 3y' = 0$
- Assume series solution for y
  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - Convert $x^3 \cdot y$ to series expansion
    $$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$
  - Shift index using $k- > k-3$
    $$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$
  - Convert y' to series expansion
    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$
  - Shift index using $k- > k+1$
    $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
    $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-1}$$
  - Shift index using $k- > k+1$
    $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r)\,x^{-1+r} + a_1(1+r)\,(3+r)\,x^r + a_2(2+r)\,(4+r)\,x^{1+r} + a_3(3+r)\,(5+r)\,x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$
- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$
- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$
- Shift index using $k- >k+3$

  $a_{k+4}(k+4+r)\,(k+6+r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$
- Recursion relation for $r = -2$

  $a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$
- Solution for $r = -2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$
- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$
- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{4b_k}{(k+4)(k+6)}\right.$$

1.386.3 Maple trace

```
Methods for second order ODEs:
```

1.386.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+3*x*diff(y(x),x)+4*x^4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x^2) + c_2 \cos(x^2)}{x^2}$$

1.386.5 Mathematica DSolve solution

Solving time : 0.071 (sec)
Leaf size : 41

```
DSolve[{x^2*D[y[x],{x,2}]+3*x*D[y[x],x]+4*x^4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1 e^{-ix^2} - ic_2 e^{ix^2}}{4x^2}$$

1.387 problem 398

Internal problem ID [8525]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 398
Date solved : `Monday, October 21, 2024 at 05:09:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = \left(x^2 + 3\right) y$$

1.387.1 Solved as second order ode using Kovacic algorithm

Time used: 0.222 (sec)

Writing the ode as

$$y'' + \left(-x^2 - 3\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -x^2 - 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 + 3}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 3 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(x^2 + 3\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 734: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{2x} - \frac{9}{8x^3} + \frac{27}{16x^5} - \frac{405}{128x^7} + \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} + \frac{72171}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 3}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2 + 3) + (0) \\ &= x^2 + 3 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{3}{1} - 1\right) &= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{3}{1} - 1\right) &= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 + 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + (x) \\ &= x \\ &= x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2(x)(1) + \left((1) + (x)^2 - \left(x^2 + 3\right)\right) &= 0 \\ -2a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int x dx} \\
&= (x)\, \mathrm{e}^{\frac{x^2}{2}} \\
&= x\, \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= x\, \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\, \mathrm{e}^{\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= x\, \mathrm{e}^{\frac{x^2}{2}} \int \frac{1}{x^2 \mathrm{e}^{x^2}} \, dx \\
&= x\, \mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi}\, \mathrm{erf}\,(x) \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\, \mathrm{e}^{\frac{x^2}{2}} \right) + c_2 \left(x\, \mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi}\, \mathrm{erf}\,(x) \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.387.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' = (x^2+3)\,y$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + (-x^2-3)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\,x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\,x^k$

  Rewrite ODE with series expansions

  $2a_2 - 3a_0 + (6a_3 - 3a_1)\,x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - 3a_k - a_{k-2})\,x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 3a_0 = 0, 6a_3 - 3a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\,a_{k+2} - 3a_k - a_{k-2} = 0$

- Shift index using $k- >k+2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 3a_{k+2} - a_k = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=\frac{3a_{k+2}+a_k}{k^2+7k+12}, a_2=\frac{3a_0}{2}, a_3=\frac{a_1}{2}\right]$$

1.387.3 Maple trace

```
Methods for second order ODEs:
```

1.387.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x) = (x^2+3)*y(x),
       y(x),singsol=all)
```

$$y=x\left(c_2\sqrt{\pi}\ \mathrm{erf}\left(x\right)+c_1\right)\mathrm{e}^{\frac{x^2}{2}}+\mathrm{e}^{-\frac{x^2}{2}}c_2$$

1.387.5 Mathematica DSolve solution

Solving time : `0.107` (sec)
Leaf size : 46

```
DSolve[{D[y[x],{x,2}]==(x^2+3)*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to e^{-\frac{x^2}{2}}\left(-\sqrt{\pi}c_2e^{x^2}x\mathrm{erf}(x)+c_1e^{x^2}x-c_2\right)$$

1.388 problem 399

Internal problem ID [8526]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 399
Date solved : `Monday, October 21, 2024 at 05:09:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 2xy' + (x^2 + 1) y = 0$$

1.388.1 Solved as second order ode using Kovacic algorithm

Time used: 0.092 (sec)

Writing the ode as

$$y'' + 2xy' + (x^2 + 1) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x \\ C &= x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 736: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{1}\,dx} \\
&= z_1 e^{-\frac{x^2}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} (x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.388.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 2xy' + (x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- > k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- > k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + a_0 + (6a_3 + 3a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(2k+1) + a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + a_0 = 0, 6a_3 + 3a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} + 2a_k k + a_k + a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $\left((k + 2)^2 + 3k + 8\right) a_{k+4} + 2a_{k+2}(k + 2) + a_{k+2} + a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2ka_{k+2}+a_k+5a_{k+2}}{k^2+7k+12}, a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{2}\right]$

1.388.3 Maple trace

```
Methods for second order ODEs:
```

1.388.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+2*x*diff(y(x),x)+(x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-\frac{x^2}{2}}(c_2 x + c_1)$$

1.388.5 Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 22

```
DSolve[{D[y[x],{x,2}]+2*x*D[y[x],x]+(x^2+1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{x^2}{2}}(c_2 x + c_1)$$

1.389 problem 400

Internal problem ID [8527]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 400
Date solved : `Monday, October 21, 2024 at 05:09:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^3y'' + y' - \frac{y}{x} = 0$$

1.389.1 Solved as second order ode using Kovacic algorithm

Time used: 0.228 (sec)

Writing the ode as

$$x^3y'' + y' - \frac{y}{x} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3 \\ B &= 1 \\ C &= -\frac{1}{x} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-2x^2 + 1}{4x^6} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -2x^2 + 1 \\
t &= 4x^6
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-2x^2 + 1}{4x^6}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 738: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^6$. There is a pole at $x = 0$ of order 6. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_2^v \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{1}{2x^4} + \frac{1}{4x^6}$$

There is pole in r at $x = 0$ of order 6, hence $v = 3$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^3} - \frac{1}{2x} - \frac{x}{4} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 3$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^3} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^3}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^4}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $-\frac{1}{2}$. Therefore

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^3} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} + 3\right) &= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} + 3\right) &= 2 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-2x^2 + 1}{4x^6}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	6	$\frac{1}{2x^3}$	1	2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to

determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = 1$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x^3} + \frac{1}{x} + (-)\,(0) \\ &= \frac{1}{2x^3} + \frac{1}{x} \\ &= \frac{1}{2x^3} + \frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x^3} + \frac{1}{x}\right)(0) + \left(\left(-\frac{3}{2x^4} - \frac{1}{x^2} \right) + \left(\frac{1}{2x^3} + \frac{1}{x} \right)^2 - \left(\frac{-2x^2 + 1}{4x^6} \right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x^3}+\frac{1}{x}\right) dx} \\
&= x\,\mathrm{e}^{-\frac{1}{4x^2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{x^3}\, dx} \\
&= z_1 e^{\frac{1}{4x^2}} \\
&= z_1 \left(\mathrm{e}^{\frac{1}{4x^2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{x^3}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{1}{2x^2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{2x}\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2 \left(x \left(\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{2x}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.389.2 Maple step by step solution

1.389.3 Maple trace

```
Methods for second order ODEs:
```

1.389.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 19

```
dsolve(x^3*diff(diff(y(x),x),x)+diff(y(x),x)-1/x*y(x) = 0,
       y(x),singsol=all)
```

$$y = x\left(c_1 + c_2 \operatorname{erf}\left(\frac{i\sqrt{2}}{2x}\right)\right)$$

1.389.5 Mathematica DSolve solution

Solving time : 0.126 (sec)
Leaf size : 34

```
DSolve[{x^3*D[y[x],{x,2}]+ D[y[x],x]-1/x*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x - \sqrt{\frac{\pi}{2}} c_2 x \operatorname{erfi}\left(\frac{1}{\sqrt{2}x}\right)$$

1.390 problem 401

Internal problem ID [8528]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 401
Date solved : `Monday, October 21, 2024 at 05:09:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

1.390.1 Solved as second order ode using Kovacic algorithm

Time used: 0.159 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 739: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.390.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 4xy' + (4x^2 - 1)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - ○ Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - ○ Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$
  - ○ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$$

1.390.3 Maple trace

```
Methods for second order ODEs:
```

1.390.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{\sqrt{x}}$$

1.390.5 Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

1.391 problem 402

Internal problem ID [8529]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 402
Date solved : `Monday, October 21, 2024 at 05:09:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 0$$

1.391.1 Solved as second order ode using Kovacic algorithm

Time used: 0.113 (sec)

Writing the ode as

$$4x^2y'' + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2+4x \\ C &= 4x^2-4x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 741: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2x - \ln(x)}}{(y_1)^2}\, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{e^x}{\sqrt{x}}\right) + c_2\left(\frac{e^x}{\sqrt{x}}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.391.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) + (-8x^2 + 4x)\,y' + (4x^2 - 4x - 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-4x-1)y}{4x^2} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(2x-1)y'}{x} + \frac{(4x^2-4x-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2-4x-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4x(2x-1)\,y' + (4x^2 - 4x - 1)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\, x^r + \left(a_1(3+2r)\,(1+2r) - 4a_0(1+2r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k -$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

$$a_1(3+2r)\,(1+2r) - 4a_0(1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = \frac{4a_0}{3+2r}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(4k^2 + 8kr + 4r^2 - 1) + (-8k - 8r + 4)\, a_{k-1} + 4a_{k-2} = 0$$

- Shift index using $k- >k + 2$

$$a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + (-8k - 12 - 8r)\, a_{k+1} + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{4(2ka_{k+1} + 2ra_{k+1} - a_k + 3a_{k+1})}{4k^2 + 8kr + 4r^2 + 16k + 16r + 15}$$

- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}$
- Solution for $r=-\frac{1}{2}$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1=2a_0\right]$$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=\frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}$
- Solution for $r=\frac{1}{2}$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=\frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}, a_1=a_0\right]$$
- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1=2a_0, b_{k+2}=\frac{4(2kb_{k+1}-b_k+4b_{k+1}}{4k^2+20k+24}\right.$$

1.391.3 Maple trace

```
Methods for second order ODEs:
```

1.391.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 15

```
dsolve(4*x^2*diff(diff(y(x),x),x)+(-8*x^2+4*x)*diff(y(x),x)+(4*x^2-4*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{e^{x}(c_2 x+c_1)}{\sqrt{x}}$$

1.391.5 Mathematica DSolve solution

Solving time : `0.047` (sec)
Leaf size : 21

```
DSolve[{4*x^2*D[y[x],{x,2}]+(-8*x^2+4*x)*D[y[x],x]+(4*x^2-4*x-1)*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^x(c_2x + c_1)}{\sqrt{x}}$$

1.392 problem 404

Internal problem ID [8530]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 404
Date solved : `Monday, October 21, 2024 at 05:09:16 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' - y' + y = 0$$

1.392.1 Solved as second order ode using Kovacic algorithm

Time used: 0.198 (sec)

Writing the ode as

$$y'' - y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-3}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -3 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -\frac{3z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 743: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1 \left(e^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)\right) + c_2 \left(\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

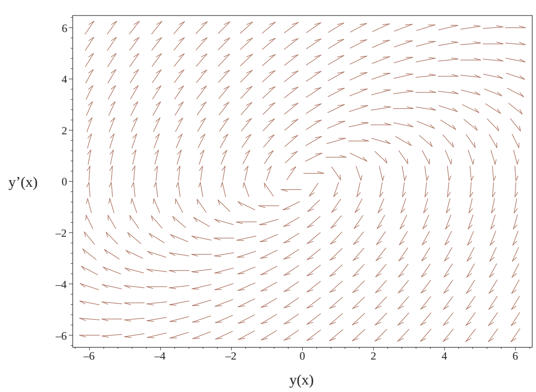

Figure 1: Slope field plot
$y'' - y' + y = 0$

1.392.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Characteristic polynomial of ODE

  $r^2 - r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1 \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{I\sqrt{3}}{2}, \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y = C1 y_1(x) + C2 y_2(x)$

- Substitute in solutions

  $y = C1\, e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\, e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

1.392.3 Maple trace

```
Methods for second order ODEs:
```

1.392.4 Maple dsolve solution

Solving time : `0.000` (sec)
Leaf size : 28

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{\frac{x}{2}}\left(c_1 \sin\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

1.392.5 Mathematica DSolve solution

Solving time : 0.034 (sec)
Leaf size : 42

```
DSolve[{D[y[x],{x,2}]-D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x/2}\left(c_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

1.393 problem 405

Internal problem ID [8531]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 405
Date solved : `Monday, October 21, 2024 at 05:09:17 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^{2}-1\right) y''-2 x y'+2 y=0$$

1.393.1 Solved as second order ode using Kovacic algorithm

Time used: 0.213 (sec)

Writing the ode as

$$\left(x^{2}-1\right) y''-2 x y'+2 y=0 \tag{1}$$

$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}-1 \\ B &= -2 x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2 A} d x}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= (x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 745: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 0\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4(x-1)} + \frac{3}{4(x+1)^2} + \frac{3}{4(x-1)^2} + \frac{3}{4(x+1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{\left(x^2-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \frac{3}{2(x+1)} + (-)(0) \\
&= -\frac{1}{2(x-1)} + \frac{3}{2(x+1)} \\
&= \frac{x-2}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{3}{2(x+1)}\right)(0) + \left(\left(\frac{1}{2(x-1)^2} - \frac{3}{2(x+1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{3}{2(x+1)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(x-1)} + \frac{3}{2(x+1)}\right) dx} \\ &= \frac{(x+1)^{3/2}}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x}{x^2-1} \, dx} \\ &= z_1 e^{\frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2}} \\ &= z_1 \left(\sqrt{x-1}\,\sqrt{x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (x+1)^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x-1)+\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x\,e^{\ln(x-1)+\ln(x+1)}}{(x+1)^3\,(x-1)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x+1)^2\right) + c_2\left((x+1)^2\left(-\frac{x\,e^{\ln(x-1)+\ln(x+1)}}{(x+1)^3\,(x-1)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.393.2 Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d}{dx}y'\right) - 2xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2y}{x^2-1} + \frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2xy'}{x^2-1} + \frac{2y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{2x}{x^2-1}, P_3(x) = \frac{2}{x^2-1}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = -1$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d}{dx}y'\right) - 2xy' + 2y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r-1) + a_k(k+r-1)(k+r-2)\right) u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $((-2k-2r-2)\, a_{k+1} + a_k(k+r-2))(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r-2)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{2(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{4}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - u + \frac{1}{4}u^2\right)$
- Revert the change of variables $u = x + 1$

  $\left[y = \frac{a_0(x-1)^2}{4}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k}{2(k+3)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$
- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$
- Combine solutions and rename parameters

  $\left[y = \frac{a_0(x-1)^2}{4} + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$

1.393.3 Maple trace

```
Methods for second order ODEs:
```

1.393.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve((x^2-1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2 x^2 + c_1 x + c_2$$

1.393.5 Mathematica DSolve solution

Solving time : 0.127 (sec)
Leaf size : 39

```
DSolve[{(x^2-1)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x^2-1}(c_1(x-1)^2 + c_2 x)}{\sqrt{1-x^2}}$$

1.394 problem 406

Internal problem ID [8532]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 406
Date solved : `Monday, October 21, 2024 at 05:09:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(x+2)\,y' + (x+2)\,y = 0$$

1.394.1 Solved as second order ode using Kovacic algorithm

Time used: 0.099 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2 - 2x\right)y' + (x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 2x \\ C &= x + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 747: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-2x}{x^2}\, dx} \\
&= z_1 e^{\frac{x}{2}+\ln(x)} \\
&= z_1 \left(x\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{x+2\ln(x)}}{x^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{e^{x+2\ln(x)}}{x^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.394.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x(x+2)\,y' + (x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x+2)y}{x^2} + \frac{(x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x+2)y'}{x} + \frac{(x+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x+2}{x}, P_3(x) = \frac{x+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - x(x+2)\,y' + (x+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^{m} \cdot y$ to series expansion for $m = 0..1$

    $x^{m} \cdot y = \sum_{k=0}^{\infty} a_{k} x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^{m} \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^{m} \cdot y'$ to series expansion for $m = 1..2$

    $x^{m} \cdot y' = \sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^{m} \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^{2} \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^{2} \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+r)\,(-2+r)\, x^{r} + \left(\sum_{k=1}^{\infty} \left(a_{k}(k+r-1)\,(k+r-2) - a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$
- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_{k}(k+r-1) - a_{k-1}) = 0$
- Shift index using $k->k+1$

  $(k+r-1)\,(a_{k+1}(k+r) - a_{k}) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_{k}}{k+r}$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_{k}}{k+1}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_{k} x^{k+1}, a_{k+1} = \frac{a_{k}}{k+1}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_{k}}{k+2}$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k}{k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+2}\right]$$

1.394.3 Maple trace

```
Methods for second order ODEs:
```

1.394.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 12

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(x+2)*diff(y(x),x)+(x+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x(c_1 + c_2 e^x)$$

1.394.5 Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 16

```
DSolve[{x^2*D[y[x],{x,2}]-x*(x+2)*D[y[x],x]+(x+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x(c_2 e^x + c_1)$$

1.395 problem 407

Internal problem ID [8533]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 407
Date solved : `Monday, October 21, 2024 at 05:09:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x+1)\,y''-(x+2)\,y'+y=0$$

1.395.1 Solved as second order ode using Kovacic algorithm

Time used: 0.240 (sec)

Writing the ode as

$$(x+1)\,y''+(-x-2)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x+1 \\ B &= -x-2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 2}{4(x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 2$$
$$t = 4(x+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 2}{4(x+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 749: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+1)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x+1)^2} - \frac{1}{2(x+1)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^2} - \frac{1}{x^3} + \frac{3}{4x^4} - \frac{3}{4x^5} + \frac{1}{x^6} - \frac{1}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 2}{4x^2 + 8x + 4} \\ &= Q + \frac{R}{4x^2 + 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 1}{4x^2 + 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 1}{4x^2 + 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 2}{4(x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x+1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x+1)} + \frac{1}{2} \\
&= \frac{x}{2x+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2(x+1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x+1)^2}\right) + \left(-\frac{1}{2(x+1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2+2}{4(x+1)^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x+1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x+1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x-2}{x+1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x+1)}{2}} \\
&= z_1 \left(\sqrt{x+1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-2}{x+1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+\ln(x+1)}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{(x+2)\,\mathrm{e}^{x+\ln(x+1)}\mathrm{e}^{-2x}}{x+1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-\frac{(x+2)\,\mathrm{e}^{x+\ln(x+1)}\mathrm{e}^{-2x}}{x+1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.395.2 Maple step by step solution

Let's solve

$(x+1)\left(\frac{d}{dx}y'\right) - (x+2)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x+1} + \frac{(x+2)y'}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x+2)y'}{x+1} + \frac{y}{x+1} = 0$

☐ Check to see if $x_0 = -1$ is a regular singular point

- Define functions
  $\left[P_2(x)=-\frac{x+2}{x+1}, P_3(x)=\frac{1}{x+1}\right]$
- $(x+1)\cdot P_2(x)$ is analytic at $x=-1$
  $$\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=-1$$
- $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$
  $$\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$$
- $x=-1$is a regular singular point
  Check to see if $x_0=-1$ is a regular singular point
  $x_0=-1$

- Multiply by denominators
  $(x+1)\left(\frac{d}{dx}y'\right)+(-x-2)\,y'+y=0$
- Change variables using $x=u-1$ so that the regular singular point is at $u=0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$
- Assume series solution for $y(u)$
  $$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

□ Rewrite ODE with series expansions
- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$$
- Shift index using $k->k+1$
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Revert the change of variables $u=x+1$

  $\left[y=\sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Revert the change of variables $u=x+1$

  $\left[y=\sum_{k=0}^{\infty} a_k(x+1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k(x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x+1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

1.395.3 Maple trace

```
Methods for second order ODEs:
```

1.395.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve((x+1)*diff(diff(y(x),x),x)-(x+2)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1(x+2) + c_2\,\mathrm{e}^x$$

1.395.5 Mathematica DSolve solution

Solving time : 0.231 (sec)
Leaf size : 29

```
DSolve[{(x+1)*D[y[x],{x,2}]-(x+2)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1 e^{x+1} - 2c_2(x+2)}{\sqrt{2e}}$$

1.396 problem 408

Internal problem ID [8534]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 408
Date solved : `Monday, October 21, 2024 at 05:09:19 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''+2xy'-2y=0$$

1.396.1 Solved as second order ode using Kovacic algorithm

Time used: 0.216 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''+2xy'-2y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= (x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 751: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 0\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x+1)} + \frac{3}{4(x+1)^2} + \frac{3}{4(x-1)^2} - \frac{3}{4(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{\left(x^2-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x-1\right)} + \frac{3}{2\left(x+1\right)} + (-)\left(0\right) \\ &= -\frac{1}{2\left(x-1\right)} + \frac{3}{2\left(x+1\right)} \\ &= \frac{x-2}{x^2-1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{3}{2(x+1)}\right)(0) + \left(\left(\frac{1}{2(x-1)^2} - \frac{3}{2(x+1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{3}{2(x+1)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{3}{2(x+1)}\right) dx} \\ &= \frac{(x+1)^{3/2}}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2x}{-x^2+1} \, dx} \\ &= z_1 e^{\frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2}} \\ &= z_1 \left(\sqrt{x-1}\,\sqrt{x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (x+1)^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x-1)+\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x\,\mathrm{e}^{\ln(x-1)+\ln(x+1)}}{(x+1)^3\,(x-1)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x+1)^2\right) + c_2\left((x+1)^2\left(-\frac{x\,\mathrm{e}^{\ln(x-1)+\ln(x+1)}}{(x+1)^3\,(x-1)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.396.2 Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d}{dx}y'\right) + 2xy' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2y}{x^2-1} + \frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2xy'}{x^2-1} + \frac{2y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x}{x^2-1}, P_3(x) = \frac{2}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.\left((x+1)\cdot P_2(x)\right)\right|_{x=-1} = -1$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- $x = -1$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0 = -1$

- Multiply by denominators
  $(x^2-1)\left(\frac{d}{dx}y'\right) - 2xy' + 2y = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$
  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
- Shift index using $k->k+1-m$
  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$
  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k->k+2-m$
  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions
  $-2a_0r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r-1) + a_k(k+r-1)(k+r-2)\right) u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $\left((-2k-2r-2)\, a_{k+1} + a_k(k+r-2)\right)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r-2)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{2(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{4}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - u + \frac{1}{4}u^2\right)$
- Revert the change of variables $u = x + 1$

  $\left[y = \frac{a_0(x-1)^2}{4}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k}{2(k+3)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$
- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$
- Combine solutions and rename parameters

  $\left[y = \frac{a_0(x-1)^2}{4} + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$

1.396.3 Maple trace

```
Methods for second order ODEs:
```

1.396.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)+2*x*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2 x^2 + c_1 x + c_2$$

1.396.5 Mathematica DSolve solution

Solving time : 0.1 (sec)
Leaf size : 39

```
DSolve[{(1-x^2)*D[y[x],{x,2}]+2*x*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x^2-1}(c_1(x-1)^2+c_2x)}{\sqrt{1-x^2}}$$

1.397 problem 409

Internal problem ID [8535]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 409
Date solved : `Monday, October 21, 2024 at 05:09:20 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-2xy'+2y=0$$

1.397.1 Solved as second order ode using Kovacic algorithm

Time used: 0.261 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 3}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 - 3$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 3}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 753: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{4(x-1)} - \frac{1}{4(x+1)^2} - \frac{5}{4(x+1)} - \frac{1}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^2-3}{\left(x^2-1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(x+1)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)^2 - \left(\frac{2x^2-3}{(x^2-1)^2}\right.\right.
$$

$$
-\frac{2a_0}{x^2-1} =
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, \mathrm{e}^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= (x) \sqrt{(x-1)(x+1)} \\
&= x\sqrt{x^2-1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{1}{x}+\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}\right) + c_2\left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}\left(\frac{1}{x}+\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.397.2 Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d}{dx}y'\right)-2xy'+2y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = \frac{2y}{x^2-1}-\frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{2xy'}{x^2-1}-\frac{2y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{2}{x^2-1}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=1$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$(x^2-1)\left(\frac{d}{dx}y'\right)+2xy'-2y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty}a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)$ to series expansion for $m =1..2$

$$u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- Shift index using $k - >k +2-m$

$$u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$-2 a_{0} r^{2} u^{-1+r}+\left(\sum_{k =0}^{\infty}\left(-2 a_{k +1} \left(k +1+r \right)^{2}+a_{k} \left(k +r +2\right) \left(k +r -1\right)\right) u^{k +r}\right)=0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation
  $-2 r^{2}=0$
- Values of r that satisfy the indicial equation
  $r =0$
- Each term in the series must be 0, giving the recursion relation
  $-2 a_{k +1} \left(k +1\right)^{2}+a_{k} \left(k +2\right) \left(k -1\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k +1}=\frac{a_{k} \left(k +2\right) \left(k -1\right)}{2 \left(k +1\right)^{2}}$
- Recursion relation for $r =0$; series terminates at $k =1$
  $a_{k +1}=\frac{a_{k} \left(k +2\right) \left(k -1\right)}{2 \left(k +1\right)^{2}}$
- Apply recursion relation for $k =0$
  $a_{1}=-a_{0}$
- Terminating series solution of the ODE for $r =0$. Use reduction of order to find the second li
  $y \left(u \right)=a_{0}\cdot \left(-u +1\right)$
- Revert the change of variables $u =x +1$
  $\left[y =-a_{0} x \right]$

1.397.3 Maple trace

```
Methods for second order ODEs:
```

1.397.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\ln(x-1) c_2 x}{2} - \frac{\ln(x+1) c_2 x}{2} + c_1 x + c_2$$

1.397.5 Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 33

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

1.398 problem 410

Internal problem ID [8536]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 410
Date solved : `Monday, October 21, 2024 at 05:09:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

1.398.1 Solved as second order ode using Kovacic algorithm

Time used: 0.161 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 755: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.398.2 Maple step by step solution

Let's solve
$x^2\left(\frac{d}{dx}y'\right) + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions
  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$
- ○ $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$
- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$
- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point
$x_0 = 0$

- Multiply by denominators
  $4x^2\left(\frac{d}{dx}y'\right) + 4xy' + (4x^2 - 1)y = 0$
- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0
  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$
  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

1.398.3 Maple trace

```
Methods for second order ODEs:
```

1.398.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1 \sin(x)+c_2 \cos(x)}{\sqrt{x}}$$

1.398.5 Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 33

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_1 x-\frac{1}{2}c_2(x\log(1-x)-x\log(x+1)+2)$$

1.399 problem 411

Internal problem ID [8537]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 411
Date solved : `Monday, October 21, 2024 at 05:09:22 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^2-1\right)y''-6xy'+12y=0$$

1.399.1 Solved as second order ode using Kovacic algorithm

Time used: 0.252 (sec)

Writing the ode as

$$\left(x^2-1\right)y''-6xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= -6x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{15}{(x^2-1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 15 \\
t &= (x^2-1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{15}{(x^2-1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 757: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 0\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4(x-1)^2} - \frac{15}{4(x-1)} + \frac{15}{4(x+1)} + \frac{15}{4(x+1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{\left(x^2 - 1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2\left(x-1\right)} + \frac{5}{2\left(x+1\right)} + (-)\left(0\right) \\ &= -\frac{3}{2\left(x-1\right)} + \frac{5}{2\left(x+1\right)} \\ &= \frac{x-4}{x^2-1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2\,(x-1)} + \frac{5}{2\,(x+1)}\right)(0) + \left(\left(\frac{3}{2\,(x-1)^2} - \frac{5}{2\,(x+1)^2}\right) + \left(-\frac{3}{2\,(x-1)} + \frac{5}{2\,(x+1)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{3}{2(x-1)} + \frac{5}{2(x+1)}\right) dx} \\
&= \frac{(x+1)^{5/2}}{(x-1)^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-6x}{x^2-1} \, dx} \\
&= z_1 e^{\frac{3 \ln(x-1)}{2} + \frac{3 \ln(x+1)}{2}} \\
&= z_1 \left((x-1)^{3/2} (x+1)^{3/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (x+1)^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-6x}{x^2-1}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{3\ln(x-1)+3\ln(x+1)}}{(y_1)^2}\,dx$$
$$= y_1 \left(-\frac{x(x^2+1)\,\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7\,(x-1)^3}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left((x+1)^4\right) + c_2\left((x+1)^4\left(-\frac{x(x^2+1)\,\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7\,(x-1)^3}\right)\right)$$

Will add steps showing solving for IC soon.

1.399.2 Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d}{dx}y'\right) - 6xy' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{12y}{x^2-1} + \frac{6xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{6xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{6x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = -3$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- $x=-1$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0 = -1$

- Multiply by denominators
  $(x^2-1)\left(\frac{d}{dx}y'\right) - 6xy' + 12y = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$
  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-6u+6)\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
  - Shift index using $k- >k+1-m$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$
    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$
  - Shift index using $k- >k+2-m$
    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions
  $-2a_0 r(-4+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r-3) + a_k(k+r-3)(k+r-4)\right) u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r(-4+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 4\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-3)\left((-2k-2r-2)\, a_{k+1} + a_k(k+r-4)\right) = 0$
- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r-4)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 4$

  $a_{k+1} = \frac{a_k(k-4)}{2(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -2a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{3a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{3a_0}{2}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2}{3}$
- Express in terms of a_0

  $a_3 = -\frac{a_0}{2}$
- Apply recursion relation for $k = 3$

  $a_4 = -\frac{a_3}{8}$
- Express in terms of a_0

  $a_4 = \frac{a_0}{16}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 2u + \frac{3}{2}u^2 - \frac{1}{2}u^3 + \frac{1}{16}u^4\right)$
- Revert the change of variables $u = x + 1$

  $\left[y = \frac{a_0(x-1)^4}{16}\right]$
- Recursion relation for $r = 4$

  $a_{k+1} = \frac{a_k k}{2(k+5)}$
- Solution for $r = 4$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+4}, a_{k+1} = \frac{a_k k}{2(k+5)}\right]$
- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+4}, a_{k+1} = \frac{a_k k}{2(k+5)}\right]$
- Combine solutions and rename parameters

  $\left[y = \frac{a_0(x-1)^4}{16} + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+4}\right), b_{k+1} = \frac{b_k k}{2(k+5)}\right]$

1.399.3 Maple trace

```
Methods for second order ODEs:
```

1.399.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 25

```
dsolve((x^2-1)*diff(diff(y(x),x),x)-6*x*diff(y(x),x)+12*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2 x^4 + c_1 x^3 + 6c_2 x^2 + c_1 x + c_2$$

1.399.5 Mathematica DSolve solution

Solving time : `0.18` (sec)
Leaf size : 45

```
DSolve[{(x^2-1)*D[y[x],{x,2}]-6*x*D[y[x],x]+12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{x^2-1}(c_2 x(x^2+1) + c_1(x-1)^4)}{\sqrt{1-x^2}}$$

1.400 problem 412

Internal problem ID [8538]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 412
Date solved : `Monday, October 21, 2024 at 05:09:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+3\right) y''-7xy'+16y=0$$

1.400.1 Solved as second order ode using Kovacic algorithm

Time used: 0.384 (sec)

Writing the ode as

$$\left(x^2+3\right) y''-7xy'+16y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+3 \\ B &= -7x \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 234}{4\left(x^2+3\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 234$$
$$t = 4\left(x^2+3\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 234}{4\left(x^2+3\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 759: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+3)^2$. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{77}{16\left(x - i\sqrt{3}\right)^2} + \frac{77}{16\left(x + i\sqrt{3}\right)^2} + \frac{79i\sqrt{3}}{48\left(x - i\sqrt{3}\right)} - \frac{79i\sqrt{3}}{48\left(x + i\sqrt{3}\right)}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4} \end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 234}{4\left(x^2+3\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 234}{4\left(x^2+3\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$i\sqrt{3}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$
$-i\sqrt{3}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{7}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{7}{4\left(x-i\sqrt{3}\right)} - \frac{7}{4\left(x+i\sqrt{3}\right)} + (-)\,(0) \\
&= -\frac{7}{4\left(x-i\sqrt{3}\right)} - \frac{7}{4\left(x+i\sqrt{3}\right)} \\
&= -\frac{7x}{2x^2+6}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(-\frac{7}{4\left(x-i\sqrt{3}\right)} - \frac{7}{4\left(x+i\sqrt{3}\right)} \right) \left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{7}{4\left(x-i\sqrt{3}\right)^2} \right. \right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{27}{8}, a_1 = 0, a_2 = -9, a_3 = 0 \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 9x^2 + \frac{27}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 - 9x^2 + \frac{27}{8}\right) e^{\int \left(-\frac{7}{4\left(x - i\sqrt{3}\right)} - \frac{7}{4\left(x + i\sqrt{3}\right)}\right) dx} \\
&= \left(x^4 - 9x^2 + \frac{27}{8}\right) \frac{1}{\left(\left(i\sqrt{3} - x\right)\left(x + i\sqrt{3}\right)\right)^{7/4}} \\
&= \frac{8x^4 - 72x^2 + 27}{8\left(-x^2 - 3\right)^{7/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-7x}{x^2+3} \, dx} \\
&= z_1 e^{\frac{7 \ln\left(x^2+3\right)}{4}} \\
&= z_1 \left(\left(x^2 + 3\right)^{7/4}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(\frac{1}{2} + \frac{i}{2}\right) \left(x^4 - 9x^2 + \frac{27}{8}\right) \sqrt{2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-7x}{x^2+3} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{7 \ln\left(x^2+3\right)}{2}}}{\left(y_1\right)^2} \, dx \\
&= y_1 (\text{Expression too large to display})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(\frac{1}{2}+\frac{i}{2}\right)\left(x^4-9x^2+\frac{27}{8}\right)\sqrt{2}\right) \\
&\quad + c_2\left(\left(\frac{1}{2}+\frac{i}{2}\right)\left(x^4-9x^2+\frac{27}{8}\right)\sqrt{2}(\text{Expression too large to display})\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.400.2 Maple step by step solution

1.400.3 Maple trace

```
Methods for second order ODEs:
```

1.400.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 65

```
dsolve((x^2+3)*diff(diff(y(x),x),x)-7*x*diff(y(x),x)+16*y(x) = 0,
       y(x),singsol=all)
```

$$
\begin{aligned}
y &= 4\left(x^4-9x^2+\frac{27}{8}\right)c_2\ln\left(\sqrt{x^2+3}-x\right) \\
&\quad + \frac{5(10x^3-33x)\,c_2\sqrt{x^2+3}}{6} + \left(x^4-9x^2+\frac{27}{8}\right)\left(c_1+\frac{25c_2}{3}\right)
\end{aligned}
$$

1.400.5 Mathematica DSolve solution

Solving time : `0.341` (sec)
Leaf size : 492

```
DSolve[{(x^2+3)*D[y[x],{x,2}]-7*x*D[y[x],x]+16*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{24}c_2\Big(12960x^2\text{RootSum}\Big[7838208000\#1^4 - 188281584000\#1^2 - 241544908800\#1$$
$$+18453344881\&, \#1\log\Big(-411757211968704000\#1^3 - 1660632746069808000\#1^2 + 10138703825167113 \ldots$$
$$+ 5248800x^2\text{RootSum}\Big[21088072057248000000\#1^4 - 30882886815600000\#1^2$$
$$+ 97825688064000\#1$$
$$+18453344881\&, \#1\log\Big(27353083060732502808000000\#1^3 - 27238528617410025720000\#1^2 - 410617 \ldots$$
$$- 4860\text{RootSum}\Big[7838208000\#1^4 - 188281584000\#1^2 - 241544908800\#1$$
$$+18453344881\&, \#1\log\Big(-411757211968704000\#1^3 - 1660632746069808000\#1^2 + 10138703825167113 \ldots$$
$$- 1968300\text{RootSum}\Big[21088072057248000000\#1^4 - 30882886815600000\#1^2$$
$$+ 97825688064000\#1$$
$$+18453344881\&, \#1\log\Big(27353083060732502808000000\#1^3 - 27238528617410025720000\#1^2 - 410617 \ldots$$
$$- 1440x^4\text{RootSum}\Big[7838208000\#1^4 - 188281584000\#1^2 - 241544908800\#1$$
$$+18453344881\&, \#1\log\Big(-411757211968704000\#1^3 - 1660632746069808000\#1^2 + 10138703825167113 \ldots$$
$$- 583200x^4\text{RootSum}\Big[21088072057248000000\#1^4 - 30882886815600000\#1^2$$
$$+ 97825688064000\#1$$
$$+18453344881\&, \#1\log\Big(27353083060732502808000000\#1^3 - 27238528617410025720000\#1^2 - 410617 \ldots$$
$$+ 165\sqrt{x^2+3}x + 216x^2\log\left(\sqrt{x^2+3}-x\right) - 81\log\left(\sqrt{x^2+3}-x\right)$$
$$- 24x^4\log\left(\sqrt{x^2+3}-x\right) - 50\sqrt{x^2+3}x^3\Big) + c_1\left(x^4 - 9x^2 + \frac{27}{8}\right)$$

1.401 problem 413

Internal problem ID [8539]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 413
Date solved : `Monday, October 21, 2024 at 05:09:23 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^2-1\right) y''+8xy'+12y=0$$

1.401.1 Solved as second order ode using Kovacic algorithm

Time used: 0.207 (sec)

Writing the ode as

$$\left(x^2-1\right) y''+8xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= 8x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 8$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 760: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x+1} + \frac{2}{(x+1)^2} + \frac{2}{(x-1)^2} - \frac{2}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8}{\left(x^2-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	2	-1
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x-1} + \frac{2}{x+1} + (-)\,(0) \\ &= -\frac{1}{x-1} + \frac{2}{x+1} \\ &= \frac{x-3}{x^2-1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-1} + \frac{2}{x+1}\right)(0) + \left(\left(\frac{1}{(x-1)^2} - \frac{2}{(x+1)^2}\right) + \left(-\frac{1}{x-1} + \frac{2}{x+1}\right)^2 - \left(\frac{8}{(x^2-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{x-1} + \frac{2}{x+1}\right) dx} \\ &= \frac{(x+1)^2}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x}{x^2-1}\, dx} \\ &= z_1 e^{-2\ln(x-1) - 2\ln(x+1)} \\ &= z_1 \left(\frac{1}{(x-1)^2 (x+1)^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(x-1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x}{x^2-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4\ln(x-1)-4\ln(x+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{(x+1)(3x^2+1)(x-1)^4 \,\mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{(x-1)^3}\right) + c_2 \left(\frac{1}{(x-1)^3}\left(-\frac{(x+1)(3x^2+1)(x-1)^4 \,\mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.401.2 Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d}{dx}y'\right) + 8xy' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{12y}{x^2-1} - \frac{8xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{8xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{8x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left.((x+1)\cdot P_2(x))\right|_{x=-1} = 4$

- ◦ $(x+1)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$
- ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$
- Multiply by denominators

  $(x^2-1)\left(\frac{d}{dx}y'\right) + 8xy' + 12y = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (8u-8)\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
- ◦ Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- ◦ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$
- ◦ Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r+4) + a_k(k+r+4)(k+r+3)\right) u^{k+r}\right) =$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$
- Each term in the series must be 0, giving the recursion relation

  $\left((-2k-2r-2)\, a_{k+1} + a_k(k+r+3)\right)(k+r+4) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+3)}{2(k+1+r)}$
- Recursion relation for $r = -3$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$
- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+3)}{2(k+1)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$
- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

1.401.3 Maple trace

```
Methods for second order ODEs:
```

1.401.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 29

```
dsolve((x^2-1)*diff(diff(y(x),x),x)+8*x*diff(y(x),x)+12*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 x^3 + 3c_1 x^2 + 3c_2 x + c_1}{(x^2-1)^3}$$

1.401.5 Mathematica DSolve solution

Solving time : 0.068 (sec)
Leaf size : 37

```
DSolve[{(x^2-1)*D[y[x],{x,2}]+8*x*D[y[x],x]+12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3c_1(x-1)^3 - c_2(3x^2+1)}{3\left(x^2-1\right)^3}$$

1.402 problem 414

Internal problem ID [8540]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 414
Date solved : `Monday, October 21, 2024 at 05:09:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3y'' + xy' - 4y = 0$$

1.402.1 Solved as second order ode using Kovacic algorithm

Time used: 0.244 (sec)

Writing the ode as

$$3y'' + xy' - 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 54}{36} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 54 \\ t &= 36 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{36} + \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 762: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} + \frac{9}{2x} - \frac{243}{4x^3} + \frac{6561}{4x^5} - \frac{885735}{16x^7} + \frac{33480783}{16x^9} - \frac{2711943423}{32x^{11}} + \frac{115063885233}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+54}{36} \\
&= Q + \frac{R}{36} \\
&= \left(\frac{x^2}{36} + \frac{3}{2}\right) + (0) \\
&= \frac{x^2}{36} + \frac{3}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{3}{2}\right) - (0) \\
&= \frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{6} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{6}} - 1\right) &= 4 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{6}} - 1\right) &= -5
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{36} + \frac{3}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{6}$	4	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{6}\right) \\ &= \frac{x}{6} \\ &= \frac{x}{6} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{x}{6}\right)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{1}{6}\right) + \left(\frac{x}{6}\right)^2 - \left(\frac{x^2}{36} + \frac{3}{2}\right)\right) = 0$$

$$-\frac{a_3x^3}{3} + \frac{2(18 - a_2)\,x^2}{3} + (-a_1 + 6a_3)\,x - \frac{4a_0}{3} + 2a_2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 27, a_1 = 0, a_2 = 18, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 + 18x^2 + 27$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x^4 + 18x^2 + 27\right) \mathrm{e}^{\int \frac{x}{6} dx} \\
&= \left(x^4 + 18x^2 + 27\right) \mathrm{e}^{\frac{x^2}{12}} \\
&= \left(x^4 + 18x^2 + 27\right) \mathrm{e}^{\frac{x^2}{12}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{3}\, dx} \\
&= z_1 e^{-\frac{x^2}{12}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{12}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 + 18x^2 + 27
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{3}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{6}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{6}}}{\left(x^4 + 18x^2 + 27\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 + 18x^2 + 27\right) + c_2 \left(x^4 + 18x^2 + 27 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{6}}}{\left(x^4 + 18x^2 + 27\right)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.402.2 Maple trace

```
Methods for second order ODEs:
```

1.402.3 Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 47

```
dsolve(3*diff(diff(y(x),x),x)+x*diff(y(x),x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y = xc_1(x^2 + 15)\sqrt{6}\,e^{-\frac{x^2}{6}} + (x^4 + 18x^2 + 27)\left(\operatorname{erf}\left(\frac{\sqrt{6}\,x}{6}\right)\sqrt{\pi}\,c_1 + c_2\right)$$

1.402.4 Mathematica DSolve solution

Solving time : 0.033 (sec)
Leaf size : 43

```
DSolve[{3*D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{6}}\,\text{HermiteH}\left(-5, \frac{x}{\sqrt{6}}\right) + \frac{1}{27}c_2(x^4 + 18x^2 + 27)$$

1.403 problem 415

Internal problem ID [8541]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 415
Date solved : `Monday, October 21, 2024 at 05:09:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$5y'' - 2xy' + 10y = 0$$

1.403.1 Solved as second order ode using Kovacic algorithm

Time used: 0.275 (sec)

Writing the ode as

$$5y'' - 2xy' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 5 \\ B &= -2x \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 55}{25} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 55$$
$$t = 25$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{25} - \frac{11}{5}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 763: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{5} - \frac{11}{2x} - \frac{605}{8x^3} - \frac{33275}{16x^5} - \frac{9150625}{128x^7} - \frac{704598125}{256x^9} - \frac{116258690625}{1024x^{11}} - \frac{10048072546875}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{5}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{5} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{25}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 55}{25} \\
&= Q + \frac{R}{25} \\
&= \left(\frac{x^2}{25} - \frac{11}{5}\right) + (0) \\
&= \frac{x^2}{25} - \frac{11}{5}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{11}{5}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{11}{5}\right) - (0) \\
&= -\frac{11}{5}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{5} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{11}{5}}{\frac{1}{5}} - 1\right) &&= -6 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{11}{5}}{\frac{1}{5}} - 1\right) &&= 5
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{25} - \frac{11}{5}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{5}$	-6	5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 5$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 5 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{5}\right) \\ &= -\frac{x}{5} \\ &= -\frac{x}{5} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(20x^3 + 12x^2a_4 + 6xa_3 + 2a_2\right) + 2\left(-\frac{x}{5}\right)\left(5x^4 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(-\frac{1}{5}\right) + \left(-\frac{x}{5}\right)^2 - \left(\frac{x^2}{25}\right.\right.$$

$$\frac{2a_4x^4}{5} + \frac{4(25+a_3)\,x^3}{5} + \frac{6(a_2+10a_4)\,x^2}{5} + \frac{2(4a_1+15a_3)\,x}{5} + 2a_0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = \frac{375}{4}, a_2 = 0, a_3 = -25, a_4 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^5 - 25x^3 + \frac{375}{4}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^5 - 25x^3 + \frac{375}{4}x\right) \mathrm{e}^{\int -\frac{x}{5} dx} \\
&= \left(x^5 - 25x^3 + \frac{375}{4}x\right) \mathrm{e}^{-\frac{x^2}{10}} \\
&= \frac{(4x^5 - 100x^3 + 375x)\, \mathrm{e}^{-\frac{x^2}{10}}}{4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{5} \, dx} \\
&= z_1 e^{\frac{x^2}{10}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{10}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^5 - 25x^3 + \frac{375}{4}x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2x}{5} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{\frac{x^2}{5}}}{(y_1)^2} dx$$
$$= y_1 \left(\int \frac{e^{\frac{x^2}{5}}}{\left(x^5 - 25x^3 + \frac{375}{4}x\right)^2} dx \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(x^5 - 25x^3 + \frac{375}{4}x \right) + c_2 \left(x^5 - 25x^3 + \frac{375}{4}x \left(\int \frac{e^{\frac{x^2}{5}}}{\left(x^5 - 25x^3 + \frac{375}{4}x\right)^2} dx \right) \right)$$

Will add steps showing solving for IC soon.

1.403.2 Maple step by step solution

Let's solve

$5\frac{d}{dx}y' - 2xy' + 10y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{2xy'}{5} - 2y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2xy'}{5} + 2y = 0$

- Multiply by denominators

  $5\frac{d}{dx}y' - 2xy' + 10y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ○ Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- ◦ Convert $\frac{d}{dx}y'$ to series expansion

$$\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- ◦ Shift index using $k- >k+2$

$$\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(5a_{k+2}(k+2)\,(k+1) - 2a_k(k-5)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$5(k^2+3k+2)\, a_{k+2} - 2a_k(k-5) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2a_k(k-5)}{5(k^2+3k+2)}\right]$$

1.403.3 Maple trace

```
Methods for second order ODEs:
```

1.403.4 Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 31

```
dsolve(5*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+10*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2 \,\text{hypergeom}\left(\left[-\frac{5}{2}\right], \left[\frac{1}{2}\right], \frac{x^2}{5}\right) + \frac{4xc_1\left(x^4 - 25x^2 + \frac{375}{4}\right)}{375}$$

1.403.5 Mathematica DSolve solution

Solving time : 0.192 (sec)
Leaf size : 138

```
DSolve[{5*D[y[x],{x,2}]-2*x*D[y[x],x]+10*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{200}\sqrt{\frac{\pi}{5}}c_2\sqrt{x^2}\left(4x^4-100x^2+375\right)\text{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{5}}\right)+\frac{32c_1x^5}{25\sqrt{5}} - \frac{32c_1x^3}{\sqrt{5}}-\frac{9}{20}c_2e^{\frac{x^2}{5}}x^2+c_2e^{\frac{x^2}{5}}+\frac{1}{50}c_2e^{\frac{x^2}{5}}x^4+24\sqrt{5}c_1x$$

1.404 problem 416

Internal problem ID [8542]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 416
Date solved : `Monday, October 21, 2024 at 05:09:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2y' - 3xy = 0$$

1.404.1 Solved as second order ode using Kovacic algorithm

Time used: 0.271 (sec)

Writing the ode as

$$y'' - x^2y' - 3xy = 0 \quad (1)$$
$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= -3x \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3 + 8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x(x^3 + 8) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3 + 8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 765: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} + \frac{2}{x} - \frac{4}{x^4} + \frac{16}{x^7} - \frac{80}{x^{10}} + \frac{448}{x^{13}} - \frac{2688}{x^{16}} + \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x(x^3+8)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^4 + 2x\right) + (0) \\
&= \frac{1}{4}x^4 + 2x
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 2. Now b can be found.

$$\begin{aligned}
b &= (2) - (0) \\
&= 2
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^2}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 2\right) &= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 2\right) &= -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3+8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-4	$\frac{x^2}{2}$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^{+} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x^2}{2}\right) \\ &= \frac{x^2}{2} \\ &= \frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{x^2}{2}\right)(1) + \left((x) + \left(\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3+8)}{4}\right)\right) &= 0 \\ -x a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, \mathrm{e}^{\int \frac{x^2}{2} dx} \\
&= (x) \, \mathrm{e}^{\frac{x^3}{6}} \\
&= x \, \mathrm{e}^{\frac{x^3}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2}{1} \, dx} \\
&= z_1 e^{\frac{x^3}{6}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^3}{6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x^3}{3}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x^3}{3}} x\right) + c_2 \left(\mathrm{e}^{\frac{x^3}{3}} x \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.404.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - x^2 y' - 3xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  - Shift index using $k- >k-1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

$\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_{k-1}(k+2))\, x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)(ka_{k+2} - a_{k-1} + a_{k+2}) = 0$

- Shift index using $k- >k+1$

  $(k+3)((k+1)a_{k+3} - a_k + a_{k+3}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k+2}, 2a_2 = 0\right]$

1.404.3 Maple trace

```
Methods for second order ODEs:
```

1.404.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 58

```
dsolve(diff(diff(y(x),x),x)-x^2*diff(y(x),x)-3*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{9\,\mathrm{WhittakerM}\left(\frac{1}{3}, \frac{5}{6}, \frac{x^3}{3}\right) \mathrm{e}^{\frac{x^3}{6}} c_2\, x^3 + 9c_1\, x^2 \mathrm{e}^{\frac{x^3}{3}} + 5\, 3^{2/3} c_2 (x^3)^{1/3} (x^3+2)}{9x}$$

1.404.5 Mathematica DSolve solution

Solving time : 0.094 (sec)
Leaf size : 51

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]-3*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{9} e^{\frac{x^3}{3}} \left(9 c_1 x - 3^{2/3} c_2 \sqrt[3]{x^3} \Gamma\left(-\frac{1}{3}, \frac{x^3}{3}\right)\right)$$

1.405 problem 417

Internal problem ID [8543]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 417
Date solved : `Monday, October 21, 2024 at 05:09:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''+2xy'-2y=0$$

1.405.1 Solved as second order ode using Kovacic algorithm

Time used: 0.294 (sec)

Writing the ode as

$$\left(x^2+1\right) y''+2xy'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{2x^2 + 3}{(x^2 + 1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 2x^2 + 3 \\
t &= (x^2 + 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{2x^2 + 3}{(x^2 + 1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 767: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{5i}{4(x-i)} + \frac{5i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+3}{(x^2+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 + 3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} + (0) \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} \\
&= \frac{x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)(1) + \left(\left(-\frac{1}{2(x-i)^2} - \frac{1}{2(x+i)^2}\right) + \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)^2 - \left(\frac{2x^2+3}{(x^2+1)^2}\right.\right.
$$

$$
-\frac{2(x^2+1)\,a_0}{(-x+i)^2\,(x+i)^2}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right) dx} \\
&= (x)\sqrt{(-x+i)(x+i)} \\
&= x\sqrt{-x^2-1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x^2+1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = ix$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\arctan\left(x\right) + \frac{1}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(ix) + c_2\left(ix\left(\arctan\left(x\right) + \frac{1}{x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.405.2 Maple step by step solution

1.405.3 Maple trace

```
Methods for second order ODEs:
```

1.405.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+2*x*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1x + \arctan(x)\, xc_2 + c_2$$

1.405.5 Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 48

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+2*x*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}i(2c_1x - c_2x\log(1-ix) + c_2x\log(1+ix) + 2ic_2)$$

1.406 problem 418

Internal problem ID [8544]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 418
Date solved : `Monday, October 21, 2024 at 05:09:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' - 2y = 0$$

1.406.1 Solved as second order ode using Kovacic algorithm

Time used: 0.230 (sec)

Writing the ode as

$$y'' + xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 768: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{5}{2x} - \frac{25}{4x^3} + \frac{125}{4x^5} - \frac{3125}{16x^7} + \frac{21875}{16x^9} - \frac{328125}{32x^{11}} + \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} + \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} + \frac{5}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{5}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{5}{2}\right) - (0) \\
&= \frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	2	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(\frac{x}{2}\right)(2x + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{5}{2}\right)\right) &= 0 \\ -a_1 x - 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^2 + 1) \, \mathrm{e}^{\int \frac{x}{2} dx} \\
&= (x^2 + 1) \, \mathrm{e}^{\frac{x^2}{4}} \\
&= (x^2 + 1) \, \mathrm{e}^{\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2 + 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{(x^2 + 1)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^2+1) + c_2 \left(x^2+1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{(x^2+1)^2} dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.406.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_k(k-2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} + a_k(k-2) = 0$

- Recursion relation; series terminates at $k = 2$

  $a_{k+2} = -\frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

$a_2 = a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind
  $y = A_2 x^2 + A_1 x + a_0$

1.406.3 Maple trace

```
Methods for second order ODEs:
```

1.406.4 Maple dsolve solution

Solving time : `0.023` (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-\frac{x^2}{2}} \sqrt{2}\, c_1 x + (x^2 + 1) \left(\sqrt{\pi}\ \mathrm{erf}\left(\frac{\sqrt{2}\, x}{2}\right) c_1 + c_2\right)$$

1.406.5 Mathematica DSolve solution

Solving time : `0.029` (sec)
Leaf size : 35

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{2}} \operatorname{HermiteH}\left(-3, \frac{x}{\sqrt{2}}\right) + c_2(x^2 + 1)$$

1.407 problem 419

Internal problem ID [8545]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 419
Date solved : `Monday, October 21, 2024 at 05:09:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^{2}-6x+10\right)y''-4(x-3)\,y'+6y=0$$

1.407.1 Solved as second order ode using Kovacic algorithm

Time used: 0.300 (sec)

Writing the ode as

$$\left(x^{2}-6x+10\right)y''+(-4x+12)\,y'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}-6x+10\\ B &= -4x+12\\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-8}{(x^2 - 6x + 10)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -8$$
$$t = (x^2 - 6x + 10)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{8}{(x^2 - 6x + 10)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 770: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 0\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 6x + 10)^2$. There is a pole at $x = 3 + i$ of order 2. There is a pole at $x = 3 - i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x - 3 - i)^2} + \frac{2}{(x - 3 + i)^2} + \frac{2i}{x - 3 - i} - \frac{2i}{x - 3 + i}$$

For the pole at $x = 3 + i$ let b be the coefficient of $\frac{1}{(x-3-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1\end{aligned}$$

For the pole at $x = 3 - i$ let b be the coefficient of $\frac{1}{(x-3+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{8}{(x^2 - 6x + 10)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$3+i$	2	0	2	-1
$3-i$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x-3-i} + \frac{2}{x-3+i} + (-)(0) \\
&= -\frac{1}{x-3-i} + \frac{2}{x-3+i} \\
&= \frac{x-3-3i}{x^2-6x+10}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right)(0) + \left(\left(\frac{1}{(x-3-i)^2} - \frac{2}{(x-3+i)^2}\right) + \left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right)^2\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right) dx} \\
&= \frac{\left(x^2 - 6x + 10\right)^2}{(ix - 3i + 1)^3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x+12}{x^2-6x+10} \, dx} \\
&= z_1 e^{\ln\left(x^2-6x+10\right)} \\
&= z_1 \left(x^2 - 6x + 10\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2 - 6x + 10\right)^3}{(ix - 3i + 1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x+12}{x^2-6x+10} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{2\ln(x^2-6x+10)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^2-6x+\frac{26}{3}}{(x-3+i)^3} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2-6x+10)^3}{(ix-3i+1)^3} \right) + c_2 \left(\frac{(x^2-6x+10)^3}{(ix-3i+1)^3} \left(\frac{x^2-6x+\frac{26}{3}}{(x-3+i)^3} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.407.2 Maple step by step solution

1.407.3 Maple trace

```
Methods for second order ODEs:
```

1.407.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 31

```
dsolve((x^2-6*x+10)*diff(diff(y(x),x),x)-4*(x-3)*diff(y(x),x)+6*y(x) = 0,
       y(x),singsol=all)
```

$$
y = c_1\, x^3 + c_2\, x^2 + 6(-5c_1 - c_2)\, x + 60c_1 + \frac{26c_2}{3}
$$

1.407.5 Mathematica DSolve solution

Solving time : `0.121` (sec)
Leaf size : 36

```
DSolve[{(x^2-6*x+10)*D[y[x],{x,2}]-4*(x-3)*D[y[x],x]+6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{3}i\left(c_2\left(3x^2 - 18x + 26\right) + 3c_1(x - (3 + i))^3\right)$$

1.408 problem 420

Internal problem ID [8546]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 420
Date solved : `Monday, October 21, 2024 at 05:09:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+6x\right)y''+(3x+9)\,y'-3y=0$$

1.408.1 Solved as second order ode using Kovacic algorithm

Time used: 0.228 (sec)

Writing the ode as

$$\left(x^2+6x\right)y''+(3x+9)\,y'-3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+6x\\B&=3x+9\\C&=-3\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 + 90x - 27}{4\left(x^2 + 6x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15x^2 + 90x - 27$$
$$t = 4\left(x^2 + 6x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 + 90x - 27}{4\left(x^2 + 6x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 771: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 6x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -6$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{11}{16(x+6)} - \frac{3}{16(x+6)^2} - \frac{3}{16x^2} + \frac{11}{16x}$$

For the pole at $x = -6$ let b be the coefficient of $\frac{1}{(x+6)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 + 90x - 27}{4(x^2 + 6x)^2}$$

Since the gcd$(s, t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{15x^2 + 90x - 27}{4\left(x^2 + 6x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-6	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4\,(x+6)} + \frac{3}{4x} + (0) \\
&= \frac{3}{4\,(x+6)} + \frac{3}{4x} \\
&= \frac{\frac{3x}{2} + \frac{9}{2}}{x\,(x+6)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4\,(x+6)} + \frac{3}{4x}\right)(1) + \left(\left(-\frac{3}{4\,(x+6)^2} - \frac{3}{4x^2}\right) + \left(\frac{3}{4\,(x+6)} + \frac{3}{4x}\right)^2 - \left(\frac{15x^2 + 90x - 27}{4\,(x^2+6x)^2}\right)\right) = \frac{9 - 3a_0}{x\,(x+6)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+3)\, e^{\int \left(\frac{3}{4(x+6)} + \frac{3}{4x}\right) dx} \\
&= (x+3)\,(x(x+6))^{3/4} \\
&= (x+3)\,(x(x+6))^{3/4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x+9}{x^2+6x}\,dx} \\
&= z_1 e^{-\frac{3\ln(x(x+6))}{4}} \\
&= z_1\left(\frac{1}{\left(x\left(x+6\right)\right)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x+3
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x+9}{x^2+6x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x(x+6))}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{\left(x+6\right)x(2x^2+12x+9)}{81\left(x+3\right)\left(x\left(x+6\right)\right)^{3/2}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x+3) + c_2\left(x+3\left(-\frac{\left(x+6\right)x(2x^2+12x+9)}{81\left(x+3\right)\left(x\left(x+6\right)\right)^{3/2}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.408.2 Maple step by step solution

Let's solve

$(x^2+6x)\left(\frac{d}{dx}y'\right)+(3x+9)\,y'-3y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{3y}{x(x+6)}-\frac{3(x+3)y'}{x(x+6)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{3(x+3)y'}{x(x+6)}-\frac{3y}{x(x+6)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3(x+3)}{x(x+6)},P_3(x)=-\frac{3}{x(x+6)}\right]$

  ◦ $(x+6)\cdot P_2(x)$ is analytic at $x=-6$

  $\left((x+6)\cdot P_2(x)\right)\bigg|_{x=-6}=\frac{3}{2}$

  ◦ $(x+6)^2\cdot P_3(x)$ is analytic at $x=-6$

  $\left((x+6)^2\cdot P_3(x)\right)\bigg|_{x=-6}=0$

  ◦ $x=-6$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-6$

- Multiply by denominators

  $x(x+6)\left(\frac{d}{dx}y'\right)+(3x+9)\,y'-3y=0$

- Change variables using $x=u-6$ so that the regular singular point is at $u=0$

  $(u^2-6u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(3u-9)\left(\frac{d}{du}y(u)\right)-3y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-3a_0r(1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-3a_{k+1}(k+1+r)\,(2k+3+2r) + a_k(k+r+3)\,(k+r-1)\right) u^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-3r(1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-6(k+1+r)\left(k+\frac{3}{2}+r\right) a_{k+1} + a_k(k+r+3)\,(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r+3)(k+r-1)}{3(k+1+r)(2k+3+2r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$$a_{k+1} = \frac{a_k(k+3)(k-1)}{3(k+1)(2k+3)}$$

- Apply recursion relation for $k = 0$

$$a_1 = -\frac{a_0}{3}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot \left(1 - \frac{u}{3}\right)$$

- Revert the change of variables $u = x + 6$

$$\left[y = a_0\left(-1 - \frac{x}{3}\right)\right]$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

- Revert the change of variables $u = x + 6$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+6)^{k-\frac{1}{2}}, a_{k+1}=\frac{a_k \left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y=a_0\left(-1-\frac{x}{3}\right)+\left(\sum_{k=0}^{\infty} b_k (x+6)^{k-\frac{1}{2}}\right), b_{k+1}=\frac{b_k \left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

1.408.3 Maple trace

```
Methods for second order ODEs:
```

1.408.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 30

```
dsolve((x^2+6*x)*diff(diff(y(x),x),x)+(3*x+9)*diff(y(x),x)-3*y(x) = 0,
       y(x),singsol=all)
```

$$y=c_1(x+3)+\frac{c_2(2x^2+12x+9)}{\sqrt{x}\,\sqrt{x+6}}$$

1.408.5 Mathematica DSolve solution

Solving time : 0.119 (sec)
Leaf size : 82

```
DSolve[{(x^2+6*x)*D[y[x],{x,2}]+(3*x+9)*D[y[x],x]-3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{9\sqrt{\pi}c_2\sqrt[4]{-x(x+6)}Q_{\frac{3}{2}}^{\frac{1}{2}}\left(\frac{x}{3}+1\right)+\sqrt{6}c_1(2x^2+12x+9)}{9\sqrt{\pi}\sqrt[4]{-x^2}\sqrt{x+6}}$$

1.409 problem 421

Internal problem ID [8547]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 421
Date solved : `Monday, October 21, 2024 at 05:09:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t y'' + \left(t^{2}-1\right) y' + t^{3} y = 0$$

1.409.1 Solved as second order ode using Kovacic algorithm

Time used: 0.364 (sec)

Writing the ode as

$$t y'' + \left(t^{2}-1\right) y' + t^{3} y = 0 \tag{1}$$

$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= t^{2}-1 \\ C &= t^{3} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3t^4 + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3t^4 + 3 \\ t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{-3t^4 + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 773: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3t^2}{4} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{1} a_i t^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{i\sqrt{3}\,t}{2} - \frac{i\sqrt{3}}{4t^3} - \frac{i\sqrt{3}}{16t^7} - \frac{i\sqrt{3}}{32t^{11}} - \frac{5i\sqrt{3}}{256t^{15}} - \frac{7i\sqrt{3}}{512t^{19}} - \frac{21i\sqrt{3}}{2048t^{23}} - \frac{33i\sqrt{3}}{4096t^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i\sqrt{3}}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= \frac{i\sqrt{3}\,t}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{3t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-3t^4 + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(-\frac{3t^2}{4}\right) + \left(\frac{3}{4t^2}\right) \\ &= -\frac{3t^2}{4} + \frac{3}{4t^2} \end{aligned}$$

We see that the coefficient of the term t in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{i\sqrt{3}\,t}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{i\sqrt{3}}{2}} - 1\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{i\sqrt{3}}{2}} - 1\right) &= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-3t^4 + 3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{i\sqrt{3}\,t}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2t} + (-)\left(\frac{i\sqrt{3}\,t}{2}\right) \\
&= -\frac{1}{2t} - \frac{i\sqrt{3}\,t}{2} \\
&= \frac{-i\sqrt{3}\,t^2 - 1}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{2t} - \frac{i\sqrt{3}\,t}{2}\right)(0) + \left(\left(\frac{1}{2t^2} - \frac{i\sqrt{3}}{2}\right) + \left(-\frac{1}{2t} - \frac{i\sqrt{3}\,t}{2}\right)^2 - \left(\frac{-3t^4+3}{4t^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2t} - \frac{i\sqrt{3}\,t}{2}\right) dt} \\
&= \frac{\mathrm{e}^{-\frac{i\sqrt{3}\,t^2}{4}}}{\sqrt{t}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2-1}{t} \, dt} \\
&= z_1 e^{-\frac{t^2}{4} + \frac{\ln(t)}{2}} \\
&= z_1 \left(\sqrt{t}\,\mathrm{e}^{-\frac{t^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{t^2\left(1+i\sqrt{3}\right)}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-1}{t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-\frac{t^2}{2}+\ln(t)}}{(y_1)^2}\, dt \\
&= y_1 \left(-\frac{i\sqrt{3}\,\mathrm{e}^{-\frac{t^2}{2}+\ln(t)}\mathrm{e}^{\frac{t^2\left(1+i\sqrt{3}\right)}{2}}}{3t} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t^2\left(1+i\sqrt{3}\right)}{4}} \right) + c_2 \left(\mathrm{e}^{-\frac{t^2\left(1+i\sqrt{3}\right)}{4}} \left(-\frac{i\sqrt{3}\,\mathrm{e}^{-\frac{t^2}{2}+\ln(t)}\mathrm{e}^{\frac{t^2\left(1+i\sqrt{3}\right)}{2}}}{3t} \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.409.2 Maple step by step solution

Let's solve

$t\left(\frac{d}{dt}y'\right) + \left(t^2-1\right)y' + t^3 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -t^2 y - \frac{\left(t^2-1\right)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y' + \frac{(t^2-1)y'}{t} + t^2 y = 0$

- ☐ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(t) = \frac{t^2-1}{t}, P_3(t) = t^2\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$

    $(t \cdot P_2(t))\bigg|_{t=0} = -1$
  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $(t^2 \cdot P_3(t))\bigg|_{t=0} = 0$
  - $t = 0$is a regular singular point

    Check to see if $t_0 = 0$ is a regular singular point

    $t_0 = 0$
- Multiply by denominators

  $t\left(\frac{d}{dt}y'\right) + (t^2-1)\,y' + t^3 y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $t^3 \cdot y$ to series expansion

    $t^3 \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+3}$
  - Shift index using $k- >k-3$

    $t^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} t^{k+r}$
  - Convert $t^m \cdot y'$ to series expansion for $m = 0..2$

    $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,t^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,t^{k+r}$
  - Convert $t \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

    $t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,t^{k+r-1}$
  - Shift index using $k- >k+1$

$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-2+r)\, t^{-1+r} + a_1(1+r)(-1+r)\, t^r + (a_2(2+r)\, r + a_0 r)\, t^{1+r} + (a_3(3+r)(1+r) + a_1(1+$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- The coefficients of each power of t must be 0
  $[a_1(1+r)(-1+r) = 0, a_2(2+r)\, r + a_0 r = 0, a_3(3+r)(1+r) + a_1(1+r) = 0]$
- Solve for the dependent coefficient(s)
  $\left\{a_1 = 0, a_2 = -\frac{a_0}{2+r}, a_3 = 0\right\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k+r-1) + a_{k-1}(k+r-1) + a_{k-3} = 0$
- Shift index using $k- > k+3$
  $a_{k+4}(k+4+r)(k+2+r) + a_{k+2}(k+2+r) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+4} = -\frac{k a_{k+2} + r a_{k+2} + a_k + 2a_{k+2}}{(k+4+r)(k+2+r)}$
- Recursion relation for $r = 0$
  $a_{k+4} = -\frac{k a_{k+2} + a_k + 2a_{k+2}}{(k+4)(k+2)}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+4} = -\frac{k a_{k+2} + a_k + 2a_{k+2}}{(k+4)(k+2)}, a_1 = 0, a_2 = -\frac{a_0}{2}, a_3 = 0\right]$
- Recursion relation for $r = 2$
  $a_{k+4} = -\frac{k a_{k+2} + a_k + 4a_{k+2}}{(k+6)(k+4)}$
- Solution for $r = 2$
  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+4} = -\frac{k a_{k+2} + a_k + 4a_{k+2}}{(k+6)(k+4)}, a_1 = 0, a_2 = -\frac{a_0}{4}, a_3 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+4} = -\frac{k a_{k+2} + a_k + 2a_{k+2}}{(k+4)(k+2)}, a_1 = 0, a_2 = -\frac{a_0}{2}, a_3 = 0, b_{k+4} = -\frac{k b_k}{}\right.$

1.409.3 Maple trace

```
Methods for second order ODEs:
```

1.409.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 34

```
dsolve(t*diff(diff(y(t),t),t)+(t^2-1)*diff(y(t),t)+t^3*y(t) = 0,
       y(t),singsol=all)
```

$$y = \mathrm{e}^{-\frac{t^2}{4}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t^2}{4}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t^2}{4}\right)\right)$$

1.409.5 Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 48

```
DSolve[{t*D[y[t],{t,2}]+(t^2-1)*D[y[t],t]+t^3*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^{-\frac{t^2}{4}} \left(c_2 \cos\left(\frac{\sqrt{3}t^2}{4}\right) + c_1 \sin\left(\frac{\sqrt{3}t^2}{4}\right)\right)$$

1.410 problem 422

Internal problem ID [8548]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 422
Date solved : `Monday, October 21, 2024 at 05:09:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' - t(t+2)\,y' + (t+2)\,y = 0$$

1.410.1 Solved as second order ode using Kovacic algorithm

Time used: 0.100 (sec)

Writing the ode as

$$t^2y'' + \left(-t^2 - 2t\right) y' + (t+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 - 2t \\ C &= t + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 775: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-t^2-2t}{t^2}\,dt}\\ &= z_1 e^{\frac{t}{2}+\ln(t)}\\ &= z_1\left(t\,\mathrm{e}^{\frac{t}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-2t}{t^2}\,dt}}{(y_1)^2}\,dt\\ &= y_1 \int \frac{e^{t+2\ln(t)}}{(y_1)^2}\,dt\\ &= y_1\left(\frac{\mathrm{e}^{t+2\ln(t)}}{t^2}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(t) + c_2\left(t\left(\frac{e^{t+2\ln(t)}}{t^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.410.2 Maple step by step solution

Let's solve
$t^2\left(\frac{d}{dt}y'\right) - t(t+2)\,y' + (t+2)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dt}y'$
- Isolate 2nd derivative
  $\frac{d}{dt}y' = -\frac{(t+2)y}{t^2} + \frac{(t+2)y'}{t}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dt}y' - \frac{(t+2)y'}{t} + \frac{(t+2)y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = -\frac{t+2}{t}, P_3(t) = \frac{t+2}{t^2}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$
    $$\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -2$$
  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$
    $$\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 2$$
  - $t = 0$is a regular singular point
    Check to see if $t_0 = 0$ is a regular singular point
    $t_0 = 0$

- Multiply by denominators
  $t^2\left(\frac{d}{dt}y'\right) - t(t+2)\,y' + (t+2)\,y = 0$
- Assume series solution for y
  $$y = \sum_{k=0}^{\infty} a_k t^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $t^m \cdot y$ to series expansion for $m = 0..1$

    $t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$
  - Shift index using $k- >k - m$

    $t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$
  - Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

    $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$

    $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$
  - Convert $t^2 \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

    $t^2 \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-2) - a_{k-1}(k+r-2)\right) t^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_k(k+r-1) - a_{k-1}) = 0$
- Shift index using $k- >k + 1$

  $(k+r-1)\,(a_{k+1}(k+r) - a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+r}$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+2}$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+2}\right]$$

1.410.3 Maple trace

```
Methods for second order ODEs:
```

1.410.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve(t^2*diff(diff(y(t),t),t)-t*(t+2)*diff(y(t),t)+(t+2)*y(t) = 0,
       y(t),singsol=all)
```

$$y = t(c_1 + c_2\,\mathrm{e}^{t})$$

1.410.5 Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 16

```
DSolve[{t^2*D[y[t],{t,2}]-t*(t+2)*D[y[t],t]+(t+2)*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to t(c_2 e^t + c_1)$$

1.411 problem 423

Internal problem ID [8549]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 423
Date solved : `Monday, October 21, 2024 at 05:09:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\,y''-xy'+y=0$$

1.411.1 Solved as second order ode using Kovacic algorithm

Time used: 0.259 (sec)

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1\\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 6 \\ t &= 4(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 777: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4} - \frac{1}{2(x-1)} + \frac{3}{4(x-1)^2}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(x-1)^2}\right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{e^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1}\, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{x\, e^{x+\ln(x-1)} e^{-2x}}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(e^x) + c_2\left(e^x\left(-\frac{x\, e^{x+\ln(x-1)} e^{-2x}}{x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.411.2 Maple step by step solution

Let's solve

$(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$
- $(x-1)\cdot P_2(x)$ is analytic at $x=1$
  $$\left.\left((x-1)\cdot P_2(x)\right)\right|_{x=1} = -1$$
- $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$
  $$\left.\left((x-1)^2\cdot P_3(x)\right)\right|_{x=1} = 0$$
- $x=1$is a regular singular point
  Check to see if $x_0 = 1$ is a regular singular point
  $x_0 = 1$

- Multiply by denominators
  $(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$
- Change variables using $x = u+1$ so that the regular singular point is at $u=0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions
- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$$
- Shift index using $k->k+1$
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r-1) - a_k(k+r-1)\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

1.411.3 Maple trace

```
Methods for second order ODEs:
```

1.411.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2 \,\mathrm{e}^{x}$$

1.411.5 Mathematica DSolve solution

Solving time : 0.049 (sec)
Leaf size : 17

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2 x$$

1.412 problem 424

Internal problem ID [8550]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 424
Date solved : `Monday, October 21, 2024 at 05:09:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - \left(x - \frac{3}{16}\right)y = 0$$

1.412.1 Solved as second order ode using Kovacic algorithm

Time used: 0.187 (sec)

Writing the ode as

$$x^2y'' + \left(-x + \frac{3}{16}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -x + \frac{3}{16} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 16x - 3$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{16x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 779: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 1 \\
&= 1
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{1, 2, 3\}
\end{aligned}
$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1-16x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+4\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+4\sqrt{x}}{4x} dx} \\ &= x^{1/4} \mathrm{e}^{2\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{1/4} \mathrm{e}^{2\sqrt{x}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{1/4} \mathrm{e}^{2\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= x^{1/4} \mathrm{e}^{2\sqrt{x}} \int \frac{1}{\sqrt{x}\, \mathrm{e}^{4\sqrt{x}}} \, dx \\ &= x^{1/4} \mathrm{e}^{2\sqrt{x}} \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2} \right) \end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1\left(x^{1/4}e^{2\sqrt{x}}\right) + c_2\left(x^{1/4}e^{2\sqrt{x}}\left(-\frac{e^{-4\sqrt{x}}}{2}\right)\right)$$

Will add steps showing solving for IC soon.

1.412.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - \left(x - \frac{3}{16}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{y(16x-3)}{16x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y(16x-3)}{16x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{16x-3}{16x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{3}{16}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $16x^2\left(\frac{d}{dx}y'\right) + (-16x+3)y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - Shift index using $k->k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$$

    Rewrite ODE with series expansions

    $$a_0(-1+4r)(-3+4r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(4k+4r-1)(4k+4r-3) - 16a_{k-1}\right) x^{k+r}\right) = 0$$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+4r)(-3+4r) = 0$$
- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$$
- Each term in the series must be 0, giving the recursion relation

  $$16\left(k+r-\frac{1}{4}\right)\left(k+r-\frac{3}{4}\right) a_k - 16a_{k-1} = 0$$
- Shift index using $k->k+1$

  $$16\left(k+\frac{3}{4}+r\right)\left(k+\frac{1}{4}+r\right) a_{k+1} - 16a_k = 0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{16a_k}{(4k+3+4r)(4k+1+4r)}$$
- Recursion relation for $r = \frac{1}{4}$

  $$a_{k+1} = \frac{16a_k}{(4k+4)(4k+2)}$$
- Solution for $r = \frac{1}{4}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = \frac{16a_k}{(4k+4)(4k+2)}\right]$$
- Recursion relation for $r = \frac{3}{4}$

  $$a_{k+1} = \frac{16a_k}{(4k+6)(4k+4)}$$
- Solution for $r = \frac{3}{4}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = \frac{16a_k}{(4k+6)(4k+4)}\right]$$
- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1} = \frac{16a_k}{(4k+4)(4k+2)}, b_{k+1} = \frac{16b_k}{(4k+6)(4k+4)}\right]$$

1.412.3 Maple trace

```
Methods for second order ODEs:
```

1.412.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve(x^2*diff(diff(y(x),x),x)-(x-3/16)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^{1/4}\left(c_1 \sinh\left(2\sqrt{x}\right) + c_2 \cosh\left(2\sqrt{x}\right)\right)$$

1.412.5 Mathematica DSolve solution

Solving time : 0.07 (sec)
Leaf size : 41

```
DSolve[{x^2*D[y[x],{x,2}]-(x-1875/10000)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-2\sqrt{x}} \sqrt[4]{x}\left(2c_1 e^{4\sqrt{x}} - c_2\right)$$

1.413 problem 425

Internal problem ID [8551]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 425
Date solved : `Monday, October 21, 2024 at 05:09:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

1.413.1 Solved as second order ode using Kovacic algorithm

Time used: 0.163 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 781: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.413.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$

  ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 \left(\frac{d}{dx} y'\right) + 4x y' + (4x^2 - 1) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)(-1+2r) x^r + a_1(3+2r)(1+2r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)(2k+2r-1) + 4a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+2r)(-1+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

$a_1(3+2r)(1+2r) = 0$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k->k+2$

$a_{k+2}\left(4(k+2)^2 + 8(k+2) r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

$a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

1.413.3 Maple trace

```
Methods for second order ODEs:
```

1.413.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1 \sin(x)+c_2 \cos(x)}{\sqrt{x}}$$

1.413.5 Mathematica DSolve solution

Solving time : 0.049 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-25/100)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{e^{-ix}(2c_1-ic_2e^{2ix})}{2\sqrt{x}}$$

1.414 problem 426

Internal problem ID [8552]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 426
Date solved : `Monday, October 21, 2024 at 05:09:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' - t(t+2)\,y' + (t+2)\,y = 0$$

1.414.1 Solved as second order ode using Kovacic algorithm

Time used: 0.102 (sec)

Writing the ode as

$$t^2y'' + \left(-t^2 - 2t\right)y' + (t+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 - 2t \\ C &= t + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 783: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t^2-2t}{t^2}\, dt} \\
&= z_1 e^{\frac{t}{2}+\ln(t)} \\
&= z_1 \left(t\,\mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-2t}{t^2}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{t+2\ln(t)}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{\mathrm{e}^{t+2\ln(t)}}{t^2}\right)
\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1(t) + c_2\left(t\left(\frac{e^{t+2\ln(t)}}{t^2}\right)\right)$$

Will add steps showing solving for IC soon.

1.414.2 Maple step by step solution

Let's solve

$t^2\left(\frac{d}{dt}y'\right) - t(t+2)\,y' + (t+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{(t+2)y}{t^2} + \frac{(t+2)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' - \frac{(t+2)y'}{t} + \frac{(t+2)y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(t) = -\frac{t+2}{t}, P_3(t) = \frac{t+2}{t^2}\right]$

  - $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left.(t \cdot P_2(t))\right|_{t=0} = -2$

  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left.(t^2 \cdot P_3(t))\right|_{t=0} = 2$

  - $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $t^2\left(\frac{d}{dt}y'\right) - t(t+2)\,y' + (t+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $t^m \cdot y$ to series expansion for $m = 0..1$

    $$t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$$
  - Shift index using $k- >k - m$

    $$t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$$
  - Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

    $$t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$$
  - Shift index using $k- >k + 1 - m$

    $$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$
  - Convert $t^2 \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

    $$t^2 \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)\,(-2+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-2) - a_{k-1}(k+r-2)\right) t^{k+r}\right) = 0$$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_k(k+r-1) - a_{k-1}) = 0$
- Shift index using $k- >k + 1$

  $(k+r-1)\,(a_{k+1}(k+r) - a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+r}$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+2}$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+2}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+2}\right]$$

1.414.3 Maple trace

```
Methods for second order ODEs:
```

1.414.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 12

```
dsolve(t^2*diff(diff(y(t),t),t)-t*(t+2)*diff(y(t),t)+(t+2)*y(t) = 0,
       y(t),singsol=all)
```

$$y = t(c_1 + c_2\,\mathrm{e}^t)$$

1.414.5 Mathematica DSolve solution

Solving time : `0.031` (sec)
Leaf size : 16

```
DSolve[{t^2*D[y[t],{t,2}]-t*(t+2)*D[y[t],t]+(t+2)*y[t] == 0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to t(c_2 e^t + c_1)$$

1.415 problem 427

Internal problem ID [8553]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 427
Date solved : `Monday, October 21, 2024 at 05:09:35 PM`
CAS classification : `[_Laguerre]`

Solve

$$ty'' - (1+t)\,y' + y = 0$$

1.415.1 Solved as second order ode using Kovacic algorithm

Time used: 0.232 (sec)

Writing the ode as

$$ty'' + (-1-t)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -1-t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 2t + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 785: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4t^2} - \frac{1}{2t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - (\alpha_{c_1}^{-}) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2t} \\
&= \frac{t-1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2 - 2t + 3}{4t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\
&= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-1-t}{t} \, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(t)}{2}} \\
&= z_1 \left(\sqrt{t}\, \mathrm{e}^{\frac{t}{2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1-t}{t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\, dt \\ &= y_1 \left(-\frac{(1+t)\, e^{t+\ln(t)} e^{-2t}}{t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^t\right) + c_2 \left(e^t \left(-\frac{(1+t)\, e^{t+\ln(t)} e^{-2t}}{t}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.415.2 Maple step by step solution

Let's solve

$t\left(\frac{d}{dt}y'\right) - (1+t)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{y}{t} + \frac{(1+t)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' - \frac{(1+t)y'}{t} + \frac{y}{t} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(t) = -\frac{1+t}{t}, P_3(t) = \frac{1}{t}\right]$
- $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -1$
- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 0$
- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $t\left(\frac{d}{dt}y'\right) + (-1-t)\,y' + y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $t^m \cdot y'$ to series expansion for $m = 0..1$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$
- Convert $t \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

  $t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r-1}$
- Shift index using $k->k+1$

  $t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) - a_k(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y=\sum_{k=0}^{\infty} a_k t^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$
  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k t^k\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

1.415.3 Maple trace

```
Methods for second order ODEs:
```

1.415.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 13

```
dsolve(t*diff(diff(y(t),t),t)-(1+t)*diff(y(t),t)+y(t) = 0,
       y(t),singsol=all)
```

$$y = c_2\,\mathrm{e}^{t}+c_1 t+c_1$$

1.415.5 Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 19

```
DSolve[{t*D[y[t],{t,2}]-(1+t)*D[y[t],t]+y[t] == 0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_1 e^t - c_2(t+1)$$

1.416 problem 428

Internal problem ID [8554]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 428
Date solved : `Monday, October 21, 2024 at 05:09:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-t)\,y''+ty'-y=0$$

1.416.1 Solved as second order ode using Kovacic algorithm

Time used: 0.251 (sec)

Writing the ode as

$$(1-t)\,y''+ty'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1-t\\ B&=t\\ C&=-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 4t + 6}{4(-1+t)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 4t + 6$$
$$t = 4(-1+t)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 4t + 6}{4(-1+t)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 787: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+t)^2$. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(-1+t)} + \frac{3}{4(-1+t)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(-1+t)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{t^3} + \frac{11}{4t^4} + \frac{21}{4t^5} + \frac{15}{2t^6} + \frac{6}{t^7} - \frac{117}{16t^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 - 4t + 6}{4t^2 - 8t + 4} \\ &= Q + \frac{R}{4t^2 - 8t + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 5}{4t^2 - 8t + 4}\right) \\ &= \frac{1}{4} + \frac{-2t + 5}{4t^2 - 8t + 4}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 4t + 6}{4\left(-1 + t\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(-1+t)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(-1+t)} + \frac{1}{2} \\
&= \frac{t-2}{2t-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(-1+t)^2}\right) + \left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right)^2 - \left(\frac{t^2-4t+6}{4(-1+t)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right) dt} \\
&= \frac{e^{\frac{t}{2}}}{\sqrt{-1+t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t}{1-t} \, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(-1+t)}{2}} \\
&= z_1 \left(\sqrt{-1+t}\, e^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{1-t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{t+\ln(-1+t)}}{(y_1)^2}\, dt \\
&= y_1 \left(-\frac{t\,\mathrm{e}^{t+\ln(-1+t)}\mathrm{e}^{-2t}}{-1+t}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^t\right) + c_2\left(\mathrm{e}^t\left(-\frac{t\,\mathrm{e}^{t+\ln(-1+t)}\mathrm{e}^{-2t}}{-1+t}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.416.2 Maple step by step solution

Let's solve

$(1-t)\left(\frac{d}{dt}y'\right) + ty' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{y}{-1+t} + \frac{ty'}{-1+t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' - \frac{ty'}{-1+t} + \frac{y}{-1+t} = 0$

☐ Check to see if $t_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(t) = -\frac{t}{-1+t}, P_3(t) = \frac{1}{-1+t}\right]$
- $(-1+t) \cdot P_2(t)$ is analytic at $t = 1$
  $$\left((-1+t) \cdot P_2(t)\right)\bigg|_{t=1} = -1$$
- $(-1+t)^2 \cdot P_3(t)$ is analytic at $t = 1$
  $$\left((-1+t)^2 \cdot P_3(t)\right)\bigg|_{t=1} = 0$$
- $t = 1$is a regular singular point
  Check to see if $t_0 = 1$ is a regular singular point
  $t_0 = 1$

- Multiply by denominators
  $(-1+t)\left(\frac{d}{dt}y'\right) - ty' + y = 0$
- Change variables using $t = u + 1$ so that the regular singular point is at $u = 0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions
- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$
- Shift index using $k- >k+1-m$
  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$
- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$
- Shift index using $k- >k+1$
  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Revert the change of variables $u=-1+t$
  $\left[y=\sum_{k=0}^{\infty} a_k(-1+t)^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$
  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Revert the change of variables $u=-1+t$
  $\left[y=\sum_{k=0}^{\infty} a_k(-1+t)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k(-1+t)^k\right)+\left(\sum_{k=0}^{\infty} b_k(-1+t)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

1.416.3 Maple trace

```
Methods for second order ODEs:
```

1.416.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((1-t)*diff(diff(y(t),t),t)+t*diff(y(t),t)-y(t) = 0,
       y(t),singsol=all)
```

$$y = c_1 t + c_2 \, e^{t}$$

1.416.5 Mathematica DSolve solution

Solving time : 0.05 (sec)
Leaf size : 17

```
DSolve[{(1-t)*D[y[t],{t,2}]+t*D[y[t],t]-y[t] == 0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_1 e^t - c_2 t$$

1.417 problem 429

Internal problem ID [8555]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 429
Date solved : `Monday, October 21, 2024 at 05:09:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

1.417.1 Solved as second order ode using Kovacic algorithm

Time used: 0.161 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 789: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.417.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 \left(\frac{d}{dx} y'\right) + 4x y' + (4x^2 - 1) y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$$

1.417.3 Maple trace

```
Methods for second order ODEs:
```

1.417.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{\sqrt{x}}$$

1.417.5 Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-25/100)*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

1.418 problem 430

Internal problem ID [8556]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 430
Date solved : `Monday, October 21, 2024 at 05:09:38 PM`
CAS classification : `[_Laguerre]`

Solve

$$ty'' - (1+t)\,y' + y = 0$$

1.418.1 Solved as second order ode using Kovacic algorithm

Time used: 0.229 (sec)

Writing the ode as

$$ty'' + (-1-t)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -1-t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= t^2 - 2t + 3 \\
t &= 4t^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 791: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 - 2t + 3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2t} \\
&= \frac{t-1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2 - 2t + 3}{4t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\
&= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1-t}{t}\, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(t)}{2}} \\
&= z_1 \left(\sqrt{t}\, \mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1-t}{t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\, dt \\ &= y_1 \left(-\frac{(1+t)\,\mathrm{e}^{t+\ln(t)}\mathrm{e}^{-2t}}{t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^t\right) + c_2 \left(\mathrm{e}^t \left(-\frac{(1+t)\,\mathrm{e}^{t+\ln(t)}\mathrm{e}^{-2t}}{t}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.418.2 Maple step by step solution

Let's solve

$t\left(\frac{d}{dt}y'\right) - (1+t)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{y}{t} + \frac{(1+t)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' - \frac{(1+t)y'}{t} + \frac{y}{t} = 0$

- □ Check to see if $t_0 = 0$ is a regular singular point

- Define functions
  $\left[P_2(t) = -\frac{1+t}{t}, P_3(t) = \frac{1}{t}\right]$
- $t \cdot P_2(t)$ is analytic at $t = 0$
  $(t \cdot P_2(t))\bigg|_{t=0} = -1$
- $t^2 \cdot P_3(t)$ is analytic at $t = 0$
  $(t^2 \cdot P_3(t))\bigg|_{t=0} = 0$
- $t = 0$is a regular singular point
  Check to see if $t_0 = 0$ is a regular singular point
  $t_0 = 0$

- Multiply by denominators
  $t\left(\frac{d}{dt}y'\right) + (-1-t)\, y' + y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions
- Convert $t^m \cdot y'$ to series expansion for $m = 0..1$
  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$
- Shift index using $k- > k+1-m$
  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$
- Convert $t \cdot \left(\frac{d}{dt}y'\right)$ to series expansion
  $t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, t^{k+r-1}$
- Shift index using $k- > k+1$
  $t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions
  $a_0 r(-2+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\,(a_{k+1}(k+1+r)-a_k)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y=\sum_{k=0}^{\infty} a_k t^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$
  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k t^k\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

1.418.3 Maple trace

```
Methods for second order ODEs:
```

1.418.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 13

```
dsolve(t*diff(diff(y(t),t),t)-(1+t)*diff(y(t),t)+y(t) = 0,
       y(t),singsol=all)
```

$$y=c_2\,\mathrm{e}^{t}+c_1 t+c_1$$

1.418.5 Mathematica DSolve solution

Solving time : 0.041 (sec)
Leaf size : 19

```
DSolve[{t*D[y[t],{t,2}]-(1+t)*D[y[t],t]+y[t] ==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_1 e^t - c_2(t+1)$$

1.419 problem 431

Internal problem ID [8557]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 431
Date solved : `Monday, October 21, 2024 at 05:09:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-t)\,y''+ty'-y=0$$

1.419.1 Solved as second order ode using Kovacic algorithm

Time used: 0.259 (sec)

Writing the ode as

$$(1-t)\,y''+ty'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-t \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 4t + 6}{4(-1+t)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 4t + 6$$
$$t = 4(-1+t)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 4t + 6}{4(-1+t)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 793: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+t)^2$. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2\left(-1+t\right)} + \frac{3}{4\left(-1+t\right)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(-1+t)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{t^3} + \frac{11}{4t^4} + \frac{21}{4t^5} + \frac{15}{2t^6} + \frac{6}{t^7} - \frac{117}{16t^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 4t + 6}{4t^2 - 8t + 4} \\ &= Q + \frac{R}{4t^2 - 8t + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 5}{4t^2 - 8t + 4}\right) \\ &= \frac{1}{4} + \frac{-2t + 5}{4t^2 - 8t + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 4t + 6}{4\left(-1+t\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(-1+t\right)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2\left(-1+t\right)} + \frac{1}{2} \\ &= \frac{t-2}{2t-2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{2\left(-1+t\right)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2\left(-1+t\right)^2}\right) + \left(-\frac{1}{2\left(-1+t\right)} + \frac{1}{2}\right)^2 - \left(\frac{t^2-4t+6}{4\left(-1+t\right)^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(-1+t)} + \frac{1}{2}\right) dt} \\ &= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{-1+t}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{t}{1-t} \, dt} \\ &= z_1 e^{\frac{t}{2} + \frac{\ln(-1+t)}{2}} \\ &= z_1 \left(\sqrt{-1+t}\, \mathrm{e}^{\frac{t}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{1-t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{t+\ln(-1+t)}}{(y_1)^2}\, dt \\
&= y_1 \left(-\frac{t\,\mathrm{e}^{t+\ln(-1+t)}\mathrm{e}^{-2t}}{-1+t}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{t}\right) + c_2 \left(\mathrm{e}^{t}\left(-\frac{t\,\mathrm{e}^{t+\ln(-1+t)}\mathrm{e}^{-2t}}{-1+t}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.419.2 Maple step by step solution

Let's solve

$(1-t)\left(\frac{d}{dt}y'\right) + ty' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{y}{-1+t} + \frac{ty'}{-1+t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' - \frac{ty'}{-1+t} + \frac{y}{-1+t} = 0$

□ Check to see if $t_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(t) = -\frac{t}{-1+t}, P_3(t) = \frac{1}{-1+t}\right]$
- $(-1+t) \cdot P_2(t)$ is analytic at $t = 1$
  $$\left((-1+t) \cdot P_2(t)\right)\bigg|_{t=1} = -1$$
- $(-1+t)^2 \cdot P_3(t)$ is analytic at $t = 1$
  $$\left((-1+t)^2 \cdot P_3(t)\right)\bigg|_{t=1} = 0$$
- $t = 1$is a regular singular point
  Check to see if $t_0 = 1$ is a regular singular point
  $t_0 = 1$

- Multiply by denominators
  $(-1+t)\left(\frac{d}{dt}y'\right) - ty' + y = 0$
- Change variables using $t = u + 1$ so that the regular singular point is at $u = 0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$
- Shift index using $k- > k+1-m$
  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$
- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$
- Shift index using $k- > k+1$
  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

  Rewrite ODE with series expansions
  $$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$
- Revert the change of variables $u = -1 + t$

  $\left[y = \sum_{k=0}^{\infty} a_k (-1+t)^k, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Revert the change of variables $u = -1 + t$

  $\left[y = \sum_{k=0}^{\infty} a_k (-1+t)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (-1+t)^k\right) + \left(\sum_{k=0}^{\infty} b_k (-1+t)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

1.419.3 Maple trace

```
Methods for second order ODEs:
```

1.419.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((1-t)*diff(diff(y(t),t),t)+t*diff(y(t),t)-y(t) = 0,
       y(t),singsol=all)
```

$$y = c_1 t + c_2 \, \mathrm{e}^{t}$$

1.419.5 Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 17

```
DSolve[{(1-t)*D[y[t],{t,2}]+t*D[y[t],t]-y[t] ==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_1 e^t - c_2 t$$

1.420 problem 432

Internal problem ID [8558]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 432
Date solved : `Monday, October 21, 2024 at 05:09:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

1.420.1 Solved as second order ode using Kovacic algorithm

Time used: 0.242 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \quad (1)$$
$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 795: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 6}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{3}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \,\mathrm{e}^{\int -\frac{x}{2} dx} \\
&= (x) \,\mathrm{e}^{-\frac{x^2}{4}} \\
&= x \,\mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{2}} x\left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)\right)$$

Will add steps showing solving for IC soon.

1.420.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions
  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - ○ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - ○ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_k(k+2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

1.420.3 Maple trace

```
Methods for second order ODEs:
```

1.420.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -x\left(c_2\pi \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) - c_1\right) \mathrm{e}^{-\frac{x^2}{2}} + i\sqrt{\pi}\,\sqrt{2}\,c_2$$

1.420.5 Mathematica DSolve solution

Solving time : 0.069 (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}} \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2} c_1 e^{-\frac{x^2}{2}} x + c_2$$

1.421 problem 433

Internal problem ID [8559]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 433
Date solved : `Monday, October 21, 2024 at 05:09:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x^2+1)\,y''-4xy'+6y=0$$

1.421.1 Solved as second order ode using Kovacic algorithm

Time used: 0.279 (sec)

Writing the ode as

$$(x^2+1)\,y''-4xy'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-8}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -8$$
$$t = \left(x^2+1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{8}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 797: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x-i)^2} + \frac{2}{(x+i)^2} + \frac{2i}{x-i} - \frac{2i}{x+i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{8}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	2	-1
$-i$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x-i} + \frac{2}{x+i} + (-)\,(0) \\ &= -\frac{1}{x-i} + \frac{2}{x+i} \\ &= \frac{x-3i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-i} + \frac{2}{x+i}\right)(0) + \left(\left(\frac{1}{(x-i)^2} - \frac{2}{(x+i)^2}\right) + \left(-\frac{1}{x-i} + \frac{2}{x+i}\right)^2 - \left(-\frac{8}{(x^2+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{x-i} + \frac{2}{x+i}\right) dx} \\ &= \frac{(x^2+1)^2}{(ix+1)^3} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2+1}\, dx} \\ &= z_1 e^{\ln(x^2+1)} \\ &= z_1 (x^2+1) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^3}{(ix+1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-4x}{x^2+1}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{2\ln(x^2+1)}}{(y_1)^2}\,dx$$

$$= y_1 \left(\frac{x^2-\frac{1}{3}}{(x+i)^3}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{(x^2+1)^3}{(ix+1)^3}\right) + c_2 \left(\frac{(x^2+1)^3}{(ix+1)^3}\left(\frac{x^2-\frac{1}{3}}{(x+i)^3}\right)\right)$$

Will add steps showing solving for IC soon.

1.421.2 Maple step by step solution

1.421.3 Maple trace

```
Methods for second order ODEs:
```

1.421.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 21

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-4*x*diff(y(x),x)+6*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2\,x^3 - 3c_1\,x^2 - 3c_2 x + c_1$$

1.421.5 Mathematica DSolve solution

Solving time : 0.098 (sec)
Leaf size : 33

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-4*x*D[y[x],x]+6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{3} i \left(c_2 \left(3x^2 - 1\right) + 3c_1 (x - i)^3\right)$$

1.422 problem 434

Internal problem ID [8560]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 434
Date solved : `Monday, October 21, 2024 at 05:09:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-x)\, y'' + x y' - y = 0$$

1.422.1 Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$(1-x)\, y'' + x y' - y = 0 \tag{1}$$

$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(-1 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(-1 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(-1 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 798: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+x)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(-1+x)} + \frac{3}{4(-1+x)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4\left(-1 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(-1+x)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(-1+x)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(-1+x)^2}\right) + \left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(-1+x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{-1+x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1-x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(-1+x)}{2}} \\
&= z_1 \left(\sqrt{-1+x}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1-x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(-1+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{x\,\mathrm{e}^{x+\ln(-1+x)}\mathrm{e}^{-2x}}{-1+x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-\frac{x\,\mathrm{e}^{x+\ln(-1+x)}\mathrm{e}^{-2x}}{-1+x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.422.2 Maple step by step solution

Let's solve

$(1-x)\left(\frac{d}{dx}y'\right) + xy' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{-1+x} + \frac{xy'}{-1+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{xy'}{-1+x} + \frac{y}{-1+x} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{x}{-1+x}, P_3(x) = \frac{1}{-1+x}\right]$

- ◦ $(-1+x)\cdot P_2(x)$ is analytic at $x=1$

  $$\left((-1+x)\cdot P_2(x)\right)\bigg|_{x=1} = -1$$

- ◦ $(-1+x)^2\cdot P_3(x)$ is analytic at $x=1$

  $$\left((-1+x)^2\cdot P_3(x)\right)\bigg|_{x=1} = 0$$

- ◦ $x=1$is a regular singular point

  Check to see if $x_0=1$ is a regular singular point

  $x_0=1$

- Multiply by denominators

  $(-1+x)\left(\frac{d}{dx}y'\right) - xy' + y = 0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- ◦ Shift index using $k->k+1-m$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- ◦ Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$

- ◦ Shift index using $k->k+1$

  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+1+r}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{k+1}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$

- Revert the change of variables $u=-1+x$

  $\left[y=\sum_{k=0}^{\infty} a_k(-1+x)^k, a_{k+1}=\frac{a_k}{k+1}\right]$

- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+3}$

- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$

- Revert the change of variables $u=-1+x$

  $\left[y=\sum_{k=0}^{\infty} a_k(-1+x)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k(-1+x)^k\right)+\left(\sum_{k=0}^{\infty} b_k(-1+x)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

1.422.3 Maple trace

```
Methods for second order ODEs:
```

1.422.4 Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 12

```
dsolve((1-x)*diff(diff(y(x),x),x)+x*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2 \,\mathrm{e}^{x}$$

1.422.5 Mathematica DSolve solution

Solving time : `0.047` (sec)
Leaf size : 17

```
DSolve[{(1-x)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2 x$$

1.423 problem 435

Internal problem ID [8561]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 435
Date solved : `Monday, October 21, 2024 at 05:09:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2y'' + xy' + 3y = 0$$

1.423.1 Solved as second order ode using Kovacic algorithm

Time used: 0.273 (sec)

Writing the ode as

$$2y'' + xy' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 20}{16} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 20$$
$$t = 16$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{16} - \frac{5}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 800: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{5}{2x} - \frac{25}{2x^3} - \frac{125}{x^5} - \frac{3125}{2x^7} - \frac{21875}{x^9} - \frac{328125}{x^{11}} - \frac{5156250}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 20}{16} \\ &= Q + \frac{R}{16} \\ &= \left(\frac{x^2}{16} - \frac{5}{4}\right) + (0) \\ &= \frac{x^2}{16} - \frac{5}{4} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{4}\right) - (0) \\ &= -\frac{5}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{4}}{\frac{1}{4}} - 1\right) &&= -3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{4}}{\frac{1}{4}} - 1\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{16} - \frac{5}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{4}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{4}\right) \\ &= -\frac{x}{4} \\ &= -\frac{x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{x}{4}\right)(2x + a_1) + \left(\left(-\frac{1}{4}\right) + \left(-\frac{x}{4}\right)^2 - \left(\frac{x^2}{16} - \frac{5}{4}\right)\right) = 0$$

$$2 + \frac{a_1 x}{2} + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 2\right) \mathrm{e}^{\int -\frac{x}{4} dx} \\
&= \left(x^2 - 2\right) \mathrm{e}^{-\frac{x^2}{8}} \\
&= \left(x^2 - 2\right) \mathrm{e}^{-\frac{x^2}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{2} \, dx} \\
&= z_1 e^{-\frac{x^2}{8}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{4}} \left(x^2 - 2\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{4}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{\left(x^2 - 2\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^2}{4}}\left(x^2-2\right)\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{4}}\left(x^2-2\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{\left(x^2-2\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.423.2 Maple step by step solution

Let's solve

$2\frac{d}{dx}y' + xy' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{xy'}{2} - \frac{3y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{xy'}{2} + \frac{3y}{2} = 0$

- Multiply by denominators

  $2\frac{d}{dx}y' + xy' + 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(2a_{k+2}(k+2)\,(k+1)+a_k(k+3)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $(2k^2+6k+4)\, a_{k+2}+a_k(k+3)=0$
- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k(k+3)}{2(k^2+3k+2)}\right]$

1.423.3 Maple trace

```
Methods for second order ODEs:
```

1.423.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 32

```
dsolve(2*diff(diff(y(x),x),x)+x*diff(y(x),x)+3*y(x) = 0,
       y(x),singsol=all)
```

$$y=\left(x^2-2\right)\left(c_1\sqrt{\pi}\,\mathrm{erfi}\left(\frac{x}{2}\right)+c_2\right)\mathrm{e}^{-\frac{x^2}{4}}-2c_1x$$

1.423.5 Mathematica DSolve solution

Solving time : `0.119` (sec)
Leaf size : 61

```
DSolve[{2*D[y[x],{x,2}]+x*D[y[x],x]+3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{1}{8}e^{-\frac{x^2}{4}}\left(\sqrt{\pi}c_2\left(x^2-2\right)\mathrm{erfi}\left(\frac{x}{2}\right)+8c_1\left(x^2-2\right)-2c_2e^{\frac{x^2}{4}}x\right)$$

1.424 problem 436

Internal problem ID [8562]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 436
Date solved : `Monday, October 21, 2024 at 05:09:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

1.424.1 Solved as second order ode using Kovacic algorithm

Time used: 0.241 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 802: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 6}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{3}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, e^{\int -\frac{x}{2} dx} \\
&= (x) \, e^{-\frac{x^2}{4}} \\
&= x \, e^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(e^{-\frac{x^2}{4}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{e^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\, x}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} x \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.424.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

1.424.3 Maple trace

```
Methods for second order ODEs:
```

1.424.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -x\left(c_2 \,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)\pi - c_1\right)\mathrm{e}^{-\frac{x^2}{2}} + i\sqrt{\pi}\,\sqrt{2}\,c_2$$

1.424.5 Mathematica DSolve solution

Solving time : 0.069 (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}}c_2 e^{-\frac{x^2}{2}}\sqrt{x^2}\mathrm{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2}c_1 e^{-\frac{x^2}{2}}x + c_2$$

1.425 problem 437

Internal problem ID [8563]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 437
Date solved : `Monday, October 21, 2024 at 05:09:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-x)\,y''+xy'-y=0$$

1.425.1 Solved as second order ode using Kovacic algorithm

Time used: 0.246 (sec)

Writing the ode as

$$(1-x)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(-1+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(-1+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(-1+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 804: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(-1+x)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\left(-1+x\right)^{2}} - \frac{1}{2\left(-1+x\right)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4\left(-1 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(-1+x)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(-1+x)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(-1+x)^2}\right) + \left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(-1+x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(-1+x)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{-1+x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1-x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(-1+x)}{2}} \\
&= z_1 \left(\sqrt{-1+x}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1-x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(-1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{x\,\mathrm{e}^{x+\ln(-1+x)}\mathrm{e}^{-2x}}{-1+x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-\frac{x\,\mathrm{e}^{x+\ln(-1+x)}\mathrm{e}^{-2x}}{-1+x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.425.2 Maple step by step solution

Let's solve

$(1-x)\left(\frac{d}{dx}y'\right) + xy' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{-1+x} + \frac{xy'}{-1+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{xy'}{-1+x} + \frac{y}{-1+x} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x}{-1+x}, P_3(x) = \frac{1}{-1+x}\right]$
- $(-1+x)\cdot P_2(x)$ is analytic at $x=1$
  $$\left((-1+x)\cdot P_2(x)\right)\bigg|_{x=1} = -1$$
- $(-1+x)^2\cdot P_3(x)$ is analytic at $x=1$
  $$\left((-1+x)^2\cdot P_3(x)\right)\bigg|_{x=1} = 0$$
- $x=1$is a regular singular point
  Check to see if $x_0=1$ is a regular singular point
  $x_0=1$

- Multiply by denominators
  $(-1+x)\left(\frac{d}{dx}y'\right) - xy' + y = 0$
- Change variables using $x=u+1$ so that the regular singular point is at $u=0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$
- Shift index using $k->k+1$
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+1+r}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{k+1}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$

- Revert the change of variables $u=-1+x$

  $\left[y=\sum_{k=0}^{\infty} a_k(-1+x)^k, a_{k+1}=\frac{a_k}{k+1}\right]$

- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+3}$

- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$

- Revert the change of variables $u=-1+x$

  $\left[y=\sum_{k=0}^{\infty} a_k(-1+x)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k(-1+x)^k\right)+\left(\sum_{k=0}^{\infty} b_k(-1+x)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

1.425.3 Maple trace

```
Methods for second order ODEs:
```

1.425.4 Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 12

```
dsolve((1-x)*diff(diff(y(x),x),x)+x*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2 \,\mathrm{e}^{x}$$

1.425.5 Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 17

```
DSolve[{(1-x)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2 x$$

1.426 problem 438

Internal problem ID [8564]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 438
Date solved : `Monday, October 21, 2024 at 05:09:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

1.426.1 Solved as second order ode using Kovacic algorithm

Time used: 0.230 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 806: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) = 0$$

$$a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= (x)\,\mathrm{e}^{\int -\frac{x}{2} dx} \\
&= (x)\,\mathrm{e}^{-\frac{x^2}{4}} \\
&= x\,\mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{2}} x\left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.426.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

1.426.3 Maple trace

```
Methods for second order ODEs:
```

1.426.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -x\left(c_2\pi\ \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) - c_1\right)\mathrm{e}^{-\frac{x^2}{2}} + i\sqrt{\pi}\,\sqrt{2}\,c_2$$

1.426.5 Mathematica DSolve solution

Solving time : 0.068 (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}}c_2e^{-\frac{x^2}{2}}\sqrt{x^2}\operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2}c_1e^{-\frac{x^2}{2}}x + c_2$$

1.427 problem 439

Internal problem ID [8565]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 439
Date solved : `Monday, October 21, 2024 at 05:09:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(-x^2+4\right) y''+x y'+2 y=0$$

1.427.1 Solved as second order ode using Kovacic algorithm

Time used: 1.022 (sec)

Writing the ode as

$$\left(-x^2+4\right) y''+x y'+2 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+4 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{11x^2 - 24}{4(x^2 - 4)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 11x^2 - 24$$
$$t = 4(x^2 - 4)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{11x^2 - 24}{4(x^2 - 4)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 808: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 4)^2$. There is a pole at $x = 2$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{17}{32(x-2)} - \frac{17}{32(x+2)} + \frac{5}{16(x+2)^2} + \frac{5}{16(x-2)^2}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{11x^2 - 24}{4(x^2 - 4)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{11}{4}$. Hence

$$\begin{aligned}E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\ &= \{2\}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
2	2	$\{-1, 2, 5\}$
-2	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1=-1,\ e_2=-1,\ e_\infty=2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\ &= \frac{1}{2}(2-(-1+(-1)))\\ &= 2\end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{-1}{(x-(2))}+\frac{-1}{(x-(-2))}\right)\\ &= -\frac{1}{2(x-2)}-\frac{1}{2(x+2)}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=2$ such that

$$p'''+3\theta p''+\left(3\theta^2+3\theta'-4r\right)p'+\left(\theta''+3\theta\theta'+\theta^3-4r\theta-2r'\right)p=0 \tag{1A}$$

Since $d=2$, then letting

$$p=x^2+a_1x+a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{11x^2 a_1 + 16(6 + a_0)\, x + 36a_1}{\left(x^2 - 4\right)^2} = 0$$

And solving for p gives

$$p = x^2 - 6$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{2x}{x^2 - 6} - \frac{1}{2\,(x-2)} - \frac{1}{2\,(x+2)}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{2x}{x^2 - 6} - \frac{1}{2\,(x-2)} - \frac{1}{2\,(x+2)}\right) w + \frac{-11x^4 + 74x^2 - 128}{4x^6 - 56x^4 + 256x^2 - 384} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{3}\, x^2 \sqrt{x^2 - 4} + x^3 - 8\sqrt{3}\,\sqrt{x^2 - 4} - 2x}{2\left(x^2 - 6\right)(x-2)\,(x+2)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{2\sqrt{3}\, x^2 \sqrt{x^2-4} + x^3 - 8\sqrt{3}\,\sqrt{x^2-4} - 2x}{2\left(x^2-6\right)(x-2)(x+2)} dx} \\ &= \frac{\sqrt{x^2 - 6}\,\left(x + \sqrt{x^2 - 4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\left(\sqrt{2}\,\sqrt{3}\,x - 4\right)\sqrt{2}}{2\sqrt{x^2-4}}\right)}{2} - \frac{\operatorname{arctanh}\left(\frac{\left(4 + \sqrt{2}\,\sqrt{3}\,x\right)\sqrt{2}}{2\sqrt{x^2-4}}\right)}{2}}}{(x+2)^{1/4}\,(x-2)^{1/4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{-x^2+4}\, dx} \\ &= z_1 e^{\frac{\ln\left(x^2-4\right)}{4}} \\ &= z_1 \left(\left(x^2 - 4\right)^{1/4}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{-x^2+4}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x^2-4)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\sqrt{x^2-4}\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}} \mathrm{e}^{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}}{x^2-6} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}{2}}\right) + c_2\left(\sqrt{x^2-6}\left(x\right.\right. \\
&\quad \left.+\sqrt{x^2-4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+x\sqrt{6}}{\sqrt{2x^2-8}}\right)}{2}} \left(\int \frac{\sqrt{x^2-4}\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}} \mathrm{e}^{\operatorname{arctanh}\left(\frac{x\sqrt{6}-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}}}{x^2-6}\right.
\end{aligned}$$

Will add steps showing solving for IC soon.

1.427.2 Maple step by step solution

Let's solve

$(-x^2+4)\left(\frac{d}{dx}y'\right)+xy'+2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{2y}{x^2-4}+\frac{xy'}{x^2-4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{xy'}{x^2-4}-\frac{2y}{x^2-4}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x}{x^2-4}, P_3(x)=-\frac{2}{x^2-4}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=-\frac{1}{2}$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $(x^2-4)\left(\frac{d}{dx}y'\right)-xy'-2y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(u^2-4u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-u+2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r(-3+2r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(2k-1+2r) + a_k(k^2+2kr+r^2-2k-2r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-4(k+1+r)\left(k-\frac{1}{2}+r\right)a_{k+1} + \left(k^2+(2r-2)\,k+r^2-2r-2\right)a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(k^2+2kr+r^2-2k-2r-2)\,a_k}{2(k+1+r)(2k-1+2r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{(k^2-2k-2)\,a_k}{2(k+1)(2k-1)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{(k^2-2k-2)\,a_k}{2(k+1)(2k-1)}\right]$$

- Revert the change of variables $u = x+2$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+2)^k, a_{k+1} = \frac{(k^2-2k-2)\,a_k}{2(k+1)(2k-1)}\right]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = \frac{\left(k^2+k-\frac{11}{4}\right)a_k}{2\left(k+\frac{5}{2}\right)(2k+2)}$$

- Solution for $r = \frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1} = \frac{\left(k^2+k-\frac{11}{4}\right)a_k}{2\left(k+\frac{5}{2}\right)(2k+2)}\right]$$

- Revert the change of variables $u = x+2$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+2)^{k+\frac{3}{2}}, a_{k+1}=\frac{\left(k^2+k-\frac{11}{4}\right) a_k}{2\left(k+\frac{5}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x+2)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x+2)^{k+\frac{3}{2}}\right), a_{k+1}=\frac{\left(k^2-2k-2\right) a_k}{2(k+1)(2k-1)}, b_{k+1}=\frac{\left(k^2+k-\frac{11}{4}\right) b_k}{2\left(k+\frac{5}{2}\right)(2k+2)}\right]$$

1.427.3 Maple trace

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Methods for second order ODEs:
```

1.427.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 37

```
dsolve((-x^2+4)*diff(diff(y(x),x),x)+x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y=\left(x^2-4\right)^{3/4}\left(\text{LegendreP}\left(-\frac{1}{2}+\sqrt{3}, \frac{3}{2}, \frac{x}{2}\right) c_1+\text{LegendreQ}\left(-\frac{1}{2}+\sqrt{3}, \frac{3}{2}, \frac{x}{2}\right) c_2\right)$$

1.427.5 Mathematica DSolve solution

Solving time : 0.076 (sec)
Leaf size : 58

```
DSolve[{(4-x^2)*D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\left(x^2-4\right)^{3/4}\left(c_1 P_{-\frac{1}{2}+\sqrt{3}}^{\frac{3}{2}}\left(\frac{x}{2}\right)+c_2 Q_{-\frac{1}{2}+\sqrt{3}}^{\frac{3}{2}}\left(\frac{x}{2}\right)\right)$$

1.428 problem 440

Internal problem ID [8566]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 440
Date solved : Monday, October 21, 2024 at 05:09:48 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$4x^2y'' - 4xy' + \left(-16x^2+3\right)y = 0$$

1.428.1 Solved as second order ode using Kovacic algorithm

Time used: 0.113 (sec)

Writing the ode as

$$4x^2y'' - 4xy' + \left(-16x^2+3\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 810: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x)}}{\left(y_1\right)^2}\, dx \\ &= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,\mathrm{e}^{-2x}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.428.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) - 4xy' + \left(-16x^2+3\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(16x^2-3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'}{x} - \frac{(16x^2-3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4xy' + \left(-16x^2+3\right)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1 + 2r)\,(-3 + 2r)\, x^r + a_1(1 + 2r)\,(-1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r - 1)\,(2k + 2r - 3) - 1$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1 + 2r)\,(-3 + 2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$
- Each term must be 0
  $a_1(1 + 2r)\,(-1 + 2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $4\left(k + r - \frac{3}{2}\right)\left(k + r - \frac{1}{2}\right) a_k - 16a_{k-2} = 0$
- Shift index using $k- > k + 2$
  $4\left(k + \frac{1}{2} + r\right)\left(k + \frac{3}{2} + r\right) a_{k+2} - 16a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{16a_k}{(2k+1+2r)(2k+3+2r)}$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}$
- Solution for $r = \frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=\frac{16a_k}{(2k+2)(2k+4)}, a_1=0\right]$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+2}=\frac{16a_k}{(2k+4)(2k+6)}$$

- Solution for $r=\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2}=\frac{16a_k}{(2k+4)(2k+6)}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2}=\frac{16a_k}{(2k+2)(2k+4)}, a_1=0, b_{k+2}=\frac{16b_k}{(2k+4)(2k+6)}, b_1=0\right]$$

1.428.3 Maple trace

```
Methods for second order ODEs:
```

1.428.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 21

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(-16*x^2+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\sqrt{x}\,(c_1\sinh(2x)+c_2\cosh(2x))$$

1.428.5 Mathematica DSolve solution

Solving time : `0.054` (sec)
Leaf size : 32

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(3-16*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{1}{4}e^{-2x}\sqrt{x}\left(c_2e^{4x}+4c_1\right)$$

1.429 problem 441

Internal problem ID [8567]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 441
Date solved : `Monday, October 21, 2024 at 05:09:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\,y''-xy'+y=0$$

1.429.1 Solved as second order ode using Kovacic algorithm

Time used: 0.250 (sec)

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 812: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{x\, e^{x+\ln(x-1)} e^{-2x}}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(e^x) + c_2\left(e^x\left(-\frac{x\, e^{x+\ln(x-1)} e^{-2x}}{x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.429.2 Maple step by step solution

Let's solve

$(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$
- $(x-1)\cdot P_2(x)$ is analytic at $x=1$
  $$\left((x-1)\cdot P_2(x)\right)\bigg|_{x=1} = -1$$
- $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$
  $$\left((x-1)^2\cdot P_3(x)\right)\bigg|_{x=1} = 0$$
- $x=1$is a regular singular point
  Check to see if $x_0=1$ is a regular singular point
  $x_0=1$

- Multiply by denominators
  $$(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$$
- Change variables using $x=u+1$ so that the regular singular point is at $u=0$
  $$u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$
- Shift index using $k->k+1$
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

  Rewrite ODE with series expansions
  $$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

$a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

1.429.3 Maple trace

```
Methods for second order ODEs:
```

1.429.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2 \,\mathrm{e}^{x}$$

1.429.5 Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 17

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2 x$$

1.430 problem 442

Internal problem ID [8568]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 442
Date solved : `Monday, October 21, 2024 at 05:09:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0$$

1.430.1 Solved as second order ode using Kovacic algorithm

Time used: 0.152 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 814: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))\end{aligned}$$

Will add steps showing solving for IC soon.

1.430.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term must be 0

  $a_1 r(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) + a_{k-2} = 0$
- Shift index using $k- > k+2$

  $a_{k+2}(k+1+r)\,(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$

$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

1.430.3 Maple trace

```
Methods for second order ODEs:
```

1.430.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+(x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x(c_1 \sin(x) + c_2 \cos(x))$$

1.430.5 Mathematica DSolve solution

Solving time : `0.042` (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

1.431 problem 444

Internal problem ID [8569]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 444
Date solved : `Monday, October 21, 2024 at 05:09:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^{2}-2x\right)y''+\left(-x^{2}+2\right)y'+\left(2x-2\right)y=0$$

1.431.1 Solved as second order ode using Kovacic algorithm

Time used: 0.325 (sec)

Writing the ode as

$$\left(x^{2}-2x\right)y''+\left(-x^{2}+2\right)y'+\left(2x-2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}-2x \\ B &= -x^{2}+2 \\ C &= 2x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4(x^2 - 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 8x^3 + 24x^2 - 24x + 12$$
$$t = 4(x^2 - 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4(x^2 - 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 816: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{4x} + \frac{3}{4(x-2)^2} - \frac{1}{4(x-2)} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{2}{x^3} + \frac{11}{x^4} + \frac{42}{x^5} + \frac{132}{x^6} + \frac{348}{x^7} + \frac{711}{x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \\
&= Q + \frac{R}{4x^4 - 16x^3 + 16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}\right) \\
&= \frac{1}{4} + \frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to

determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{+} = -1$ then

$$\begin{aligned} d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + (+)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{2x} - \frac{1}{2\,(x-2)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2x} - \frac{1}{2\,(x-2)} + \frac{1}{2} \\ &= -\frac{1}{2x} - \frac{1}{2x-4} + \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{1}{2\,(x-2)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2x^2} + \frac{1}{2\,(x-2)^2}\right) + \left(-\frac{1}{2x} - \frac{1}{2\,(x-2)} + \frac{1}{2}\right)^2 - \left(\frac{x^4 - 8x^3 +}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-2}\,\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+2}{x^2-2x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-2)}{2} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{x}}{\sqrt{x\,(x-2)}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^2-2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{\mathrm{e}^{x+\ln(x-2)+\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{x\,\mathrm{e}^{x+\ln(x-2)+\ln(x)}\mathrm{e}^{-2x}}{x-2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{x}}{\sqrt{x\,(x-2)}}\right) + c_2 \left(\frac{\sqrt{x-2}\,\sqrt{x}\,\mathrm{e}^{x}}{\sqrt{x\,(x-2)}} \left(-\frac{x\,\mathrm{e}^{x+\ln(x-2)+\ln(x)}\mathrm{e}^{-2x}}{x-2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.431.2 Maple step by step solution

Let's solve

$(x^2-2x)\left(\frac{d}{dx}y'\right)+(-x^2+2)\,y'+(2x-2)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{2(x-1)y}{x(x-2)}+\frac{(x^2-2)y'}{x(x-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{(x^2-2)y'}{x(x-2)}+\frac{2(x-1)y}{x(x-2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x^2-2}{x(x-2)},P_3(x)=\frac{2(x-1)}{x(x-2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x-2)\left(\frac{d}{dx}y'\right)+(-x^2+2)\,y'+(2x-2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx} y'\right)$ to series expansion for $m = 1..2$

  $$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

  $$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0 r(-2+r)\, x^{-1+r} + \left(-2a_1(1+r)(-1+r) + a_0(1+r)(-2+r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(-2a_{k+1}(k+r+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term must be 0

  $-2a_1(1+r)(-1+r) + a_0(1+r)(-2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-2) - 2k^2 a_{k+1} + (-4r a_{k+1} - a_{k-1})\, k - 2r^2 a_{k+1} - a_{k-1} r + 3a_{k-1} + 2a_{k+1} =$

- Shift index using $k- > k+1$

  $a_{k+1}(k+2+r)(k+r-1) - 2(k+1)^2 a_{k+2} + (-4r a_{k+2} - a_k)(k+1) - 2r^2 a_{k+2} - r a_k + 3a_k +$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{k^2 a_{k+1} + 2kr a_{k+1} + r^2 a_{k+1} - k a_k + k a_{k+1} - r a_k + r a_{k+1} + 2a_k - 2a_{k+1}}{2(k^2+2kr+r^2+2k+2r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{k^2 a_{k+1} - k a_k + k a_{k+1} + 2a_k - 2a_{k+1}}{2(k^2+2k)}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{k^2 a_{k+1} - k a_k + k a_{k+1} + 2a_k - 2a_{k+1}}{2(k^2+2k)}$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{k^2 a_{k+1} - k a_k + 5k a_{k+1} + 4a_{k+1}}{2(k^2+6k+8)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{k^2 a_{k+1} - k a_k + 5k a_{k+1} + 4a_{k+1}}{2(k^2+6k+8)}, -6a_1 = 0\right]$$

1.431.3 Maple trace

```
Methods for second order ODEs:
```

1.431.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve((x^2-2*x)*diff(diff(y(x),x),x)+(-x^2+2)*diff(y(x),x)+(2*x-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1\, x^2 + c_2\, e^x$$

1.431.5 Mathematica DSolve solution

Solving time : 0.071 (sec)
Leaf size : 18

```
DSolve[{(x^2-2*x)*D[y[x],{x,2}]+(2-x^2)*D[y[x],x]+(2*x-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x^2 + c_1 e^x$$

1.432 problem 445

Internal problem ID [8570]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 445
Date solved : `Monday, October 21, 2024 at 05:09:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2x+1)\,y''-2y'-(2x+3)\,y=0$$

1.432.1 Solved as second order ode using Kovacic algorithm

Time used: 0.241 (sec)

Writing the ode as

$$(2x+1)\,y''-2y'+(-2x-3)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x+1 \\ B &= -2 \\ C &= -2x-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 8x + 6}{(2x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 8x + 6 \\ t &= (2x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 8x + 6}{(2x+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 818: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x+1)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x+\frac{1}{2}\right)^2} + \frac{1}{x+\frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 + 8x + 6}{(2x + 1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} + (-)(1) \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} - 1 \\
&= -\frac{2(x+1)}{2x+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2+8x+6}{(2x+1)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right) dx} \\
&= \frac{\mathrm{e}^{-x}}{\sqrt{2x+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{2x+1} \, dx} \\
&= z_1 e^{\frac{\ln(2x+1)}{2}} \\
&= z_1\left(\sqrt{2x+1}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{2x+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(2x+1)}}{(y_1)^2}\, dx \\ &= y_1\left(x\,\mathrm{e}^{2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(x\,\mathrm{e}^{2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.432.2 Maple step by step solution

Let's solve

$(2x+1)\left(\frac{d}{dx}y'\right) - 2y' - (2x+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(2x+3)y}{2x+1} + \frac{2y'}{2x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y'}{2x+1} - \frac{(2x+3)y}{2x+1} = 0$

□ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2}{2x+1}, P_3(x) = -\frac{2x+3}{2x+1}\right]$

- $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{1}{2}}=-1$
- $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\Big|_{x=-\frac{1}{2}}=0$
- $x=-\frac{1}{2}$is a regular singular point

  Check to see if $x_0=-\frac{1}{2}$ is a regular singular point

  $x_0=-\frac{1}{2}$

- Multiply by denominators

  $(2x+1)\left(\frac{d}{dx}y'\right)-2y'+(-2x-3)\,y=0$
- Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

  $2u\left(\frac{d}{du}\frac{d}{du}y(u)\right)-2\frac{d}{du}y(u)+(-2u-2)\,y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$
- Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$
- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1}$
- Shift index using $k->k+1$

  $\frac{d}{du}y(u)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,u^{k+r}$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$
- Shift index using $k->k+1$

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$2a_0 r(-2+r)\, u^{-1+r} + (2a_1(1+r)(-1+r) - 2a_0)\, u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+r)(k+r-1) - 2a_k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term must be 0

  $2a_1(1+r)(-1+r) - 2a_0 = 0$

- Each term in the series must be 0, giving the recursion relation

  $2a_{k+1}(k+1+r)(k+r-1) - 2a_k - 2a_{k-1} = 0$

- Shift index using $k- >k+1$

  $2a_{k+2}(k+2+r)(k+r) - 2a_{k+1} - 2a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}$

- Solution for $r = 2$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$$

- Revert the change of variables $u = x + \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$$

1.432.3 Maple trace

```
Methods for second order ODEs:
```

1.432.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 16

```
dsolve((2*x+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)-(2*x+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \,\mathrm{e}^{-x} + c_2 x \,\mathrm{e}^{x}$$

1.432.5 Mathematica DSolve solution

Solving time : 0.08 (sec)
Leaf size : 29

```
DSolve[{(2*x+1)*D[y[x],{x,2}]-2*D[y[x],x]-(2*x+3)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x-\frac{1}{2}}\left(c_2 e^{2x+1} x + c_1\right)$$

1.433 problem 446

Internal problem ID [8571]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 446
Date solved : `Monday, October 21, 2024 at 05:09:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + \left(-8x^2 + 4x\right) y' + \left(4x^2 - 4x - 1\right) y = 0$$

1.433.1 Solved as second order ode using Kovacic algorithm

Time used: 0.113 (sec)

Writing the ode as

$$4x^2y'' + \left(-8x^2 + 4x\right) y' + \left(4x^2 - 4x - 1\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2 + 4x \\ C &= 4x^2 - 4x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 820: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\,dx} \\ &= z_1 e^{x-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{\sqrt{x}}\right) + c_2 \left(\frac{e^x}{\sqrt{x}}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.433.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-4x-1)y}{4x^2} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(2x-1)y'}{x} + \frac{(4x^2-4x-1)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2-4x-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4x(2x-1)\, y' + (4x^2 - 4x - 1)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - Shift index using $k- >k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$
  - Shift index using $k- >k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)(-1+2r)\, x^r + \left(a_1(3+2r)(1+2r) - 4a_0(1+2r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)(2k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)(-1+2r) = 0$$
- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$
- Each term must be 0

  $$a_1(3+2r)(1+2r) - 4a_0(1+2r) = 0$$
- Solve for the dependent coefficient(s)

  $$a_1 = \frac{4a_0}{3+2r}$$
- Each term in the series must be 0, giving the recursion relation

  $$a_k(4k^2 + 8kr + 4r^2 - 1) + (-8k - 8r + 4)\, a_{k-1} + 4a_{k-2} = 0$$
- Shift index using $k- >k+2$

  $$a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + (-8k - 12 - 8r)\, a_{k+1} + 4a_k = 0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{4(2ka_{k+1} + 2ra_{k+1} - a_k + 3a_{k+1})}{4k^2 + 8kr + 4r^2 + 16k + 16r + 15}$$

- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}$
- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}, a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1=2a_0\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=\frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=\frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}, a_1=a_0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1=2a_0, b_{k+2}=\frac{4(2kb_{k+1}-b_k+4b_{k+1}}{4k^2+20k+24}\right.$

1.433.3 Maple trace

```
Methods for second order ODEs:
```

1.433.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 15

```
dsolve(4*x^2*diff(diff(y(x),x),x)+(-8*x^2+4*x)*diff(y(x),x)+(4*x^2-4*x-1)*y(x) = 0,
    y(x),singsol=all)
```

$$y=\frac{e^{x}(c_2 x+c_1)}{\sqrt{x}}$$

1.433.5 Mathematica DSolve solution

Solving time : `0.045` (sec)
Leaf size : 21

```
DSolve[{4*x^2*D[y[x],{x,2}]+(4*x-8*x^2)*D[y[x],x]+(4*x^2-4*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^x(c_2x + c_1)}{\sqrt{x}}$$

1.434 problem 447

Internal problem ID [8572]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 447
Date solved : `Monday, October 21, 2024 at 05:09:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + (4x^2 + 2)\, y = 0$$

1.434.1 Solved as second order ode using Kovacic algorithm

Time used: 0.087 (sec)

Writing the ode as

$$y'' + 4xy' + (4x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 822: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\ &= z_1 e^{-x^2} \\ &= z_1 \left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.434.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 4xy' + (4x^2 + 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k + 2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $\left((k + 2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k + 2) + 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2 = -a_0, a_3 = -a_1\right]$

1.434.3 Maple trace

```
Methods for second order ODEs:
```

1.434.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(4*x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-x^2}(c_2 x + c_1)$$

1.434.5 Mathematica DSolve solution

Solving time : `0.037` (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x^2}(c_2 x + c_1)$$

1.435 problem 448

Internal problem ID [8573]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 448
Date solved : `Monday, October 21, 2024 at 05:09:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 2x(x-1)\,y' + \left(x^2-2x+2\right)y = 0$$

1.435.1 Solved as second order ode using Kovacic algorithm

Time used: 0.098 (sec)

Writing the ode as

$$x^2y'' + \left(2x^2-2x\right)y' + \left(x^2-2x+2\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2-2x \\ C &= x^2-2x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 824: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2-2x}{x^2}\,dx} \\
&= z_1 e^{-x+\ln(x)} \\
&= z_1\left(x\,\mathrm{e}^{-x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x+2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-x}\right) + c_2 \left(x\,\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.435.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx}y'\right) + 2x(x-1)\,y' + (x^2-2x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2-2x+2)y}{x^2} - \frac{2(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2(x-1)y'}{x} + \frac{(x^2-2x+2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2(x-1)}{x}, P_3(x) = \frac{x^2-2x+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 \left(\frac{d}{dx}y'\right) + 2x(x-1)\,y' + (x^2-2x+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$
  - Shift index using $k- >k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)\,(-2+r)\, x^r + (a_1 r(-1+r) + 2a_0(-1+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)\,(k+r-2) +\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+r)\,(-2+r) = 0$$
- Values of r that satisfy the indicial equation

  $$r \in \{1, 2\}$$
- Each term must be 0

  $$a_1 r(-1+r) + 2a_0(-1+r) = 0$$
- Solve for the dependent coefficient(s)

  $$a_1 = -\frac{2a_0}{r}$$
- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r-1)\,(k+r-2) + 2a_{k-1}k + 2a_{k-1}r + a_{k-2} - 4a_{k-1} = 0$$
- Shift index using $k- >k + 2$

  $$a_{k+2}(k+1+r)\,(k+r) + 2a_{k+1}(k+2) + 2a_{k+1}r + a_k - 4a_{k+1} = 0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{2ka_{k+1} + 2a_{k+1}r + a_k}{(k+1+r)(k+r)}$$
- Recursion relation for $r = 1$

  $$a_{k+2} = -\frac{2ka_{k+1} + a_k + 2a_{k+1}}{(k+2)(k+1)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2k a_{k+1} + a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = -2a_0\right]$$

- Recursion relation for $r = 2$

$$a_{k+2} = -\frac{2k a_{k+1} + a_k + 4a_{k+1}}{(k+3)(k+2)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{2k a_{k+1} + a_k + 4a_{k+1}}{(k+3)(k+2)}, a_1 = -a_0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{2k a_{k+1} + a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = -2a_0, b_{k+2} = -\frac{2k b_{k+1} + b_k + 4b_{k-}}{(k+3)(k+2)}\right.$$

1.435.3 Maple trace

```
Methods for second order ODEs:
```

1.435.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)+2*x*(x-1)*diff(y(x),x)+(x^2-2*x+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{-x} x (c_2 x + c_1)$$

1.435.5 Mathematica DSolve solution

Solving time : `0.048` (sec)
Leaf size : 19

```
DSolve[{x^2*D[y[x],{x,2}]+2*x*(x-1)*D[y[x],x]+(x^2-2*x+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x} x (c_2 x + c_1)$$

1.436 problem 449

Internal problem ID [8574]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 449
Date solved : `Monday, October 21, 2024 at 05:09:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(2x-1)\,y' + \left(x^2-x-1\right)y = 0$$

1.436.1 Solved as second order ode using Kovacic algorithm

Time used: 0.176 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^2+x\right)y' + \left(x^2-x-1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2+x \\ C &= x^2-x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 826: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx}\\ &= e^{\int -\frac{1}{2x} dx}\\ &= \frac{1}{\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}\\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x^2+x}{x^2} \, dx}\\ &= z_1 e^{x - \frac{\ln(x)}{2}}\\ &= z_1 \left(\frac{e^x}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^3 e^{2x-\ln(x)} e^{-2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{x}\right) + c_2 \left(\frac{e^x}{x}\left(\frac{x^3 e^{2x-\ln(x)} e^{-2x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.436.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x(2x-1)\,y' + \left(x^2-x-1\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2-x-1)y}{x^2} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(2x-1)y'}{x} + \frac{(x^2-x-1)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x^2-x-1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -1$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - x(2x-1)\,y' + (x^2 - x - 1)\,y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-1+r)\, x^r + \left(a_1(2+r)\, r - a_0(1+2r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)\,(k+r-1) - a_{k-1}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term must be 0

  $a_1(2+r)\, r - a_0(1+2r) = 0$

- Solve for the dependent coefficient(s)

$a_1 = \frac{a_0(1+2r)}{r(2+r)}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-1) + (-2k-2r+1) a_{k-1} + a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(k+3+r)(k+r+1) + (-2k-3-2r) a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1}+2ra_{k+1}-a_k+3a_{k+1}}{(k+3+r)(k+r+1)}$

- Recursion relation for $r = -1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+a_{k+1}}{(k+2)k}$

- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+a_{k+1}}{(k+2)k}$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+5a_{k+1}}{(k+4)(k+2)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{2ka_{k+1}-a_k+5a_{k+1}}{(k+4)(k+2)}, a_1 = a_0\right]$

1.436.3 Maple trace

```
Methods for second order ODEs:
```

1.436.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(2*x-1)*diff(y(x),x)+(x^2-x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{x}(c_2 x^2 + c_1)}{x}$$

1.436.5 Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 23

```
DSolve[{x^2*D[y[x],{x,2}]-x*(2*x-1)*D[y[x],x]+(x^2-x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x}\left(\frac{c_1}{x} + \frac{c_2 x}{2}\right)$$

1.437 problem 450

Internal problem ID [8575]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 450
Date solved : `Monday, October 21, 2024 at 05:09:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-2x)\,y''+2y'+(2x-3)\,y=0$$

1.437.1 Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$(1-2x)\,y''+2y'+(2x-3)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1-2x\\ B&=2\\ C&=2x-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 8x + 6}{(-1 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 - 8x + 6$$
$$t = (-1 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 8x + 6}{(-1 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 828: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (-1 + 2x)^2$. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x - \frac{1}{2}\right)^2} - \frac{1}{x - \frac{1}{2}}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} + \frac{1}{4x^3} + \frac{11}{32x^4} + \frac{21}{64x^5} + \frac{15}{64x^6} + \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 8x + 6}{4x^2 - 4x + 1} \\ &= Q + \frac{R}{4x^2 - 4x + 1} \\ &= (1) + \left(\frac{-4x + 5}{4x^2 - 4x + 1}\right) \\ &= 1 + \frac{-4x + 5}{4x^2 - 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 - 8x + 6}{(-1 + 2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{2\left(x - \frac{1}{2}\right)} + (1) \\ &= -\frac{1}{2\left(x - \frac{1}{2}\right)} + 1 \\ &= \frac{-2 + 2x}{-1 + 2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{2\left(x - \frac{1}{2}\right)} + 1\right)(0) + \left(\left(\frac{1}{2\left(x - \frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x - \frac{1}{2}\right)} + 1\right)^2 - \left(\frac{4x^2 - 8x + 6}{\left(-1 + 2x\right)^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2\left(x - \frac{1}{2}\right)} + 1\right) dx} \\ &= \frac{\mathrm{e}^{x}}{\sqrt{-1 + 2x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2}{1 - 2x} \, dx} \\ &= z_1 e^{\frac{\ln(1 - 2x)}{2}} \\ &= z_1\left(\sqrt{1 - 2x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{1-2x}\,\mathrm{e}^{x}}{\sqrt{-1+2x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1-2x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(1-2x)}}{(y_1)^2}\,dx \\ &= y_1\left(-x\,\mathrm{e}^{-2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\sqrt{1-2x}\,\mathrm{e}^{x}}{\sqrt{-1+2x}}\right) + c_2\left(\frac{\sqrt{1-2x}\,\mathrm{e}^{x}}{\sqrt{-1+2x}}\left(-x\,\mathrm{e}^{-2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.437.2 Maple step by step solution

Let's solve
$(1-2x)\left(\frac{d}{dx}y'\right) + 2y' + (2x-3)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = \frac{(2x-3)y}{-1+2x} + \frac{2y'}{-1+2x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{2y'}{-1+2x} - \frac{(2x-3)y}{-1+2x} = 0$

□ Check to see if $x_0 = \frac{1}{2}$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{2}{-1+2x}, P_3(x) = -\frac{2x-3}{-1+2x}\right]$
- $\left(x - \frac{1}{2}\right) \cdot P_2(x)$ is analytic at $x = \frac{1}{2}$
  $\left(\left(x - \frac{1}{2}\right) \cdot P_2(x)\right)\bigg|_{x=\frac{1}{2}} = -1$
- $\left(x - \frac{1}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = \frac{1}{2}$
  $\left(\left(x - \frac{1}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=\frac{1}{2}} = 0$
- $x = \frac{1}{2}$is a regular singular point
  Check to see if $x_0 = \frac{1}{2}$ is a regular singular point
  $x_0 = \frac{1}{2}$

- Multiply by denominators
  $(-1 + 2x)\left(\frac{d}{dx}y'\right) - 2y' + (-2x + 3)\, y = 0$
- Change variables using $x = u + \frac{1}{2}$ so that the regular singular point is at $u = 0$
  $2u\left(\frac{d}{du}\frac{d}{du}y(u)\right) - 2\frac{d}{du}y(u) + (-2u + 2)\, y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions
- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$
  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
- Shift index using $k- >k - m$
  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
- Convert $\frac{d}{du}y(u)$ to series expansion
  $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k (k + r)\, u^{k+r-1}$
- Shift index using $k- >k + 1$
  $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\, u^{k+r}$
- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k + r)(k + r - 1)\, u^{k+r-1}$

- Shift index using $k->k+1$

  $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum\limits_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(-2+r)\,u^{-1+r}+(2a_1(1+r)\,(-1+r)+2a_0)\,u^r+\left(\sum\limits_{k=1}^{\infty}(2a_{k+1}(k+1+r)\,(k+r-1)+2a_k$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-2+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$
- Each term must be 0

  $2a_1(1+r)\,(-1+r)+2a_0=0$
- Each term in the series must be 0, giving the recursion relation

  $2a_{k+1}(k+1+r)\,(k+r-1)+2a_k-2a_{k-1}=0$
- Shift index using $k->k+1$

  $2a_{k+2}(k+2+r)\,(k+r)+2a_{k+1}-2a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{-a_{k+1}+a_k}{(k+2+r)(k+r)}$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{-a_{k+1}+a_k}{(k+2)k}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$

  $a_{k+2}=\frac{-a_{k+1}+a_k}{(k+2)k}$
- Recursion relation for $r=2$

  $a_{k+2}=\frac{-a_{k+1}+a_k}{(k+4)(k+2)}$
- Solution for $r=2$

  $\left[y(u)=\sum\limits_{k=0}^{\infty}a_ku^{k+2},a_{k+2}=\frac{-a_{k+1}+a_k}{(k+4)(k+2)},6a_1+2a_0=0\right]$
- Revert the change of variables $u=x-\frac{1}{2}$

  $\left[y=\sum\limits_{k=0}^{\infty}a_k\left(x-\frac{1}{2}\right)^{k+2},a_{k+2}=\frac{-a_{k+1}+a_k}{(k+4)(k+2)},6a_1+2a_0=0\right]$

1.437.3 Maple trace

```
Methods for second order ODEs:
```

1.437.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 16

```
dsolve((1-2*x)*diff(diff(y(x),x),x)+2*diff(y(x),x)+(2*x-3)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \,\mathrm{e}^{x} + c_2 \,\mathrm{e}^{-x} x$$

1.437.5 Mathematica DSolve solution

Solving time : `0.267` (sec)
Leaf size : 48

```
DSolve[{(1-2*x)*D[y[x],{x,2}]+2*D[y[x],x]+(2*x-3)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x-\frac{1}{2}}\sqrt{1-2x}(c_1 e^{2x} - ec_2 x)}{\sqrt{2x-1}}$$

1.438 problem 451

Internal problem ID [8576]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 451
Date solved : `Monday, October 21, 2024 at 05:09:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2xy'' + (4x+1)\,y' + (2x+1)\,y = 0$$

1.438.1 Solved as second order ode using Kovacic algorithm

Time used: 0.168 (sec)

Writing the ode as

$$2xy'' + (4x+1)\,y' + (2x+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= 4x+1 \\ C &= 2x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 830: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{4x} + (-)(0) \\
&= \frac{1}{4x} \\
&= \frac{1}{4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x}\right)(0) + \left(\left(-\frac{1}{4x^2}\right) + \left(\frac{1}{4x}\right)^2 - \left(-\frac{3}{16x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= e^{\int \frac{1}{4x} dx} \\ &= x^{1/4} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x+1}{2x}\, dx} \\ &= z_1 e^{-x-\frac{\ln(x)}{4}} \\ &= z_1 \left(\frac{e^{-x}}{x^{1/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x+1}{2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x-\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(2x\,\mathrm{e}^{-2x-\frac{\ln(x)}{2}}\,\mathrm{e}^{2x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(2x\,\mathrm{e}^{-2x-\frac{\ln(x)}{2}}\,\mathrm{e}^{2x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.438.2 Maple step by step solution

Let's solve

$2x\left(\frac{d}{dx}y'\right) + (4x+1)\,y' + (2x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(2x+1)y}{2x} - \frac{(4x+1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(4x+1)y'}{2x} + \frac{(2x+1)y}{2x} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{4x+1}{2x}, P_3(x) = \frac{2x+1}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x\left(\frac{d}{dx}y'\right) + (4x+1)\,y' + (2x+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k + 1$

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+2r)\, x^{-1+r} + \left(a_1(1+r)\,(1+2r) + a_0(1+4r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+1+2r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(1+r)\,(1+2r) + a_0(1+4r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+\frac{1}{2}+r\right)(k+1+r)\,a_{k+1}+4a_k k+4a_k r+a_k+2a_{k-1}=0$
- Shift index using $k->k+1$

  $2\left(k+\frac{3}{2}+r\right)(k+2+r)\,a_{k+2}+4a_{k+1}(k+1)+4ra_{k+1}+a_{k+1}+2a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4ka_{k+1}+4ra_{k+1}+2a_k+5a_{k+1}}{(2k+3+2r)(k+2+r)}$
- Recursion relation for $r=0$

  $a_{k+2}=-\frac{4ka_{k+1}+2a_k+5a_{k+1}}{(2k+3)(k+2)}$
- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+2}=-\frac{4ka_{k+1}+2a_k+5a_{k+1}}{(2k+3)(k+2)}, a_1+a_0=0\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-\frac{4ka_{k+1}+2a_k+7a_{k+1}}{(2k+4)\left(k+\frac{5}{2}\right)}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{4ka_{k+1}+2a_k+7a_{k+1}}{(2k+4)\left(k+\frac{5}{2}\right)}, 3a_1+3a_0=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^k\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4ka_{k+1}+2a_k+5a_{k+1}}{(2k+3)(k+2)}, a_1+a_0=0, b_{k+2}=-\frac{4kb_{k+1}+2b_k+7b_{k+1}}{(2k+4)\left(k+\frac{5}{2}\right)}\right.$

1.438.3 Maple trace

```
Methods for second order ODEs:
```

1.438.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve(2*x*diff(diff(y(x),x),x)+(4*x+1)*diff(y(x),x)+(2*x+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y=e^{-x}\left(c_1+\sqrt{x}\,c_2\right)$$

1.438.5 Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 23

```
DSolve[{2*x*D[y[x],{x,2}]+(4*x+1)*D[y[x],x]+(2*x+1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x}\left(2c_2\sqrt{x} + c_1\right)$$

1.439 problem 452

Internal problem ID [8577]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 452
Date solved : `Monday, October 21, 2024 at 05:09:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (2x + 1)\, y' + (x + 1)\, y = 0$$

1.439.1 Solved as second order ode using Kovacic algorithm

Time used: 0.176 (sec)

Writing the ode as

$$xy'' + (-2x - 1)\, y' + (x + 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x - 1 \\ C &= x + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 832: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0 \\ 0 = 0\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x-1}{x} \, dx} \\ &= z_1 e^{x + \frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\, e^x\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-1}{x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{2x+\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x \, e^{2x+\ln(x)} e^{-2x}}{2} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (e^{x}) + c_2 \left(e^{x} \left(\frac{x \, e^{2x+\ln(x)} e^{-2x}}{2} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.439.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) - (2x+1)\, y' + (x+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x+1)y}{x} + \frac{(2x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(2x+1)y'}{x} + \frac{(x+1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x+1}{x}, P_3(x) = \frac{x+1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$
- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + (-2x-1)\,y' + (x+1)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
  - Shift index using $k- >k+1$

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\,x^{-1+r} + \left(a_1(1+r)\,(-1+r) - a_0(-1+2r)\right)x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1)\right.\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term must be 0
  $a_1(1+r)(-1+r) - a_0(-1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k+r-1) + a_k(-2k-2r+1) + a_{k-1} = 0$
- Shift index using $k->k+1$
  $a_{k+2}(k+2+r)(k+r) + a_{k+1}(-2k-1-2r) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2ka_{k+1}+2ra_{k+1}-a_k+a_{k+1}}{(k+2+r)(k+r)}$
- Recursion relation for $r=0$
  $a_{k+2} = \frac{2ka_{k+1}-a_k+a_{k+1}}{(k+2)k}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$
  $a_{k+2} = \frac{2ka_{k+1}-a_k+a_{k+1}}{(k+2)k}$
- Recursion relation for $r=2$
  $a_{k+2} = \frac{2ka_{k+1}-a_k+5a_{k+1}}{(k+4)(k+2)}$
- Solution for $r=2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2ka_{k+1}-a_k+5a_{k+1}}{(k+4)(k+2)}, 3a_1 - 3a_0 = 0\right]$

1.439.3 Maple trace

```
Methods for second order ODEs:
```

1.439.4 Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 14

```
dsolve(x*diff(diff(y(x),x),x)-(2*x+1)*diff(y(x),x)+(x+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^x (c_2 x^2 + c_1)$$

1.439.5 Mathematica DSolve solution

Solving time : 0.038 (sec)
Leaf size : 23

```
DSolve[{x*D[y[x],{x,2}]-(2*x+1)*D[y[x],x]+(x+1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^x\left(c_2x^2 + 2c_1\right)$$

1.440 problem 453

Internal problem ID [8578]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 453
Date solved : `Monday, October 21, 2024 at 05:09:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4x(x+1)\,y' + (2x+3)\,y = 0$$

1.440.1 Solved as second order ode using Kovacic algorithm

Time used: 0.131 (sec)

Writing the ode as

$$4x^2y'' + \left(-4x^2 - 4x\right)y' + (2x+3)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 - 4x \\ C &= 2x + 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 834: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x^2-4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\,\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{x+\ln(x)}}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\left(\frac{e^{x+\ln(x)}}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.440.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) - 4x(x+1)\,y' + (2x+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(2x+3)y}{4x^2} + \frac{(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x+1)y'}{x} + \frac{(2x+3)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x+1}{x}, P_3(x) = \frac{2x+3}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4x(x+1)\,y' + (2x+3)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)(2k+2r-3) - 2a_{k-1}(2k+2r-3)\right) x^{k+r}\right) =$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)(-3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $4\left(\left(k+r-\frac{1}{2}\right) a_k - a_{k-1}\right)\left(k+r-\frac{3}{2}\right) = 0$
- Shift index using $k->k+1$

  $4\left(\left(k+\frac{1}{2}+r\right) a_{k+1} - a_k\right)\left(k+r-\frac{1}{2}\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k}{2k+1+2r}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{2a_k}{2k+2}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k}{2k+2}\right]$
- Recursion relation for $r = \frac{3}{2}$

$a_{k+1} = \frac{2a_k}{2k+4}$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{2a_k}{2k+2}, b_{k+1} = \frac{2b_k}{2k+4}\right]$$

1.440.3 Maple trace

```
Methods for second order ODEs:
```

1.440.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x*(x+1)*diff(y(x),x)+(2*x+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\,(c_2\,e^x + c_1)$$

1.440.5 Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 20

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*(x+1)*D[y[x],x]+(2*x+3)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{x}(c_2 e^x + c_1)$$

1.441 problem 454

Internal problem ID [8579]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 454
Date solved : `Monday, October 21, 2024 at 05:09:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (2 - 2x)\, y' + (x - 2)\, y = 0$$

1.441.1 Solved as second order ode using Kovacic algorithm

Time used: 0.095 (sec)

Writing the ode as

$$xy'' + (2 - 2x)\, y' + (x - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 - 2x \\ C &= x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 836: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2-2x}{x}\,dx} \\
&= z_1 e^{x-\ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^{x}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{x}}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2-2x}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{x}\right) + c_2 \left(\frac{e^x}{x}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.441.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + (2-2x)\,y' + (x-2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x-2)y}{x} + \frac{2(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(x-1)y'}{x} + \frac{(x-2)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(x-1)}{x}, P_3(x) = \frac{x-2}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + (2-2x)\,y' + (x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$
  - Shift index using $k- >k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$
  - Convert $x \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

    $$x \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$
  - Shift index using $k- >k + 1$

    $$x \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + (a_1(1+r)\,(2+r) - 2a_0(1+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k+2+r) - 2a_k\right.$$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term must be 0

  $a_1(1+r)\,(2+r) - 2a_0(1+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+2+r) - 2a_k k - 2a_k r - 2a_k + a_{k-1} = 0$
- Shift index using $k- >k + 1$

  $a_{k+2}(k+2+r)\,(k+3+r) - 2a_{k+1}(k+1) - 2r a_{k+1} - 2a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2k a_{k+1} + 2r a_{k+1} - a_k + 4a_{k+1}}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$

$a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}, 2a_1 - 2a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0, b_{k+2} = \frac{2kb_{k+1}-b_k+4b_{k+1}}{(k+2)(k+3)}, 2b_1 - 2b\right.$

1.441.3 Maple trace

```
Methods for second order ODEs:
```

1.441.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(x*diff(diff(y(x),x),x)+(2-2*x)*diff(y(x),x)+(x-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{x}(c_1 x + c_2)}{x}$$

1.441.5 Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 19

```
DSolve[{x*D[y[x],{x,2}]+(2-2*x)*D[y[x],x]+(x-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{x}(c_2 x + c_1)}{x}$$

1.442 problem 455

Internal problem ID [8580]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 455
Date solved : `Monday, October 21, 2024 at 05:09:59 PM`
CAS classification :
`[[_Emden, _Fowler], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']]`

Solve

$$x^2y'' - 2xy' + 2y = 0$$

1.442.1 Solved as second order ode using Kovacic algorithm

Time used: 0.075 (sec)

Writing the ode as

$$x^2y'' - 2xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 838: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2(x(x))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.442.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - 2xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y'}{x} + \frac{2y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2\left(\frac{d}{dx}y'\right) - 2xy' + 2y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y' = \left(\frac{d}{dt}\frac{d}{dt}y(t)\right)t'(x)^2 + \left(\frac{d}{dx}t'(x)\right)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $\frac{d}{dx}y' = \frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2\frac{d}{dt}y(t) + 2y(t) = 0$

- Simplify

$\frac{d}{dt}\frac{d}{dt}y(t) - 3\frac{d}{dt}y(t) + 2y(t) = 0$

- Characteristic polynomial of ODE
  $r^2 - 3r + 2 = 0$
- Factor the characteristic polynomial
  $(r-1)(r-2) = 0$
- Roots of the characteristic polynomial
  $r = (1, 2)$
- 1st solution of the ODE
  $y_1(t) = e^t$
- 2nd solution of the ODE
  $y_2(t) = e^{2t}$
- General solution of the ODE
  $y(t) = C1 y_1(t) + C2 y_2(t)$
- Substitute in solutions
  $y(t) = C1\, e^t + C2\, e^{2t}$
- Change variables back using $t = \ln(x)$
  $y = C2\, x^2 + C1 x$
- Simplify
  $y = x(C2 x + C1)$

1.442.3 Maple trace

```
Methods for second order ODEs:
```

1.442.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 11

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = x(c_1 x + c_2)$$

1.442.5 Mathematica DSolve solution

Solving time : 0.017 (sec)
Leaf size : 14

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x(c_2x + c_1)$$

1.443 problem 456

Internal problem ID [8581]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 456
Date solved : `Monday, October 21, 2024 at 05:10:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (2x+2)\,y' + (x+2)\,y = 0$$

1.443.1 Solved as second order ode using Kovacic algorithm

Time used: 0.158 (sec)

Writing the ode as

$$xy'' + (-2x-2)\,y' + (x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 840: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\,(0) \\ &= -\frac{1}{x} \\ &= -\frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1(x\,\mathrm{e}^x) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x\,\mathrm{e}^{2x+2\ln(x)}\mathrm{e}^{-2x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x\,\mathrm{e}^{2x+2\ln(x)}\mathrm{e}^{-2x}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.443.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) - (2x+2)\,y' + (x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$
- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x+2)y}{x} + \frac{2(x+1)y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(x+1)y'}{x} + \frac{(x+2)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{x+2}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + (-2x-2)\,y' + (x+2)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r) (k+r-1) x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r) (k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r) x^{-1+r} + (a_1(1+r)(-2+r) - 2a_0(-1+r)) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k-2+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term must be 0

$$a_1(1+r)(-2+r) - 2a_0(-1+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k-2+r) - 2a_k k - 2a_k r + 2a_k + a_{k-1} = 0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2+r)(k+r-1) - 2a_{k+1}(k+1) - 2r a_{k+1} + 2a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2k a_{k+1} + 2r a_{k+1} - a_k}{(k+2+r)(k+r-1)}$$

- Recursion relation for $r = 0$

$a_{k+2} = \frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Recursion relation for $r = 3$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}, 4a_1 - 4a_0 = 0\right]$

1.443.3 Maple trace

```
Methods for second order ODEs:
```

1.443.4 Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 14

```
dsolve(x*diff(diff(y(x),x),x)-(2*x+2)*diff(y(x),x)+(x+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^x (c_2 x^3 + c_1)$$

1.443.5 Mathematica DSolve solution

Solving time : `0.037` (sec)
Leaf size : 23

```
DSolve[{x*D[y[x],{x,2}]-(2*x+2)*D[y[x],x]+(x+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3} e^x (c_2 x^3 + 3c_1)$$

1.444 problem 457

Internal problem ID [8582]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 457
Date solved : `Monday, October 21, 2024 at 05:10:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + \left(x^2+2\right) y = 0$$

1.444.1 Solved as second order ode using Kovacic algorithm

Time used: 0.146 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 842: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx}\\ &= z_1 e^{\ln(x)}\\ &= z_1(x)\end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx\\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.444.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)(-2+r) x^r + a_1 r(-1+r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)(k+r-2) + a_{k-2}) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term must be 0

  $a_1 r(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2) + a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $a_{k+2}(k+1+r)(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$

$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

1.444.3 Maple trace

```
Methods for second order ODEs:
```

1.444.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+(x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x(c_1 \sin(x) + c_2 \cos(x))$$

1.444.5 Mathematica DSolve solution

Solving time : 0.041 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

1.445 problem 458

Internal problem ID [8583]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 458
Date solved : `Monday, October 21, 2024 at 05:10:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (4x + 1) y' + (4x + 2) y = 0$$

1.445.1 Solved as second order ode using Kovacic algorithm

Time used: 0.179 (sec)

Writing the ode as

$$xy'' + (-4x - 1) y' + (4x + 2) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -4x - 1 \\ C &= 4x + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 844: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4x-1}{x} \, dx} \\ &= z_1 e^{2x + \frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\, e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x-1}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{4x+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x\, e^{4x+\ln(x)} e^{-4x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{2x}\right) + c_2 \left(e^{2x} \left(\frac{x\, e^{4x+\ln(x)} e^{-4x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.445.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) - (4x+1)\, y' + (4x+2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2(2x+1)y}{x} + \frac{(4x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(4x+1)y'}{x} + \frac{2(2x+1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{4x+1}{x}, P_3(x) = \frac{2(2x+1)}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + (-4x-1)\,y' + (4x+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\,x^{-1+r} + \left(a_1(1+r)\,(-1+r) - 2a_0(-1+2r)\right)x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term must be 0
  $a_1(1+r)(-1+r)-2a_0(-1+2r)=0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k+r-1)+a_k(-4k-4r+2)+4a_{k-1}=0$
- Shift index using $k->k+1$
  $a_{k+2}(k+2+r)(k+r)+a_{k+1}(-4k-2-4r)+4a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{2(2ka_{k+1}+2ra_{k+1}-2a_k+a_{k+1})}{(k+2+r)(k+r)}$
- Recursion relation for $r=0$
  $a_{k+2}=\frac{2(2ka_{k+1}-2a_k+a_{k+1})}{(k+2)k}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$
  $a_{k+2}=\frac{2(2ka_{k+1}-2a_k+a_{k+1})}{(k+2)k}$
- Recursion relation for $r=2$
  $a_{k+2}=\frac{2(2ka_{k+1}-2a_k+5a_{k+1})}{(k+4)(k+2)}$
- Solution for $r=2$
  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+2}, a_{k+2}=\frac{2(2ka_{k+1}-2a_k+5a_{k+1})}{(k+4)(k+2)}, 3a_1-6a_0=0\right]$

1.445.3 Maple trace

```
Methods for second order ODEs:
```

1.445.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 16

```
dsolve(x*diff(diff(y(x),x),x)-(4*x+1)*diff(y(x),x)+(4*x+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y=e^{2x}\left(c_2\,x^2+c_1\right)$$

1.445.5 Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 25

```
DSolve[{x*D[y[x],{x,2}]-(4*x+1)*D[y[x],x]+(4*x+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{2x}\left(c_2x^2 + 2c_1\right)$$

1.446 problem 460

Internal problem ID [8584]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 460
Date solved : `Monday, October 21, 2024 at 05:10:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4xy' + \left(-16x^2+3\right)y = 0$$

1.446.1 Solved as second order ode using Kovacic algorithm

Time used: 0.112 (sec)

Writing the ode as

$$4x^2y'' - 4xy' + \left(-16x^2+3\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 846: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-2x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x}\,\mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{x}\, \mathrm{e}^{-2x}\right) + c_2 \left(\sqrt{x}\, \mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.446.2 Maple step by step solution

Let's solve

$4x^2 \left(\frac{d}{dx} y'\right) - 4xy' + (-16x^2 + 3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{(16x^2 - 3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' - \frac{y'}{x} - \frac{(16x^2 - 3)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2 - 3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 \left(\frac{d}{dx} y'\right) - 4xy' + (-16x^2 + 3)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - ◦ Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - ◦ Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$
  - ◦ Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+2r)\,(-3+2r)\,x^r + a_1(1+2r)\,(-1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-3) - 1$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+2r)\,(-3+2r) = 0$$
- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$$
- Each term must be 0

  $$a_1(1+2r)\,(-1+2r) = 0$$
- Solve for the dependent coefficient(s)

  $$a_1 = 0$$
- Each term in the series must be 0, giving the recursion relation

  $$4\left(k+r-\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right) a_k - 16 a_{k-2} = 0$$
- Shift index using $k- >k + 2$

  $$4\left(k+\frac{3}{2}+r\right)\left(k+\frac{1}{2}+r\right) a_{k+2} - 16 a_k = 0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{16 a_k}{(2k+3+2r)(2k+1+2r)}$$
- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = \frac{16 a_k}{(2k+4)(2k+2)}$$
- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+2} = \frac{16a_k}{(2k+6)(2k+4)}$$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = \frac{16a_k}{(2k+6)(2k+4)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}, a_1 = 0, b_{k+2} = \frac{16b_k}{(2k+6)(2k+4)}, b_1 = 0\right]$$

1.446.3 Maple trace

```
Methods for second order ODEs:
```

1.446.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 21

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(-16*x^2+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\,(c_1 \sinh(2x) + c_2 \cosh(2x))$$

1.446.5 Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 32

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(3-16*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4} e^{-2x} \sqrt{x} \left(c_2 e^{4x} + 4c_1\right)$$

1.447 problem 461

Internal problem ID [8585]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 461
Date solved : `Monday, October 21, 2024 at 05:10:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2x+1)\,xy'' - 2\left(2x^2-1\right)y' - 4(x+1)\,y = 0$$

1.447.1 Solved as second order ode using Kovacic algorithm

Time used: 0.266 (sec)

Writing the ode as

$$\left(2x^2+x\right)y'' + \left(-4x^2+2\right)y' + (-4x-4)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+x \\ B &= -4x^2+2 \\ C &= -4x-4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 8x + 6}{(2x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 + 8x + 6$$
$$t = (2x+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 8x + 6}{(2x+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 848: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x+1)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x+\frac{1}{2}\right)^2} + \frac{1}{x+\frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 + 8x + 6}{(2x + 1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} + (-)\,(1) \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} - 1 \\
&= -\frac{2(x+1)}{2x+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2+8x+6}{(2x+1)^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right) dx} \\
&= \frac{\mathrm{e}^{-x}}{\sqrt{2x+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x^2+2}{2x^2+x} \, dx} \\
&= z_1 e^{x + \frac{\ln(2x+1)}{2} - \ln(x)} \\
&= z_1 \left(\frac{\sqrt{2x+1}\,\mathrm{e}^x}{x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2+2}{2x^2+x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x+\ln(2x+1)-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^3 e^{2x+\ln(2x+1)-2\ln(x)}}{2x+1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^3 e^{2x+\ln(2x+1)-2\ln(x)}}{2x+1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.447.2 Maple step by step solution

Let's solve

$(2x+1)\,x\left(\frac{d}{dx}y'\right) - 2(2x^2-1)\,y' - 4(x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{4(x+1)y}{(2x+1)x} + \frac{2(2x^2-1)y'}{(2x+1)x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(2x^2-1)y'}{(2x+1)x} - \frac{4(x+1)y}{(2x+1)x} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $$\left[P_2(x) = -\frac{2(2x^2-1)}{(2x+1)x}, P_3(x) = -\frac{4(x+1)}{(2x+1)x}\right]$$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- • Multiply by denominators

  $$(2x+1)\, x\left(\frac{d}{dx}y'\right) + (-4x^2+2)\, y' + (-4x-4)\, y = 0$$

- • Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- ○ Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- ○ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..2$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- ○ Shift index using $k- >k + 2 - m$

$$x^{m} \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r)\,x^{-1+r} + \left(a_1(1+r)\,(2+r) + 2a_0(1+r)\,(-2+r)\right)x^{r} + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 0\}$$

- Each term must be 0

  $$a_1(1+r)\,(2+r) + 2a_0(1+r)\,(-2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+r+1)\,(k+2+r) + 2a_k(k+r+1)\,(k+r-2) - 4a_{k-1}(k+r) = 0$$

- Shift index using $k->k+1$

  $$a_{k+2}(k+2+r)\,(k+3+r) + 2a_{k+1}(k+2+r)\,(k+r-1) - 4a_k(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{2\left(k^2 a_{k+1} + 2kr a_{k+1} + r^2 a_{k+1} - 2k a_k + k a_{k+1} - 2r a_k + r a_{k+1} - 2a_k - 2a_{k+1}\right)}{(k+2+r)(k+3+r)}$$

- Recursion relation for $r = -1$

  $$a_{k+2} = -\frac{2\left(k^2 a_{k+1} - 2k a_k - k a_{k+1} - 2a_{k+1}\right)}{(k+1)(k+2)}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2\left(k^2 a_{k+1} - 2k a_k - k a_{k+1} - 2a_{k+1}\right)}{(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{2\left(k^2 a_{k+1} - 2k a_k + k a_{k+1} - 2a_k - 2a_{k+1}\right)}{(k+2)(k+3)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+2} = -\frac{2\left(k^2 a_{k+1} - 2k a_k + k a_{k+1} - 2a_k - 2a_{k+1}\right)}{(k+2)(k+3)}, 2a_1 - 4a_0 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+2} = -\frac{2\left(k^2 a_{k+1} - 2k a_k - k a_{k+1} - 2a_{k+1}\right)}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{2\left(k^2 b_{k+1} - 2k\right.}{(}\right.$$

1.447.3 Maple trace

```
Methods for second order ODEs:
```

1.447.4 Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 17

```
dsolve((2*x+1)*x*diff(diff(y(x),x),x)-2*(2*x^2-1)*diff(y(x),x)-4*(x+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \, \mathrm{e}^{2x} x + c_1}{x}$$

1.447.5 Mathematica DSolve solution

Solving time : `0.081` (sec)
Leaf size : 28

```
DSolve[{(2*x+1)*x*D[y[x],{x,2}]-2*(2*x^2-1)*D[y[x],x]-4*(x+1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 e^{2x+1} x + c_1}{\sqrt{e} x}$$

1.448 problem 462

Internal problem ID [8586]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 462
Date solved : `Monday, October 21, 2024 at 05:10:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^{2}-2x\right)y''+\left(-x^{2}+2\right)y'+\left(2x-2\right)y=0$$

1.448.1 Solved as second order ode using Kovacic algorithm

Time used: 0.319 (sec)

Writing the ode as

$$\left(x^{2}-2x\right)y''+\left(-x^{2}+2\right)y'+\left(2x-2\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}-2x \\ B &= -x^{2}+2 \\ C &= 2x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 8x^3 + 24x^2 - 24x + 12$$
$$t = 4\left(x^2 - 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 850: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4x^2} + \frac{3}{4(x-2)^2} - \frac{1}{4(x-2)} - \frac{3}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{2}{x^3} + \frac{11}{x^4} + \frac{42}{x^5} + \frac{132}{x^6} + \frac{348}{x^7} + \frac{711}{x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \\
&= Q + \frac{R}{4x^4 - 16x^3 + 16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}\right) \\
&= \frac{1}{4} + \frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to

determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{+} = -1$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} - \frac{1}{2\,(x-2)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2x} - \frac{1}{2\,(x-2)} + \frac{1}{2} \\
&= -\frac{1}{2x} - \frac{1}{2x-4} + \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{1}{2\,(x-2)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2x^2} + \frac{1}{2\,(x-2)^2}\right) + \left(-\frac{1}{2x} - \frac{1}{2\,(x-2)} + \frac{1}{2}\right)^2 - \left(\frac{x^4 - 8x^3 +}{4\,(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x}-\frac{1}{2(x-2)}+\frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}\,\sqrt{x-2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+2}{x^2-2x} \, dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{2}+\frac{\ln(x-2)}{2}} \\
&= z_1 \left(\sqrt{x}\,\sqrt{x-2}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x}\,\sqrt{x-2}\,\mathrm{e}^{x}}{\sqrt{x\,(x-2)}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^2-2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x+\ln(x)+\ln(x-2)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{x\,\mathrm{e}^{x+\ln(x)+\ln(x-2)}\mathrm{e}^{-2x}}{x-2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}\,\sqrt{x-2}\,\mathrm{e}^{x}}{\sqrt{x\,(x-2)}}\right) + c_2 \left(\frac{\sqrt{x}\,\sqrt{x-2}\,\mathrm{e}^{x}}{\sqrt{x\,(x-2)}} \left(-\frac{x\,\mathrm{e}^{x+\ln(x)+\ln(x-2)}\mathrm{e}^{-2x}}{x-2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.448.2 Maple step by step solution

Let's solve

$\left(x^2-2x\right)\left(\frac{d}{dx}y'\right)+\left(-x^2+2\right)y'+\left(2x-2\right)y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{2(x-1)y}{x(x-2)}+\frac{(x^2-2)y'}{x(x-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{(x^2-2)y'}{x(x-2)}+\frac{2(x-1)y}{x(x-2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{x^2-2}{x(x-2)}, P_3(x)=\frac{2(x-1)}{x(x-2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x-2)\left(\frac{d}{dx}y'\right)+\left(-x^2+2\right)y'+\left(2x-2\right)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx} y'\right)$ to series expansion for $m = 1..2$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r(-2+r)\, x^{-1+r} + (-2a_1(1+r)\,(-1+r) + a_0(1+r)\,(-2+r))\, x^r + \left(\sum_{k=1}^{\infty} (-2a_{k+1}(k+r+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$-2a_1(1+r)\,(-1+r) + a_0(1+r)\,(-2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+1)\,(k+r-2) - 2k^2 a_{k+1} + (-4r a_{k+1} - a_{k-1})\, k - 2r^2 a_{k+1} - a_{k-1} r + 3a_{k-1} + 2a_{k+1} =$$

- Shift index using $k- >k+1$

$$a_{k+1}(k+2+r)\,(k+r-1) - 2(k+1)^2 a_{k+2} + (-4r a_{k+2} - a_k)\,(k+1) - 2r^2 a_{k+2} - r a_k + 3a_k +$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{k^2 a_{k+1} + 2kr a_{k+1} + r^2 a_{k+1} - k a_k + k a_{k+1} - r a_k + r a_{k+1} + 2a_k - 2a_{k+1}}{2(k^2 + 2kr + r^2 + 2k + 2r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{k^2 a_{k+1} - k a_k + k a_{k+1} + 2a_k - 2a_{k+1}}{2(k^2 + 2k)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = \frac{k^2 a_{k+1} - k a_k + k a_{k+1} + 2a_k - 2a_{k+1}}{2(k^2 + 2k)}$$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{k^2 a_{k+1} - k a_k + 5k a_{k+1} + 4a_{k+1}}{2(k^2+6k+8)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{k^2 a_{k+1} - k a_k + 5k a_{k+1} + 4a_{k+1}}{2(k^2+6k+8)}, -6a_1 = 0\right]$$

1.448.3 Maple trace

```
Methods for second order ODEs:
```

1.448.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 14

```
dsolve((x^2-2*x)*diff(diff(y(x),x),x)+(-x^2+2)*diff(y(x),x)+(2*x-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1\, x^2 + c_2\, e^x$$

1.448.5 Mathematica DSolve solution

Solving time : `0.066` (sec)
Leaf size : 18

```
DSolve[{(x^2-2*x)*D[y[x],{x,2}]+(2-x^2)*D[y[x],x]+(2*x-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x^2 + c_1 e^x$$

1.449 problem 463

Internal problem ID [8587]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 463
Date solved : `Monday, October 21, 2024 at 05:10:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (4x+1)\,y' + (4x+2)\,y = 0$$

1.449.1 Solved as second order ode using Kovacic algorithm

Time used: 0.177 (sec)

Writing the ode as

$$xy'' + (-4x-1)\,y' + (4x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -4x-1 \\ C &= 4x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 852: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3}{4x^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4x-1}{x} \, dx} \\ &= z_1 e^{2x + \frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\, e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-4x-1}{x}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{4x+\ln(x)}}{(y_1)^2}\,dx$$

$$= y_1 \left(\frac{x\,e^{4x+\ln(x)}e^{-4x}}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1\left(e^{2x}\right) + c_2\left(e^{2x}\left(\frac{x\,e^{4x+\ln(x)}e^{-4x}}{2}\right)\right)$$

Will add steps showing solving for IC soon.

1.449.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) - (4x+1)\,y' + (4x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2(2x+1)y}{x} + \frac{(4x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(4x+1)y'}{x} + \frac{2(2x+1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{4x+1}{x}, P_3(x) = \frac{2(2x+1)}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + (-4x - 1)\, y' + (4x + 2)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ◦ Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

  - ◦ Shift index using $k- >k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$$

  - ◦ Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r-1}$$

  - ◦ Shift index using $k- >k + 1$

    $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\, (k + r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2 + r)\, x^{-1+r} + \left(a_1(1 + r)\,(-1 + r) - 2a_0(-1 + 2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k + 1 + r)\,(k + r - 1\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term must be 0
  $a_1(1+r)(-1+r)-2a_0(-1+2r)=0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k+r-1)+a_k(-4k-4r+2)+4a_{k-1}=0$
- Shift index using $k->k+1$
  $a_{k+2}(k+2+r)(k+r)+a_{k+1}(-4k-2-4r)+4a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{2(2ka_{k+1}+2ra_{k+1}-2a_k+a_{k+1})}{(k+2+r)(k+r)}$
- Recursion relation for $r=0$
  $a_{k+2}=\frac{2(2ka_{k+1}-2a_k+a_{k+1})}{(k+2)k}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$
  $a_{k+2}=\frac{2(2ka_{k+1}-2a_k+a_{k+1})}{(k+2)k}$
- Recursion relation for $r=2$
  $a_{k+2}=\frac{2(2ka_{k+1}-2a_k+5a_{k+1})}{(k+4)(k+2)}$
- Solution for $r=2$
  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+2}, a_{k+2}=\frac{2(2ka_{k+1}-2a_k+5a_{k+1})}{(k+4)(k+2)}, 3a_1-6a_0=0\right]$

1.449.3 Maple trace

```
Methods for second order ODEs:
```

1.449.4 Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 16

```
dsolve(x*diff(diff(y(x),x),x)-(4*x+1)*diff(y(x),x)+(4*x+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y=e^{2x}\left(c_2\,x^2+c_1\right)$$

1.449.5 Mathematica DSolve solution

Solving time : 0.04 (sec)
Leaf size : 25

```
DSolve[{x*D[y[x],{x,2}]-(4*x+1)*D[y[x],x]+(4*x+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{2x}\left(c_2x^2 + 2c_1\right)$$

1.450 problem 464

Internal problem ID [8588]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 464
Date solved : `Monday, October 21, 2024 at 05:10:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(3x-1)\,y''-(3x+2)\,y'-(6x-8)\,y=0$$

1.450.1 Solved as second order ode using Kovacic algorithm

Time used: 0.264 (sec)

Writing the ode as

$$(3x-1)\,y''+(-3x-2)\,y'+(-6x+8)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=3x-1\\ B&=-3x-2\\ C&=-6x+8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{81x^2 - 108x + 54}{4\left(3x - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 81x^2 - 108x + 54 \\ t &= 4(3x - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{81x^2 - 108x + 54}{4\left(3x - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 854: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x-1)^2$. There is a pole at $x = \frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{3}{4\left(x - \frac{1}{3}\right)^2} - \frac{3}{2\left(x - \frac{1}{3}\right)}$$

For the pole at $x = \frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{1}{2x} + \frac{1}{9x^3} + \frac{11}{108x^4} + \frac{7}{108x^5} + \frac{5}{162x^6} + \frac{2}{243x^7} - \frac{13}{3888x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{81x^2 - 108x + 54}{36x^2 - 24x + 4} \\ &= Q + \frac{R}{36x^2 - 24x + 4} \\ &= \left(\frac{9}{4}\right) + \left(\frac{-54x + 45}{36x^2 - 24x + 4}\right) \\ &= \frac{9}{4} + \frac{-54x + 45}{36x^2 - 24x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -54. Dividing this by leading coefficient in t which is 36 gives $-\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{81x^2 - 108x + 54}{4\left(3x - 1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{1}{3}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-\frac{1}{3}\right)} + \left(\frac{3}{2}\right) \\
&= -\frac{1}{2\left(x-\frac{1}{3}\right)} + \frac{3}{2} \\
&= \frac{9x-6}{6x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(x-\frac{1}{3}\right)} + \frac{3}{2}\right)(0) + \left(\left(\frac{1}{2\left(x-\frac{1}{3}\right)^2}\right) + \left(-\frac{1}{2\left(x-\frac{1}{3}\right)} + \frac{3}{2}\right)^2 - \left(\frac{81x^2-108x+54}{4\left(3x-1\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2\left(x-\frac{1}{3}\right)} + \frac{3}{2}\right) dx} \\
&= \frac{e^{\frac{3x}{2}}}{\sqrt{3x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x-2}{3x-1}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(3x-1)}{2}} \\
&= z_1\left(\sqrt{3x-1}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-2}{3x-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+\ln(3x-1)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(3x-1)}\mathrm{e}^{-4x}}{3x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}\left(-\frac{x\,\mathrm{e}^{x+\ln(3x-1)}\mathrm{e}^{-4x}}{3x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.450.2 Maple step by step solution

Let's solve

$(3x-1)\left(\frac{d}{dx}y'\right)-(3x+2)\,y'-(6x-8)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{2(3x-4)y}{3x-1}+\frac{(3x+2)y'}{3x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{(3x+2)y'}{3x-1}-\frac{2(3x-4)y}{3x-1}=0$

□ Check to see if $x_0=\frac{1}{3}$ is a regular singular point

- ○ Define functions

$\left[P_2(x)=-\frac{3x+2}{3x-1},P_3(x)=-\frac{2(3x-4)}{3x-1}\right]$

- ○ $\left(x-\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=\frac{1}{3}$

$\left(\left(x-\frac{1}{3}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{1}{3}}=-1$

- ○ $\left(x-\frac{1}{3}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{1}{3}$

$\left(\left(x-\frac{1}{3}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{1}{3}}=0$

- ○ $x=\frac{1}{3}$is a regular singular point

Check to see if $x_0=\frac{1}{3}$ is a regular singular point

$x_0=\frac{1}{3}$

- Multiply by denominators

$(3x-1)\left(\frac{d}{dx}y'\right)+(-3x-2)\,y'+(-6x+8)\,y=0$

- Change variables using $x=u+\frac{1}{3}$ so that the regular singular point is at $u=0$

$3u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-3u-3)\left(\frac{d}{du}y(u)\right)+(-6u+6)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^{m}\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m}u^{k+r}$

- Convert $u^{m}\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,u^{k+r-1}$

- Shift index using $k->k+1$

  $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $3a_{0}r(-2+r)\,u^{-1+r}+\left(3a_{1}(1+r)\,(-1+r)-3a_{0}(-2+r)\right)u^{r}+\left(\sum_{k=1}^{\infty}\left(3a_{k+1}(k+1+r)\,(k+r-\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $3r(-2+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$
- Each term must be 0

  $3a_{1}(1+r)\,(-1+r)-3a_{0}(-2+r)=0$
- Each term in the series must be 0, giving the recursion relation

  $3a_{k+1}(k+1+r)\,(k+r-1)+a_{k}(-3k-3r+6)-6a_{k-1}=0$
- Shift index using $k->k+1$

  $3a_{k+2}(k+2+r)\,(k+r)+a_{k+1}(-3k+3-3r)-6a_{k}=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{ka_{k+1}+ra_{k+1}+2a_{k}-a_{k+1}}{(k+2+r)(k+r)}$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{ka_{k+1}+2a_{k}-a_{k+1}}{(k+2)k}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$

  $a_{k+2}=\frac{ka_{k+1}+2a_{k}-a_{k+1}}{(k+2)k}$
- Recursion relation for $r=2$

$a_{k+2} = \frac{k a_{k+1} + 2 a_k + a_{k+1}}{(k+4)(k+2)}$

- Solution for $r = 2$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{k a_{k+1} + 2 a_k + a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$$

- Revert the change of variables $u = x - \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{1}{3}\right)^{k+2}, a_{k+2} = \frac{k a_{k+1} + 2 a_k + a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$$

1.450.3 Maple trace

```
Methods for second order ODEs:
```

1.450.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 18

```
dsolve((3*x-1)*diff(diff(y(x),x),x)-(3*x+2)*diff(y(x),x)-(6*x-8)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \,\mathrm{e}^{2x} + c_2 x \,\mathrm{e}^{-x}$$

1.450.5 Mathematica DSolve solution

Solving time : `0.126` (sec)
Leaf size : 35

```
DSolve[{(3*x-1)*D[y[x],{x,2}]-(3*x+2)*D[y[x],x]-(6*x-8)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x-\frac{1}{2}} \left(c_1 e^{3x} + 2 e c_2 x\right)}{\sqrt{2}}$$

1.451 problem 465

Internal problem ID [8589]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 465
Date solved : `Monday, October 21, 2024 at 05:10:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x+1)^2 y'' - 2(x+1) y' - \left(x^2+2x-1\right) y = 0$$

1.451.1 Solved as second order ode using Kovacic algorithm

Time used: 0.103 (sec)

Writing the ode as

$$(x+1)^2 y'' + (-2x-2) y' + \left(-x^2-2x+1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x+1)^2 \\ B &= -2x-2 \\ C &= -x^2-2x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 856: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{(x+1)^2}\,dx} \\ &= z_1 e^{\ln(x+1)} \\ &= z_1(x+1) \end{aligned}$$

Which simplifies to

$$y_1 = (x+1)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{(x+1)^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x+1)}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x+1)\,\mathrm{e}^{-x}\right) + c_2\left((x+1)\,\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.451.2 Maple step by step solution

Let's solve

$(x+1)^2\left(\frac{d}{dx}y'\right) - 2(x+1)\,y' - (x^2+2x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(x^2+2x-1)y}{(x+1)^2} + \frac{2y'}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y'}{x+1} - \frac{(x^2+2x-1)y}{(x+1)^2} = 0$

□ Check to see if $x_0 = -1$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2}{x+1}, P_3(x) = -\frac{x^2+2x-1}{(x+1)^2}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = -2$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 2$

  - $x=-1$is a regular singular point

  Check to see if $x_0 = -1$ is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x+1)^2\left(\frac{d}{dx}y'\right) + (-2x-2)\,y' + (-x^2-2x+1)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $u^2\left(\frac{d}{du}\frac{d}{du}y(u)\right) - 2u\left(\frac{d}{du}y(u)\right) + (-u^2+2)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion

    $u\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r}$

  - Convert $u^2\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

    $u^2\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\,u^r+a_1r(-1+r)\,u^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r-1)\,(k+r-2)-a_{k-2}\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{1,2\}$

- Each term must be 0

  $a_1r(-1+r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2)-a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)\,(k+r)-a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r=1$

  $a_{k+2}=\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+1}, a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$

  $a_{k+2} = \frac{a_k}{(k+3)(k+2)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+2}, a_{k+2} = \frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+2}\right), a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = \frac{b_k}{(k+3)(k+2)}, b_1 =\right.$

1.451.3 Maple trace

```
Methods for second order ODEs:
```

1.451.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 17

```
dsolve((x+1)^2*diff(diff(y(x),x),x)-2*(x+1)*diff(y(x),x)-(x^2+2*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = (x+1)(c_1 \sinh(x) + c_2 \cosh(x))$$

1.451.5 Mathematica DSolve solution

Solving time : 0.192 (sec)
Leaf size : 147

```
DSolve[{(x+1)^2*D[y[x],{x,2}]-2*(x+1)*x*D[y[x],x]-(x^2+2*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 2^{\frac{1}{2}i\left(\sqrt{7}+i\right)}e^{-\left(\left(\sqrt{2}-1\right)(x+1)\right)}(x+1)^{\frac{1}{2}i\left(\sqrt{7}+i\right)}\left(c_1 \text{HypergeometricU}\left(\frac{1}{2}\left(1-\sqrt{2}+i\sqrt{7}\right),1+i\sqrt{7},2\sqrt{2}(x+1)\right) + c_2 L_{\frac{1}{2}\left(-1+\sqrt{2}-i\sqrt{7}\right)}^{i\sqrt{7}}\left(2\sqrt{2}(x+1)\right)\right)$$

1.452 problem 466

Internal problem ID [8590]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 466
Date solved : `Monday, October 21, 2024 at 05:10:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 0$$

1.452.1 Solved as second order ode using Kovacic algorithm

Time used: 0.114 (sec)

Writing the ode as

$$4x^2y'' + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2+4x \\ C &= 4x^2-4x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 858: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\,dx} \\
&= z_1 e^{x-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{e^x}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^x}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{\sqrt{x}}\right) + c_2 \left(\frac{e^x}{\sqrt{x}}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.452.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-4x-1)y}{4x^2} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(2x-1)y'}{x} + \frac{(4x^2-4x-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2-4x-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4x(2x-1)\, y' + (4x^2 - 4x - 1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+2r)\,(-1+2r)\, x^r + (a_1(3+2r)\,(1+2r) - 4a_0(1+2r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k -\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) - 4a_0(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{4a_0}{3+2r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + (-8k - 8r + 4)\, a_{k-1} + 4a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + (-8k - 12 - 8r)\, a_{k+1} + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4(2ka_{k+1}+2ra_{k+1}-a_k+3a_{k+1})}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1 = 2a_0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}, a_1 = a_0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1 = 2a_0, b_{k+2} = \frac{4(2kb_{k+1}-b_k+4b_{k+1}}{4k^2+20k+24}\right.$

1.452.3 Maple trace

```
Methods for second order ODEs:
```

1.452.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 15

```
dsolve(4*x^2*diff(diff(y(x),x),x)+(-8*x^2+4*x)*diff(y(x),x)+(4*x^2-4*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{x}(c_2 x + c_1)}{\sqrt{x}}$$

1.452.5 Mathematica DSolve solution

Solving time : `0.043` (sec)
Leaf size : 21

```
DSolve[{4*x^2*D[y[x],{x,2}]+(4*x-8*x^2)*D[y[x],x]+(4*x^2-4*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^x(c_2x + c_1)}{\sqrt{x}}$$

1.453 problem 467

Internal problem ID [8591]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 467
Date solved : `Monday, October 21, 2024 at 05:10:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + (4x^2 + 2)\, y = 0$$

1.453.1 Solved as second order ode using Kovacic algorithm

Time used: 0.086 (sec)

Writing the ode as

$$y'' + 4xy' + (4x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 860: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\ &= z_1 e^{-x^2} \\ &= z_1 \left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.453.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 4xy' + (4x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ◦ Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ◦ Shift index using $k->k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + 2a_0 + (6a_3 + 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2 = -a_0, a_3 = -a_1\right]$

1.453.3 Maple trace

```
Methods for second order ODEs:
```

1.453.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(4*x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-x^2}(c_2 x + c_1)$$

1.453.5 Mathematica DSolve solution

Solving time : `0.04` (sec)
Leaf size : 21

```
DSolve[{4*x^2*D[y[x],{x,2}]+(4*x-8*x^2)*D[y[x],x]+(4*x^2-4*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^x(c_2 x + c_1)}{\sqrt{x}}$$

1.454 problem 468

Internal problem ID [8592]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 468
Date solved : `Monday, October 21, 2024 at 05:10:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2x+1)\,y''-2y'-(2x+3)\,y=0$$

1.454.1 Solved as second order ode using Kovacic algorithm

Time used: 0.239 (sec)

Writing the ode as

$$(2x+1)\,y''-2y'+(-2x-3)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x+1\\ B &= -2\\ C &= -2x-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 8x + 6}{(2x + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 + 8x + 6$$
$$t = (2x + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 8x + 6}{(2x + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 862: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x+1)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x+\frac{1}{2}\right)^2} + \frac{1}{x+\frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{1}{1} - 0\right) &= \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 + 8x + 6}{(2x+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} + (-)(1) \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} - 1 \\
&= -\frac{2(x+1)}{2x+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2+8x+6}{(2x+1)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right) dx} \\
&= \frac{\mathrm{e}^{-x}}{\sqrt{2x+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{2x+1}\, dx} \\
&= z_1 e^{\frac{\ln(2x+1)}{2}} \\
&= z_1\left(\sqrt{2x+1}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{2x+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(2x+1)}}{(y_1)^2}\, dx \\ &= y_1 \left(x\, \mathrm{e}^{2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(x\, \mathrm{e}^{2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.454.2 Maple step by step solution

Let's solve

$(2x+1)\left(\frac{d}{dx}y'\right) - 2y' - (2x+3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(2x+3)y}{2x+1} + \frac{2y'}{2x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y'}{2x+1} - \frac{(2x+3)y}{2x+1} = 0$

□ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2}{2x+1}, P_3(x) = -\frac{2x+3}{2x+1}\right]$

- $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{1}{2}}=-1$
- $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\Big|_{x=-\frac{1}{2}}=0$
- $x=-\frac{1}{2}$is a regular singular point

  Check to see if $x_0=-\frac{1}{2}$ is a regular singular point

  $x_0=-\frac{1}{2}$

- Multiply by denominators

  $(2x+1)\left(\frac{d}{dx}y'\right)-2y'+(-2x-3)\,y=0$
- Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

  $2u\left(\frac{d}{du}\frac{d}{du}y(u)\right)-2\frac{d}{du}y(u)+(-2u-2)\,y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$
- Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$
- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1}$
- Shift index using $k->k+1$

  $\frac{d}{du}y(u)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,u^{k+r}$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$
- Shift index using $k->k+1$

$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$

Rewrite ODE with series expansions

$2a_0 r(-2+r)\, u^{-1+r} + \left(2a_1(1+r)(-1+r) - 2a_0\right) u^r + \left(\sum_{k=1}^{\infty} \left(2a_{k+1}(k+1+r)(k+r-1) - 2a_k\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term must be 0

  $2a_1(1+r)(-1+r) - 2a_0 = 0$
- Each term in the series must be 0, giving the recursion relation

  $2a_{k+1}(k+1+r)(k+r-1) - 2a_k - 2a_{k-1} = 0$
- Shift index using $k- >k+1$

  $2a_{k+2}(k+2+r)(k+r) - 2a_{k+1} - 2a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2+r)(k+r)}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$
- Recursion relation for $r = 2$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$
- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$

1.454.3 Maple trace

```
Methods for second order ODEs:
```

1.454.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve((2*x+1)*diff(diff(y(x),x),x)-2*diff(y(x),x)-(2*x+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \,\mathrm{e}^{-x} + c_2 x \,\mathrm{e}^{x}$$

1.454.5 Mathematica DSolve solution

Solving time : 0.067 (sec)
Leaf size : 29

```
DSolve[{(2*x+1)*D[y[x],{x,2}]-2*D[y[x],x]-(2*x+3)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x-\frac{1}{2}} \left(c_2 e^{2x+1} x + c_1\right)$$

1.455 problem 469

Internal problem ID [8593]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 469
Date solved : `Monday, October 21, 2024 at 05:10:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (2x+2)\,y' + (x+2)\,y = 0$$

1.455.1 Solved as second order ode using Kovacic algorithm

Time used: 0.154 (sec)

Writing the ode as

$$xy'' + (-2x-2)\,y' + (x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 864: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\,(0) \\ &= -\frac{1}{x} \\ &= -\frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x-2}{x} \, dx} \\
&= z_1 e^{x + \ln(x)} \\
&= z_1 (x \, \mathrm{e}^x)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{2x + 2\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x \, \mathrm{e}^{2x + 2\ln(x)} \mathrm{e}^{-2x}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x\,\mathrm{e}^{2x+2\ln(x)}\mathrm{e}^{-2x}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.455.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) - (2x+2)\,y' + (x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x+2)y}{x} + \frac{2(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(x+1)y'}{x} + \frac{(x+2)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{x+2}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + (-2x-2)\,y' + (x+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-3+r)\, x^{-1+r} + \left(a_1 (1+r)\,(-2+r) - 2a_0 (-1+r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1} (k+1+r)\,(k-2+r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 3\}$$

- Each term must be 0

  $$a_1 (1+r)\,(-2+r) - 2a_0 (-1+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1} (k+1+r)\,(k-2+r) - 2a_k k - 2a_k r + 2a_k + a_{k-1} = 0$$

- Shift index using $k- >k + 1$

  $$a_{k+2} (k+2+r)\,(k+r-1) - 2a_{k+1} (k+1) - 2r a_{k+1} + 2a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{2k a_{k+1} + 2r a_{k+1} - a_k}{(k+2+r)(k+r-1)}$$

- Recursion relation for $r = 0$

$a_{k+2} = \frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$
- Recursion relation for $r = 3$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}$
- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}, 4a_1 - 4a_0 = 0\right]$

1.455.3 Maple trace

```
Methods for second order ODEs:
```

1.455.4 Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 14

```
dsolve(x*diff(diff(y(x),x),x)-(2*x+2)*diff(y(x),x)+(x+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{x} \left(c_2\, x^3 + c_1\right)$$

1.455.5 Mathematica DSolve solution

Solving time : `0.053` (sec)
Leaf size : 29

```
DSolve[{x*D[y[x],{x,2}]-(2*x+2)*D[y[x],x]+(x+2)*y[x]==6*x^3*Exp[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{6} e^{x} \left(9x^4 + 2c_2 x^3 + 6c_1\right)$$

1.456 problem 470

Internal problem ID [8594]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 470
Date solved : `Monday, October 21, 2024 at 05:10:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0$$

1.456.1 Solved as second order ode using Kovacic algorithm

Time used: 0.149 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 866: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.456.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 2$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - Shift index using $k->k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+r)(-2+r) = 0$$
- Values of r that satisfy the indicial equation

  $$r \in \{1, 2\}$$
- Each term must be 0

  $$a_1 r(-1+r) = 0$$
- Solve for the dependent coefficient(s)

  $$a_1 = 0$$
- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r-1)(k+r-2) + a_{k-2} = 0$$
- Shift index using $k->k+2$

  $$a_{k+2}(k+1+r)(k+r) + a_k = 0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$$
- Recursion relation for $r = 1$

  $$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$$
- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$$
- Recursion relation for $r = 2$

$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

1.456.3 Maple trace

```
Methods for second order ODEs:
```

1.456.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+(x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x(c_1 \sin(x) + c_2 \cos(x))$$

1.456.5 Mathematica DSolve solution

Solving time : `0.041` (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

1.457 problem 472

Internal problem ID [8595]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 472
Date solved : `Monday, October 21, 2024 at 05:10:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4xy' + \left(-16x^2 + 3\right) y = 0$$

1.457.1 Solved as second order ode using Kovacic algorithm

Time used: 0.114 (sec)

Writing the ode as

$$4x^2y'' - 4xy' + \left(-16x^2 + 3\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2 + 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 868: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,\mathrm{e}^{-2x}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.457.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) - 4xy' + \left(-16x^2+3\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(16x^2-3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'}{x} - \frac{(16x^2-3)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4xy' + \left(-16x^2+3\right)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)(-3+2r) x^r + a_1(1+2r)(-1+2r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)(2k+2r-3) - 1$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(1+2r)(-1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right) a_k - 16a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$4\left(k+\frac{3}{2}+r\right)\left(k+\frac{1}{2}+r\right) a_{k+2} - 16a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{16a_k}{(2k+3+2r)(2k+1+2r)}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=\frac{16a_k}{(2k+4)(2k+2)}, a_1=0\right]$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+2}=\frac{16a_k}{(2k+6)(2k+4)}$$

- Solution for $r=\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2}=\frac{16a_k}{(2k+6)(2k+4)}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2}=\frac{16a_k}{(2k+4)(2k+2)}, a_1=0, b_{k+2}=\frac{16b_k}{(2k+6)(2k+4)}, b_1=0\right]$$

1.457.3 Maple trace

```
Methods for second order ODEs:
```

1.457.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 21

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(-16*x^2+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\sqrt{x}\left(c_1 \sinh(2x)+c_2 \cosh(2x)\right)$$

1.457.5 Mathematica DSolve solution

Solving time : `0.046` (sec)
Leaf size : 32

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(3-16*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{1}{4}e^{-2x}\sqrt{x}\left(c_2 e^{4x}+4c_1\right)$$

1.458 problem 473

Internal problem ID [8596]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 473
Date solved : `Monday, October 21, 2024 at 05:10:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4xy' + \left(4x^2+3\right)y = 0$$

1.458.1 Solved as second order ode using Kovacic algorithm

Time used: 0.159 (sec)

Writing the ode as

$$4x^2y'' - 4xy' + \left(4x^2+3\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= 4x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 870: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,\cos(x)\right) + c_2\left(\sqrt{x}\,\cos(x)\,(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.458.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) - 4xy' + (4x^2+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2+3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'}{x} + \frac{(4x^2+3)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{4x^2+3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4xy' + (4x^2+3)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\, x^r + a_1(1+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-3) + 4\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1+2r)\,(-1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right) a_k + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $4\left(k+\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right) a_{k+2} + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(2k+1+2r)(2k+3+2r)}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+2)(2k+4)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{(2k+2)(2k+4)}, a_1 = 0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+4)(2k+6)}$
- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{4a_k}{(2k+4)(2k+6)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -\frac{4a_k}{(2k+2)(2k+4)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(2k+4)(2k+6)}, b_1 = 0\right]$

1.458.3 Maple trace

```
Methods for second order ODEs:
```

1.458.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 17

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(4*x^2+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\,(c_1 \sin(x) + c_2 \cos(x))$$

1.458.5 Mathematica DSolve solution

Solving time : `0.049` (sec)
Leaf size : 39

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2+3)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-ix} \sqrt{x} \left(2c_1 - i c_2 e^{2ix}\right)$$

1.459 problem 474

Internal problem ID [8597]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 474
Date solved : `Monday, October 21, 2024 at 05:10:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' - \left(x^2-2\right)y = 0$$

1.459.1 Solved as second order ode using Kovacic algorithm

Time used: 0.093 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(-x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= -x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 872: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-x}\right) + c_2 \left(x\,\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.459.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) - 2 x y' - \left(x^2 - 2\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{\left(x^2-2\right) y}{x^2} + \frac{2 y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' - \frac{2 y'}{x} - \frac{\left(x^2-2\right) y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = -\frac{x^2-2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 2$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 \left(\frac{d}{dx} y'\right) - 2 x y' + \left(-x^2 + 2\right) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)(-2+r) x^r + a_1 r(-1+r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)(k+r-2) - a_{k-2}) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{1, 2\}$

- Each term must be 0

$a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$a_k(k+r-1)(k+r-2) - a_{k-2} = 0$

- Shift index using $k- >k + 2$

$a_{k+2}(k+1+r)(k+r) - a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = \frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

$a_{k+2} = \frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$$

- Recursion relation for $r=2$

  $a_{k+2}=\frac{a_k}{(k+3)(k+2)}$

- Solution for $r=2$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$$

1.459.3 Maple trace

```
Methods for second order ODEs:
```

1.459.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*diff(y(x),x)-(x^2-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y=x(c_1\sinh(x)+c_2\cosh(x))$$

1.459.5 Mathematica DSolve solution

Solving time : `0.04` (sec)
Leaf size : 25

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]-(x^2-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_1e^{-x}x+\frac{1}{2}c_2e^{x}x$$

1.460 problem 475

Internal problem ID [8598]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 475
Date solved : `Monday, October 21, 2024 at 05:10:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2x(x+1)\,y' + \left(x^2+2x+2\right)y = 0$$

1.460.1 Solved as second order ode using Kovacic algorithm

Time used: 0.098 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^2-2x\right)y' + \left(x^2+2x+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2-2x \\ C &= x^2+2x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 874: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-2x}{x^2}\,dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x\,\mathrm{e}^x) \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\,e^x) + c_2(x\,e^x(x))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.460.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - 2x(x+1)\,y' + (x^2+2x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+2x+2)y}{x^2} + \frac{2(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(x+1)y'}{x} + \frac{(x^2+2x+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{x^2+2x+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 2$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - 2x(x+1)\,y' + (x^2+2x+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\,x^r + (a_1 r(-1+r) - 2a_0(-1+r))\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)\,(k+r-2) -\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term must be 0

  $a_1 r(-1+r) - 2a_0(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = \frac{2a_0}{r}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) - 2a_{k-1}k - 2a_{k-1}r + a_{k-2} + 4a_{k-1} = 0$
- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)\,(k+r) - 2a_{k+1}(k+2) - 2a_{k+1}r + a_k + 4a_{k+1} = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1}+2a_{k+1}r-a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+2)(k+1)}$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{2ka_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0\right]$$

- Recursion relation for $r = 2$

  $$a_{k+2} = \frac{2ka_{k+1} - a_k + 4a_{k+1}}{(k+3)(k+2)}$$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2ka_{k+1} - a_k + 4a_{k+1}}{(k+3)(k+2)}, a_1 = a_0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = \frac{2ka_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0, b_{k+2} = \frac{2kb_{k+1} - b_k + 4b_{k+1}}{(k+3)(k+2)}, b_1 =$$

1.460.3 Maple trace

```
Methods for second order ODEs:
```

1.460.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 13

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*(x+1)*diff(y(x),x)+(x^2+2*x+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^x x(c_2 x + c_1)$$

1.460.5 Mathematica DSolve solution

Solving time : `0.202` (sec)
Leaf size : 41

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2*x+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{ix} x(c_1 \text{HypergeometricU}(-i, 0, -2ix) + c_2 L_i^{-1}(-2ix))$$

1.461 problem 476

Internal problem ID [8599]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 476
Date solved : `Monday, October 21, 2024 at 05:10:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2x(x+2)\,y' + \left(x^2+4x+6\right)y = 0$$

1.461.1 Solved as second order ode using Kovacic algorithm

Time used: 0.096 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^2-4x\right)y' + \left(x^2+4x+6\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2-4x \\ C &= x^2+4x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 876: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-4x}{x^2}\,dx} \\ &= z_1 e^{x+2\ln(x)} \\ &= z_1\left(x^2 \mathrm{e}^x\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x+4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2 \mathrm{e}^x\right) + c_2\left(x^2 \mathrm{e}^x (x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.461.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - 2x(x+2)\,y' + (x^2+4x+6)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+4x+6)y}{x^2} + \frac{2(x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(x+2)y'}{x} + \frac{(x^2+4x+6)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(x+2)}{x}, P_3(x) = \frac{x^2+4x+6}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -4$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 6$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - 2x(x+2)\,y' + (x^2+4x+6)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-3+r)\, x^r + (a_1(-1+r)\,(-2+r) - 2a_0(-2+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 3\}$

- Each term must be 0

  $a_1(-1+r)\,(-2+r) - 2a_0(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{2a_0}{-1+r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)\,(k+r-3) - 2a_{k-1}k - 2a_{k-1}r + a_{k-2} + 6a_{k-1} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}(k+r)\,(k+r-1) - 2a_{k+1}(k+2) - 2a_{k+1}r + a_k + 6a_{k+1} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1} + 2a_{k+1}r - a_k - 2a_{k+1}}{(k+r)(k+r-1)}$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{2ka_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2k a_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0\right]$$

- Recursion relation for $r = 3$

$$a_{k+2} = \frac{2k a_{k+1} - a_k + 4a_{k+1}}{(k+3)(k+2)}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{2k a_{k+1} - a_k + 4a_{k+1}}{(k+3)(k+2)}, a_1 = a_0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = \frac{2k a_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0, b_{k+2} = \frac{2k b_{k+1} - b_k + 4b_{k+1}}{(k+3)(k+2)}, b_1 =$$

1.461.3 Maple trace

```
Methods for second order ODEs:
```

1.461.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*(x+2)*diff(y(x),x)+(x^2+4*x+6)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^x x^2 (c_2 x + c_1)$$

1.461.5 Mathematica DSolve solution

Solving time : `0.043` (sec)
Leaf size : 19

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*(x+2)*D[y[x],x]+(x^2+4*x+6)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^x x^2 (c_2 x + c_1)$$

1.462 problem 477

Internal problem ID [8600]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 477
Date solved : `Monday, October 21, 2024 at 05:10:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 4xy' + (x^2 + 6)\,y = 0$$

1.462.1 Solved as second order ode using Kovacic algorithm

Time used: 0.149 (sec)

Writing the ode as

$$x^2y'' - 4xy' + (x^2 + 6)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 878: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\, dx} \\ &= z_1 e^{2\ln(x)} \\ &= z_1\left(x^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2 \cos(x)\right) + c_2 \left(x^2 \cos(x) \left(\tan(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.462.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) - 4xy' + (x^2 + 6) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(x^2+6)y}{x^2} + \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' - \frac{4y'}{x} + \frac{(x^2+6)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{4}{x}, P_3(x) = \frac{x^2+6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 6$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 \left(\frac{d}{dx} y'\right) - 4xy' + (x^2 + 6) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-3+r)\, x^r + a_1(-1+r)\,(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-3) + a_{k-2})\, x^{k}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 3\}$

- Each term must be 0

  $a_1(-1+r)\,(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)\,(k+r-3) + a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}(k+r)\,(k+r-1) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+r)(k+r-1)}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 3$

$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

1.462.3 Maple trace

```
Methods for second order ODEs:
```

1.462.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(x^2+6)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^2(c_1 \sin(x) + c_2 \cos(x))$$

1.462.5 Mathematica DSolve solution

Solving time : `0.043` (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]-4*x*D[y[x],x]+(x^2+6)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-ix} x^2 \left(2c_1 - ic_2 e^{2ix}\right)$$

1.463 problem 478

Internal problem ID [8601]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 478
Date solved : `Monday, October 21, 2024 at 05:10:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\,y''-xy'+y=0$$

1.463.1 Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 880: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2} \right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2} \right)(0) + \left(\left(\frac{1}{2(x-1)^2} \right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2} \right)^2 - \left(\frac{x^2 - 4x + 6}{4(x-1)^2} \right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2} \right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{x\, e^{x+\ln(x-1)} e^{-2x}}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(e^x) + c_2\left(e^x \left(-\frac{x\, e^{x+\ln(x-1)} e^{-2x}}{x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.463.2 Maple step by step solution

Let's solve

$(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$
- $(x-1)\cdot P_2(x)$ is analytic at $x=1$
  $$\left((x-1)\cdot P_2(x)\right)\Big|_{x=1} = -1$$
- $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$
  $$\left((x-1)^2\cdot P_3(x)\right)\Big|_{x=1} = 0$$
- $x=1$is a regular singular point
  Check to see if $x_0 = 1$ is a regular singular point
  $x_0 = 1$

- Multiply by denominators
  $(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$
- Change variables using $x = u+1$ so that the regular singular point is at $u=0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$
- Shift index using $k->k+1$
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)(a_{k+1}(k+1+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

1.463.3 Maple trace

```
Methods for second order ODEs:
```

1.463.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2 \,\mathrm{e}^{x}$$

1.463.5 Mathematica DSolve solution

Solving time : `0.049` (sec)
Leaf size : 17

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2 x$$

1.464 problem 479

Internal problem ID [8602]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 479
Date solved : `Monday, October 21, 2024 at 05:10:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4x(x+1)\,y' + (2x+3)\,y = 0$$

1.464.1 Solved as second order ode using Kovacic algorithm

Time used: 0.122 (sec)

Writing the ode as

$$4x^2y'' + \left(-4x^2 - 4x\right)y' + (2x+3)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 - 4x \\ C &= 2x+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 882: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^2-4x}{4x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{x+\ln(x)}}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\left(\frac{e^{x+\ln(x)}}{x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.464.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) - 4x(x+1)\,y' + (2x+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(2x+3)y}{4x^2} + \frac{(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x+1)y'}{x} + \frac{(2x+3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x+1}{x}, P_3(x) = \frac{2x+3}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{3}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4x(x+1)\,y' + (2x+3)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + 2r)\,(-3 + 2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k + 2r - 1)\,(2k + 2r - 3) - 2a_{k-1}(2k + 2r - 3)\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + 2r)\,(-3 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k + r - \frac{3}{2}\right)\left(\left(k + r - \frac{1}{2}\right) a_k - a_{k-1}\right) = 0$

- Shift index using $k- >k + 1$

  $4\left(k + r - \frac{1}{2}\right)\left(\left(k + \frac{1}{2} + r\right) a_{k+1} - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k}{2k+1+2r}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{2a_k}{2k+2}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k}{2k+2}\right]$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = \frac{2a_k}{2k+4}$$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{2a_k}{2k+2}, b_{k+1} = \frac{2b_k}{2k+4}\right]$$

1.464.3 Maple trace

```
Methods for second order ODEs:
```

1.464.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x*(x+1)*diff(y(x),x)+(2*x+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\,(c_2\,e^x + c_1)$$

1.464.5 Mathematica DSolve solution

Solving time : 0.036 (sec)
Leaf size : 20

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x*(x+1)*D[y[x],x]+(2*x+3)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{x}(c_2 e^x + c_1)$$

1.465 problem 480

Internal problem ID [8603]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 480
Date solved : `Monday, October 21, 2024 at 05:10:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(3x-1)\, y'' - (3x+2)\, y' - (6x-8)\, y = 0$$

1.465.1 Solved as second order ode using Kovacic algorithm

Time used: 0.265 (sec)

Writing the ode as

$$(3x-1)\, y'' + (-3x-2)\, y' + (-6x+8)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x-1 \\ B &= -3x-2 \\ C &= -6x+8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{81x^2 - 108x + 54}{4(3x - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 81x^2 - 108x + 54$$
$$t = 4(3x - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{81x^2 - 108x + 54}{4(3x - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 884: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x-1)^2$. There is a pole at $x = \frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{3}{4\left(x - \frac{1}{3}\right)^2} - \frac{3}{2\left(x - \frac{1}{3}\right)}$$

For the pole at $x = \frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{1}{2x} + \frac{1}{9x^3} + \frac{11}{108x^4} + \frac{7}{108x^5} + \frac{5}{162x^6} + \frac{2}{243x^7} - \frac{13}{3888x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{81x^2 - 108x + 54}{36x^2 - 24x + 4} \\ &= Q + \frac{R}{36x^2 - 24x + 4} \\ &= \left(\frac{9}{4}\right) + \left(\frac{-54x + 45}{36x^2 - 24x + 4}\right) \\ &= \frac{9}{4} + \frac{-54x + 45}{36x^2 - 24x + 4}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -54. Dividing this by leading coefficient in t which is 36 gives $-\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{81x^2 - 108x + 54}{4\left(3x - 1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{1}{3}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-\frac{1}{3}\right)} + \left(\frac{3}{2}\right) \\
&= -\frac{1}{2\left(x-\frac{1}{3}\right)} + \frac{3}{2} \\
&= \frac{9x-6}{6x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(x-\frac{1}{3}\right)} + \frac{3}{2}\right)(0) + \left(\left(\frac{1}{2\left(x-\frac{1}{3}\right)^2}\right) + \left(-\frac{1}{2\left(x-\frac{1}{3}\right)} + \frac{3}{2}\right)^2 - \left(\frac{81x^2-108x+54}{4\left(3x-1\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2\left(x-\frac{1}{3}\right)} + \frac{3}{2}\right) dx} \\
&= \frac{e^{\frac{3x}{2}}}{\sqrt{3x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x-2}{3x-1}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(3x-1)}{2}} \\
&= z_1\left(\sqrt{3x-1}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-2}{3x-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+\ln(3x-1)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{x\,\mathrm{e}^{x+\ln(3x-1)}\mathrm{e}^{-4x}}{3x-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}\left(-\frac{x\,\mathrm{e}^{x+\ln(3x-1)}\mathrm{e}^{-4x}}{3x-1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.465.2 Maple step by step solution

Let's solve

$(3x-1)\left(\frac{d}{dx}y'\right)-(3x+2)\,y'-(6x-8)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{2(3x-4)y}{3x-1}+\frac{(3x+2)y'}{3x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{(3x+2)y'}{3x-1}-\frac{2(3x-4)y}{3x-1}=0$

□ Check to see if $x_0=\frac{1}{3}$ is a regular singular point

- ○ Define functions

$\left[P_2(x)=-\frac{3x+2}{3x-1}, P_3(x)=-\frac{2(3x-4)}{3x-1}\right]$

- ○ $\left(x-\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=\frac{1}{3}$

$\left(\left(x-\frac{1}{3}\right)\cdot P_2(x)\right)\Big|_{x=\frac{1}{3}}=-1$

- ○ $\left(x-\frac{1}{3}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{1}{3}$

$\left(\left(x-\frac{1}{3}\right)^2\cdot P_3(x)\right)\Big|_{x=\frac{1}{3}}=0$

- ○ $x=\frac{1}{3}$is a regular singular point

Check to see if $x_0=\frac{1}{3}$ is a regular singular point

$x_0=\frac{1}{3}$

- Multiply by denominators

$(3x-1)\left(\frac{d}{dx}y'\right)+(-3x-2)\,y'+(-6x+8)\,y=0$

- Change variables using $x=u+\frac{1}{3}$ so that the regular singular point is at $u=0$

$3u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-3u-3)\left(\frac{d}{du}y(u)\right)+(-6u+6)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^{m}\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m}u^{k+r}$
- Convert $u^{m}\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,u^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,u^{k+r-1}$
- Shift index using $k->k+1$

  $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $3a_{0}r(-2+r)\,u^{-1+r}+\left(3a_{1}(1+r)\,(-1+r)-3a_{0}(-2+r)\right)u^{r}+\left(\sum_{k=1}^{\infty}\left(3a_{k+1}(k+1+r)\,(k+r-\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $3r(-2+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$
- Each term must be 0

  $3a_{1}(1+r)\,(-1+r)-3a_{0}(-2+r)=0$
- Each term in the series must be 0, giving the recursion relation

  $3a_{k+1}(k+1+r)\,(k+r-1)+a_{k}(-3k-3r+6)-6a_{k-1}=0$
- Shift index using $k->k+1$

  $3a_{k+2}(k+2+r)\,(k+r)+a_{k+1}(-3k+3-3r)-6a_{k}=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{ka_{k+1}+ra_{k+1}+2a_{k}-a_{k+1}}{(k+2+r)(k+r)}$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{ka_{k+1}+2a_{k}-a_{k+1}}{(k+2)k}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$

  $a_{k+2}=\frac{ka_{k+1}+2a_{k}-a_{k+1}}{(k+2)k}$
- Recursion relation for $r=2$

$a_{k+2} = \frac{k a_{k+1}+2 a_k+a_{k+1}}{(k+4)(k+2)}$

- Solution for $r = 2$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{k a_{k+1}+2 a_k+a_{k+1}}{(k+4)(k+2)}, 9 a_1 = 0\right]$$

- Revert the change of variables $u = x - \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k \left(x-\frac{1}{3}\right)^{k+2}, a_{k+2} = \frac{k a_{k+1}+2 a_k+a_{k+1}}{(k+4)(k+2)}, 9 a_1 = 0\right]$$

1.465.3 Maple trace

```
Methods for second order ODEs:
```

1.465.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 18

```
dsolve((3*x-1)*diff(diff(y(x),x),x)-(3*x+2)*diff(y(x),x)-(6*x-8)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \,\mathrm{e}^{2x} + c_2 x \,\mathrm{e}^{-x}$$

1.465.5 Mathematica DSolve solution

Solving time : 0.103 (sec)
Leaf size : 35

```
DSolve[{(3*x-1)*D[y[x],{x,2}]-(3*x+2)*D[y[x],x]-(6*x-8)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x-\frac{1}{2}}\left(c_1 e^{3x} + 2 e c_2 x\right)}{\sqrt{2}}$$

1.466 problem 481

Internal problem ID [8604]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 481
Date solved : `Monday, October 21, 2024 at 05:10:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2+x)\,y''+xy'+3y=0$$

1.466.1 Solved as second order ode using Kovacic algorithm

Time used: 0.323 (sec)

Writing the ode as

$$(2+x)\,y''+xy'+3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2+x\\B&=x\\C&=3\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 12x - 20}{4\left(2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 12x - 20$$
$$t = 4(2+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 12x - 20}{4\left(2+x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 886: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2+x)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(2+x)^2} - \frac{4}{2+x}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{4}{x} - \frac{6}{x^2} - \frac{72}{x^3} - \frac{556}{x^4} - \frac{5440}{x^5} - \frac{55088}{x^6} - \frac{586688}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 12x - 20}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-16x - 24}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{-16x - 24}{4x^2 + 16x + 16}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -16. Dividing this by leading coefficient in t which is 4 gives -4. Now b can be found.

$$\begin{aligned} b &= (-4) - (0) \\ &= -4 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-4}{\frac{1}{2}} - 0\right) &&= -4 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-4}{\frac{1}{2}} - 0\right) &&= 4 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 12x - 20}{4\,(2 + x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-4	4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 4$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= 4 - (2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{2}{2+x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{2}{2+x} - \frac{1}{2} \\
&= -\frac{x-2}{2(2+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{2}{2+x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{2}{(2+x)^2} \right) + \left(\frac{2}{2+x} - \frac{1}{2} \right)^2 - \left(\frac{x^2 - 12x - 20}{4(2+x)^2} \right) \right) = 0$$

$$\frac{(a_1 + 6)\, x + 2a_0 + 2a_1 + 4}{2+x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4, a_1 = -6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 6x + 4$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x^2 - 6x + 4)\, e^{\int \left(\frac{2}{2+x} - \frac{1}{2}\right) dx} \\
&= (x^2 - 6x + 4)\, e^{-\frac{x}{2} + 2\ln(2+x)} \\
&= (x^2 - 6x + 4)\,(2+x)^2\, e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{2+x}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\ln(2+x)} \\
&= z_1\left((2+x)\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (2+x)^3\,\mathrm{e}^{-x}\left(x^2-6x+4\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{2+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+2\ln(2+x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{x}\left(x^4-x^3-18x^2-22x+8\right)}{240\left(x^2-6x+4\right)(2+x)^3}-\frac{\mathrm{e}^{-2}\,\mathrm{Ei}_1\left(-2-x\right)}{240}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left((2+x)^3\,\mathrm{e}^{-x}\left(x^2-6x+4\right)\right)+c_2\Bigg((2+x)^3\,\mathrm{e}^{-x}\left(x^2-6x\right. \\
&\qquad \left.+4\right)\left(-\frac{\mathrm{e}^{x}\left(x^4-x^3-18x^2-22x+8\right)}{240\left(x^2-6x+4\right)(2+x)^3}-\frac{\mathrm{e}^{-2}\,\mathrm{Ei}_1\left(-2-x\right)}{240}\right)\Bigg)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.466.2 Maple step by step solution

Let's solve

$(2+x)\left(\frac{d}{dx}y'\right)+xy'+3y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{3y}{2+x}-\frac{xy'}{2+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{xy'}{2+x}+\frac{3y}{2+x}=0$

□ Check to see if $x_0=-2$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{x}{2+x}, P_3(x)=\frac{3}{2+x}\right]$

- ◦ $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((2+x)\cdot P_2(x)\right)\bigg|_{x=-2}=-2$

- ◦ $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((2+x)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

- ◦ $x=-2$is a regular singular point

Check to see if $x_0=-2$ is a regular singular point

$x_0=-2$

- Multiply by denominators

$(2+x)\left(\frac{d}{dx}y'\right)+xy'+3y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)+3y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k->k+1$

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r+3)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 3\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r+3) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k(k+r+3)}{(k+1+r)(k-2+r)}$
- Recursion relation for $r = 0$
  $a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-2)}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$
  $a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-2)}$
- Recursion relation for $r = 3$
  $a_{k+1} = -\frac{a_k(k+6)}{(k+4)(k+1)}$
- Solution for $r = 3$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+6)}{(k+4)(k+1)}\right]$
- Revert the change of variables $u = 2 + x$
  $\left[y = \sum_{k=0}^{\infty} a_k(2+x)^{k+3}, a_{k+1} = -\frac{a_k(k+6)}{(k+4)(k+1)}\right]$

1.466.3 Maple trace

```
Methods for second order ODEs:
```

1.466.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 72

```
dsolve((2+x)*diff(diff(y(x),x),x)+x*diff(y(x),x)+3*y(x) = 0,
       y(x),singsol=all)
```

$$
\begin{aligned}
y = {} & c_2\,\mathrm{e}^{-2-x}(2+x)^3\left(x^2-6x+4\right)\mathrm{Ei}_1\left(-2-x\right) \\
& + c_1(2+x)^3\,\mathrm{e}^{-x}\left(x^2-6x+4\right)+c_2\left(x^4-x^3-18x^2-22x+8\right)
\end{aligned}
$$

1.466.5 Mathematica DSolve solution

Solving time : 0.207 (sec)
Leaf size : 81

```
DSolve[{(2+x)*D[y[x],{x,2}]+x*D[y[x],x]+3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to \frac{1}{960}e^{-x-1}\big(& c_2\left(x^2-6x+4\right)(x+2)^3\,\mathrm{ExpIntegralEi}(x+2) \\
& + 3840c_1\left(x^2-6x+4\right)(x+2)^3 - c_2e^{x+2}\left(x^4-x^3-18x^2-22x+8\right)\big)
\end{aligned}
$$

1.467 problem 482

Internal problem ID [8605]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 482
Date solved : `Monday, October 21, 2024 at 05:10:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-x)\,y''+x(4+x)\,y'+(2-x)\,y=0$$

1.467.1 Solved as second order ode using Kovacic algorithm

Time used: 0.248 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(x^2+4x\right)y'+(2-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+x^2 \\ B &= x^2+4x \\ C &= 2-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x + 36}{4x\left(-1 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x + 36 \\ t &= 4x(-1 + x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x + 36}{4x\left(-1 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 888: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(-1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{x} + \frac{35}{4\left(-1+x\right)^2} - \frac{9}{-1+x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+36}{4x\left(-1+x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x+36}{4x\left(-1+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
1	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} - \frac{5}{2\,(-1+x)} + (-)\,(0) \\
&= \frac{1}{x} - \frac{5}{2\,(-1+x)} \\
&= \frac{1}{x} - \frac{5}{-2+2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{1}{x} - \frac{5}{2\,(-1+x)}\right)(2x + a_1) + \left(\left(-\frac{1}{x^2} + \frac{5}{2\,(-1+x)^2}\right) + \left(\frac{1}{x} - \frac{5}{2\,(-1+x)}\right)^2 - \left(\frac{-x+36}{4x\,(-1+x)^2}\right.\right.
$$

$$
\frac{(a_1-6)\,x + 4a_0 - 2a_1}{x\,(-1+x)}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 3, a_1 = 6\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + 6x + 3
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + 6x + 3\right) e^{\int \left(\frac{1}{x} - \frac{5}{2(-1+x)}\right) dx} \\
&= \left(x^2 + 6x + 3\right) e^{-\frac{5\ln(-1+x)}{2} + \ln(x)} \\
&= \frac{\left(x^2 + 6x + 3\right) x}{(-1+x)^{5/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+4x}{-x^3+x^2}\,dx} \\
&= z_1 e^{\frac{5\ln(-1+x)}{2}-2\ln(x)} \\
&= z_1\left(\frac{(-1+x)^{5/2}}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2+6x+3}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+4x}{-x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(-1+x)-4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{4\left(-38x-\frac{69}{2}\right)}{9\left(x^2+6x+3\right)}+\ln\left(x\right)+\frac{1}{9x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2+6x+3}{x}\right)+c_2\left(\frac{x^2+6x+3}{x}\left(-\frac{4\left(-38x-\frac{69}{2}\right)}{9\left(x^2+6x+3\right)}+\ln\left(x\right)+\frac{1}{9x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.467.2 Maple step by step solution

1.467.3 Maple trace

```
Methods for second order ODEs:
```

1.467.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 49

```
dsolve(x^2*(1-x)*diff(diff(y(x),x),x)+x*(4+x)*diff(y(x),x)+(2-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{3xc_2(x^2 + 6x + 3) \ln (x) + c_1 x^3 + (6c_1 + 51c_2) x^2 + (3c_1 + 48c_2) x + c_2}{x^2}$$

1.467.5 Mathematica DSolve solution

Solving time : 0.112 (sec)
Leaf size : 53

```
DSolve[{x^2*(1-x)*D[y[x],{x,2}]+x*(4+x)*D[y[x],x]+(2-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3c_1x(x^2 + 6x + 3) - c_2(51x^2 + 3(x^2 + 6x + 3) x \log(x) + 48x + 1)}{3x^2}$$

1.468 problem 483

Internal problem ID [8606]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 483
Date solved : `Monday, October 21, 2024 at 05:10:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+x(1+2x)\,y'-(4+6x)\,y=0$$

1.468.1 Solved as second order ode using Kovacic algorithm

Time used: 0.253 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(2x^2+x\right)y'+(-6x-4)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x)\\ B &= 2x^2+x\\ C &= -6x-4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{24x^2 + 40x + 15}{4(x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 24x^2 + 40x + 15$$
$$t = 4(x^2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{24x^2 + 40x + 15}{4(x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 889: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{2x} - \frac{1}{4(1+x)^2} - \frac{5}{2(1+x)} + \frac{15}{4x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{24x^2 + 40x + 15}{4(x^2 + x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=6$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=3\\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-2\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{24x^2+40x+15}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=3$ then

$$\begin{aligned}d &= \alpha_\infty^+-\left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right)\\ &= 3-(3)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2+2x} + \frac{5}{2x} + (0) \\
&= \frac{1}{2+2x} + \frac{5}{2x} \\
&= \frac{6x+5}{2x\,(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2+2x} + \frac{5}{2x}\right)(0) + \left(\left(-\frac{1}{2\,(1+x)^2} - \frac{5}{2x^2}\right) + \left(\frac{1}{2+2x} + \frac{5}{2x}\right)^2 - \left(\frac{24x^2+40x+15}{4\,(x^2+x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{2+2x} + \frac{5}{2x}\right) dx} \\
&= \sqrt{1+x}\, x^{5/2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2+x}{x^2(1+x)}\, dx} \\
&= z_1 e^{-\frac{\ln(x(1+x))}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x\,(1+x)}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\sqrt{1+x}\, x^{5/2}}{\sqrt{x(1+x)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+x}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x(1+x))}}{(y_1)^2}\, dx \\ &= y_1 \left(-\ln(1+x) - \frac{1}{4x^4} - \frac{1}{2x^2} + \ln(x) + \frac{1}{3x^3} + \frac{1}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} &y = c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{1+x}\, x^{5/2}}{\sqrt{x(1+x)}}\right) + c_2 \left(\frac{\sqrt{1+x}\, x^{5/2}}{\sqrt{x(1+x)}} \left(-\ln(1+x) - \frac{1}{4x^4} - \frac{1}{2x^2} + \ln(x) + \frac{1}{3x^3} + \frac{1}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.468.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right) + x(1+2x)\, y' - (4+6x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{2(2+3x)y}{x^2(1+x)} - \frac{(1+2x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(1+2x)y'}{x(1+x)} - \frac{2(2+3x)y}{x^2(1+x)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{1+2x}{x(1+x)}, P_3(x) = -\frac{2(2+3x)}{x^2(1+x)}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = 1$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x^2(1+x)\left(\frac{d}{dx}y'\right) + x(1+2x)\,y' + (-6x-4)\,y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - 2u^2 + u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (2u^2 - 3u + 1)\left(\frac{d}{du}y(u)\right) + (-6u + 2)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 u^{-1+r} + \left(a_1(1+r)^2 - a_0(2r^2+r-2)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k^2+4kr+2r^2+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term must be 0

$$a_1(1+r)^2 - a_0(2r^2+r-2) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2-k-6)\, a_{k-1} + (-2k^2-k+2)\, a_k + a_{k+1}(k+1)^2 = 0$$

- Shift index using $k->k+1$

$$\left((k+1)^2-k-7\right) a_k + \left(-2(k+1)^2-k+1\right) a_{k+1} + a_{k+2}(k+2)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 5ka_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 5ka_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 5ka_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}, a_1 + 2a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 5ka_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}, a_1 + 2a_0 = 0\right]$$

1.468.3 Maple trace

```
Methods for second order ODEs:
```

1.468.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 46

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)+x*(1+2*x)*diff(y(x),x)-(4+6*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x^2 + \frac{c_2(12 \ln (1+x) x^4 - 12 \ln (x) x^4 - 12x^3 + 6x^2 - 4x + 3)}{x^2}$$

1.468.5 Mathematica DSolve solution

Solving time : 0.072 (sec)
Leaf size : 52

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+x*(1+2*x)*D[y[x],x]-(4+6*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x^2 + \frac{c_2(12x^4 \log(x) - 12x^4 \log(x+1) + 12x^3 - 6x^2 + 4x - 3)}{12x^2}$$

1.469 problem 484

Internal problem ID [8607]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 484
Date solved : `Monday, October 21, 2024 at 05:10:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(2x^2+1)\,y''+x(2x^2+4)\,y'+2(-x^2+1)\,y=0$$

1.469.1 Solved as second order ode using Kovacic algorithm

Time used: 0.369 (sec)

Writing the ode as

$$\left(2x^4+x^2\right)y''+\left(2x^3+4x\right)y'+\left(-2x^2+2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^4+x^2\\ B&=2x^3+4x\\ C&=-2x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 9}{(2x^2 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^2 - 9$$
$$t = (2x^2 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 9}{(2x^2 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 891: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + 1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{21}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} + \frac{15i\sqrt{2}}{16\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{15i\sqrt{2}}{16\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 9}{(2x^2 + 1)^2}$$

Since the gcd$(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 9}{(2x^2 + 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^- + \alpha_{c_2}^-) \\
&= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)} + (-)(0) \\
&= -\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)} \\
&= -\frac{3x}{2x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)} \right)(1) + \left(\left(\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)^2} + \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)^2} \right) + \left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)}-\frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)}\right) dx} \\
&= (x) \frac{1}{\left(\left(i\sqrt{2}-2x\right)\left(2x+i\sqrt{2}\right)\right)^{3/4}} \\
&= \frac{x}{\left(-4x^2-2\right)^{3/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+4x}{2x^4+x^2} \, dx} \\
&= z_1 e^{-2\ln(x)+\frac{3\ln\left(2x^2+1\right)}{4}} \\
&= z_1 \left(\frac{\left(2x^2+1\right)^{3/4}}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{1/4}\left(4x^2+2\right)^{3/4}}{2x\left(-4x^2-2\right)^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+4x}{2x^4+x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-4\ln(x)+\frac{3\ln\left(2x^2+1\right)}{2}}}{\left(y_1\right)^2} \, dx
\end{aligned}
$$

$$
= y_1 \left(-\frac{2i(2x^4 - x^2 - 1)\sqrt{2x^2+1}\,\sqrt{2}\,\sqrt{\frac{(-4x^2-2)(4x^2+2)}{(2x^2+1)^2}}}{x\sqrt{-4x^2-2}\,\sqrt{4x^2+2}} - \frac{6i \operatorname{arcsinh}\left(\sqrt{2}\,x\right)\sqrt{\frac{(-4x^2-2)(4x^2+2)}{(2x^2+1)^2}}\,(2x^2+1)}{\sqrt{-4x^2-2}\,\sqrt{4x^2+2}} \right)
$$

Therefore the solution is

$$
y = c_1 y_1 + c_2 y_2
$$

$$
= c_1 \left(\frac{2^{1/4}(4x^2+2)^{3/4}}{2x\,(-4x^2-2)^{3/4}} \right) + c_2 \left(\frac{2^{1/4}(4x^2+2)^{3/4}}{2x\,(-4x^2-2)^{3/4}} \left(-\frac{2i(2x^4 - x^2 - 1)\sqrt{2x^2+1}\,\sqrt{2}\,\sqrt{\frac{(-4x^2-2)(4x^2+2)}{(2x^2+1)^2}}}{x\sqrt{-4x^2-2}\,\sqrt{4x^2+2}} - \frac{6i
$$

Will add steps showing solving for IC soon.

1.469.2 Maple step by step solution

Let's solve

$x^2(2x^2+1)\left(\frac{d}{dx}y'\right) + x(2x^2+4)\,y' + 2(-x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{2(x^2-1)y}{x^2(2x^2+1)} - \frac{2(x^2+2)y'}{x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2(x^2+2)y'}{x(2x^2+1)} - \frac{2(x^2-1)y}{x^2(2x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2(x^2+2)}{x(2x^2+1)}, P_3(x) = -\frac{2(x^2-1)}{x^2(2x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 2$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(2x^2+1)\left(\frac{d}{dx}y'\right) + 2x(x^2+2)\,y' + (-2x^2+2)\,y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(1+r)\,x^r + a_1(3+r)\,(2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)\,(k+r+1) + 2a_{k-2}(k+r-1\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$

- Each term must be 0

  $a_1(3+r)(2+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r+1) + 2a_{k-2}(k+r-1)(k-3+r) = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+4+r)(k+3+r) + 2a_k(k+r+1)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k(k+r+1)(k+r-1)}{(k+4+r)(k+3+r)}$
- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}$
- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = -1$; series terminates at $k = 2$

  $a_{k+2} = -\frac{2a_k k(k-2)}{(k+3)(k+2)}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k k(k-2)}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{2b_k k(k-2)}{(k+3)(k+2)}, b_1 = 0\right]$

1.469.3 Maple trace

```
Methods for second order ODEs:
```

1.469.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 42

```
dsolve(x^2*(2*x^2+1)*diff(diff(y(x),x),x)+x*(2*x^2+4)*diff(y(x),x)+2*(-x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\sqrt{2}\,(x-1)\,(x+1)\,\sqrt{2x^2+1}+x\left(3c_2 \operatorname{arcsinh}\left(\sqrt{2}\,x\right)+c_1\right)}{x^2}$$

1.469.5 Mathematica DSolve solution

Solving time : 0.153 (sec)
Leaf size : 70

```
DSolve[{x^2*(1+2*x^2)*D[y[x],{x,2}]+x*(4+2*x^2)*D[y[x],x]+2*(1-x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3c_2\text{arctanh}\left(\frac{x}{\sqrt{x^2+\frac{1}{2}}}\right)}{\sqrt{2}x} - \frac{c_2\sqrt{2x^2+1}}{x^2} + c_2\sqrt{2x^2+1} + \frac{c_1}{x}$$

1.470 problem 485

Internal problem ID [8608]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 485
Date solved : `Monday, October 21, 2024 at 05:10:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+2\right)y''+2x\left(x^2+5\right)y'+2\left(-x^2+3\right)y=0$$

1.470.1 Solved as second order ode using Kovacic algorithm

Time used: 0.488 (sec)

Writing the ode as

$$\left(x^4+2x^2\right)y''+\left(2x^3+10x\right)y'+\left(-2x^2+6\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+2x^2 \\ B &= 2x^3+10x \\ C &= -2x^2+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^4 - 5x^2 + 3}{(x^3 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^4 - 5x^2 + 3$$
$$t = (x^3 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^4 - 5x^2 + 3}{(x^3 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 893: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16\left(x - i\sqrt{2}\right)^2} + \frac{21}{16\left(x + i\sqrt{2}\right)^2} + \frac{11i\sqrt{2}}{32\left(x - i\sqrt{2}\right)} - \frac{11i\sqrt{2}}{32\left(x + i\sqrt{2}\right)} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{2x^4 - 5x^2 + 3}{\left(x^3 + 2x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^4 - 5x^2 + 3}{\left(x^3 + 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$i\sqrt{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i\sqrt{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= 2 - (0) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{3}{2x} - \frac{3}{4\left(x - i\sqrt{2}\right)} - \frac{3}{4\left(x + i\sqrt{2}\right)} + (0) \\ &= \frac{3}{2x} - \frac{3}{4\left(x - i\sqrt{2}\right)} - \frac{3}{4\left(x + i\sqrt{2}\right)} \\ &= \frac{3}{x^3 + 2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2x} - \frac{3}{4\left(x - i\sqrt{2}\right)} - \frac{3}{4\left(x + i\sqrt{2}\right)} \right)(2x + a_1) + \left(\left(-\frac{3}{2x^2} + \frac{3}{4\left(x - i\sqrt{2}\right)^2} + \frac{3}{4\left(x + i\sqrt{2}\right)^2} \right) + \right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^2 + 8) \, e^{\int \left(\frac{3}{2x} - \frac{3}{4(x - i\sqrt{2})} - \frac{3}{4(x + i\sqrt{2})}\right) dx} \\
&= (x^2 + 8) \, e^{\frac{3 \ln(x)}{2} - \frac{3 \ln(x^2+2)}{4}} \\
&= \frac{(x^2 + 8) \, x^{3/2}}{(x^2 + 2)^{3/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^3+10x}{x^4+2x^2} \, dx} \\
&= z_1 e^{-\frac{5 \ln(x)}{2} + \frac{3 \ln(x^2+2)}{4}} \\
&= z_1 \left(\frac{(x^2 + 2)^{3/4}}{x^{5/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 + 8}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+10x}{x^4+2x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-5 \ln(x) + \frac{3 \ln(x^2+2)}{2}}}{(y_1)^2} \, dx
\end{aligned}$$

$$= y_1 \left(\frac{(x^2+2)^{3/2}}{384} + \frac{\sqrt{x^2+2}}{96} - \frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)}{64} - \frac{(x^2+2)^{5/2}}{256x^2} + \frac{3\sqrt{x^2+2}}{64\,(x^2+8)} + \frac{x^2\sqrt{x^2+2}}{768} \right)$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^2+8}{x}\right) + c_2 \left(\frac{x^2+8}{x} \left(\frac{(x^2+2)^{3/2}}{384} + \frac{\sqrt{x^2+2}}{96} - \frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)}{64} - \frac{(x^2+2)^{5/2}}{256x^2} + \frac{3\sqrt{x^2+2}}{64\,(x^2+8)} + \frac{x^2\sqrt{x^2+2}}{768} \right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.470.2 Maple step by step solution

Let's solve

$x^2(x^2+2)\left(\frac{d}{dx}y'\right) + 2x(x^2+5)\,y' + 2(-x^2+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{2(x^2-3)y}{x^2(x^2+2)} - \frac{2(x^2+5)y'}{x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2(x^2+5)y'}{x(x^2+2)} - \frac{2(x^2-3)y}{x^2(x^2+2)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2(x^2+5)}{x(x^2+2)}, P_3(x) = -\frac{2(x^2-3)}{x^2(x^2+2)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 5$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 3$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2(x^2+2)\left(\frac{d}{dx}y'\right) + 2x(x^2+5)\,y' + (-2x^2+6)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k- >k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

    $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - Shift index using $k- >k+2-m$

    $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$2a_0(3+r)\,(1+r)\,x^r + 2a_1(4+r)\,(2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(2a_k(k+r+3)\,(k+r+1) + a_{k-2}(k+r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2(3+r)\,(1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-3, -1\}$

- Each term must be 0

  $2a_1(4+r)(2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $2a_k(k+r+3)(k+r+1) + a_{k-2}(k+r)(k-3+r) = 0$

- Shift index using $k- >k+2$

  $2a_{k+2}(k+5+r)(k+r+3) + a_k(k+r+2)(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+2)(k+r-1)}{2(k+5+r)(k+r+3)}$

- Recursion relation for $r = -3$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}$

- Solution for $r = -3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}, a_1 = 0\right]$

- Recursion relation for $r = -1$; series terminates at $k = 2$

  $a_{k+2} = -\frac{a_k(k+1)(k-2)}{2(k+4)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k(k+1)(k-2)}{2(k+4)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}, a_1 = 0, b_{k+2} = -\frac{b_k(k+1)(k-2)}{2(k+4)(k+2)}, b_1 = 0\right]$

1.470.3 Maple trace

```
Methods for second order ODEs:
```

1.470.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 53

```
dsolve(x^2*(x^2+2)*diff(diff(y(x),x),x)+2*x*(x^2+5)*diff(y(x),x)+2*(-x^2+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-\sqrt{x^2+2}\,(x-2)\,(x+2)\,c_2\sqrt{2}+x^2(x^2+8)\left(\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)c_2+c_1\right)}{x^3}$$

1.470.5 Mathematica DSolve solution

Solving time : 0.183 (sec)
Leaf size : 88

```
DSolve[{x^2*(2+x^2)*D[y[x],{x,2}]+2*x*(x^2+5)*D[y[x],x]+2*(3-x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-\sqrt{2}c_2(x^2+8)\,x^2\operatorname{arctanh}\left(\frac{\sqrt{x^2+2}}{\sqrt{2}}\right)+64c_1x^4+2x^2\left(c_2\sqrt{x^2+2}+256c_1\right)-8c_2\sqrt{x^2+2}}{64x^3}$$

1.471 problem 486

Internal problem ID [8609]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 486
Date solved : `Monday, October 21, 2024 at 05:10:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''+6xy'+6y=0$$

1.471.1 Solved as second order ode using Kovacic algorithm

Time used: 0.283 (sec)

Writing the ode as

$$\left(x^2+1\right) y''+6xy'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 6x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (x^2+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 895: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)}\right)(0) + \left(\left(\frac{1}{2\,(x-i)^2} - \frac{3}{2\,(x+i)^2}\right) + \left(-\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)}\right)^2 - \left(-\frac{}{(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{3/2}}{(ix+1)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{6x}{x^2+1} \, dx} \\
&= z_1 e^{-\frac{3 \ln(x^2+1)}{2}} \\
&= z_1 \left(\frac{1}{(x^2+1)^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{x}{(x+i)^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{(ix+1)^2}\right) + c_2\left(\frac{1}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.471.2 Maple step by step solution

1.471.3 Maple trace

```
Methods for second order ODEs:
```

1.471.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 24

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+6*x*diff(y(x),x)+6*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\,x^2 + c_1 x - c_2}{(x^2+1)^2}$$

1.471.5 Mathematica DSolve solution

Solving time : `0.074` (sec)
Leaf size : 29

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+6*x*D[y[x],x]+6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x - c_1 (x - i)^2}{(x^2 + 1)^2}$$

1.472 problem 487

Internal problem ID [8610]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 487
Date solved : `Monday, October 21, 2024 at 05:10:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''+2xy'-2y=0$$

1.472.1 Solved as second order ode using Kovacic algorithm

Time used: 0.291 (sec)

Writing the ode as

$$\left(x^2+1\right) y''+2xy'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 + 3}{(x^2 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2 + 3 \\ t &= (x^2 + 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 + 3}{(x^2 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 896: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{5i}{4(x-i)} + \frac{5i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+3}{(x^2+1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^2+3}{(x^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=2$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x - 2i} + \frac{1}{2x + 2i} + (0) \\
&= \frac{1}{2x - 2i} + \frac{1}{2x + 2i} \\
&= \frac{x}{x^2 + 1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right)(1) + \left(\left(-\frac{1}{2\left(x - i\right)^2} - \frac{1}{2\left(x + i\right)^2}\right) + \left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right)^2 - \left(\frac{2x^2 + 3}{\left(x^2 + 1\right)^2} - \frac{2(x^2 + 1)\, a_0}{\left(-x + i\right)^2 \left(x + i\right)^2}\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right) dx} \\
&= (x) \sqrt{\left(-x + i\right)\left(x + i\right)} \\
&= x\sqrt{-x^2 - 1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x^2+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = ix
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{1}{x} + \arctan(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(ix) + c_2\left(ix\left(\frac{1}{x} + \arctan(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.472.2 Maple step by step solution

1.472.3 Maple trace

```
Methods for second order ODEs:
```

1.472.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 14

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+2*x*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + \arctan(x)\, x c_2 + c_2$$

1.472.5 Mathematica DSolve solution

Solving time : 0.033 (sec)
Leaf size : 48

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+2*x*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} i (2c_1 x - c_2 x \log(1 - ix) + c_2 x \log(1 + ix) + 2ic_2)$$

1.473 problem 488

Internal problem ID [8611]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 488
Date solved : `Monday, October 21, 2024 at 05:10:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-8xy'+20y=0$$

1.473.1 Solved as second order ode using Kovacic algorithm

Time used: 0.284 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-8xy'+20y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -8x \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-24}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -24 \\ t &= (x^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{24}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 897: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{(x-i)^2} + \frac{6}{(x+i)^2} + \frac{6i}{x-i} - \frac{6i}{x+i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{24}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	3	-2
$-i$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x-i} + \frac{3}{x+i} + (-)(0) \\
&= -\frac{2}{x-i} + \frac{3}{x+i} \\
&= \frac{x-5i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x-i} + \frac{3}{x+i}\right)(0) + \left(\left(\frac{2}{(x-i)^2} - \frac{3}{(x+i)^2}\right) + \left(-\frac{2}{x-i} + \frac{3}{x+i}\right)^2 - \left(-\frac{24}{(x^2+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{2}{x-i} + \frac{3}{x+i}\right) dx} \\ &= \frac{(x^2+1)^3}{(ix+1)^5} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-8x}{x^2+1} \, dx} \\ &= z_1 e^{2 \ln(x^2+1)} \\ &= z_1 \left(\left(x^2+1\right)^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^5}{(ix+1)^5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x}{x^2+1}\,dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{4\ln(x^2+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^4-2x^2+\frac{1}{5}}{(x+i)^5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^5}{(ix+1)^5}\right) + c_2 \left(\frac{(x^2+1)^5}{(ix+1)^5} \left(\frac{x^4-2x^2+\frac{1}{5}}{(x+i)^5}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.473.2 Maple step by step solution

1.473.3 Maple trace

```
Methods for second order ODEs:
```

1.473.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 33

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-8*x*diff(y(x),x)+20*y(x) = 0,
       y(x),singsol=all)
```

$$
y = c_2\, x^5 + 5c_1\, x^4 - 10c_2\, x^3 - 10c_1\, x^2 + 5c_2 x + c_1
$$

1.473.5 Mathematica DSolve solution

Solving time : 0.106 (sec)
Leaf size : 38

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-8*x*D[y[x],x]+20*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{5} i c_2 \left(5x^4 - 10x^2 + 1\right) + c_1 (1 + ix)^5$$

1.474 problem 489

Internal problem ID [8612]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 489
Date solved : `Monday, October 21, 2024 at 05:10:26 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-8 x y'-12 y=0$$

1.474.1 Solved as second order ode using Kovacic algorithm

Time used: 0.200 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-8 x y'-12 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -8 x \\ C &= -12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2 A} d x}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8 \\ t &= (x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8}{(x^2 - 1)^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 898: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x+1} + \frac{2}{(x+1)^2} + \frac{2}{(x-1)^2} - \frac{2}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8}{\left(x^2-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	2	-1
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x-1} + \frac{2}{x+1} + (-)\,(0) \\
&= -\frac{1}{x-1} + \frac{2}{x+1} \\
&= \frac{x-3}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-1} + \frac{2}{x+1}\right)(0) + \left(\left(\frac{1}{(x-1)^2} - \frac{2}{(x+1)^2}\right) + \left(-\frac{1}{x-1} + \frac{2}{x+1}\right)^2 - \left(\frac{8}{(x^2-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{x-1} + \frac{2}{x+1}\right) dx} \\ &= \frac{(x+1)^2}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-8x}{-x^2+1} \, dx} \\ &= z_1 e^{-2\ln(x-1) - 2\ln(x+1)} \\ &= z_1 \left(\frac{1}{(x-1)^2 (x+1)^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(x-1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-8x}{-x^2+1}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-4\ln(x-1)-4\ln(x+1)}}{(y_1)^2}\,dx$$
$$= y_1 \left(-\frac{(x+1)\,(3x^2+1)\,(x-1)^4\,\mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{1}{(x-1)^3}\right) + c_2 \left(\frac{1}{(x-1)^3}\left(-\frac{(x+1)\,(3x^2+1)\,(x-1)^4\,\mathrm{e}^{-4\ln(x-1)-4\ln(x+1)}}{3}\right)\right)$$

Will add steps showing solving for IC soon.

1.474.2 Maple step by step solution

Let's solve
$(-x^2+1)\left(\frac{d}{dx}y'\right) - 8xy' - 12y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{12y}{x^2-1} - \frac{8xy'}{x^2-1}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + \frac{8xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{8x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$
  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$
    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = 4$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2 \cdot P_3(x)\right)\Bigg|_{x=-1} = 0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d}{dx}y'\right) + 8xy' + 12y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (8u-8)\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0 r(3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r+4) + a_k(k+r+4)(k+r+3)\right) u^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $\left((-2k-2r-2)\, a_{k+1} + a_k(k+r+3)\right)(k+r+4) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+3)}{2(k+1+r)}$

- Recursion relation for $r = -3$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$

- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+3)}{2(k+1)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k , a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

1.474.3 Maple trace

```
Methods for second order ODEs:
```

1.474.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 29

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-8*x*diff(y(x),x)-12*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 x^3 + 3c_1 x^2 + 3c_2 x + c_1}{(x^2-1)^3}$$

1.474.5 Mathematica DSolve solution

Solving time : `0.071` (sec)
Leaf size : 37

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-8*x*D[y[x],x]-12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3c_1(x-1)^3 - c_2(3x^2+1)}{3\left(x^2-1\right)^3}$$

1.475 problem 490

Internal problem ID [8613]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 490
Date solved : `Monday, October 21, 2024 at 05:10:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+1\right)y''+7xy'+2y=0$$

1.475.1 Solved as second order ode using Kovacic algorithm

Time used: 0.347 (sec)

Writing the ode as

$$\left(2x^2+1\right)y''+7xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+1 \\ B &= 7x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 6}{4(2x^2 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 + 6$$
$$t = 4(2x^2 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 6}{4(2x^2 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 900: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{7}{64\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{17i\sqrt{2}}{64\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{17i\sqrt{2}}{64\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2+6}{4\left(2x^2+1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2+6}{4\left(2x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{5}{4} - \left(\frac{1}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}} + (0) \\ &= \frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}} \\ &= \frac{x}{4x^2+2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}} \right)(1) + \left(\left(-\frac{1}{8\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{8\left(x + \frac{i\sqrt{2}}{2}\right)^2} \right) + \left(\frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int \left(\frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}} \right) dx} \\ &= (x) \left(\left(i\sqrt{2} - 2x \right) \left(2x + i\sqrt{2} \right) \right)^{1/8} \\ &= x\left(-4x^2 - 2\right)^{1/8}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x}{2x^2+1}\, dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^2+1\right)}{8}} \\
&= z_1 \left(\frac{1}{\left(2x^2+1\right)^{7/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{7/8} x (-4x^2-2)^{1/8}}{\left(4x^2+2\right)^{7/8}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x}{2x^2+1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^2+1\right)}{4}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\left(2x^2+1\right)^{7/4} x^2 \left(-4x^2-2\right)^{1/4}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2^{7/8} x (-4x^2-2)^{1/8}}{\left(4x^2+2\right)^{7/8}}\right) + c_2 \left(\frac{2^{7/8} x (-4x^2-2)^{1/8}}{\left(4x^2+2\right)^{7/8}} \left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\left(2x^2+1\right)^{7/4} x^2 \left(-4x^2-2\right)^{1/4}} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.475.2 Maple step by step solution

1.475.3 Maple trace

```
Methods for second order ODEs:
```

1.475.4 Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 37

```
dsolve((2*x^2+1)*diff(diff(y(x),x),x)+7*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \operatorname{LegendreP}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\, x\right) + c_2 \operatorname{LegendreQ}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\, x\right)}{\left(2x^2+1\right)^{3/8}}$$

1.475.5 Mathematica DSolve solution

Solving time : 0.096 (sec)
Leaf size : 66

```
DSolve[{(1+2*x^2)*D[y[x],{x,2}]+7*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 Q_{\frac{1}{4}}^{\frac{3}{4}}\left(i\sqrt{2}x\right)}{\left(2x^2+1\right)^{3/8}} + \frac{2i\sqrt[4]{2}c_1 x}{\left(2x^2+1\right)^{3/4} \operatorname{Gamma}\left(\frac{1}{4}\right)}$$

1.476 problem 491

Internal problem ID [8614]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 491
Date solved : `Monday, October 21, 2024 at 05:10:28 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-5xy'-4y=0$$

1.476.1 Solved as second order ode using Kovacic algorithm

Time used: 0.286 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-5xy'-4y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -5x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{4(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 6$$
$$t = 4(x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{4(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 901: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{16(x-1)} + \frac{5}{16(x+1)^2} + \frac{5}{16(x-1)^2} + \frac{7}{16(x+1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2+6}{4(x^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-x^2+6}{4\left(x^2-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
-1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4(x-1)} - \frac{1}{4(x+1)} + (-)(0) \\
&= -\frac{1}{4(x-1)} - \frac{1}{4(x+1)} \\
&= -\frac{x}{2x^2 - 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4(x-1)} - \frac{1}{4(x+1)}\right)(1) + \left(\left(\frac{1}{4(x-1)^2} + \frac{1}{4(x+1)^2}\right) + \left(-\frac{1}{4(x-1)} - \frac{1}{4(x+1)}\right)^2 - \Big(
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(-\frac{1}{4(x-1)} - \frac{1}{4(x+1)}\right) dx} \\
&= (x) \frac{1}{((x-1)(x+1))^{1/4}} \\
&= \frac{x}{(x^2 - 1)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x}{-x^2+1}\,dx} \\
&= z_1 e^{-\frac{5\ln(x-1)}{4}-\frac{5\ln(x+1)}{4}} \\
&= z_1 \left(\frac{1}{(x-1)^{5/4}(x+1)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{(x-1)^{5/4}(x+1)^{5/4}(x^2-1)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x-1)}{2}-\frac{5\ln(x+1)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(x^2-1)^{3/2}}{x} - x\sqrt{x^2-1} + \ln\left(x+\sqrt{x^2-1}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{(x-1)^{5/4}(x+1)^{5/4}(x^2-1)^{1/4}}\right) + c_2 \left(\frac{x}{(x-1)^{5/4}(x+1)^{5/4}(x^2-1)^{1/4}} \left(\frac{(x^2-1)^{3/2}}{x} - x\sqrt{x^2-1} +
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.476.2 Maple step by step solution

Let's solve

$\left(-x^2+1\right)\left(\frac{d}{dx}y'\right)-5xy'-4y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{4y}{x^2-1}-\frac{5xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{5xy'}{x^2-1}+\frac{4y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{5x}{x^2-1}, P_3(x)=\frac{4}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=\frac{5}{2}$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$\left(x^2-1\right)\left(\frac{d}{dx}y'\right)+5xy'+4y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$\left(u^2-2u\right)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\left(5u-5\right)\left(\frac{d}{du}y(u)\right)+4y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^{m} \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^{m} \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^{m} \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(3+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+5+2r) + a_k(k+r+2)^2\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$-r(3+2r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{0, -\frac{3}{2}\right\}$$
- Each term in the series must be 0, giving the recursion relation
$$a_k(k+r+2)^2 - 2a_{k+1}(k+1+r)\left(k+\frac{5}{2}+r\right) = 0$$
- Recursion relation that defines series solution to ODE
$$a_{k+1} = \frac{a_k(k+r+2)^2}{(k+1+r)(2k+5+2r)}$$
- Recursion relation for $r = 0$
$$a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}$$
- Solution for $r = 0$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}\right]$$
- Revert the change of variables $u = x+1$
$$\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}\right]$$
- Recursion relation for $r = -\frac{3}{2}$
$$a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}$$
- Solution for $r = -\frac{3}{2}$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$
- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k \left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k-\frac{3}{2}}\right), a_{k+1} = \frac{a_k (k+2)^2}{(k+1)(2k+5)}, b_{k+1} = \frac{b_k \left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

1.476.3 Maple trace

```
Methods for second order ODEs:
```

1.476.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 39

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-5*x*diff(y(x),x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \ln\left(x+\sqrt{x^2-1}\right) x - \sqrt{x^2-1}\, c_2 + c_1 x}{\left(x^2-1\right)^{3/2}}$$

1.476.5 Mathematica DSolve solution

Solving time : `0.096` (sec)
Leaf size : 49

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-5*x*D[y[x],x]-4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x \operatorname{arctanh}\left(\frac{x}{\sqrt{x^2-1}}\right) - c_2\sqrt{x^2-1} + c_1 x}{\left(x^2-1\right)^{3/2}}$$

1.477 problem 492

Internal problem ID [8615]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 492
Date solved : `Monday, October 21, 2024 at 05:10:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-10 x y'+28 y=0$$

1.477.1 Solved as second order ode using Kovacic algorithm

Time used: 0.348 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-10 x y'+28 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -10 x \\ C &= 28 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 33}{(x^2 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 - 33$$
$$t = (x^2 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 33}{(x^2 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 903: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4(x-i)^2} + \frac{35}{4(x+i)^2} + \frac{31i}{4(x-i)} - \frac{31i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 33}{(x^2+1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = 2\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2-33}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-i$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right)\\
&= 2-(1)\\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{5}{2\left(x - i\right)} + \frac{7}{2\left(x + i\right)} + (0) \\
&= -\frac{5}{2\left(x - i\right)} + \frac{7}{2\left(x + i\right)} \\
&= \frac{x - 6i}{x^2 + 1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{5}{2\left(x - i\right)} + \frac{7}{2\left(x + i\right)}\right)(1) + \left(\left(\frac{5}{2\left(x - i\right)^2} - \frac{7}{2\left(x + i\right)^2}\right) + \left(-\frac{5}{2\left(x - i\right)} + \frac{7}{2\left(x + i\right)}\right)^2 - \left(\frac{2x^{\cdots}}{(x^{2\cdots}}\right.\right.
$$

$$
-\frac{2(6i + a_0)\left(x^{\cdots}\right.}{\left(-x + i\right)^2\left(x \cdots\right.}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -6i\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 6i
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - 6i\right) e^{\int \left(-\frac{5}{2(x - i)} + \frac{7}{2(x + i)}\right) dx} \\
&= \left(x - 6i\right) e^{\frac{\ln\left(x^2 + 1\right)}{2} - 6i \arctan(x)} \\
&= \frac{\left(-x + 6i\right)\left(x^2 + 1\right)^{7/2}}{\left(-x + i\right)^6}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10x}{x^2+1}\,dx} \\
&= z_1 e^{\frac{5\ln\left(x^2+1\right)}{2}} \\
&= z_1\left(\left(x^2+1\right)^{5/2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+1\right)^6(-x+6i)}{(-x+i)^6}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln\left(x^2+1\right)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{724i}{2401(x+i)^4} - \frac{16i}{147(x+i)^6} - \frac{3125i}{117649(x+i)^2} + \frac{496}{1715(x+i)^5} - \frac{7432}{50421(x+i)^3} - \frac{3125}{823543(x+i)} + \frac{3125}{823543(x-6i)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x^2+1\right)^6(-x+6i)}{(-x+i)^6}\right) \\
&\quad + c_2\left(\frac{\left(x^2+1\right)^6(-x+6i)}{(-x+i)^6}\left(\frac{724i}{2401(x+i)^4} - \frac{16i}{147(x+i)^6} - \frac{3125i}{117649(x+i)^2} + \frac{496}{1715(x+i)^5} - \frac{7432}{50421(x+i)^3} - \frac{3125}{823543(x+i)} + \frac{3125}{823543(x-6i)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.477.2 Maple step by step solution

1.477.3 Maple trace

```
Methods for second order ODEs:
```

1.477.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 39

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-10*x*diff(y(x),x)+28*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 + \frac{35}{3}c_1 x^4 - 14c_1 x^2 + c_2 x^7 + 21c_2 x^5 - 105c_2 x^3 + 35c_2 x$$

1.477.5 Mathematica DSolve solution

Solving time : 0.106 (sec)
Leaf size : 40

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-10*x*D[y[x],x]+28*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{105}c_2\left(35x^4 - 42x^2 + 3\right) - c_1(x-i)^6(x+6i)$$

1.478 problem 493

Internal problem ID [8616]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 493
Date solved : `Monday, October 21, 2024 at 05:10:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

1.478.1 Solved as second order ode using Kovacic algorithm

Time used: 0.255 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 904: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 6}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{3}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) = 0$$

$$a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, e^{\int -\frac{x}{2} dx} \\ &= (x) \, e^{-\frac{x^2}{4}} \\ &= x \, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(e^{-\frac{x^2}{4}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{2}} x\left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.478.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

1.478.3 Maple trace

```
Methods for second order ODEs:
```

1.478.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -x\left(c_2\pi \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)-c_1\right)\mathrm{e}^{-\frac{x^2}{2}}+i\sqrt{\pi}\,\sqrt{2}\,c_2$$

1.478.5 Mathematica DSolve solution

Solving time : 0.068 (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}}c_2e^{-\frac{x^2}{2}}\sqrt{x^2}\operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right)+\sqrt{2}c_1e^{-\frac{x^2}{2}}x+c_2$$

1.479 problem 495

Internal problem ID [8617]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 495
Date solved : `Monday, October 21, 2024 at 05:10:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2-8x+11\right)y''-16(x-2)\,y'+36y=0$$

1.479.1 Solved as second order ode using Kovacic algorithm

Time used: 0.579 (sec)

Writing the ode as

$$\left(2x^2-8x+11\right)y''+(-16x+32)\,y'+36y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^2-8x+11\\B&=-16x+32\\C&=36\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 8x^2 - 32x - 100$$
$$t = (2x^2 - 8x + 11)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 906: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 - 8x + 11)^2$. There is a pole at $x = 2 + \frac{i\sqrt{6}}{2}$ of order 2. There is a pole at $x = 2 - \frac{i\sqrt{6}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{\left(x - 2 - \frac{i\sqrt{6}}{2}\right)^2} + \frac{6}{\left(x - 2 + \frac{i\sqrt{6}}{2}\right)^2} + \frac{5i\sqrt{6}}{3\left(x - 2 - \frac{i\sqrt{6}}{2}\right)} - \frac{5i\sqrt{6}}{3\left(x - 2 + \frac{i\sqrt{6}}{2}\right)}$$

For the pole at $x = 2 + \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x-2-\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

For the pole at $x = 2 - \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x-2+\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$2 + \frac{i\sqrt{6}}{2}$	2	0	3	-2
$2 - \frac{i\sqrt{6}}{2}$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{2}{x-2-\frac{i\sqrt{6}}{2}} + \frac{3}{x-2+\frac{i\sqrt{6}}{2}} + (0) \\
&= -\frac{2}{x-2-\frac{i\sqrt{6}}{2}} + \frac{3}{x-2+\frac{i\sqrt{6}}{2}} \\
&= \frac{-5i\sqrt{6}+2x-4}{2x^2-8x+11}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x-2-\frac{i\sqrt{6}}{2}} + \frac{3}{x-2+\frac{i\sqrt{6}}{2}} \right)(1) + \left(\left(\frac{2}{\left(x-2-\frac{i\sqrt{6}}{2}\right)^2} - \frac{3}{\left(x-2+\frac{i\sqrt{6}}{2}\right)^2} \right) + \left(-\frac{2}{x-2-\frac{i\sqrt{6}}{2}} \right. \right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = -\frac{5i\sqrt{6}}{2} - 2 \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 2 - \frac{5i\sqrt{6}}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x-2-\frac{5i\sqrt{6}}{2}\right) \mathrm{e}^{\int \left(-\frac{2}{x-2-\frac{i\sqrt{6}}{2}}+\frac{3}{x-2+\frac{i\sqrt{6}}{2}}\right) dx} \\
&= \left(x-2-\frac{5i\sqrt{6}}{2}\right) \mathrm{e}^{\frac{\ln\left(4x^2-16x+22\right)}{2}-5i\arctan\left(\frac{(2x-4)\sqrt{6}}{6}\right)} \\
&= -\frac{9\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^3\sqrt{6}}{2\left(x\sqrt{6}-2\sqrt{6}-3i\right)^5}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-16x+32}{2x^2-8x+11}\, dx} \\
&= z_1 e^{2\ln\left(2x^2-8x+11\right)} \\
&= z_1\left(\left(2x^2-8x+11\right)^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = -\frac{9\left(2x^2-8x+11\right)^5\left(5\sqrt{6}+2ix-4i\right)\sqrt{6}}{2\left(x\sqrt{6}-2\sqrt{6}-3i\right)^5}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-16x+32}{2x^2-8x+11}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{4\ln\left(2x^2-8x+11\right)}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(-\frac{10i\sqrt{6}}{27\left(2x-4+i\sqrt{6}\right)^4}+\frac{8i\sqrt{6}}{729\left(2x-4+i\sqrt{6}\right)^2}-\frac{16}{15\left(2x-4+i\sqrt{6}\right)^5}\right. \\
&\quad \left.+\frac{22}{81\left(2x-4+i\sqrt{6}\right)^3}+\frac{4}{2187\left(2x-4+i\sqrt{6}\right)}-\frac{4}{2187\left(-5i\sqrt{6}+2x-4\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(-\frac{9(2x^2-8x+11)^5\left(5\sqrt{6}+2ix-4i\right)\sqrt{6}}{2\left(x\sqrt{6}-2\sqrt{6}-3i\right)^5}\right) \\
&\quad + c_2\left(-\frac{9(2x^2-8x+11)^5\left(5\sqrt{6}+2ix-4i\right)\sqrt{6}}{2\left(x\sqrt{6}-2\sqrt{6}-3i\right)^5}\left(-\frac{10i\sqrt{6}}{27\left(2x-4+i\sqrt{6}\right)^4}\right.\right. \\
&\quad + \frac{8i\sqrt{6}}{729\left(2x-4+i\sqrt{6}\right)^2} - \frac{16}{15\left(2x-4+i\sqrt{6}\right)^5} + \frac{22}{81\left(2x-4+i\sqrt{6}\right)^3} \\
&\quad \left.\left. + \frac{4}{2187\left(2x-4+i\sqrt{6}\right)} - \frac{4}{2187\left(-5i\sqrt{6}+2x-4\right)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.479.2 Maple step by step solution

Let's solve

$(2x^2-8x+11)\left(\frac{d}{dx}y'\right)-16(x-2)\,y'+36y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{36y}{2x^2-8x+11}+\frac{16(x-2)y'}{2x^2-8x+11}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{16(x-2)y'}{2x^2-8x+11}+\frac{36y}{2x^2-8x+11}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{16(x-2)}{2x^2-8x+11}, P_3(x)=\frac{36}{2x^2-8x+11}\right]$

  ◦ $\left(x-2+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)$ is analytic at $x=2-\frac{\mathrm{I}\sqrt{6}}{2}$

  $\left(\left(x-2+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)\right)\bigg|_{x=2-\frac{\mathrm{I}\sqrt{6}}{2}}=0$

  ◦ $\left(x-2+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2\cdot P_3(x)$ is analytic at $x=2-\frac{\mathrm{I}\sqrt{6}}{2}$

$$\left(\left(x-2+\frac{I\sqrt{6}}{2}\right)^{2}\cdot P_{3}(x)\right)\bigg|_{x=2-\frac{I\sqrt{6}}{2}}=0$$

- ○ $x=2-\frac{I\sqrt{6}}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=2-\frac{I\sqrt{6}}{2}$

- Multiply by denominators

  $(2x^2-8x+11)\left(\frac{d}{dx}y'\right)+(-16x+32)\,y'+36y=0$

- Change variables using $x=u+2-\frac{I\sqrt{6}}{2}$ so that the regular singular point is at $u=0$

  $(2u^2-2\,Iu\sqrt{6})\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-16u+8\,I\sqrt{6})\left(\frac{d}{du}y(u)\right)+36y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - ○ Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - ○ Convert $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - ○ Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2\,I\sqrt{6}\,r(r-5)\,a_0u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2\,I\sqrt{6}\,(k+1+r)\,(k-4+r)\,a_{k+1}+2a_k(k+r-3)\,(k+r-\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2\,I\sqrt{6}\,r(r-5)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,5\}$

- Each term in the series must be 0, giving the recursion relation

  $-2\,I\sqrt{6}\,(k+1+r)\,(k-4+r)\,a_{k+1}+2a_k(k+r-3)\,(k+r-6)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k (k^2+2kr+r^2-9k-9r+18)\sqrt{6}}{k^2+2kr+r^2-3k-3r-4}$
- Recursion relation for $r = 0$; series terminates at $k = 3$

  $a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k (k^2-9k+18)\sqrt{6}}{k^2-3k-4}$
- Apply recursion relation for $k = 0$

  $a_1 = \frac{3\,\mathrm{I}}{4} a_0 \sqrt{6}$
- Apply recursion relation for $k = 1$

  $a_2 = \frac{5\,\mathrm{I}}{18} a_1 \sqrt{6}$
- Express in terms of a_0

  $a_2 = -\frac{5a_0}{4}$
- Apply recursion relation for $k = 2$

  $a_3 = \frac{\mathrm{I}}{9} a_2 \sqrt{6}$
- Express in terms of a_0

  $a_3 = -\frac{5\,\mathrm{I}}{36} a_0 \sqrt{6}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 + \frac{3\,\mathrm{I}\sqrt{6}\,u}{4} - \frac{5u^2}{4} - \frac{5\,\mathrm{I}\sqrt{6}\,u^3}{36}\right)$
- Revert the change of variables $u = x - 2 + \frac{\mathrm{I}\sqrt{6}}{2}$

  $\left[y = -\frac{\mathrm{I}}{72} a_0 \sqrt{6}\,(10x^3 - 60x^2 + 111x - 62)\right]$
- Recursion relation for $r = 5$; series terminates at $k = 1$

  $a_{k+1} = \frac{-\frac{\mathrm{I}}{6} a_k (k^2+k-2)\sqrt{6}}{k^2+7k+6}$
- Apply recursion relation for $k = 0$

  $a_1 = \frac{\mathrm{I}}{18} a_0 \sqrt{6}$
- Terminating series solution of the ODE for $r = 5$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 + \frac{\mathrm{I}\sqrt{6}\,u}{18}\right)$
- Revert the change of variables $u = x - 2 + \frac{\mathrm{I}\sqrt{6}}{2}$

  $\left[y = a_0 \left(\frac{5}{6} + \frac{\mathrm{I}(x-2)\sqrt{6}}{18}\right)\right]$
- Combine solutions and rename parameters

  $\left[y = -\frac{\mathrm{I} a_0 \sqrt{6}\,(10x^3-60x^2+111x-62)}{72} + b_0 \left(\frac{5}{6} + \frac{\mathrm{I}(x-2)\sqrt{6}}{18}\right)\right]$

1.479.3 Maple trace

```
Methods for second order ODEs:
```

1.479.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 55

```
dsolve((2*x^2-8*x+11)*diff(diff(y(x),x),x)-16*(x-2)*diff(y(x),x)+36*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2 x^6 - 12c_2 x^5 + \frac{165c_2 x^4}{2} + c_1 x^3 + \frac{3(-8c_1 - 1815c_2) x^2}{4} + \frac{3(37c_1 + 10890c_2) x}{10} - \frac{31c_1}{5} - \frac{16577c_2}{8}$$

1.479.5 Mathematica DSolve solution

Solving time : `1.432` (sec)
Leaf size : 91

```
DSolve[{(11-8*x+2*x^2)*D[y[x],{x,2}]-16*(x-2)*D[y[x],x]+36*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{15} i c_2 (10x^3 - 60x^2 + 111x - 62) + \frac{c_1 (2x + 5i\sqrt{6} - 4) (2(x-4)x + 11)^2 (2ix + \sqrt{6} - 4i)^3}{2 (-2ix + \sqrt{6} + 4i)^2}$$

1.480 problem 496

Internal problem ID [8618]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 496
Date solved : `Monday, October 21, 2024 at 05:10:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + (x-3)\, y' + 3y = 0$$

1.480.1 Solved as second order ode using Kovacic algorithm

Time used: 0.316 (sec)

Writing the ode as

$$y'' + (x-3)\, y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x-3 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6x - 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 908: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2} - \frac{5}{2x} - \frac{15}{2x^2} - \frac{115}{4x^3} - \frac{495}{4x^4} - \frac{2285}{4x^5} - \frac{11055}{4x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} - \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 - \frac{3}{2}x + \frac{9}{4}$$

This shows that the coefficient of 1 in the above is $\frac{9}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6x - 1}{4} \\ &= Q + \frac{R}{4} \\ &= \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right) + (0) \\ &= -\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{4}\right) - \left(\frac{9}{4}\right) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} - \frac{3}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}-\frac{3}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 2 \end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0+(-)\left(\frac{x}{2}-\frac{3}{2}\right) \\ &= \frac{3}{2}-\frac{x}{2} \\ &= \frac{3}{2}-\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x^2+a_1x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2)+2\left(\frac{3}{2}-\frac{x}{2}\right)(2x+a_1)+\left(\left(-\frac{1}{2}\right)+\left(\frac{3}{2}-\frac{x}{2}\right)^2-\left(-\frac{1}{4}+\frac{1}{4}x^2-\frac{3}{2}x\right)\right)=0$$

$$(x+3)\,a_1+6x+2a_0+2=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = -6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 6x + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 6x + 8\right) e^{\int \left(\frac{3}{2} - \frac{x}{2}\right) dx} \\
&= \left(x^2 - 6x + 8\right) e^{\frac{3}{2}x - \frac{1}{4}x^2} \\
&= \left(x^2 - 6x + 8\right) e^{-\frac{x(-6+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x-3}{1} \, dx} \\
&= z_1 e^{\frac{3}{2}x - \frac{1}{4}x^2} \\
&= z_1 \left(e^{-\frac{x(-6+x)}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{x(-6+x)}{2}} \left(x^2 - 6x + 8\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x-3}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{1}{2}x^2 + 3x}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{-\frac{1}{2}x^2 + 3x} e^{x(-6+x)}}{\left(x^2 - 6x + 8\right)^2} \, dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x(-6+x)}{2}} (x^2 - 6x + 8)\right) + c_2 \left(\mathrm{e}^{-\frac{x(-6+x)}{2}} (x^2 - 6x + 8) \left(\int \frac{\mathrm{e}^{-\frac{1}{2}x^2+3x} \mathrm{e}^{x(-6+x)}}{(x^2 - 6x + 8)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.480.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + (x-3)\,y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 3a_{k+1}(k+1) + a_k(k+3)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $k^2 a_{k+2} + (a_k - 3a_{k+1} + 3a_{k+2})\,k + 3a_k - 3a_{k+1} + 2a_{k+2} = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k k - 3a_{k+1}k + 3a_k - 3a_{k+1}}{k^2 + 3k + 2}\right]$$

1.480.3 Maple trace

```
Methods for second order ODEs:
```

1.480.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 73

```
dsolve(diff(diff(y(x),x),x)+(x-3)*diff(y(x),x)+3*y(x) = 0,
       y(x),singsol=all)
```

$$y = (x-4)\,\mathrm{e}^{-\frac{(x-3)^2}{2}} \left(\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{-(x-3)^2}}{2}\right) - 1\right)(x-2)\,c_2\sqrt{\pi} - \sqrt{2}\,\sqrt{-(x-3)^2}\,c_2 - c_1\,\mathrm{e}^{-\frac{(x-3)^2}{2}}(x-2)(x-4)$$

1.480.5 Mathematica DSolve solution

Solving time : `0.343` (sec)
Leaf size : 90

```
DSolve[{D[y[x],{x,2}]+(x-3)*D[y[x],x]+3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}e^{-\frac{1}{2}(x-6)x-8}\left(e^{7/2}\sqrt{2\pi}c_2\left(x^2-6x+8\right)\operatorname{erfi}\left(\frac{x-3}{\sqrt{2}}\right) + 4e^8 c_1\left(x^2-6x+8\right) - 2c_2 e^{\frac{1}{2}(x-4)^2+x}(x-3)\right)$$

1.481 problem 497

Internal problem ID [8619]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 497
Date solved : `Monday, October 21, 2024 at 05:10:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^{2}-8x+14\right) y''-8(x-4)\, y'+20y=0$$

1.481.1 Solved as second order ode using Kovacic algorithm

Time used: 0.308 (sec)

Writing the ode as

$$\left(x^{2}-8x+14\right) y''+(-8x+32)\, y'+20y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}-8x+14 \\ B &= -8x+32 \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{48}{(x^2 - 8x + 14)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 48$$
$$t = (x^2 - 8x + 14)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{48}{(x^2 - 8x + 14)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 910: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 8x + 14)^2$. There is a pole at $x = 4 + \sqrt{2}$ of order 2. There is a pole at $x = 4 - \sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{\left(x - 4 + \sqrt{2}\right)^2} + \frac{6}{\left(x - 4 - \sqrt{2}\right)^2} + \frac{3\sqrt{2}}{x - 4 + \sqrt{2}} - \frac{3\sqrt{2}}{x - 4 - \sqrt{2}}$$

For the pole at $x = 4 + \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-4-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

For the pole at $x = 4 - \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-4+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{48}{\left(x^2 - 8x + 14\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$4 + \sqrt{2}$	2	0	3	-2
$4 - \sqrt{2}$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c) [\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty) [\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}} + (-)(0) \\
&= -\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}} \\
&= \frac{x-4-5\sqrt{2}}{x^2-8x+14}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}}\right)(0) + \left(\left(\frac{2}{\left(x-4-\sqrt{2}\right)^2} - \frac{3}{\left(x-4+\sqrt{2}\right)^2}\right) + \left(-\frac{2}{x-4-\sqrt{2}} +
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}}\right) dx} \\
&= \frac{\left(x-4+\sqrt{2}\right)^3}{\left(-x+4+\sqrt{2}\right)^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-8x+32}{x^2-8x+14} \, dx} \\
&= z_1 e^{2 \ln(x^2-8x+14)} \\
&= z_1 \left(\left(x^2-8x+14\right)^2\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{(x^2-8x+14)^2 \left(x-4+\sqrt{2}\right)^3}{\left(-x+4+\sqrt{2}\right)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x+32}{x^2-8x+14}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{4\ln(x^2-8x+14)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{64}{5\left(x-4+\sqrt{2}\right)^5} - \frac{16}{\left(x-4+\sqrt{2}\right)^3} + \frac{16\sqrt{2}}{\left(x-4+\sqrt{2}\right)^4} + \frac{4\sqrt{2}}{\left(x-4+\sqrt{2}\right)^2} - \frac{1}{x-4+\sqrt{2}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2-8x+14)^2 \left(x-4+\sqrt{2}\right)^3}{\left(-x+4+\sqrt{2}\right)^2}\right) \\
&\quad + c_2 \left(\frac{(x^2-8x+14)^2 \left(x-4+\sqrt{2}\right)^3}{\left(-x+4+\sqrt{2}\right)^2} \left(-\frac{64}{5\left(x-4+\sqrt{2}\right)^5} - \frac{16}{\left(x-4+\sqrt{2}\right)^3} + \frac{16\sqrt{2}}{\left(x-4+\sqrt{2}\right)^4} + \frac{4\sqrt{2}}{\left(x-4+\sqrt{2}\right)^2} - \frac{1}{x-4+\sqrt{2}}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.481.2 Maple step by step solution

Let's solve

$(x^2-8x+14)\left(\frac{d}{dx}y'\right)-8(x-4)\,y'+20y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{20y}{x^2-8x+14}+\frac{8(x-4)y'}{x^2-8x+14}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{8(x-4)y'}{x^2-8x+14}+\frac{20y}{x^2-8x+14}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{8(x-4)}{x^2-8x+14},P_3(x)=\frac{20}{x^2-8x+14}\right]$

- ◦ $(x-4+\sqrt{2})\cdot P_2(x)$ is analytic at $x=4-\sqrt{2}$

$\left((x-4+\sqrt{2})\cdot P_2(x)\right)\bigg|_{x=4-\sqrt{2}}=0$

- ◦ $(x-4+\sqrt{2})^2\cdot P_3(x)$ is analytic at $x=4-\sqrt{2}$

$\left((x-4+\sqrt{2})^2\cdot P_3(x)\right)\bigg|_{x=4-\sqrt{2}}=0$

- ◦ $x=4-\sqrt{2}$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=4-\sqrt{2}$

- Multiply by denominators

$(x^2-8x+14)\left(\frac{d}{dx}y'\right)+(-8x+32)\,y'+20y=0$

- Change variables using $x=u+4-\sqrt{2}$ so that the regular singular point is at $u=0$

$(u^2-2u\sqrt{2})\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-8u+8\sqrt{2})\left(\frac{d}{du}y(u)\right)+20y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2\sqrt{2}\,(r-5)\, r a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2\sqrt{2}\,(k+r-4)\,(k+1+r)\, a_{k+1} + a_k(k+r-4)\,(k+r-5)\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2\sqrt{2}\,(r-5)\, r = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 5\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-4)\left(-2a_{k+1}(k+1+r)\sqrt{2} + a_k(k+r-5)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r-5)\sqrt{2}}{4(k+1+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 5$

$$a_{k+1} = \frac{a_k(k-5)\sqrt{2}}{4(k+1)}$$

- Apply recursion relation for $k = 0$

$$a_1 = -\frac{5a_0\sqrt{2}}{4}$$

- Apply recursion relation for $k = 1$

$$a_2 = -\frac{a_1\sqrt{2}}{2}$$

- Express in terms of a_0

$$a_2 = \frac{5a_0}{4}$$

- Apply recursion relation for $k = 2$

$$a_3 = -\frac{a_2\sqrt{2}}{4}$$

- Express in terms of a_0

$$a_3 = -\frac{5a_0\sqrt{2}}{16}$$

- Apply recursion relation for $k = 3$

$$a_4 = -\frac{a_3\sqrt{2}}{8}$$

- Express in terms of a_0

  $a_4 = \frac{5a_0}{64}$

- Apply recursion relation for $k = 4$

  $a_5 = -\frac{a_4\sqrt{2}}{20}$

- Express in terms of a_0

  $a_5 = -\frac{a_0\sqrt{2}}{256}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{5u\sqrt{2}}{4} + \frac{5u^2}{4} - \frac{5\sqrt{2}\,u^3}{16} + \frac{5u^4}{64} - \frac{\sqrt{2}\,u^5}{256}\right)$

- Revert the change of variables $u = x - 4 + \sqrt{2}$

  $\left[y = a_0\left(\frac{(-x^5+20x^4-180x^3+880x^2-2260x+2384)\sqrt{2}}{256} + \frac{5x^4}{128} - \frac{5x^3}{8} + \frac{125x^2}{32} - \frac{45x}{4} + \frac{401}{32}\right)\right]$

- Recursion relation for $r = 5$

  $a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}$

- Solution for $r = 5$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+5}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}\right]$

- Revert the change of variables $u = x - 4 + \sqrt{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k\left(x - 4 + \sqrt{2}\right)^{k+5}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0\left(\frac{(-x^5+20x^4-180x^3+880x^2-2260x+2384)\sqrt{2}}{256} + \frac{5x^4}{128} - \frac{5x^3}{8} + \frac{125x^2}{32} - \frac{45x}{4} + \frac{401}{32}\right) + \left(\sum_{k=0}^{\infty} b_k\left(x - 4 + \right.\right.\right.$

1.481.3 Maple trace

```
Methods for second order ODEs:
```

1.481.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 55

```
dsolve((x^2-8*x+14)*diff(diff(y(x),x),x)-8*(x-4)*diff(y(x),x)+20*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x^5 + c_2 x^4 + 4(-35c_1 - 4c_2) x^3 + 20(56c_1 + 5c_2) x^2 + 4(-875c_1 - 72c_2) x + 4032c_1 + \frac{1604c_2}{5}$$

1.481.5 Mathematica DSolve solution

Solving time : `0.148` (sec)
Leaf size : 77

```
DSolve[{(x^2-8*x+14)*D[y[x],{x,2}]+8*(x-4)*D[y[x],x]+20*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1 P^3_{\frac{1}{2}i\left(i+\sqrt{31}\right)}\left(\frac{x-4}{\sqrt{2}}\right) + c_2 Q^3_{\frac{1}{2}i\left(i+\sqrt{31}\right)}\left(\frac{x-4}{\sqrt{2}}\right)}{(x^2-8x+14)^{3/2}}$$

1.482 problem 498

Internal problem ID [8620]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 498
Date solved : `Monday, October 21, 2024 at 05:10:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+4x+5\right)y''-20(x+1)\,y'+60y=0$$

1.482.1 Solved as second order ode using Kovacic algorithm

Time used: 0.591 (sec)

Writing the ode as

$$\left(2x^2+4x+5\right)y''+(-20x-20)\,y'+60y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+4x+5 \\ B &= -20x-20 \\ C &= 60 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-210}{(2x^2 + 4x + 5)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -210$$
$$t = (2x^2 + 4x + 5)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{210}{(2x^2 + 4x + 5)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 912: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 0\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2+4x+5)^2$. There is a pole at $x = -1 + \frac{i\sqrt{6}}{2}$ of order 2. There is a pole at $x = -1 - \frac{i\sqrt{6}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4\left(x+1-\frac{i\sqrt{6}}{2}\right)^2} + \frac{35}{4\left(x+1+\frac{i\sqrt{6}}{2}\right)^2} + \frac{35i\sqrt{6}}{12\left(x+1-\frac{i\sqrt{6}}{2}\right)} - \frac{35i\sqrt{6}}{12\left(x+1+\frac{i\sqrt{6}}{2}\right)}$$

For the pole at $x = -1 + \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x+1-\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

For the pole at $x = -1 - \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x+1+\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{210}{\left(2x^2+4x+5\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-1+\frac{i\sqrt{6}}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-1-\frac{i\sqrt{6}}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)} + (-)(0) \\
&= -\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)} \\
&= \frac{-6i\sqrt{6}+2x+2}{2x^2+4x+5}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)}\right)(0) + \left(\left(\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)^2} - \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)^2}\right) + \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)}\right) dx} \\
&= \frac{27\sqrt{2}\,(2x^2+4x+5)^{7/2}}{\left(3+i\,(x+1)\sqrt{6}\right)^6}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-20x-20}{2x^2+4x+5}\, dx} \\
&= z_1 e^{\frac{5\ln(2x^2+4x+5)}{2}} \\
&= z_1\left((2x^2+4x+5)^{5/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = -\frac{(2x^2+4x+5)^6\sqrt{2}}{27\left(i-\frac{(x+1)\sqrt{2}\,\sqrt{3}}{3}\right)^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-20x-20}{2x^2+4x+5}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(2x^2+4x+5)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\frac{1}{2}x^5+\frac{5}{2}x^4+\frac{5}{2}x^3-\frac{5}{2}x^2-\frac{31}{8}x-\frac{7}{8}}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)^6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(-\frac{(2x^2+4x+5)^6\sqrt{2}}{27\left(i-\frac{(x+1)\sqrt{2}\,\sqrt{3}}{3}\right)^6}\right) \\
&\quad + c_2\left(-\frac{(2x^2+4x+5)^6\sqrt{2}}{27\left(i-\frac{(x+1)\sqrt{2}\,\sqrt{3}}{3}\right)^6}\left(-\frac{\frac{1}{2}x^5+\frac{5}{2}x^4+\frac{5}{2}x^3-\frac{5}{2}x^2-\frac{31}{8}x-\frac{7}{8}}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)^6}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.482.2 Maple step by step solution

Let's solve

$(2x^2+4x+5)\left(\frac{d}{dx}y'\right)-20(x+1)\,y'+60y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{60y}{2x^2+4x+5}+\frac{20(x+1)y'}{2x^2+4x+5}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{20(x+1)y'}{2x^2+4x+5}+\frac{60y}{2x^2+4x+5}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=-\frac{20(x+1)}{2x^2+4x+5},P_3(x)=\frac{60}{2x^2+4x+5}\right]$

- ○ $\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)$ is analytic at $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

$\left(\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-1-\frac{\mathrm{I}\sqrt{6}}{2}}=0$

- ○ $\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

$\left(\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-1-\frac{\mathrm{I}\sqrt{6}}{2}}=0$

- ○ $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

- Multiply by denominators

$(2x^2+4x+5)\left(\frac{d}{dx}y'\right)+(-20x-20)\,y'+60y=0$

- Change variables using $x=u-1-\frac{\mathrm{I}\sqrt{6}}{2}$ so that the regular singular point is at $u=0$

$\left(2u^2-2\,\mathrm{I}u\sqrt{6}\right)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\left(-20u+10\,\mathrm{I}\sqrt{6}\right)\left(\frac{d}{du}y(u)\right)+60y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- ◦ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- ◦ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2\,\mathrm{I}\sqrt{6}\, r(r-6)\, a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2\,\mathrm{I}\sqrt{6}\,(k+1+r)\,(k+r-5)\, a_{k+1} + 2a_k(k+r-5)\,(k+r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2\,\mathrm{I}\sqrt{6}\, r(r-6) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 6\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(\mathrm{I}a_{k+1}(k+1+r)\sqrt{6} - a_k(k+r-6)\right)(k+r-5) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k+r-6)\sqrt{6}}{k+1+r}$$

- Recursion relation for $r = 0$; series terminates at $k = 6$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k-6)\sqrt{6}}{k+1}$$

- Recursion relation that defines the terminating series solution of the ODE for $r = 0$

$$\left[y(u) = \sum_{k=0}^{5} a_k u^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k-6)\sqrt{6}}{k+1}\right]$$

- Revert the change of variables $u = x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}$

$$\left[y = \sum_{k=0}^{5} a_k \left(x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}\right)^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k-6)\sqrt{6}}{k+1}\right]$$

- Recursion relation for $r = 6$

$$a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k k\sqrt{6}}{k+7}$$

- Solution for $r = 6$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+1}=\frac{-\frac{I}{6}a_k k\sqrt{6}}{k+7}\right]$$

- Revert the change of variables $u=x+1+\frac{I\sqrt{6}}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(x+1+\frac{I\sqrt{6}}{2}\right)^{k+6}, a_{k+1}=\frac{-\frac{I}{6}a_k k\sqrt{6}}{k+7}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{5} a_k\left(x+1+\frac{I\sqrt{6}}{2}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k\left(x+1+\frac{I\sqrt{6}}{2}\right)^{k+6}\right), a_{k+1}=\frac{-\frac{I}{6}a_k(k-6)\sqrt{6}}{k+1}, b_{k+1}=\frac{-\frac{I}{6}b_k k\sqrt{6}}{k+7}\right]$$

1.482.3 Maple trace

```
Methods for second order ODEs:
```

1.482.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 65

```
dsolve((2*x^2+4*x+5)*diff(diff(y(x),x),x)-20*(x+1)*diff(y(x),x)+60*y(x) = 0,
       y(x),singsol=all)
```

$$y=c_2x^6+c_1x^5+\frac{5(2c_1-15c_2)x^4}{2}+5(c_1-20c_2)x^3+\frac{5(-4c_1-45c_2)x^2}{4}+\frac{(-31c_1+120c_2)x}{4}-\frac{7c_1}{4}+\frac{155c_2}{8}$$

1.482.5 Mathematica DSolve solution

Solving time : `1.442` (sec)
Leaf size : 83

```
DSolve[{(2*x^2+4*x+5)*D[y[x],{x,2}]-20*(x+1)*D[y[x],x]+60*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{(2x^2+4x+5)^{5/2}\left(4c_2(4x^5+20x^4+20x^3-20x^2-31x-7)+c_1\left(2ix+\sqrt{6}+2i\right)^6\right)}{(4x^2+8x+10)^{5/2}}$$

1.483 problem 499

Internal problem ID [8621]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 499
Date solved : `Monday, October 21, 2024 at 05:10:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^3+1\right) y''+7x^2y'+9xy=0$$

1.483.1 Solved as second order ode using Kovacic algorithm

Time used: 0.425 (sec)

Writing the ode as

$$\left(x^3+1\right) y''+7x^2y'+9xy=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3+1 \\ B &= 7x^2 \\ C &= 9x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x(x^3 + 8)}{4(x^3 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x(x^3 + 8)$$
$$t = 4(x^3 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{x(x^3 + 8)}{4(x^3 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 914: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 6 - 4\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+1)^2$. There is a pole at $x = -1$ of order 2. There is a pole at $x = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned}r = {} & \frac{7}{36\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2} + \frac{7}{36\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{36}+\frac{5i\sqrt{3}}{36}}{x-\frac{1}{2}-\frac{i\sqrt{3}}{2}}\\ & + \frac{-\frac{5}{36}-\frac{5i\sqrt{3}}{36}}{x-\frac{1}{2}+\frac{i\sqrt{3}}{2}} + \frac{5}{18(x+1)} + \frac{7}{36(x+1)^2}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}\end{aligned}$$

For the pole at $x = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}\end{aligned}$$

For the pole at $x = \frac{1}{2}+\frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{6}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{x(x^3+8)}{4\left(x^3+1\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{x(x^3+8)}{4\left(x^3+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
$\frac{1}{2}-\frac{i\sqrt{3}}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
$\frac{1}{2}+\frac{i\sqrt{3}}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{6(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)} + (-)(0) \\ &= -\frac{1}{6(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)} \\ &= -\frac{x^2}{2x^3+2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6(x+1)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}\right)(1) + \left(\left(\frac{1}{6(x+1)^2} + \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2} + \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, \mathrm{e}^{\int \left(-\frac{1}{6(x+1)} - \frac{1}{6\left(x - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x - \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)}\right) dx} \\ &= (x) \, \frac{1}{\left((x+1)\left(2x - 1 + i\sqrt{3}\right)\left(i\sqrt{3} - 2x + 1\right)\right)^{1/6}} \\ &= \frac{x}{\left((x+1)\left(2x - 1 + i\sqrt{3}\right)\left(i\sqrt{3} - 2x + 1\right)\right)^{1/6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{7x^2}{x^3+1} \, dx} \\ &= z_1 e^{-\frac{7 \ln\left(x^3+1\right)}{6}} \\ &= z_1 \left(\frac{1}{\left(x^3+1\right)^{7/6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{\left(x^3+1\right)^{7/6} \left(-4x^3 - 4\right)^{1/6}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{7x^2}{x^3+1}dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-\frac{7\ln(x^3+1)}{3}}}{(y_1)^2} dx$$
$$= y_1 \left(\int \frac{(-4x^3-4)^{1/3}}{x^2} dx \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x}{(x^3+1)^{7/6} (-4x^3-4)^{1/6}} \right) + c_2 \left(\frac{x}{(x^3+1)^{7/6} (-4x^3-4)^{1/6}} \left(\int \frac{(-4x^3-4)^{1/3}}{x^2} dx \right) \right)$$

Will add steps showing solving for IC soon.

1.483.2 Maple step by step solution

Let's solve

$(x^3+1)\left(\frac{d}{dx}y'\right) + 7x^2y' + 9xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{9xy}{x^3+1} - \frac{7x^2y'}{x^3+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{7x^2y'}{x^3+1} + \frac{9xy}{x^3+1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{7x^2}{x^3+1}, P_3(x) = \frac{9x}{x^3+1}\right]$

  ◦ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $\left.((x+1) \cdot P_2(x))\right|_{x=-1} = \frac{7}{3}$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $$\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $$(x^3+1)\left(\frac{d}{dx}y'\right) + 7x^2y' + 9xy = 0$$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $$(u^3-3u^2+3u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (7u^2-14u+7)\left(\frac{d}{du}y(u)\right) + (9u-9)\,y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m=0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..3$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(4+3r)\, u^{-1+r} + \left(a_1(1+r)(7+3r) - a_0(3r^2+11r+9)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)(3k+7\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(4+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{4}{3}\right\}$
- Each term must be 0
  $a_1(1+r)(7+3r) - a_0(3r^2+11r+9) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(3k+7+3r) - a_k(3k^2+6kr+3r^2+11k+11r+9) + a_{k-1}(k+2+r)^2 = 0$
- Shift index using $k->k+1$
  $a_{k+2}(k+2+r)(3k+10+3r) - a_{k+1}\left(3(k+1)^2+6(k+1)r+3r^2+11k+20+11r\right) + a_k(k+$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{k^2a_k - 3k^2a_{k+1} + 2kra_k - 6kra_{k+1} + r^2a_k - 3r^2a_{k+1} + 6ka_k - 17ka_{k+1} + 6ra_k - 17ra_{k+1} + 9a_k - 23a_{k+1}}{(k+2+r)(3k+10+3r)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{k^2a_k - 3k^2a_{k+1} + 6ka_k - 17ka_{k+1} + 9a_k - 23a_{k+1}}{(k+2)(3k+10)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k - 3k^2a_{k+1} + 6ka_k - 17ka_{k+1} + 9a_k - 23a_{k+1}}{(k+2)(3k+10)}, 7a_1 - 9a_0 = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{k^2a_k - 3k^2a_{k+1} + 6ka_k - 17ka_{k+1} + 9a_k - 23a_{k+1}}{(k+2)(3k+10)}, 7a_1 - 9a_0 = 0\right]$
- Recursion relation for $r = -\frac{4}{3}$
  $a_{k+2} = -\frac{k^2a_k - 3k^2a_{k+1} + \frac{10}{3}ka_k - 9ka_{k+1} + \frac{25}{9}a_k - \frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}$
- Solution for $r = -\frac{4}{3}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2} = -\frac{k^2a_k - 3k^2a_{k+1} + \frac{10}{3}ka_k - 9ka_{k+1} + \frac{25}{9}a_k - \frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}, -a_1 + \frac{a_0}{3} = 0\right]$
- Revert the change of variables $u = x+1$
  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{4}{3}}, a_{k+2} = -\frac{k^2a_k - 3k^2a_{k+1} + \frac{10}{3}ka_k - 9ka_{k+1} + \frac{25}{9}a_k - \frac{17}{3}a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}, -a_1 + \frac{a_0}{3} = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k(x+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{4}{3}}\right), a_{k+2} = -\frac{k^2a_k - 3k^2a_{k+1} + 6ka_k - 17ka_{k+1} + 9a_k - 23a_{k+1}}{(k+2)(3k+10)}, 7a_1\right.$

1.483.3 Maple trace

```
Methods for second order ODEs:
```

1.483.4 Maple dsolve solution

Solving time : `0.024` (sec)
Leaf size : 28

```
dsolve((x^3+1)*diff(diff(y(x),x),x)+7*x^2*diff(y(x),x)+9*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \operatorname{hypergeom}\left([1,1], \left[\frac{2}{3}\right], -x^3\right) + \frac{c_2 x}{(x^3+1)^{4/3}}$$

1.483.5 Mathematica DSolve solution

Solving time : `0.972` (sec)
Leaf size : 118

```
DSolve[{(1+x^3)*D[y[x],{x,2}]+7*x^2*D[y[x],x]+9*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-2\sqrt{3}c_2 x \arctan\left(\frac{\sqrt{3}x}{2\sqrt[3]{x^3+1}+x}\right) - 6c_2\sqrt[3]{x^3+1} - 2c_2 x \log\left(\sqrt[3]{x^3+1}-x\right) + c_2 x \log\left(\sqrt[3]{x^3+1}x + (x^3 -}{6(x^3+1)^{4/3}}$$

1.484 problem 500

Internal problem ID [8622]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 500
Date solved : `Monday, October 21, 2024 at 05:10:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^5+1\right)y''+14x^4y'+10x^3y=0$$

1.484.1 Solved as second order ode using Kovacic algorithm

Time used: 1.028 (sec)

Writing the ode as

$$\left(2x^5+1\right)y''+14x^4y'+10x^3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^5+1 \\ B &= 14x^4 \\ C &= 10x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^3(5x^5 + 6)}{(2x^5 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^3(5x^5 + 6)$$
$$t = (2x^5 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^3(5x^5 + 6)}{(2x^5 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 916: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 10 - 8 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^5+1)^2$. There is a pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ of order 2. There is a pole at $x = \frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8}$ of order 2. There is a pole at $x = -\frac{2^{4/5}}{2}$ of order 2. There is a pole at $x = \frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}$ of order 2. There is a pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \text{Expression too large to display}$$

For the pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{2^{4/5}\sqrt{5}}{8} - \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10} \end{aligned}$$

For the pole at $x = \frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8}$ let b be the coefficient of $\frac{1}{\left(x - \frac{2^{4/5}}{8} + \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10} \end{aligned}$$

For the pole at $x = -\frac{2^{4/5}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2^{4/5}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}
$$

For the pole at $x = \frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{4/5}}{8}+\frac{2^{4/5}\sqrt{5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}\,\sqrt{5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}
$$

For the pole at $x = \frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{4/5}\sqrt{5}}{8}-\frac{2^{4/5}}{8}+\frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^3(5x^5+6)}{(2x^5+1)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^3(5x^5+6)}{(2x^5+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} + \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$-\frac{2^{4/5}}{2}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{4/5}}{8} - \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{4/5}\sqrt{5}}{8} + \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^- + \alpha_{c_4}^- + \alpha_{c_5}^-\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\omega = \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3}\right) + \left((-)[\sqrt{r}]_{c_4} + \frac{\alpha_{c_4}^-}{x - c_4}\right)$$

$$= \frac{3}{10\left(x - \frac{2^{4/5}\sqrt{5}}{8} - \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)} + \frac{3}{10\left(x - \frac{2^{4/5}}{8} + \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8}\right)} + \frac{}{10(}$$

$$= \frac{3}{10\left(x - \frac{2^{4/5}\sqrt{5}}{8} - \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)} + \frac{3}{10\left(x - \frac{2^{4/5}}{8} + \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8}\right)} + \frac{}{10(}$$

$$= \frac{3x^4}{2x^5 + 1}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) and Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$z_1(x) = pe^{\int \omega \, dx}$$

$$= (x)\, e^{\int \left(\frac{3}{10\left(x - \frac{2^{4/5}\sqrt{5}}{8} - \frac{2^{4/5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{4}\right)} + \frac{3}{10\left(x - \frac{2^{4/5}}{8} + \frac{2^{4/5}\sqrt{5}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}}{8} - \frac{i2^{3/10}\sqrt{5-\sqrt{5}}\sqrt{5}}{8}\right)} + \frac{3}{10\left(x + \frac{2^{4/5}}{2}\right)} + \frac{}{10\left(x - \frac{2^{4/5}}{8}\right.}\right.}$$

$$= (x)\left(\left(2i2^{3/10}\sqrt{5 - \sqrt{5}} + 2^{4/5}\sqrt{5} + 2^{4/5} - 8x\right)\left(-i2^{3/10}\sqrt{5 - \sqrt{5}}\sqrt{5} + 2^{4/5}\sqrt{5} - i2^{3/10}\sqrt{5 - \sqrt{}}\right.\right.$$

$$= x8^{3/10}\left(\left(x + \frac{2^{4/5}}{2}\right)\left(i2^{3/10}\left(\sqrt{5} + 1\right)\sqrt{5 - \sqrt{5}} + \left(\sqrt{5} - 1\right)2^{4/5} + 8x\right)\left(i2^{3/10}\sqrt{5 - \sqrt{5}} + \frac{(-\sqrt{}}{}\right.\right.$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{14x^4}{2x^5+1}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^5+1\right)}{10}} \\
&= z_1\left(\frac{1}{\left(2x^5+1\right)^{7/10}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x 8^{3/10} \left(1024x^5+512\right)^{3/10}}{\left(2x^5+1\right)^{7/10}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{14x^4}{2x^5+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^5+1\right)}{5}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int \frac{8^{2/5}}{8x^2\left(1024x^5+512\right)^{3/5}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x 8^{3/10} \left(1024x^5+512\right)^{3/10}}{\left(2x^5+1\right)^{7/10}}\right) + c_2\left(\frac{x 8^{3/10} \left(1024x^5+512\right)^{3/10}}{\left(2x^5+1\right)^{7/10}}\left(\int \frac{8^{2/5}}{8x^2\left(1024x^5+512\right)^{3/5}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.484.2 Maple step by step solution

1.484.3 Maple trace

```
Methods for second order ODEs:
```

1.484.4 Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 30

```
dsolve((2*x^5+1)*diff(diff(y(x),x),x)+14*x^4*diff(y(x),x)+10*x^3*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 x}{(2x^5+1)^{2/5}} + c_2 \,\mathrm{hypergeom}\left(\left[\frac{1}{5}, 1\right], \left[\frac{4}{5}\right], -2x^5\right)$$

1.484.5 Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{(1+2*x^5)*D[y[x],{x,2}]+14*x^4*D[y[x],x]+10*x^3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Timed out

1.485 problem 501

Internal problem ID [8623]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 501
Date solved : `Monday, October 21, 2024 at 05:11:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + x^6 y' + 7x^5 y = 0$$

1.485.1 Solved as second order ode using Kovacic algorithm

Time used: 0.422 (sec)

Writing the ode as

$$y'' + x^6 y' + 7x^5 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x^6 \\ C &= 7x^5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^5(x^7 - 16)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^5(x^7 - 16) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^5(x^7 - 16)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 917: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 12 \\ &= -12 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -12 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -12$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{12}{2} = 6$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{6} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^6$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^6}{2} - \frac{4}{x} - \frac{16}{x^8} - \frac{128}{x^{15}} - \frac{1280}{x^{22}} - \frac{14336}{x^{29}} - \frac{172032}{x^{36}} - \frac{2162688}{x^{43}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 6$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{6} a_i x^i \\ &= \frac{x^6}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^5 = x^5$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^{12}}{4}$$

This shows that the coefficient of x^5 in the above is 0. Now we need to find the coefficient of x^5 in r. How this is done depends on if $v = 0$ or not. Since $v = 6$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^5 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^5(x^7 - 16)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^{12} - 4x^5\right) + (0) \\ &= \frac{1}{4}x^{12} - 4x^5 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -4. Now b can be found.

$$\begin{aligned} b &= (-4) - (0) \\ &= -4 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^6}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-4}{\frac{1}{2}} - 6\right) &&= -7 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-4}{\frac{1}{2}} - 6\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^5(x^7 - 16)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-12	$\frac{x^6}{2}$	-7	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^6}{2}\right) \\ &= -\frac{x^6}{2} \\ &= -\frac{x^6}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x^6}{2}\right)(1) + \left((-3x^5) + \left(-\frac{x^6}{2}\right)^2 - \left(\frac{x^5(x^7-16)}{4}\right)\right) &= 0 \\ x^5 a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, \mathrm{e}^{\int -\frac{x^6}{2} dx} \\ &= (x) \, \mathrm{e}^{-\frac{x^7}{14}} \\ &= x \, \mathrm{e}^{-\frac{x^7}{14}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^6}{1} \, dx} \\ &= z_1 e^{-\frac{x^7}{14}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^7}{14}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^7}{7}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^6}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^7}{7}}}{(y_1)^2} \, dx \end{aligned}$$

$$= y_1 \left(\frac{7^{6/7}(-1)^{1/7}\left(-\frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}\right)}{(-x^7)^{6/7}} + \frac{7\,7^{1/7}(-1)^{6/7}\mathrm{e}^{\frac{x^7}{7}}}{x} + \frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)}{(-x^7)^{6/7}}\right)}{49}\right)$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{x^7}{7}}x\right) \\ &\quad + c_2\left(\mathrm{e}^{-\frac{x^7}{7}}x\left(\frac{7^{6/7}(-1)^{1/7}\left(-\frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}\right)}{(-x^7)^{6/7}} + \frac{7\,7^{1/7}(-1)^{6/7}\mathrm{e}^{\frac{x^7}{7}}}{x} + \frac{7x^6(-1)^{6/7}\Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)}{(-x^7)^{6/7}}\right)}{49}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.485.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + x^6y' + 7x^5y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^5 \cdot y$ to series expansion

    $x^5 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+5}$

  - Shift index using $k- >k - 5$

    $x^5 \cdot y = \sum_{k=5}^{\infty} a_{k-5} x^k$

  - Convert $x^6 \cdot y'$ to series expansion

    $x^6 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+5}$

- Shift index using $k->k-5$

  $x^{6}\cdot y' = \sum_{k=5}^{\infty} a_{k-5}(k-5)\, x^{k}$
- Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_{k}k(k-1)\, x^{k-2}$
- Shift index using $k->k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$

  Rewrite ODE with series expansions

  $30a_{6}x^{4}+20a_{5}x^{3}+12a_{4}x^{2}+6a_{3}x+2a_{2}+\left(\sum_{k=5}^{\infty}\left(a_{k+2}(k+2)\,(k+1)+a_{k-5}(k+2)\right)x^{k}\right)=0$

- The coefficients of each power of x must be 0

  $[2a_{2}=0, 6a_{3}=0, 12a_{4}=0, 20a_{5}=0, 30a_{6}=0]$
- Solve for the dependent coefficient(s)

  $\{a_{2}=0, a_{3}=0, a_{4}=0, a_{5}=0, a_{6}=0\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2}+a_{k-5}+a_{k+2})=0$
- Shift index using $k->k+5$

  $(k+7)\,((k+5)\,a_{k+7}+a_{k}+a_{k+7})=0$
- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty} a_{k}x^{k}, a_{k+7}=-\frac{a_{k}}{k+6}, a_{2}=0, a_{3}=0, a_{4}=0, a_{5}=0, a_{6}=0\right]$

1.485.3 Maple trace

```
Methods for second order ODEs:
```

1.485.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 62

```
dsolve(diff(diff(y(x),x),x)+x^6*diff(y(x),x)+7*x^5*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{\left(-c_1\,\mathrm{e}^{-\frac{x^7}{7}}x - c_2 7^{1/7}\right)\left(-x^7\right)^{6/7} + x^7 c_2\,\mathrm{e}^{-\frac{x^7}{7}}\left(\Gamma\left(\frac{6}{7}\right) - \Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)\right)}{\left(-x^7\right)^{6/7}}$$

1.485.5 Mathematica DSolve solution

Solving time : 0.123 (sec)
Leaf size : 53

```
DSolve[{D[y[x],{x,2}]+x^6*D[y[x],x]+7*x^5*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{49}e^{-\frac{x^7}{7}}\left(49c_1x - 7^{6/7}c_2\sqrt[7]{-x^7}\Gamma\left(-\frac{1}{7}, -\frac{x^7}{7}\right)\right)$$

1.486 problem 502

Internal problem ID [8624]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 502
Date solved : `Tuesday, October 22, 2024 at 03:07:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^{8}+1\right) y''-16x^{7}y'+72x^{6}y=0$$

1.486.1 Solved as second order ode using Kovacic algorithm

Time used: 428.904 (sec)

Writing the ode as

$$\left(x^{8}+1\right) y''-16x^{7}y'+72x^{6}y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{8}+1 \\ B &= -16x^{7} \\ C &= 72x^{6} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-128x^6}{(x^8+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -128x^6$$
$$t = (x^8+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{128x^6}{(x^8+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 919: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 16 - 6 \\
&= 10
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^8+1)^2$. There is a pole at $x = \frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 10 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 10 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
\begin{aligned}
r = {} & \frac{2}{\left(x - \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} \\
& + \frac{2}{\left(x + \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x + \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} \\
& + \frac{2}{\left(x + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} \\
& + \frac{2}{\left(x - \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} + \frac{2\left(\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} + \frac{2\left(\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}} \\
& + \frac{2\left(-\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}} + \frac{2\left(-\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} + \frac{2\left(-\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}} \\
& + \frac{2\left(-\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}} + \frac{2\left(\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}} + \frac{2\left(\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}}
\end{aligned}
$$

For the pole at $x = \frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$

in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = 2\\ \alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x=-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = 2\\ \alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x=\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = 2\\ \alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x=\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = 2\\ \alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is $10>2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= 0\\ \alpha_\infty^{-} &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{128x^6}{(x^8+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
10	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^- + \alpha_{c_4}^- + \alpha_{c_5}^- + \alpha_{c_6}^- + \alpha_{c_7}^- + \alpha_{c_8}^+\right) \\ &= 1 - (-5) \\ &= 6 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\omega = \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + \left((-)[\sqrt{r}]_{c_4} + \frac{\alpha_{c_4}^-}{x-c_4}\right)$$

$$= -\frac{1}{x - \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} -$$

$$= -\frac{1}{x - \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} -$$

$$= \frac{\left((3x^6 - 3ix^4 - 3ix^2 - 3)\sqrt{2} - 3((-1+i)x^4 + 1 + i)(x^2+1)\right)\sqrt{2-\sqrt{2}} - 3\Big(((-1+i)x^4 + 1 + i)}{2\left(x\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(x(1+\sqrt{2})\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(-x(1+\sqrt{2})\sqrt{2-\sqrt{2}} + x^2 + 1\right)\Big(x^2}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + x^5 a_5 + x^4 a_4 + x^3 a_3 + x^2 a_2 + x a_1 + a_0 \tag{2A}$$

Substituting the above in eq. (1A) and Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{i\sqrt{2} - 1 + i}{i\sqrt{2} + 1 + i}, a_1 = \frac{\left(\frac{12}{7} - \frac{12i}{7}\right)\sqrt{2}}{\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}}, a_2 = -\frac{15(-\sqrt{2} - 1 + i)}{7\left(i\sqrt{2} + 1 + i\right)}, a_3 = \frac{32}{7\left(i\sqrt{2} + 1 + i\right)\sqrt{2}\right.$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)x^5\sqrt{2}}{\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} + \frac{15x^4(\sqrt{2} + 1 + i)}{7\left(i\sqrt{2} + 1 + i\right)} + \frac{32x^3}{7\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} - \frac{15x^2(-\sqrt{2} - 1}{7\left(i\sqrt{2} + 1 +\right.}$$

Therefore the first solution to the ode $z'' = rz$ is

$$z_1(x) = pe^{\int \omega\, dx}$$

$$= \left(x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)x^5\sqrt{2}}{\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} + \frac{15x^4(\sqrt{2} + 1 + i)}{7\left(i\sqrt{2} + 1 + i\right)} + \frac{32x^3}{7\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} - \frac{15x^2(-\sqrt{2} -}{7\left(i\sqrt{2} + 1\right.}\right.$$

$$= \left(x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)x^5\sqrt{2}}{\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} + \frac{15x^4(\sqrt{2} + 1 + i)}{7\left(i\sqrt{2} + 1 + i\right)} + \frac{32x^3}{7\left(i\sqrt{2} + 1 + i\right)\sqrt{2-\sqrt{2}}} - \frac{15x^2(-\sqrt{2} -}{7\left(i\sqrt{2} + 1\right.}\right.$$

$= \text{Expression too large to display}$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-16x^7}{x^8+1}\,dx} \\
&= z_1 e^{\ln(x^8+1)} \\
&= z_1 (x^8+1)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \text{Expression too large to display}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-16x^7}{x^8+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x^8+1)}}{(y_1)^2}\,dx \\
&= y_1(\text{Expression too large to display})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\text{Expression too large to display}) \\
&\quad + c_2(\text{Expression too large to display}(\text{Expression too large to display}))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.486.2 Maple step by step solution

1.486.3 Maple trace

```
Methods for second order ODEs:
```

1.486.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 22

```
dsolve((x^8+1)*diff(diff(y(x),x),x)-16*x^7*diff(y(x),x)+72*x^6*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{7}{9}c_1 + c_1 x^8 + c_2 x^9 - \frac{9}{7}c_2 x$$

1.486.5 Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{(1+x^8)*D[y[x],{x,2}]-16*x^7*D[y[x],x]+72*x^6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Timed out

1.487 problem 503

Internal problem ID [8625]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 503
Date solved : `Monday, October 21, 2024 at 05:18:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + x^5 y' + 6x^4 y = 0$$

1.487.1 Solved as second order ode using Kovacic algorithm

Time used: 0.479 (sec)

Writing the ode as

$$y'' + x^5 y' + 6x^4 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x^5 \\ C &= 6x^4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^4(x^6 - 14)}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^4(x^6 - 14) \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^4(x^6 - 14)}{4}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 920: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 10 \\ &= -10 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -10 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -10$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{10}{2} = 5$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{5} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^5$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^5}{2} - \frac{7}{2x} - \frac{49}{4x^7} - \frac{343}{4x^{13}} - \frac{12005}{16x^{19}} - \frac{117649}{16x^{25}} - \frac{2470629}{32x^{31}} - \frac{27176919}{32x^{37}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 5$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{5} a_i x^i \\ &= \frac{x^5}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^4 = x^4$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^{10}}{4}$$

This shows that the coefficient of x^4 in the above is 0. Now we need to find the coefficient of x^4 in r. How this is done depends on if $v = 0$ or not. Since $v = 5$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^4 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4(x^6 - 14)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^{10} - \frac{7}{2}x^4\right) + (0) \\
&= \frac{1}{4}x^{10} - \frac{7}{2}x^4
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{7}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{7}{2}\right) - (0) \\
&= -\frac{7}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^5}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{2}}{\frac{1}{2}} - 5\right) &&= -6 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{2}}{\frac{1}{2}} - 5\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4(x^6 - 14)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-10	$\frac{x^5}{2}$	-6	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^5}{2}\right) \\ &= -\frac{x^5}{2} \\ &= -\frac{x^5}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x^5}{2}\right)(1) + \left(\left(-\frac{5x^4}{2}\right) + \left(-\frac{x^5}{2}\right)^2 - \left(\frac{x^4(x^6-14)}{4}\right)\right) = 0$$

$$x^4 a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, e^{\int -\frac{x^5}{2} dx} \\ &= (x) \, e^{-\frac{x^6}{12}} \\ &= x \, e^{-\frac{x^6}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^5}{1} \, dx} \\ &= z_1 e^{-\frac{x^6}{12}} \\ &= z_1 \left(e^{-\frac{x^6}{12}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{x^6}{6}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^5}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^6}{6}}}{(y_1)^2} \, dx \end{aligned}$$

$$
= y_1 \left(\frac{6^{5/6}(-1)^{1/6}\left(-\frac{6x^5(-1)^{5/6}\Gamma\left(\frac{5}{6}\right)}{(-x^6)^{5/6}} + \frac{6\,6^{1/6}(-1)^{5/6}\mathrm{e}^{\frac{x^6}{6}}}{x} + \frac{6x^5(-1)^{5/6}\Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)}{(-x^6)^{5/6}}\right)}{36}\right)
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^6}{6}} x\right) \\
&\quad + c_2\left(\mathrm{e}^{-\frac{x^6}{6}} x \left(\frac{6^{5/6}(-1)^{1/6}\left(-\frac{6x^5(-1)^{5/6}\Gamma\left(\frac{5}{6}\right)}{(-x^6)^{5/6}} + \frac{6\,6^{1/6}(-1)^{5/6}\mathrm{e}^{\frac{x^6}{6}}}{x} + \frac{6x^5(-1)^{5/6}\Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)}{(-x^6)^{5/6}}\right)}{36}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.487.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + x^5 y' + 6x^4 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ○ Convert $x^4 \cdot y$ to series expansion

  $x^4 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+4}$

  ○ Shift index using $k- >k-4$

  $x^4 \cdot y = \sum_{k=4}^{\infty} a_{k-4} x^k$

  ○ Convert $x^5 \cdot y'$ to series expansion

  $x^5 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+4}$

- Shift index using $k->k-4$

  $x^5 \cdot y' = \sum_{k=4}^{\infty} a_{k-4}(k-4)\, x^k$

- Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k->k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=4}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-4}(k+2)\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 = 0, 6a_3 = 0, 12a_4 = 0, 20a_5 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2} + a_{k-4} + a_{k+2}) = 0$

- Shift index using $k->k+4$

  $(k+6)\,((k+4)\, a_{k+6} + a_k + a_{k+6}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{a_k}{k+5}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\right]$

1.487.3 Maple trace

```
Methods for second order ODEs:
```

1.487.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 62

```
dsolve(diff(diff(y(x),x),x)+x^5*diff(y(x),x)+6*x^4*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{\left(-c_1\,\mathrm{e}^{-\frac{x^6}{6}}x - c_2 6^{1/6}\right)\left(-x^6\right)^{5/6} + x^6 c_2\,\mathrm{e}^{-\frac{x^6}{6}}\left(\Gamma\left(\frac{5}{6}\right) - \Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)\right)}{\left(-x^6\right)^{5/6}}$$

1.487.5 Mathematica DSolve solution

Solving time : 0.137 (sec)
Leaf size : 53

```
DSolve[{D[y[x],{x,2}]+x^5*D[y[x],x]+6*x^4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{36}e^{-\frac{x^6}{6}}\left(36c_1x - 6^{5/6}c_2\sqrt[6]{-x^6}\,\Gamma\left(-\frac{1}{6}, -\frac{x^6}{6}\right)\right)$$

1.488 problem 504

Internal problem ID [8626]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 504
Date solved : `Monday, October 21, 2024 at 05:18:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1 + 3x)\, y'' + xy' + 2y = 0$$

1.488.1 Solved as second order ode using Kovacic algorithm

Time used: 30.869 (sec)

Writing the ode as

$$(1 + 3x)\, y'' + xy' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 + 3x \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 24x - 6}{4\left(1 + 3x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 24x - 6 \\
t &= 4(1 + 3x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 - 24x - 6}{4\left(1 + 3x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 922: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(1+3x)^2$. There is a pole at $x = -\frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{19}{324\left(x+\frac{1}{3}\right)^2} - \frac{37}{54\left(x+\frac{1}{3}\right)}$$

For the pole at $x = -\frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{19}{324}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{18} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{18} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} - \frac{37}{18x} - \frac{319}{27x^2} - \frac{11831}{81x^3} - \frac{2157901}{972x^4} - \frac{110035199}{2916x^5} - \frac{1501983319}{2187x^6} - \frac{85889060456}{6561x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 24x - 6}{36x^2 + 24x + 4} \\ &= Q + \frac{R}{36x^2 + 24x + 4} \\ &= \left(\frac{1}{36}\right) + \left(\frac{-\frac{74x}{3} - \frac{55}{9}}{36x^2 + 24x + 4}\right) \\ &= \frac{1}{36} + \frac{-\frac{74x}{3} - \frac{55}{9}}{36x^2 + 24x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is $-\frac{74}{3}$. Dividing this by leading coefficient in t which is 36 gives $-\frac{37}{54}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{37}{54}\right) - (0) \\ &= -\frac{37}{54} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{37}{54}}{\frac{1}{6}} - 0\right) &= -\frac{37}{18} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{37}{54}}{\frac{1}{6}} - 0\right) &= \frac{37}{18} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 24x - 6}{4\left(1+3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{3}$	2	0	$\frac{19}{18}$	$-\frac{1}{18}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$-\frac{37}{18}$	$\frac{37}{18}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{37}{18}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{37}{18} - \left(\frac{19}{18}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{19}{18\left(x + \frac{1}{3}\right)} + (-)\left(\frac{1}{6}\right) \\
&= \frac{19}{18\left(x + \frac{1}{3}\right)} - \frac{1}{6} \\
&= -\frac{-6 + x}{2(1 + 3x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{19}{18\left(x + \frac{1}{3}\right)} - \frac{1}{6}\right)(1) + \left(\left(-\frac{19}{18\left(x + \frac{1}{3}\right)^2}\right) + \left(\frac{19}{18\left(x + \frac{1}{3}\right)} - \frac{1}{6}\right)^2 - \left(\frac{x^2 - 24x - 6}{4(1 + 3x)^2}\right)\right) = 0$$

$$\frac{a_0 + 6}{1 + 3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -6 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (-6 + x)\, e^{\int \left(\frac{19}{18\left(x + \frac{1}{3}\right)} - \frac{1}{6}\right) dx} \\
&= (-6 + x)\, e^{-\frac{x}{6} + \frac{19 \ln(1 + 3x)}{18}} \\
&= (-6 + x)(1 + 3x)^{19/18}\, e^{-\frac{x}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1+3x}\,dx} \\
&= z_1 e^{-\frac{x}{6}+\frac{\ln(1+3x)}{18}} \\
&= z_1\left((1+3x)^{1/18}\,\mathrm{e}^{-\frac{x}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (1+3x)^{10/9}\,\mathrm{e}^{-\frac{x}{3}}(-6+x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1+3x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{3}+\frac{\ln(1+3x)}{9}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{x}{3}+\frac{\ln(1+3x)}{9}}\,\mathrm{e}^{\frac{2x}{3}}}{(1+3x)^{20/9}\,(-6+x)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((1+3x)^{10/9}\,\mathrm{e}^{-\frac{x}{3}}(-6+x)\right) \\
&\quad + c_2\left((1+3x)^{10/9}\,\mathrm{e}^{-\frac{x}{3}}(-6+x)\left(\int \frac{\mathrm{e}^{-\frac{x}{3}+\frac{\ln(1+3x)}{9}}\,\mathrm{e}^{\frac{2x}{3}}}{(1+3x)^{20/9}\,(-6+x)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.488.2 Maple step by step solution

Let's solve

$(1+3x)\left(\frac{d}{dx}y'\right)+xy'+2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{2y}{1+3x}-\frac{xy'}{1+3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{xy'}{1+3x}+\frac{2y}{1+3x}=0$

□ Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{x}{1+3x}, P_3(x)=\frac{2}{1+3x}\right]$

  ◦ $\left(x+\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{3}$

    $\left(\left(x+\frac{1}{3}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{3}}=-\frac{1}{9}$

  ◦ $\left(x+\frac{1}{3}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{3}$

    $\left(\left(x+\frac{1}{3}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{3}}=0$

  ◦ $x=-\frac{1}{3}$is a regular singular point

  Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

  $x_0=-\frac{1}{3}$

- Multiply by denominators

  $(1+3x)\left(\frac{d}{dx}y'\right)+xy'+2y=0$

- Change variables using $x=u-\frac{1}{3}$ so that the regular singular point is at $u=0$

  $3u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\left(u-\frac{1}{3}\right)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty}a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u \cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)$ to series expansion

  $$u \cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -1}$$

- Shift index using $k->k+1$

  $$u \cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1}^{\infty}a_{k +1} \left(k +1+r \right) \left(k +r \right) u^{k +r}$$

  Rewrite ODE with series expansions

  $$\frac{a_{0} r \left(-10+9 r \right) u^{-1+r}}{3}+\left(\sum_{k =0}^{\infty}\left(\frac{a_{k +1} \left(k +1+r \right) \left(9 k -1+9 r \right)}{3}+a_{k} \left(k +r +2\right)\right) u^{k +r}\right)=0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$\frac{r \left(-10+9 r \right)}{3}=0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{10}{9}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$3 \left(k -\frac{1}{9}+r \right) \left(k +1+r \right) a_{k +1}+a_{k} \left(k +r +2\right)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k +1}=-\frac{3 a_{k} \left(k +r +2\right)}{\left(9 k -1+9 r \right) \left(k +1+r \right)}$$

- Recursion relation for $r =0$

  $$a_{k +1}=-\frac{3 a_{k} \left(k +2\right)}{\left(9 k -1\right) \left(k +1\right)}$$

- Solution for $r =0$

  $$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k}, a_{k +1}=-\frac{3 a_{k} \left(k +2\right)}{\left(9 k -1\right) \left(k +1\right)}\right]$$

- Revert the change of variables $u =x +\frac{1}{3}$

  $$\left[y =\sum_{k =0}^{\infty}a_{k} \left(x +\frac{1}{3}\right)^{k}, a_{k +1}=-\frac{3 a_{k} \left(k +2\right)}{\left(9 k -1\right) \left(k +1\right)}\right]$$

- Recursion relation for $r =\frac{10}{9}$

  $$a_{k +1}=-\frac{3 a_{k} \left(k +\frac{28}{9}\right)}{\left(9 k +9\right) \left(k +\frac{19}{9}\right)}$$

- Solution for $r =\frac{10}{9}$

  $$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k +\frac{10}{9}}, a_{k +1}=-\frac{3 a_{k} \left(k +\frac{28}{9}\right)}{\left(9 k +9\right) \left(k +\frac{19}{9}\right)}\right]$$

- Revert the change of variables $u = x + \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{3}\right)^{k+\frac{10}{9}}, a_{k+1} = -\frac{3a_k \left(k+\frac{28}{9}\right)}{(9k+9)\left(k+\frac{19}{9}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x + \frac{1}{3}\right)^{k}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + \frac{1}{3}\right)^{k+\frac{10}{9}}\right), a_{k+1} = -\frac{3a_k(k+2)}{(9k-1)(k+1)}, b_{k+1} = -\frac{3b_k \left(k+\frac{28}{9}\right)}{(9k+9)\left(k+\frac{19}{9}\right)}\right]$$

1.488.3 Maple trace

```
Methods for second order ODEs:
```

1.488.4 Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 62

```
dsolve((1+3*x)*diff(diff(y(x),x),x)+x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\mathrm{e}^{-\frac{x}{3}}(-6+x)\, c_1 \left(\Gamma\left(-\frac{1}{9}\right) + \frac{10\Gamma\left(-\frac{10}{9}, -\frac{1}{9}-\frac{x}{3}\right)}{9}\right)\left(x+\frac{1}{3}\right)\left(-\frac{1}{9}-\frac{x}{3}\right)^{1/9}}{9} + 3c_2(-6+x)\left(x+\frac{1}{3}\right)\mathrm{e}^{-\frac{x}{3}}\left(\frac{1}{9}+\frac{x}{3}\right)^{1/9} - \frac{10c_1\,\mathrm{e}^{\frac{1}{9}}}{9}$$

1.488.5 Mathematica DSolve solution

Solving time : `1.32` (sec)
Leaf size : 124

```
DSolve[{(1+3*x)*D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{x}{3}-\frac{1}{9}}\left(1520c_1\sqrt[9]{3x+1}(3x^2-17x-6) - 2^{8/9}c_2e^{\frac{x}{3}+\frac{1}{9}}(9x^2-48x-26) + 2^{8/9}3^{7/9}c_2\sqrt[9]{-3x-1}(3x^2-17\right.}{380\ 2^{17/18}}$$

1.489 problem 505

Internal problem ID [8627]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 505
Date solved : `Monday, October 21, 2024 at 05:18:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(3x^2+x+1\right)y''+(2+15x)\,y'+12y=0$$

1.489.1 Solved as second order ode using Kovacic algorithm

Time used: 0.813 (sec)

Writing the ode as

$$\left(3x^2+x+1\right)y''+(2+15x)\,y'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2+x+1 \\ B &= 2+15x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -9x^2 - 12x - 18$$
$$t = 4\left(3x^2 + x + 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 924: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x^2 + x + 1)^2$. There is a pole at $x = -\frac{1}{6} + \frac{i\sqrt{11}}{6}$ of order 2. There is a pole at $x = -\frac{1}{6} - \frac{i\sqrt{11}}{6}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{\frac{27}{88} + \frac{3i\sqrt{11}}{88}}{\left(x + \frac{1}{6} - \frac{i\sqrt{11}}{6}\right)^2} + \frac{\frac{27}{88} - \frac{3i\sqrt{11}}{88}}{\left(x + \frac{1}{6} + \frac{i\sqrt{11}}{6}\right)^2} + \frac{57i\sqrt{11}}{242\left(x + \frac{1}{6} - \frac{i\sqrt{11}}{6}\right)} - \frac{57i\sqrt{11}}{242\left(x + \frac{1}{6} + \frac{i\sqrt{11}}{6}\right)}$$

For the pole at $x = -\frac{1}{6} + \frac{i\sqrt{11}}{6}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{6} - \frac{i\sqrt{11}}{6}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{27}{88} + \frac{3i\sqrt{11}}{88}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{1078 + 66i\sqrt{11}}}{44} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{1078 + 66i\sqrt{11}}}{44} \end{aligned}$$

For the pole at $x = -\frac{1}{6} - \frac{i\sqrt{11}}{6}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{6} + \frac{i\sqrt{11}}{6}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{27}{88} - \frac{3i\sqrt{11}}{88}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{1078 - 66i\sqrt{11}}}{44} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{1078 - 66i\sqrt{11}}}{44} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2}$$

Since the gcd$(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{6} + \frac{i\sqrt{11}}{6}$	2	0	$\frac{1}{2} + \frac{\sqrt{1078+66i\sqrt{11}}}{44}$	$\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}$
$-\frac{1}{6} - \frac{i\sqrt{11}}{6}$	2	0	$\frac{1}{2} + \frac{\sqrt{1078-66i\sqrt{11}}}{44}$	$\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x+\frac{1}{6}-\frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x+\frac{1}{6}+\frac{i\sqrt{11}}{6}} + (-)(0) \\
&= \frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x+\frac{1}{6}-\frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x+\frac{1}{6}+\frac{i\sqrt{11}}{6}} \\
&= -\frac{3x}{6x^2+2x+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x+\frac{1}{6}-\frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x+\frac{1}{6}+\frac{i\sqrt{11}}{6}} \right)(1) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{\left(x+\frac{1}{6}-\frac{i\sqrt{11}}{6}\right)^2} - \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{\left(x+\frac{1}{6}+\frac{i\sqrt{11}}{6}\right)^2} \right) + \left(\frac{1}{2} \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(\frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x+\frac{1}{6}-\frac{i\sqrt{11}}{6}} + \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x+\frac{1}{6}+\frac{i\sqrt{11}}{6}} \right) dx} \\
&= (x)\, \mathrm{e}^{-\frac{\ln\left(36x^2+12x+12\right)}{4} + \frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}} \\
&= \frac{x\sqrt{2}\, 3^{3/4} \mathrm{e}^{\frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}}}{6\left(3x^2+x+1\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2+15x}{3x^2+x+1} \, dx} \\
&= z_1 e^{-\frac{5\ln\left(3x^2+x+1\right)}{4} + \frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}}}{\left(3x^2+x+1\right)^{5/4}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{\frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} x\sqrt{2}\, 3^{3/4}}{6\left(3x^2+x+1\right)^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2+15x}{3x^2+x+1} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(3x^2+x+1\right)}{2} + \frac{\sqrt{11}\, \arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}}}{\left(y_1\right)^2} \, dx
\end{aligned}
$$

$$= y_1 \left(\int \frac{2\,\mathrm{e}^{-\frac{5\ln\left(3x^2+x+1\right)}{2}+\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} \left(3x^2+x+1\right)^3 \mathrm{e}^{-\frac{2\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} \sqrt{3}}{x^2} dx \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} x\sqrt{2}\,3^{3/4}}{6\left(3x^2+x+1\right)^{3/2}} \right)$$

$$+ c_2 \left(\frac{\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} x\sqrt{2}\,3^{3/4}}{6\left(3x^2+x+1\right)^{3/2}} \left(\int \frac{2\,\mathrm{e}^{-\frac{5\ln\left(3x^2+x+1\right)}{2}+\frac{\sqrt{11}\,\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{11}} \left(3x^2+x+1\right)^3 \mathrm{e}^{-\frac{2\sqrt{11}\,\arctan\left(\frac{(6x-}{}\right.}{11}}}{x^2} \right.\right.$$

Will add steps showing solving for IC soon.

1.489.2 Maple step by step solution

Let's solve

$(3x^2 + x + 1)\left(\frac{d}{dx}y'\right) + (2 + 15x)\,y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{12y}{3x^2+x+1} - \frac{(2+15x)y'}{3x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(2+15x)y'}{3x^2+x+1} + \frac{12y}{3x^2+x+1} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2+15x}{3x^2+x+1}, P_3(x) = \frac{12}{3x^2+x+1}\right]$

- ◦ $\left(x + \frac{1}{6} + \frac{\mathrm{I}\sqrt{11}}{6}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{6} - \frac{\mathrm{I}\sqrt{11}}{6}$

  $\left(\left(x + \frac{1}{6} + \frac{\mathrm{I}\sqrt{11}}{6}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}} = 0$

- ◦ $\left(x + \frac{1}{6} + \frac{\mathrm{I}\sqrt{11}}{6}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{6} - \frac{\mathrm{I}\sqrt{11}}{6}$

$$\left(\left(x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}\right)^{2}\cdot P_{3}(x)\right)\Bigg|_{x=-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}}=0$$

- ◦ $x=-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}$

- Multiply by denominators

  $(3x^2+x+1)\left(\frac{d}{dx}y'\right)+(2+15x)\,y'+12y=0$

- Change variables using $x=u-\frac{1}{6}-\frac{\mathrm{I}\sqrt{11}}{6}$ so that the regular singular point is at $u=0$

  $(3u^2-\mathrm{I}u\sqrt{11})\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\left(-\frac{1}{2}+15u-\frac{5\,\mathrm{I}\sqrt{11}}{2}\right)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - ◦ Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - ◦ Convert $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - ◦ Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $\frac{\mathrm{I}\sqrt{11}\left(\mathrm{I}\sqrt{11}-33-22r\right)r a_0 u^{-1+r}}{22}+\left(\sum_{k=0}^{\infty}\left(\frac{\mathrm{I}\sqrt{11}\left(\mathrm{I}\sqrt{11}-22k-55-22r\right)(k+1+r)a_{k+1}}{22}+3a_k(k+r+2)^2\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\frac{\mathrm{I}}{22}\sqrt{11}\left(\mathrm{I}\sqrt{11}-33-22r\right)r=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right\}$

- Each term in the series must be 0, giving the recursion relation

$$3a_k(k+r+2)^2 - a_{k+1}(k+1+r)\left(\frac{1}{2}+\mathrm{I}\left(k+r+\frac{5}{2}\right)\sqrt{11}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{6a_k\left(k^2+2kr+r^2+4k+4r+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}kr\sqrt{11}+2\,\mathrm{I}\sqrt{11}\,r^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}r\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k+r+1}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}\right]$$

- Revert the change of variables $u=x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}$

$$\left[y=\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}\right)^k, a_{k+1}=\frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}\right]$$

- Recursion relation for $r=-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}$

$$a_{k+1}=\frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2+\frac{2\,\mathrm{I}\sqrt{11}}{11}\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\,\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+\frac{111\,\mathrm{I}\sqrt{11}}{22}+k-\frac{1}{2}}$$

- Solution for $r=-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}, a_{k+1}=\frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2+\frac{2\,\mathrm{I}\sqrt{11}}{11}\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\,\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}}$$

- Revert the change of variables $u=x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}$

$$\left[y=\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}\right)^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}, a_{k+1}=\frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2\right.}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\,\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}\left(\right.}$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}\right)^k\right)+\left(\sum_{k=0}^{\infty}b_k\left(x+\frac{1}{6}+\frac{\mathrm{I}\sqrt{11}}{6}\right)^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}\right), a_{k+1}=\frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}}\right.$$

1.489.3 Maple trace

```
Methods for second order ODEs:
```

1.489.4 Maple dsolve solution

Solving time : 0.085 (sec)
Leaf size : 163

```
dsolve((3*x^2+x+1)*diff(diff(y(x),x),x)+(2+15*x)*diff(y(x),x)+12*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{\frac{\sqrt{11}\arctan\left(\frac{(6x+1)\sqrt{11}}{11}\right)}{22}}\left(c_1\left(i\sqrt{11}-6x-1\right)^{3/2}\left(-36x^2-12x-12\right)^{-\frac{1}{4}+\frac{i\sqrt{11}}{44}}\operatorname{hypergeom}\left(\left[\frac{1}{2}+\frac{i\sqrt{11}}{22},\frac{1}{2}+\frac{i\sqrt{11}}{22}\right.\right.\right.}{}$$

1.489.5 Mathematica DSolve solution

Solving time : 5.054 (sec)
Leaf size : 93

```
DSolve[{(1+x+3*x^2)*D[y[x],{x,2}]+(2+15*x)*D[y[x],x]+12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{xe^{\frac{\arctan\left(\frac{6x+1}{\sqrt{11}}\right)}{\sqrt{11}}}\left(c_2\int_1^x \frac{e^{-\frac{\arctan\left(\frac{6K[1]+1}{\sqrt{11}}\right)}{\sqrt{11}}}\sqrt{3K[1]^2+K[1]+1}}{K[1]^2}dK[1]+c_1\right)}{\left(3x^2+x+1\right)^{3/2}}$$

1.490 problem 506

Internal problem ID [8628]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 506
Date solved : `Monday, October 21, 2024 at 05:18:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2+x)\,y''+(1+x)\,y'+3y=0$$

1.490.1 Solved as second order ode using Kovacic algorithm

Time used: 0.334 (sec)

Writing the ode as

$$(2+x)\,y''+(1+x)\,y'+3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2+x\\ B &= 1+x\\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10x - 21}{4(2+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 10x - 21$$
$$t = 4(2+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 10x - 21}{4(2+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 926: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2+x)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{7}{2\,(2+x)} + \frac{3}{4\,(2+x)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{7}{2x} - \frac{9}{2x^2} - \frac{97}{2x^3} - \frac{1291}{4x^4} - \frac{11103}{4x^5} - \frac{98061}{4x^6} - \frac{913053}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10x - 21}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-14x - 25}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{-14x - 25}{4x^2 + 16x + 16} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -14. Dividing this by leading coefficient in t which is 4 gives $-\frac{7}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{2}\right) - (0) \\ &= -\frac{7}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{2}}{\frac{1}{2}} - 0\right) &&= \frac{7}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 10x - 21}{4\left(2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{7}{2}$	$\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{7}{2} - \left(\frac{3}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{2\,(2+x)} + (-)\left(\frac{1}{2}\right) \\
&= \frac{3}{2\,(2+x)} - \frac{1}{2} \\
&= -\frac{-1+x}{2\,(2+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2\,(2+x)} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{3}{2\,(2+x)^2}\right) + \left(\frac{3}{2\,(2+x)} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 10x - 21}{4\,(2+x)^2}\right)\right) = 0$$

$$\frac{(a_1+4)\,x + 2a_0 + a_1 + 4}{2+x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = -4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 4x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 4x\right) e^{\int \left(\frac{3}{2(2+x)} - \frac{1}{2}\right) dx} \\
&= \left(x^2 - 4x\right) e^{-\frac{x}{2} + \frac{3\ln(2+x)}{2}} \\
&= x(x-4)\,(2+x)^{3/2}\, e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+x}{2+x}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\frac{\ln(2+x)}{2}} \\
&= z_1\left(\sqrt{2+x}\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (2+x)^2\,\mathrm{e}^{-x}x(x-4)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+x}{2+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+\ln(2+x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^x}{288\,(2+x)^2}-\frac{11\,\mathrm{e}^x}{864\,(2+x)}-\frac{\mathrm{e}^{-2}\,\mathrm{Ei}_1\left(-2-x\right)}{48}-\frac{\mathrm{e}^x}{3456\,(x-4)}-\frac{\mathrm{e}^x}{128x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left((2+x)^2\,\mathrm{e}^{-x}x(x-4)\right)+c_2\left((2+x)^2\,\mathrm{e}^{-x}x(x-4)\left(-\frac{\mathrm{e}^x}{288\,(2+x)^2}-\frac{11\,\mathrm{e}^x}{864\,(2+x)}\right.\right. \\
&\qquad \left.\left.-\frac{\mathrm{e}^{-2}\,\mathrm{Ei}_1\left(-2-x\right)}{48}-\frac{\mathrm{e}^x}{3456\,(x-4)}-\frac{\mathrm{e}^x}{128x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.490.2 Maple step by step solution

Let's solve

$(2+x)\left(\frac{d}{dx}y'\right)+(1+x)\,y'+3y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{3y}{2+x}-\frac{(1+x)y'}{2+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(1+x)y'}{2+x}+\frac{3y}{2+x}=0$

□ Check to see if $x_0=-2$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{1+x}{2+x}, P_3(x)=\frac{3}{2+x}\right]$

  ◦ $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((2+x)\cdot P_2(x)\right)\Big|_{x=-2}=-1$

  ◦ $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((2+x)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

Check to see if $x_0=-2$ is a regular singular point

$x_0=-2$

- Multiply by denominators

$(2+x)\left(\frac{d}{dx}y'\right)+(1+x)\,y'+3y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-1+u)\left(\frac{d}{du}y(u)\right)+3y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k->k+1$

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) + a_k(k+r+3)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(k+r-1) + a_k(k+r+3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r+3)}{(k+1+r)(k+r-1)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-1)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

$$a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-1)}$$

- Recursion relation for $r = 2$

$$a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}\right]$$

- Revert the change of variables $u = 2+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k+2}, a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}\right]$$

1.490.3 Maple trace

```
Methods for second order ODEs:
```

1.490.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 59

```
dsolve((2+x)*diff(diff(y(x),x),x)+(1+x)*diff(y(x),x)+3*y(x) = 0,
       y(x),singsol=all)
```

$$y = xc_2 \,\mathrm{e}^{-2-x}(x-4)\,(2+x)^2 \,\mathrm{Ei}_1\,(-2-x)+c_1(2+x)^2 \,\mathrm{e}^{-x}x(x-4)+c_2\left(x^3-x^2-10x-6\right)$$

1.490.5 Mathematica DSolve solution

Solving time : 0.279 (sec)
Leaf size : 99

```
DSolve[{(2+x)*D[y[x],{x,2}]+(1+x)*D[y[x],x]+3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x-1}(c_2(x-4)(x+2)^2x\,\mathrm{ExpIntegralEi}(x+2)+384c_1x^4-c_2e^{x+2}x^3+x^2(c_2e^{x+2}-4608c_1)+x(10c_2e$$
$$96\sqrt{2}$$

1.491 problem 507

Internal problem ID [8629]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 507
Date solved : `Monday, October 21, 2024 at 05:18:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(4+x)\,y''+(2+x)\,y'+2y=0$$

1.491.1 Solved as second order ode using Kovacic algorithm

Time used: 0.326 (sec)

Writing the ode as

$$(4+x)\,y''+(2+x)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=4+x\\ B&=2+x\\ C&=2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x - 24}{4(4+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x - 24$$
$$t = 4(4+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x - 24}{4(4+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 928: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(4+x)^2$. There is a pole at $x = -4$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(4+x)^2} - \frac{3}{4+x}$$

For the pole at $x = -4$ let b be the coefficient of $\frac{1}{(4+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{x} + \frac{5}{x^2} - \frac{34}{x^3} + \frac{59}{x^4} - \frac{586}{x^5} + \frac{370}{x^6} - \frac{12484}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x - 24}{4x^2 + 32x + 64} \\ &= Q + \frac{R}{4x^2 + 32x + 64} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-12x - 40}{4x^2 + 32x + 64}\right) \\ &= \frac{1}{4} + \frac{-12x - 40}{4x^2 + 32x + 64} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -12. Dividing this by leading coefficient in t which is 4 gives -3. Now b can be found.

$$\begin{aligned} b &= (-3) - (0) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-3}{\frac{1}{2}} - 0\right) &= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-3}{\frac{1}{2}} - 0\right) &= 3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x - 24}{4(4 + x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-4	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-3	3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 3$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= 3 - (2) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{2}{4+x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{2}{4+x} - \frac{1}{2} \\
&= -\frac{x}{2(4+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{4+x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{2}{(4+x)^2} \right) + \left(\frac{2}{4+x} - \frac{1}{2} \right)^2 - \left(\frac{x^2 - 4x - 24}{4(4+x)^2} \right) \right) = 0$$

$$\frac{a_0}{4+x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{2}{4+x} - \frac{1}{2}\right) dx} \\
&= (x)\, e^{-\frac{x}{2} + 2\ln(4+x)} \\
&= x(4+x)^2\, e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2+x}{4+x}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\ln(4+x)} \\
&= z_1\left((4+x)\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (4+x)^3\,\mathrm{e}^{-x} x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2+x}{4+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+2\ln(4+x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{x}}{48\,(4+x)^3} - \frac{5\,\mathrm{e}^{x}}{192\,(4+x)^2} - \frac{29\,\mathrm{e}^{x}}{768\,(4+x)} - \frac{\mathrm{e}^{-4}\,\mathrm{Ei}_1(-4-x)}{24} - \frac{\mathrm{e}^{x}}{256x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((4+x)^3\,\mathrm{e}^{-x} x\right) + c_2\left((4+x)^3\,\mathrm{e}^{-x} x\left(-\frac{\mathrm{e}^{x}}{48\,(4+x)^3} - \frac{5\,\mathrm{e}^{x}}{192\,(4+x)^2} - \frac{29\,\mathrm{e}^{x}}{768\,(4+x)} - \frac{\mathrm{e}^{-4}\,\mathrm{Ei}_1(-4-x)}{24} - \frac{\mathrm{e}^{x}}{256x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.491.2 Maple step by step solution

Let's solve

$(4+x)\left(\frac{d}{dx}y'\right)+(2+x)\,y'+2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{2y}{4+x}-\frac{(2+x)y'}{4+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(2+x)y'}{4+x}+\frac{2y}{4+x}=0$

□ Check to see if $x_0=-4$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{2+x}{4+x},P_3(x)=\frac{2}{4+x}\right]$

  ◦ $(4+x)\cdot P_2(x)$ is analytic at $x=-4$

    $\left((4+x)\cdot P_2(x)\right)\Big|_{x=-4}=-2$

  ◦ $(4+x)^2\cdot P_3(x)$ is analytic at $x=-4$

    $\left((4+x)^2\cdot P_3(x)\right)\Big|_{x=-4}=0$

  ◦ $x=-4$is a regular singular point

  Check to see if $x_0=-4$ is a regular singular point

  $x_0=-4$

- Multiply by denominators

  $(4+x)\left(\frac{d}{dx}y'\right)+(2+x)\,y'+2y=0$

- Change variables using $x=u-4$ so that the regular singular point is at $u=0$

  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-2+u)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k->k+1$

$$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r+2)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 3\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r+2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k(k+r+2)}{(k+1+r)(k-2+r)}$
- Recursion relation for $r = 0$
  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-2)}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$
  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-2)}$
- Recursion relation for $r = 3$
  $a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}$
- Solution for $r = 3$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}\right]$
- Revert the change of variables $u = 4 + x$
  $\left[y = \sum_{k=0}^{\infty} a_k(4+x)^{k+3}, a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}\right]$

1.491.3 Maple trace

```
Methods for second order ODEs:
```

1.491.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 53

```
dsolve((4+x)*diff(diff(y(x),x),x)+(2+x)*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = xc_2 \,\mathrm{e}^{-4-x}(4+x)^3 \,\mathrm{Ei}_1\left(-4-x\right) + c_1(4+x)^3 \,\mathrm{e}^{-x} x + c_2\left(x^3+9x^2+22x+6\right)$$

1.491.5 Mathematica DSolve solution

Solving time : `0.165` (sec)
Leaf size : 97

```
DSolve[{(4+x)*D[y[x],{x,2}]+(2+x)*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{24}e^{-x-4}\left(c_2 x(x+4)^3 \,\mathrm{ExpIntegralEi}(x+4) + e^4\left(24c_1x^4+x^3(288c_1-c_2e^x)+9x^2(128c_1-c_2e^x)+2x(768c_1-11c_2e^x)-6c_2e^x\right)\right)$$

1.492 problem 508

Internal problem ID [8630]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 508
Date solved : `Monday, October 21, 2024 at 05:18:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+3x\right)y''+10(1+x)\,y'+8y=0$$

1.492.1 Solved as second order ode using Kovacic algorithm

Time used: 0.312 (sec)

Writing the ode as

$$\left(2x^2+3x\right)y''+(10x+10)\,y'+8y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+3x \\ B &= 10x+10 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6x + 10}{(2x^2 + 3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 6x + 10$$
$$t = (2x^2 + 3x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6x + 10}{(2x^2 + 3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 930: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{22}{27x} + \frac{10}{9x^2} - \frac{5}{36\left(x + \frac{3}{2}\right)^2} + \frac{22}{27\left(x + \frac{3}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{10}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{2}{3} \end{aligned}$$

For the pole at $x = -\frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 6x + 10}{(2x^2 + 3x)^2}$$

Since the gcd$(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 6x + 10}{(2x^2 + 3x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{3}$	$-\frac{2}{3}$
$-\frac{3}{2}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{2}{3x} + \frac{1}{6x+9} + (-)\,(0) \\ &= -\frac{2}{3x} + \frac{1}{6x+9} \\ &= -\frac{x+2}{x\,(2x+3)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{3x} + \frac{1}{6x+9}\right)(1) + \left(\left(\frac{2}{3x^2} - \frac{1}{6\left(x+\frac{3}{2}\right)^2}\right) + \left(-\frac{2}{3x} + \frac{1}{6x+9}\right)^2 - \left(\frac{-x^2+6x+10}{\left(2x^2+3x\right)^2}\right)\right) = 0$$

$$\frac{-4+2a_0}{x\,(2x+3)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= (x+2)\, e^{\int \left(-\frac{2}{3x} + \frac{1}{6x+9}\right) dx} \\ &= (x+2)\, e^{-\frac{2\ln(x)}{3} + \frac{\ln(2x+3)}{6}} \\ &= \frac{(x+2)\,(2x+3)^{1/6}}{x^{2/3}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{10x+10}{2x^2+3x}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{3}-\frac{5\ln(2x+3)}{6}} \\
&= z_1\left(\frac{1}{x^{5/3}\left(2x+3\right)^{5/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x+2}{x^{7/3}\left(2x+3\right)^{2/3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{10x+10}{2x^2+3x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{10\ln(x)}{3}-\frac{5\ln(2x+3)}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{10\ln(x)}{3}-\frac{5\ln(2x+3)}{3}}x^{14/3}(2x+3)^{4/3}}{\left(x+2\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{x+2}{x^{7/3}\left(2x+3\right)^{2/3}}\right) + c_2\left(\frac{x+2}{x^{7/3}\left(2x+3\right)^{2/3}}\left(\int \frac{\mathrm{e}^{-\frac{10\ln(x)}{3}-\frac{5\ln(2x+3)}{3}}x^{14/3}(2x+3)^{4/3}}{\left(x+2\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.492.2 Maple step by step solution

Let's solve

$(2x^2+3x)\left(\frac{d}{dx}y'\right)+10(1+x)\,y'+8y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{8y}{x(2x+3)}-\frac{10(1+x)y'}{x(2x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{10(1+x)y'}{x(2x+3)}+\frac{8y}{x(2x+3)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{10(1+x)}{x(2x+3)}, P_3(x)=\frac{8}{x(2x+3)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=\frac{10}{3}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(2x+3)\left(\frac{d}{dx}y'\right)+(10x+10)\,y'+8y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y'$ to series expansion for $m=0..1$

    $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=1..2$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0r(7+3r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(3k+10+3r)+2a_k(k+r+2)^2\right)x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(7+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{7}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(3k+10+3r)+2a_k(k+r+2)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{2a_k(k+r+2)^2}{(k+1+r)(3k+10+3r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}\right]$$

- Recursion relation for $r = -\frac{7}{3}$

$$a_{k+1} = -\frac{2a_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}$$

- Solution for $r = -\frac{7}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{7}{3}}, a_{k+1} = -\frac{2a_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{7}{3}}\right), a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}, b_{k+1} = -\frac{2b_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$$

1.492.3 Maple trace

```
Methods for second order ODEs:
```

1.492.4 Maple dsolve solution

Solving time : 0.097 (sec)
Leaf size : 31

```
dsolve((2*x^2+3*x)*diff(diff(y(x),x),x)+10*(1+x)*diff(y(x),x)+8*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1(x+2)}{\left(1+\frac{2x}{3}\right)^{2/3} x^{7/3}} + c_2 \,\text{hypergeom}\left(\left[2,2\right], \left[\frac{10}{3}\right], -\frac{2x}{3}\right)$$

1.492.5 Mathematica DSolve solution

Solving time : 0.374 (sec)
Leaf size : 245

```
DSolve[{(3*x+2*x^2)*D[y[x],{x,2}]+10*(1+x)*D[y[x],x]+8*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{-2\ 2^{2/3}\sqrt{3}c_2(x+2)\arctan\left(\frac{\sqrt{3}\sqrt[3]{2x+3}}{2\sqrt[3]{2}\sqrt[3]{x}+\sqrt[3]{2x+3}}\right)+2^{2/3}c_2x\log\left(2^{2/3}x^{2/3}+\sqrt[3]{2}\sqrt[3]{2x+3}\sqrt[3]{x}+(2x+3)^{2}\right.}{}$$

1.493 problem 509

Internal problem ID [8631]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 509
Date solved : `Monday, October 21, 2024 at 05:18:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - (6 - 7x)\,y' + 8y = 0$$

1.493.1 Solved as second order ode using Kovacic algorithm

Time used: 0.302 (sec)

Writing the ode as

$$x^2y'' + (-6 + 7x)\,y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -6 + 7x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 60x + 36}{4x^4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^2 - 60x + 36$$
$$t = 4x^4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 60x + 36}{4x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 932: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned}
\alpha_c^{+} &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\
\alpha_c^{-} &= \frac{1}{2}\left(-\frac{b}{a} + v\right)
\end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{9}{x^4} + \frac{3}{4x^2} - \frac{15}{x^3}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{3}{x^2} - \frac{5}{2x} - \frac{11}{12} - \frac{55x}{72} - \frac{671x^2}{864} - \frac{4565x^3}{5184} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{3}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = 3$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -15. Therefore

$$\begin{aligned} b &= (-15) - (0) \\ &= -15 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{3}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{-15}{3} + 2\right) &= -\frac{3}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{-15}{3} + 2\right) &= \frac{7}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 60x + 36}{4x^4}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 60x + 36}{4x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{3}{x^2}$	$-\frac{3}{2}$	$\frac{7}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{x^2} - \frac{3}{2x} + (-)(0) \\
&= \frac{3}{x^2} - \frac{3}{2x} \\
&= -\frac{3(-2 + x)}{2x^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{3}{x^2}-\frac{3}{2x}\right)(1)+\left(\left(-\frac{6}{x^3}+\frac{3}{2x^2}\right)+\left(\frac{3}{x^2}-\frac{3}{2x}\right)^2-\left(\frac{3x^2-60x+36}{4x^4}\right)\right)=0$$

$$\frac{6+3a_0}{x^2}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=-2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=-2+x$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (-2+x)\, e^{\int \left(\frac{3}{x^2}-\frac{3}{2x}\right) dx} \\
&= (-2+x)\, e^{-\frac{3}{x}-\frac{3\ln(x)}{2}} \\
&= \frac{(-2+x)\, e^{-\frac{3}{x}}}{x^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6+7x}{x^2}\, dx} \\
&= z_1 e^{-\frac{3}{x}-\frac{7\ln(x)}{2}} \\
&= z_1\left(\frac{e^{-\frac{3}{x}}}{x^{7/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{e^{-\frac{6}{x}}(-2+x)}{x^5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6+7x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{6}{x}-7\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^2 \mathrm{e}^{\frac{6}{x}}}{2} + 7x\,\mathrm{e}^{\frac{6}{x}} + 54\,\mathrm{Ei}_1\left(-\frac{6}{x}\right) + \frac{12\,\mathrm{e}^{\frac{6}{x}}}{\frac{6}{x}-3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\frac{6}{x}}(-2+x)}{x^5}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{6}{x}}(-2+x)}{x^5}\left(\frac{x^2 \mathrm{e}^{\frac{6}{x}}}{2} + 7x\,\mathrm{e}^{\frac{6}{x}} + 54\,\mathrm{Ei}_1\left(-\frac{6}{x}\right) + \frac{12\,\mathrm{e}^{\frac{6}{x}}}{\frac{6}{x}-3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.493.2 Maple step by step solution

1.493.3 Maple trace

```
Methods for second order ODEs:
```

1.493.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 50

```
dsolve(x^2*diff(diff(y(x),x),x)-(6-7*x)*diff(y(x),x)+8*y(x) = 0,
       y(x),singsol=all)
```

$$
y = \frac{108 c_2\,\mathrm{e}^{-\frac{6}{x}}(-2+x)\,\mathrm{Ei}_1\left(-\frac{6}{x}\right) + c_1\,\mathrm{e}^{-\frac{6}{x}}(-2+x) + c_2 x(x^2+12x-36)}{x^5}
$$

1.493.5 Mathematica DSolve solution

Solving time : 0.183 (sec)
Leaf size : 59

```
DSolve[{x^2*D[y[x],{x,2}]-(6-7*x)*D[y[x],x]+8*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-6/x}\left(-108c_2(x-2)\operatorname{ExpIntegralEi}\left(\frac{6}{x}\right)+c_2e^{6/x}x(x^2+12x-36)+2c_1(x-2)\right)}{2x^5}$$

1.494 problem 510

Internal problem ID [8632]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 510
Date solved : `Monday, October 21, 2024 at 05:18:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2x^2+x+1)\,y''+(1+7x)\,y'+2y=0$$

1.494.1 Solved as second order ode using Kovacic algorithm

Time used: 1.020 (sec)

Writing the ode as

$$(2x^2+x+1)\,y''+(1+7x)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+x+1 \\ B &= 1+7x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 - 2x + 5}{4(2x^2 + x + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 - 2x + 5$$
$$t = 4(2x^2 + x + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 - 2x + 5}{4(2x^2 + x + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 933: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\&= 4 - 2\\&= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x + 1)^2$. There is a pole at $x = -\frac{1}{4} + \frac{i\sqrt{7}}{4}$ of order 2. There is a pole at $x = -\frac{1}{4} - \frac{i\sqrt{7}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{29}{224} + \frac{9i\sqrt{7}}{224}}{\left(x + \frac{1}{4} - \frac{i\sqrt{7}}{4}\right)^2} + \frac{-\frac{29}{224} - \frac{9i\sqrt{7}}{224}}{\left(x + \frac{1}{4} + \frac{i\sqrt{7}}{4}\right)^2} - \frac{8i\sqrt{7}}{49\left(x + \frac{1}{4} - \frac{i\sqrt{7}}{4}\right)} + \frac{8i\sqrt{7}}{49\left(x + \frac{1}{4} + \frac{i\sqrt{7}}{4}\right)}$$

For the pole at $x = -\frac{1}{4} + \frac{i\sqrt{7}}{4}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{4}-\frac{i\sqrt{7}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{224} + \frac{9i\sqrt{7}}{224}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{3\sqrt{42 + 14i\sqrt{7}}}{56}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{3\sqrt{42 + 14i\sqrt{7}}}{56}\end{aligned}$$

For the pole at $x = -\frac{1}{4} - \frac{i\sqrt{7}}{4}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{4}+\frac{i\sqrt{7}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{224} - \frac{9i\sqrt{7}}{224}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{3\sqrt{42 - 14i\sqrt{7}}}{56}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{3\sqrt{42 - 14i\sqrt{7}}}{56}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 - 2x + 5}{4\left(2x^2 + x + 1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 - 2x + 5}{4\left(2x^2 + x + 1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{4} + \frac{i\sqrt{7}}{4}$	2	0	$\frac{1}{2} + \frac{3\sqrt{42+14i\sqrt{7}}}{56}$	$\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}$
$-\frac{1}{4} - \frac{i\sqrt{7}}{4}$	2	0	$\frac{1}{2} + \frac{3\sqrt{42-14i\sqrt{7}}}{56}$	$\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{5}{4} - \left(\frac{1}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x + \frac{1}{4} - \frac{i\sqrt{7}}{4}} + \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x + \frac{1}{4} + \frac{i\sqrt{7}}{4}} + (0) \\
&= \frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x + \frac{1}{4} - \frac{i\sqrt{7}}{4}} + \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x + \frac{1}{4} + \frac{i\sqrt{7}}{4}} \\
&= \frac{x + 1}{4x^2 + 2x + 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x + \frac{1}{4} - \frac{i\sqrt{7}}{4}} + \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x + \frac{1}{4} + \frac{i\sqrt{7}}{4}} \right)(1) + \left(\left(-\frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{\left(x + \frac{1}{4} - \frac{i\sqrt{7}}{4}\right)^2} - \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{\left(x + \frac{1}{4} + \frac{i\sqrt{7}}{4}\right)^2} \right) + \left(\frac{\frac{1}{2} - 3\sqrt{}}{x + \frac{1}{4}} \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= (x+1)\,\mathrm{e}^{\int \left(\frac{\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x+\frac{1}{4}-\frac{i\sqrt{7}}{4}}+\frac{\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x+\frac{1}{4}+\frac{i\sqrt{7}}{4}}\right)dx} \\
&= (x+1)\,\mathrm{e}^{\frac{\ln\left(16x^2+8x+8\right)}{8}+\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}} \\
&= (x+1)\,2^{3/8}\left(2x^2+x+1\right)^{1/8}\mathrm{e}^{\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+7x}{2x^2+x+1}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^2+x+1\right)}{8}+\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}} \\
&= z_1\left(\frac{\mathrm{e}^{\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{28}}}{\left(2x^2+x+1\right)^{7/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}}(x+1)\,2^{3/8}}{\left(2x^2+x+1\right)^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+7x}{2x^2+x+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^2+x+1\right)}{4}+\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{7\ln\left(2x^2+x+1\right)}{4}+\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}}\left(2x^2+x+1\right)^{3/2}\mathrm{e}^{-\frac{3\sqrt{7}\,\arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{7}}\,2^{1/4}}{2\,(x+1)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{e^{\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}} (x+1)\, 2^{3/8}}{(2x^2+x+1)^{3/4}} \right)$$

$$+ c_2 \left(\frac{e^{\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}} (x+1)\, 2^{3/8}}{(2x^2+x+1)^{3/4}} \left(\int \frac{e^{-\frac{7\ln\left(2x^2+x+1\right)}{4}+\frac{3\sqrt{7}\, \arctan\left(\frac{(4x+1)\sqrt{7}}{7}\right)}{14}} (2x^2+x+1)^{3/2} e^{-\frac{3\sqrt{7}\, \arctan\left(\frac{(4x}{7}}}{2(x+1)^2}\right.\right.$$

Will add steps showing solving for IC soon.

1.494.2 Maple step by step solution

Let's solve

$(2x^2+x+1)\left(\frac{d}{dx}y'\right) + (1+7x)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2y}{2x^2+x+1} - \frac{(1+7x)y'}{2x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(1+7x)y'}{2x^2+x+1} + \frac{2y}{2x^2+x+1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1+7x}{2x^2+x+1}, P_3(x) = \frac{2}{2x^2+x+1}\right]$

  ◦ $\left(x+\frac{1}{4}+\frac{\mathrm{I}\sqrt{7}}{4}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{4}-\frac{\mathrm{I}\sqrt{7}}{4}$

    $\left(\left(x+\frac{1}{4}+\frac{\mathrm{I}\sqrt{7}}{4}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{1}{4}-\frac{\mathrm{I}\sqrt{7}}{4}} = 0$

  ◦ $\left(x+\frac{1}{4}+\frac{\mathrm{I}\sqrt{7}}{4}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{4}-\frac{\mathrm{I}\sqrt{7}}{4}$

    $\left(\left(x+\frac{1}{4}+\frac{\mathrm{I}\sqrt{7}}{4}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{1}{4}-\frac{\mathrm{I}\sqrt{7}}{4}} = 0$

  ◦ $x = -\frac{1}{4}-\frac{\mathrm{I}\sqrt{7}}{4}$ is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -\frac{1}{4} - \frac{\mathrm{I}\sqrt{7}}{4}$

- Multiply by denominators

  $(2x^2+x+1)\left(\frac{d}{dx}y'\right)+(1+7x)\,y'+2y=0$

- Change variables using $x = u - \frac{1}{4} - \frac{\mathrm{I}\sqrt{7}}{4}$ so that the regular singular point is at $u = 0$

  $(2u^2-\mathrm{I}u\sqrt{7})\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\left(-\frac{3}{4}+7u-\frac{7\,\mathrm{I}\sqrt{7}}{4}\right)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $\frac{\mathrm{I}\sqrt{7}\,r\left(3\,\mathrm{I}\sqrt{7}-21-28r\right)a_0u^{-1+r}}{28}+\left(\sum_{k=0}^{\infty}\left(\frac{\mathrm{I}\sqrt{7}\,(k+1+r)\left(3\,\mathrm{I}\sqrt{7}-28k-49-28r\right)a_{k+1}}{28}+a_k(k+r+2)\,(2k+2r+1)\right)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\frac{\mathrm{I}}{28}\sqrt{7}\,r\left(3\,\mathrm{I}\sqrt{7}-21-28r\right)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-\mathrm{I}\left(k+r+\frac{7}{4}\right)a_{k+1}(k+1+r)\sqrt{7}+\frac{(-3k-3r-3)a_{k+1}}{4}+2(k+r+2)\,a_k\left(k+r+\frac{1}{2}\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{4a_k\,(2k^2+4kr+2r^2+5k+5r+2)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+8\,\mathrm{I}\sqrt{7}\,kr+4\,\mathrm{I}\sqrt{7}\,r^2+11\,\mathrm{I}\sqrt{7}\,k+11\,\mathrm{I}\sqrt{7}\,r+7\,\mathrm{I}\sqrt{7}+3k+3r}$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{4a_k(2k^2+5k+2)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}+3k}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{4a_k(2k^2+5k+2)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}+3k}\right]$$

- Revert the change of variables $u = x + \frac{1}{4} + \frac{\mathrm{I}\sqrt{7}}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{4} + \frac{\mathrm{I}\sqrt{7}}{4}\right)^k, a_{k+1} = \frac{4a_k(2k^2+5k+2)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}+3k}\right]$$

- Recursion relation for $r = \frac{3\,\mathrm{I}\sqrt{7}}{28} - \frac{3}{4}$

$$a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\,\mathrm{I}\sqrt{7}}{28}-\frac{7}{4}\right)}{\frac{3}{4}+4\,\mathrm{I}\sqrt{7}\,k^2+8\,\mathrm{I}\sqrt{7}\,k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+4\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\,\mathrm{I}\sqrt{7}\,k+11\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+\frac{205\,\mathrm{I}\sqrt{7}}{28}+3k}$$

- Solution for $r = \frac{3\,\mathrm{I}\sqrt{7}}{28} - \frac{3}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}}, a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\,\mathrm{I}\sqrt{7}}{28}-\frac{7}{4}\right)}{\frac{3}{4}+4\,\mathrm{I}\sqrt{7}\,k^2+8\,\mathrm{I}\sqrt{7}\,k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+4\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\,\mathrm{I}\sqrt{7}\,k+11\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+$$

- Revert the change of variables $u = x + \frac{1}{4} + \frac{\mathrm{I}\sqrt{7}}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{4} + \frac{\mathrm{I}\sqrt{7}}{4}\right)^{k+\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}}, a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\,\mathrm{I}\sqrt{7}}{28}\right.}{\frac{3}{4}+4\,\mathrm{I}\sqrt{7}\,k^2+8\,\mathrm{I}\sqrt{7}\,k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+4\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\,\mathrm{I}\sqrt{7}\,k+11\,\mathrm{I}\sqrt{7}}$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x + \frac{1}{4} + \frac{\mathrm{I}\sqrt{7}}{4}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + \frac{1}{4} + \frac{\mathrm{I}\sqrt{7}}{4}\right)^{k+\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}}\right), a_{k+1} = \frac{4a_k(2k^2+5k+2)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}}\right.$$

1.494.3 Maple trace

```
Methods for second order ODEs:
```

1.494.4 Maple dsolve solution

Solving time : 0.030 (sec)
Leaf size : 77

```
dsolve((2*x^2+x+1)*diff(diff(y(x),x),x)+(1+7*x)*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \operatorname{hypergeom}\left(\left[\frac{1}{2}, 2\right], \left[\frac{\left(7\sqrt{7}-3i\right)\sqrt{7}}{28}\right], \frac{1}{2}+\frac{i(-4x-1)\sqrt{7}}{14}\right) + c_2\left(4x+1+i\sqrt{7}\right)^{\frac{3i\sqrt{7}}{28}-\frac{3}{4}}\left(i\sqrt{7}-4x-1\right)^{-\frac{3i\sqrt{7}}{28}-\frac{3}{4}}(x+1)$$

1.494.5 Mathematica DSolve solution

Solving time : 3.459 (sec)
Leaf size : 102

```
DSolve[{(1+x+2*x^2)*D[y[x],{x,2}]+(1+7*x)*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{(x+1)e^{\frac{3\arctan\left(\frac{4x+1}{\sqrt{7}}\right)}{2\sqrt{7}}}\left(c_2\int_1^x \frac{e^{-\frac{3\arctan\left(\frac{4K[1]+1}{\sqrt{7}}\right)}{2\sqrt{7}}}}{(K[1]+1)^2\sqrt[4]{2K[1]^2+K[1]+1}}dK[1]+c_1\right)}{(2x^2+x+1)^{3/4}}$$

1.495 problem 511

Internal problem ID [8633]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 511
Date solved : `Monday, October 21, 2024 at 05:18:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(3+x)\, y''+(1+2x)\, y'-(2-x)\, y=0$$

1.495.1 Solved as second order ode using Kovacic algorithm

Time used: 0.195 (sec)

Writing the ode as

$$(3+x)\, y''+(1+2x)\, y'+(x-2)\, y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3+x \\ B &= 1+2x \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35}{4(3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35$$
$$t = 4(3+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35}{4(3+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 935: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3+x)^2$. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4\left(3+x\right)^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(3+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35}{4\left(3+x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{35}{4 \left(3+x \right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-3	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{5}{2} - \left(-\frac{5}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c) [\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty) [\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2 \left(3+x \right)} + (-) \left(0\right) \\
&= -\frac{5}{2 \left(3+x \right)} \\
&= -\frac{5}{2 \left(3+x \right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2\,(3+x)}\right)(0) + \left(\left(\frac{5}{2\,(3+x)^2}\right) + \left(-\frac{5}{2\,(3+x)}\right)^2 - \left(\frac{35}{4\,(3+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{5}{2(3+x)} dx} \\ &= \frac{1}{(3+x)^{5/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{1+2x}{3+x} \, dx} \\ &= z_1 e^{-x + \frac{5 \ln(3+x)}{2}} \\ &= z_1 \left((3+x)^{5/2} \, e^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+2x}{3+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x+5\ln(3+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x(x^5+18x^4+135x^3+540x^2+1215x+1458)\,\mathrm{e}^{-2x+5\ln(3+x)}\mathrm{e}^{2x}}{6\,(3+x)^5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{x(x^5+18x^4+135x^3+540x^2+1215x+1458)\,\mathrm{e}^{-2x+5\ln(3+x)}\mathrm{e}^{2x}}{6\,(3+x)^5}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.495.2 Maple step by step solution

Let's solve

$(3+x)\left(\frac{d}{dx}y'\right)+(1+2x)\,y'-(2-x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x-2)y}{3+x} - \frac{(1+2x)y'}{3+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(1+2x)y'}{3+x} + \frac{(x-2)y}{3+x} = 0$

□ Check to see if $x_0=-3$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1+2x}{3+x}, P_3(x)=\frac{x-2}{3+x}\right]$

  - $(3+x)\cdot P_2(x)$ is analytic at $x=-3$

    $\left.\left((3+x)\cdot P_2(x)\right)\right|_{x=-3} = -5$

  - $(3+x)^2\cdot P_3(x)$ is analytic at $x=-3$

$$\left((3+x)^2 \cdot P_3(x)\right)\Big|_{x=-3} = 0$$

- $x=-3$is a regular singular point

  Check to see if $x_0 = -3$ is a regular singular point

  $x_0 = -3$

- Multiply by denominators

  $(3+x)\left(\frac{d}{dx}y'\right) + (1+2x)\,y' + (x-2)\,y = 0$

- Change variables using $x = u-3$ so that the regular singular point is at $u=0$

  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-5+2u)\left(\frac{d}{du}y(u)\right) + (u-5)\,y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$$

- Shift index using $k->k+1$

  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-6+r)\,u^{-1+r} + \left(a_1(1+r)\,(-5+r) + a_0(-5+2r)\right)u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k-5+r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-6+r) = 0$

- Values of r that satisfy the indicial equation
  $r \in \{0, 6\}$
- Each term must be 0
  $a_1(1+r)(-5+r) + a_0(-5+2r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k-5+r) + 2a_k k + 2a_k r - 5a_k + a_{k-1} = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2+r)(k-4+r) + 2a_{k+1}(k+1) + 2r a_{k+1} - 5a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2k a_{k+1} + 2r a_{k+1} + a_k - 3a_{k+1}}{(k+2+r)(k-4+r)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{2k a_{k+1} + a_k - 3a_{k+1}}{(k+2)(k-4)}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 4$
  $a_{k+2} = -\frac{2k a_{k+1} + a_k - 3a_{k+1}}{(k+2)(k-4)}$
- Recursion relation for $r = 6$
  $a_{k+2} = -\frac{2k a_{k+1} + a_k + 9a_{k+1}}{(k+8)(k+2)}$
- Solution for $r = 6$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+2} = -\frac{2k a_{k+1} + a_k + 9a_{k+1}}{(k+8)(k+2)}, 7a_1 + 7a_0 = 0\right]$
- Revert the change of variables $u = 3 + x$
  $\left[y = \sum_{k=0}^{\infty} a_k (3+x)^{k+6}, a_{k+2} = -\frac{2k a_{k+1} + a_k + 9a_{k+1}}{(k+8)(k+2)}, 7a_1 + 7a_0 = 0\right]$

1.495.3 Maple trace

```
Methods for second order ODEs:
```

1.495.4 Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 33

```
dsolve((3+x)*diff(diff(y(x),x),x)+(1+2*x)*diff(y(x),x)-(2-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{-x}\left(\left(x^2+3x+9\right)\left(x^2+9x+27\right)(x+6)\, c_2 x + c_1\right)$$

1.495.5 Mathematica DSolve solution

Solving time : 0.075 (sec)
Leaf size : 29

```
DSolve[{(3+x)*D[y[x],{x,2}]+(1+2*x)*D[y[x],x]-(2-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{6} e^{-x-3} \left(c_2 (x+3)^6 + 6 c_1\right)$$

1.496 problem 512

Internal problem ID [8634]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 512
Date solved : `Monday, October 21, 2024 at 05:18:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 3xy' + (2x^2 + 4) y = 0$$

1.496.1 Solved as second order ode using Kovacic algorithm

Time used: 0.279 (sec)

Writing the ode as

$$y'' + 3xy' + (2x^2 + 4) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3x \\ C &= 2x^2 + 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 10$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 937: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{5}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) = 0$$
$$a_1 x + 2a_0 + 2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= (x^2 - 1)\, \mathrm{e}^{\int -\frac{x}{2}dx} \\
&= (x^2 - 1)\, \mathrm{e}^{-\frac{x^2}{4}} \\
&= (x^2 - 1)\, \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{1}\, dx} \\
&= z_1 e^{-\frac{3x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (x^2 - 1)\, \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{3x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{3x^2}{2}}\, \mathrm{e}^{2x^2}}{(x^2 - 1)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\left(x^2-1\right) e^{-x^2}\right) + c_2\left(\left(x^2-1\right) e^{-x^2} \left(\int \frac{e^{-\frac{3x^2}{2}} e^{2x^2}}{\left(x^2-1\right)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.496.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 3xy' + (2x^2+4)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2 + 4a_0 + (6a_3 + 7a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(3k+4) + 2a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $[2a_2 + 4a_0 = 0, 6a_3 + 7a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = -2a_0, a_3 = -\frac{7a_1}{6}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + 3a_k k + 4a_k + 2a_{k-2} = 0$
- Shift index using $k- >k+2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} + 3a_{k+2}(k+2) + 4a_{k+2} + 2a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{3ka_{k+2}+2a_k+10a_{k+2}}{k^2+7k+12}, a_2 = -2a_0, a_3 = -\frac{7a_1}{6}\right]$

1.496.3 Maple trace

```
Methods for second order ODEs:
```

1.496.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 45

```
dsolve(diff(diff(y(x),x),x)+3*x*diff(y(x),x)+(2*x^2+4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = -2\,\mathrm{e}^{-\frac{x^2}{2}} c_1 x + \mathrm{e}^{-x^2}(x-1)\,(x+1)\left(c_1\sqrt{2}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{\pi} + c_2\right)$$

1.496.5 Mathematica DSolve solution

Solving time : 0.192 (sec)
Leaf size : 63

```
DSolve[{D[y[x],{x,2}]+3*x*D[y[x],x]+(4+2*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}e^{-x^2}\left(\sqrt{2\pi}c_2(x^2-1)\,\mathrm{erfi}\left(\frac{x}{\sqrt{2}}\right)+4c_1(x^2-1)-2c_2e^{\frac{x^2}{2}}x\right)$$

1.497 problem 513

Internal problem ID [8635]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 513
Date solved : `Monday, October 21, 2024 at 05:18:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2+4x)\,y''-4y'-(6+4x)\,y=0$$

1.497.1 Solved as second order ode using Kovacic algorithm

Time used: 0.254 (sec)

Writing the ode as

$$(2+4x)\,y''-4y'+(-4x-6)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2+4x\\B&=-4\\C&=-4x-6\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4x^2 + 8x + 6}{(1 + 2x)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4x^2 + 8x + 6 \\
t &= (1 + 2x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{4x^2 + 8x + 6}{(1 + 2x)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 939: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+2x)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x+\frac{1}{2}\right)^2} + \frac{1}{x+\frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 + 8x + 6}{(1 + 2x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{2\left(x+\frac{1}{2}\right)} + (-)(1) \\ &= -\frac{1}{2\left(x+\frac{1}{2}\right)} - 1 \\ &= -\frac{2(x+1)}{1+2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2+8x+6}{(1+2x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right) dx} \\ &= \frac{\mathrm{e}^{-x}}{\sqrt{1+2x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{2+4x} \, dx} \\ &= z_1 e^{\frac{\ln(1+2x)}{2}} \\ &= z_1\left(\sqrt{1+2x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{2+4x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(1+2x)}}{(y_1)^2}\, dx \\ &= y_1 \left(x\, e^{2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x} \left(x\, e^{2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.497.2 Maple step by step solution

Let's solve

$(2+4x)\left(\frac{d}{dx}y'\right) - 4y' - (6+4x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(2x+3)y}{1+2x} + \frac{2y'}{1+2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y'}{1+2x} - \frac{(2x+3)y}{1+2x} = 0$

□ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2}{1+2x}, P_3(x) = -\frac{2x+3}{1+2x}\right]$

- $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}}=-1$
- $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}}=0$
- $x=-\frac{1}{2}$is a regular singular point

  Check to see if $x_0=-\frac{1}{2}$ is a regular singular point

  $x_0=-\frac{1}{2}$

- Multiply by denominators

  $(1+2x)\left(\frac{d}{dx}y'\right)-2y'+(-2x-3)\,y=0$
- Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

  $2u\left(\frac{d}{du}\frac{d}{du}y(u)\right)-2\frac{d}{du}y(u)+(-2u-2)\,y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$
- Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$
- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1}$
- Shift index using $k->k+1$

  $\frac{d}{du}y(u)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,u^{k+r}$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$
- Shift index using $k->k+1$

$$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$2a_0r(-2+r)\,u^{-1+r}+\left(2a_1(1+r)\,(-1+r)-2a_0\right)u^r+\left(\sum_{k=1}^{\infty}\left(2a_{k+1}(k+1+r)\,(k+r-1)-2a_k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $2r(-2+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{0,2\}$
- Each term must be 0
  $2a_1(1+r)\,(-1+r)-2a_0=0$
- Each term in the series must be 0, giving the recursion relation
  $2a_{k+1}(k+1+r)\,(k+r-1)-2a_k-2a_{k-1}=0$
- Shift index using $k->k+1$
  $2a_{k+2}(k+2+r)\,(k+r)-2a_{k+1}-2a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+2+r)(k+r)}$
- Recursion relation for $r=0$
  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+2)k}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$
  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+2)k}$
- Recursion relation for $r=2$
  $a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)}$
- Solution for $r=2$
  $$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+2},a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)},6a_1-2a_0=0\right]$$
- Revert the change of variables $u=x+\frac{1}{2}$
  $$\left[y=\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{2}\right)^{k+2},a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)},6a_1-2a_0=0\right]$$

1.497.3 Maple trace

```
Methods for second order ODEs:
```

1.497.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 16

```
dsolve((2+4*x)*diff(diff(y(x),x),x)-4*diff(y(x),x)-(6+4*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \, \mathrm{e}^{-x} + c_2 x \, \mathrm{e}^{x}$$

1.497.5 Mathematica DSolve solution

Solving time : 0.081 (sec)
Leaf size : 29

```
DSolve[{(2+4*x)*D[y[x],{x,2}]-4*D[y[x],x]-(6+4*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x-\frac{1}{2}}\left(c_2 e^{2x+1} x + c_1\right)$$

1.498 problem 514

Internal problem ID [8636]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 514
Date solved : `Monday, October 21, 2024 at 05:18:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 3xy' + \left(2x^2 + 5\right) y = 0$$

1.498.1 Solved as second order ode using Kovacic algorithm

Time used: 0.280 (sec)

Writing the ode as

$$y'' - 3xy' + \left(2x^2 + 5\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3x \\ C &= 2x^2 + 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 26}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 26 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{13}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 941: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{13}{2x} - \frac{169}{4x^3} - \frac{2197}{4x^5} - \frac{142805}{16x^7} - \frac{2599051}{16x^9} - \frac{101362989}{32x^{11}} - \frac{2070701061}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 26}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{13}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{13}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{13}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{13}{2}\right) - (0) \\
&= -\frac{13}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{13}{2}}{\frac{1}{2}} - 1\right) &= -7 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{13}{2}}{\frac{1}{2}} - 1\right) &= 6
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{13}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-7	6

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 6$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 6 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(30x^4 + 20x^3 a_5 + 12x^2 a_4 + 6x a_3 + 2a_2\right) + 2\left(-\frac{x}{2}\right)\left(6x^5 + 5x^4 a_5 + 4x^3 a_4 + 3x^2 a_3 + 2x a_2 + a_1\right) + \left(\left(-\frac{1}{2}\right.\right.$$

$$a_5 x^5 + 2(15 + a_4) x^4 + (3a_3 + 20a_5) x^3 + 4(a_2 + 3a_4) x^2 + (5$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -15, a_1 = 0, a_2 = 45, a_3 = 0, a_4 = -15, a_5 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 - 15x^4 + 45x^2 - 15$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= \left(x^6 - 15x^4 + 45x^2 - 15\right) e^{\int -\frac{x}{2} dx} \\
&= \left(x^6 - 15x^4 + 45x^2 - 15\right) e^{-\frac{x^2}{4}} \\
&= \left(x^6 - 15x^4 + 45x^2 - 15\right) e^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3x}{1} \, dx} \\
&= z_1 e^{\frac{3x^2}{4}} \\
&= z_1 \left(e^{\frac{3x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{x^2}{2}} \left(x^6 - 15x^4 + 45x^2 - 15\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{3x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{\frac{3x^2}{2}} e^{-x^2}}{\left(x^6 - 15x^4 + 45x^2 - 15\right)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right)\right) \\
&\quad + c_2\left(\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right)\left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}}\,\mathrm{e}^{-x^2}}{\left(x^6-15x^4+45x^2-15\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.498.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - 3xy' + (2x^2+5)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2+5a_0+(6a_3+2a_1)\,x+\left(\sum_{k=2}^{\infty}\left(a_{k+2}(k+2)\,(k+1)-a_k(3k-5)+2a_{k-2}\right)x^k\right)=0$$

- The coefficients of each power of x must be 0

  $[2a_2+5a_0=0, 6a_3+2a_1=0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2=-\frac{5a_0}{2}, a_3=-\frac{a_1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\,a_{k+2}-3a_kk+5a_k+2a_{k-2}=0$
- Shift index using $k->k+2$

  $\left((k+2)^2+3k+8\right)a_{k+4}-3a_{k+2}(k+2)+5a_{k+2}+2a_k=0$
- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty}a_kx^k, a_{k+4}=\frac{3ka_{k+2}-2a_k+a_{k+2}}{k^2+7k+12}, a_2=-\frac{5a_0}{2}, a_3=-\frac{a_1}{3}\right]$

1.498.3 Maple trace

```
Methods for second order ODEs:
```

1.498.4 Maple dsolve solution

Solving time : `0.011` (sec)
Leaf size : 62

```
dsolve(diff(diff(y(x),x),x)-3*x*diff(y(x),x)+(2*x^2+5)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\left(x^6-15x^4+45x^2-15\right)\left(c_1\sqrt{\pi}\,\sqrt{2}\,\operatorname{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)+c_2\right)\mathrm{e}^{\frac{x^2}{2}}-2\,\mathrm{e}^{x^2}c_1x\left(x^2-11\right)\left(x^2-3\right)$$

1.498.5 Mathematica DSolve solution

Solving time : `0.318` (sec)
Leaf size : 95

```
DSolve[{D[y[x],{x,2}]-3*x*D[y[x],x]+(5+2*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{\frac{x^2}{2}}\left(\sqrt{2\pi}c_2(x^6-15x^4+45x^2-15)\operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right)-2c_2e^{\frac{x^2}{2}}x(x^4-14x^2+33)+1440c_1(x^6-15x^4+45x^2-}{1440}$$

1.499 problem 515

Internal problem ID [8637]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 515
Date solved : `Monday, October 21, 2024 at 05:18:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2y'' + 5xy' + \left(2x^2+4\right)y = 0$$

1.499.1 Solved as second order ode using Kovacic algorithm

Time used: 0.215 (sec)

Writing the ode as

$$2y'' + 5xy' + \left(2x^2+4\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 5x \\ C &= 2x^2+4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 - 12}{16} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9x^2 - 12$$
$$t = 16$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2}{16} - \frac{3}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 943: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3x}{4} - \frac{1}{2x} - \frac{1}{6x^3} - \frac{1}{9x^5} - \frac{5}{54x^7} - \frac{7}{81x^9} - \frac{7}{81x^{11}} - \frac{22}{243x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{3x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^2 - 12}{16} \\ &= Q + \frac{R}{16} \\ &= \left(\frac{9x^2}{16} - \frac{3}{4}\right) + (0) \\ &= \frac{9x^2}{16} - \frac{3}{4} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{4}\right) - (0) \\ &= -\frac{3}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{3x}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{4}}{\frac{3}{4}} - 1\right) &&= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{4}}{\frac{3}{4}} - 1\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^2}{16} - \frac{3}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{3x}{4}$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$, and since there are no poles then

$$\begin{aligned}
d &= \alpha_\infty^- \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{3x}{4}\right) \\
&= -\frac{3x}{4} \\
&= -\frac{3x}{4}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{3x}{4}\right)(0) + \left(\left(-\frac{3}{4}\right) + \left(-\frac{3x}{4}\right)^2 - \left(\frac{9x^2}{16} - \frac{3}{4}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{3x}{4} dx} \\
&= \mathrm{e}^{-\frac{3x^2}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x}{2} dx} \\
&= z_1 e^{-\frac{5x^2}{8}} \\
&= z_1 \left(\mathrm{e}^{-\frac{5x^2}{8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{2} dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{\mathrm{e}^{-\frac{5x^2}{4}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{i\sqrt{\pi}\,\sqrt{3}\,\operatorname{erf}\left(\frac{i\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}\left(-\frac{i\sqrt{\pi}\,\sqrt{3}\,\operatorname{erf}\left(\frac{i\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.499.2 Maple step by step solution

Let's solve

$2\frac{d}{dx}y' + 5xy' + (2x^2 + 4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = (-x^2 - 2)\, y - \frac{5xy'}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{5xy'}{2} + (x^2 + 2)\, y = 0$

- Multiply by denominators

  $2\frac{d}{dx}y' + 5xy' + (2x^2 + 4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k + 2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$4a_2 + 4a_0 + (12a_3 + 9a_1)\, x + \left(\sum_{k=2}^{\infty} (2a_{k+2}(k+2)\,(k+1) + a_k(5k+4) + 2a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $[4a_2 + 4a_0 = 0, 12a_3 + 9a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = -a_0, a_3 = -\frac{3a_1}{4}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(2k^2 + 6k + 4)\, a_{k+2} + 5a_k k + 4a_k + 2a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $\left(2(k+2)^2 + 6k + 16\right) a_{k+4} + 5a_{k+2}(k+2) + 4a_{k+2} + 2a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{5ka_{k+2}+2a_k+14a_{k+2}}{2(k^2+7k+12)}, a_2 = -a_0, a_3 = -\frac{3a_1}{4}\right]$

1.499.3 Maple trace

```
Methods for second order ODEs:
```

1.499.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 22

```
dsolve(2*diff(diff(y(x),x),x)+5*x*diff(y(x),x)+(2*x^2+4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{-x^2} \left(c_1 + \operatorname{erf}\left(\frac{i\sqrt{3}\, x}{2}\right) c_2\right)$$

1.499.5 Mathematica DSolve solution

Solving time : 0.077 (sec)
Leaf size : 42

```
DSolve[{2*D[y[x],{x,2}]+5*x*D[y[x],x]+(4+2*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3}e^{-x^2}\left(\sqrt{3\pi}c_2\operatorname{erfi}\left(\frac{\sqrt{3}x}{2}\right)+3c_1\right)$$

1.500 problem 516

Internal problem ID [8638]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 516
Date solved : `Monday, October 21, 2024 at 05:18:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0$$

1.500.1 Solved as second order ode using Kovacic algorithm

Time used: 0.090 (sec)

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 945: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\
&= z_1 e^{-x^2} \\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.500.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 4xy' + (4x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

- ∘ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ∘ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ∘ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ∘ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ∘ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + 2a_0 + (6a_3 + 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2 = -a_0, a_3 = -a_1\right]$

1.500.3 Maple trace

```
Methods for second order ODEs:
```

1.500.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(4*x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-x^2}(c_2 x + c_1)$$

1.500.5 Mathematica DSolve solution

Solving time : `0.036` (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(2+4*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x^2}(c_2 x + c_1)$$

1.501 problem 517

Internal problem ID [8639]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 517
Date solved : `Monday, October 21, 2024 at 05:18:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + (4x^2 + 2)\, y = 0$$

1.501.1 Solved as second order ode using Kovacic algorithm

Time used: 0.089 (sec)

Writing the ode as

$$y'' + 4xy' + (4x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 947: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\
&= z_1 e^{-x^2} \\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.501.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 4xy' + (4x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- > k + 2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $\left((k + 2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k + 2) + 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2 = -a_0, a_3 = -a_1\right]$

1.501.3 Maple trace

```
Methods for second order ODEs:
```

1.501.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(4*x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-x^2}(c_2 x + c_1)$$

1.501.5 Mathematica DSolve solution

Solving time : `0.033` (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(2+4*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x^2}(c_2 x + c_1)$$

1.502 problem 518

Internal problem ID [8640]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 518
Date solved : `Monday, October 21, 2024 at 05:18:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+x+1\right)y''+x\left(11x^2+11x+9\right)y'+\left(7x^2+10x+6\right)y=0$$

1.502.1 Solved as second order ode using Kovacic algorithm

Time used: 1.077 (sec)

Writing the ode as

$$\left(2x^4+2x^3+2x^2\right)y''+\left(11x^3+11x^2+9x\right)y'+\left(7x^2+10x+6\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^4+2x^3+2x^2\\B&=11x^3+11x^2+9x\\C&=7x^2+10x+6\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 21x^4 + 18x^3 + 27x^2 - 2x - 3 \\ t &= 16\left(x^3 + x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 949: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{1}{4x} + \frac{-\frac{5}{24} + \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{24} - \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{1}{8} - \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{1}{8} + \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} + \frac{i\sqrt{3}}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{6 + 6i\sqrt{3}}}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{6 + 6i\sqrt{3}}}{12} \end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} - \frac{i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6+6i\sqrt{3}}}{12}$
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{7}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+} + \alpha_{c_3}^{+}\right) \\
&= \frac{7}{4} - \left(\frac{7}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} + (0) \\
&= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \\
&= \frac{7x^2 + 3x + 1}{4x\left(x^2 + x + 1\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} \right) + \left(\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right) dx} \\
&= 2\left(x^2 + x + 1\right)^{3/4} \sqrt{2}\, x^{1/4} \mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{11x^3 + 11x^2 + 9x}{2x^4 + 2x^3 + 2x^2} \, dx} \\
&= z_1 e^{-\frac{\ln\left(x^2 + x + 1\right)}{4} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6} - \frac{9\ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{\left(x^2 + x + 1\right)^{1/4} x^{9/4}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2\sqrt{x^2 + x + 1}\, \mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \sqrt{2}}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3 + 11x^2 + 9x}{2x^4 + 2x^3 + 2x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{\ln\left(x^2 + x + 1\right)}{2} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3} - \frac{9\ln(x)}{2}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{\ln\left(x^2 + x + 1\right)}{2} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3} - \frac{9\ln(x)}{2}} x^4 \mathrm{e}^{\frac{2\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2 + 8x + 8} dx \right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1\left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\right)$$
$$+c_2\left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\left(\int \frac{\mathrm{e}^{-\frac{\ln\left(x^2+x+1\right)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}-\frac{9\ln(x)}{2}}x^4\mathrm{e}^{\frac{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2+8x+8}\right.\right.$$

Will add steps showing solving for IC soon.

1.502.2 Maple step by step solution

Let's solve

$2x^2(x^2+x+1)\left(\frac{d}{dx}y'\right) + x(11x^2+11x+9)\,y' + (7x^2+10x+6)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(7x^2+10x+6)y}{2x^2(x^2+x+1)} - \frac{(11x^2+11x+9)y'}{2x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(11x^2+11x+9)y'}{2x(x^2+x+1)} + \frac{(7x^2+10x+6)y}{2x^2(x^2+x+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{11x^2+11x+9}{2x(x^2+x+1)}, P_3(x) = \frac{7x^2+10x+6}{2x^2(x^2+x+1)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{9}{2}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 3$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2(x^2+x+1)\left(\frac{d}{dx}y'\right)+x(11x^2+11x+9)\,y'+(7x^2+10x+6)\,y=0$
- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m=0..2$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m=1..3$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m=2..4$

    $x^m \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(3+2r)\,x^r+\left(a_1(3+r)\,(5+2r)+a_0(5+2r)\,(2+r)\right)x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(2k+$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(3+2r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{-2,-\frac{3}{2}\right\}$
- Each term must be 0

  $a_1(3+r)\,(5+2r)+a_0(5+2r)\,(2+r)=0$
- Solve for the dependent coefficient(s)

  $a_1=-\frac{(2+r)a_0}{3+r}$

- Each term in the series must be 0, giving the recursion relation

$$2((a_k+a_{k-2}+a_{k-1})\,k+(a_k+a_{k-2}+a_{k-1})\,r+2a_k-a_{k-2}+a_{k-1})\left(k+r+\frac{3}{2}\right)=0$$

- Shift index using $k->k+2$

$$2((a_{k+2}+a_k+a_{k+1})\,(k+2)+(a_{k+2}+a_k+a_{k+1})\,r+2a_{k+2}-a_k+a_{k+1})\left(k+\frac{7}{2}+r\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{ka_k+ka_{k+1}+ra_k+ra_{k+1}+a_k+3a_{k+1}}{k+4+r}$$

- Recursion relation for $r=-2$

$$a_{k+2}=-\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2}$$

- Solution for $r=-2$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k-2},a_{k+2}=-\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2},a_1=0\right]$$

- Recursion relation for $r=-\frac{3}{2}$

$$a_{k+2}=-\frac{ka_k+ka_{k+1}-\frac{1}{2}a_k+\frac{3}{2}a_{k+1}}{k+\frac{5}{2}}$$

- Solution for $r=-\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k-\frac{3}{2}},a_{k+2}=-\frac{ka_k+ka_{k+1}-\frac{1}{2}a_k+\frac{3}{2}a_{k+1}}{k+\frac{5}{2}},a_1=-\frac{a_0}{3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-2}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k-\frac{3}{2}}\right),a_{k+2}=-\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2},a_1=0,b_{k+2}=-\frac{kb_k+kb_{k+1}-\frac{1}{2}b_k+}{k+\frac{5}{2}}\right.$$

1.502.3 Maple trace

```
Methods for second order ODEs:
```

1.502.4 Maple dsolve solution

Solving time : `0.109` (sec)
Leaf size : 231

```
dsolve(2*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+x*(11*x^2+11*x+9)*diff(y(x),x)+(7*x^2+10*x
       y(x),singsol=all)
```

$$y=\frac{e^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}\left(2x+1+i\sqrt{3}\right)^{\frac{5\sqrt{3}+3i}{6\sqrt{3}+6i}}\left(i\sqrt{3}-2x-1\right)^{\frac{64i\sqrt{3}+2368}{\left(\sqrt{3}+i\right)^3\left(i-\sqrt{3}\right)^4\left(13\sqrt{3}+9i\right)}}\left(\operatorname{HeunG}\left(\frac{\sqrt{3}+i}{i-\sqrt{3}},0,0,\frac{5}{2},\frac{1}{2},\right.\right.}{x^{5/2}\left(x^2+x+1\right)}$$

1.502.5 Mathematica DSolve solution

Solving time : 1.061 (sec)
Leaf size : 93

```
DSolve[{2*x^2*(1+x+x^2)*D[y[x],{x,2}]+x*(9+11*x+11*x^2)*D[y[x],x]+(6+10*x+7*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x^2+x+1}e^{-\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}}\left(c_2\int_1^x \frac{e^{\frac{\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}{\sqrt{3}}}}{\sqrt{K[1]}(K[1]^2+K[1]+1)^{3/2}}dK[1]+c_1\right)}{x^2}$$

1.503 problem 519

Internal problem ID [8641]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 519
Date solved : `Monday, October 21, 2024 at 05:18:56 PM`
CAS classification :
`[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]`

Solve

$$3x^2y'' + 2x(-2x^2 + x + 1)\, y' + (-8x^2 + 2x)\, y = 0$$

1.503.1 Solved as second order ode using Kovacic algorithm

Time used: 0.377 (sec)

Writing the ode as

$$3x^2y'' + (-4x^3 + 2x^2 + 2x)\, y' + (-8x^2 + 2x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= -4x^3 + 2x^2 + 2x \\ C &= -8x^2 + 2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^4 - 4x^3 + 15x^2 - 4x - 2$$
$$t = 9x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 951: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 9x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{4x^2}{9} - \frac{4x}{9} + \frac{5}{3} - \frac{2}{9x^2} - \frac{4}{9x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{3} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{2x}{3} - \frac{1}{3} + \frac{7}{6x} + \frac{1}{4x^2} - \frac{17}{16x^3} - \frac{31}{32x^4} + \frac{85}{64x^5} + \frac{353}{128x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{2}{3}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= -\frac{1}{3} + \frac{2x}{3}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{9} - \frac{4}{9}x + \frac{4}{9}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{9}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2} \\ &= Q + \frac{R}{9x^2} \\ &= \left(\frac{4}{9}x^2 - \frac{4}{9}x + \frac{5}{3}\right) + \left(\frac{-4x - 2}{9x^2}\right) \\ &= \frac{4x^2}{9} - \frac{4x}{9} + \frac{5}{3} + \frac{-4x - 2}{9x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $\frac{5}{3}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{3}\right) - \left(\frac{1}{9}\right) \\ &= \frac{14}{9} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= -\frac{1}{3} + \frac{2x}{3} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{14}{9}}{\frac{2}{3}} - 1\right) &&= \frac{2}{3} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{14}{9}}{\frac{2}{3}} - 1\right) &&= -\frac{5}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-\frac{1}{3} + \frac{2x}{3}$	$\frac{2}{3}$	$-\frac{5}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{2}{3}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\
&= \frac{2}{3} - \left(\frac{2}{3}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{2}{3x} + \left(-\frac{1}{3} + \frac{2x}{3}\right) \\
&= \frac{2}{3x} - \frac{1}{3} + \frac{2x}{3} \\
&= \frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right)(0) + \left(\left(-\frac{2}{3x^2} + \frac{2}{3}\right) + \left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right)^2 - \left(\frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}\right)\right) = 0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right) dx} \\
&= x^{2/3} e^{\frac{x(x-1)}{3}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^3+2x^2+2x}{3x^2}\, dx} \\
&= z_1 e^{\frac{x^2}{3} - \frac{x}{3} - \frac{\ln(x)}{3}} \\
&= z_1 \left(\frac{e^{\frac{x(x-1)}{3}}}{x^{1/3}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^{1/3} \mathrm{e}^{\frac{2x(x-1)}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3+2x^2+2x}{3x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{2x^2}{3}-\frac{2x}{3}-\frac{2\ln(x)}{3}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{\frac{2x^2}{3}-\frac{2x}{3}-\frac{2\ln(x)}{3}} \mathrm{e}^{-\frac{4x(x-1)}{3}}}{x^{2/3}} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{1/3} \mathrm{e}^{\frac{2x(x-1)}{3}} \right) + c_2 \left(x^{1/3} \mathrm{e}^{\frac{2x(x-1)}{3}} \left(\int \frac{\mathrm{e}^{\frac{2x^2}{3}-\frac{2x}{3}-\frac{2\ln(x)}{3}} \mathrm{e}^{-\frac{4x(x-1)}{3}}}{x^{2/3}} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.503.2 Maple step by step solution

Let's solve

$3x^2 \left(\frac{d}{dx} y' \right) + 2x(-2x^2 + x + 1)\, y' + (-8x^2 + 2x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{2(4x-1)y}{3x} + \frac{2(2x^2-x-1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' - \frac{2(2x^2-x-1)y'}{3x} - \frac{2(4x-1)y}{3x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{2(2x^2-x-1)}{3x}, P_3(x) = -\frac{2(4x-1)}{3x}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{2}{3}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3\left(\frac{d}{dx}y'\right)x + (-4x^2 + 2x + 2)\, y' + (-8x + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- ◦ Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

- ◦ Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$

- ◦ Shift index using $k- >k + 1$

$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-1+3r)\, x^{-1+r} + (a_1(1+r)(2+3r) + 2a_0(1+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(3k+2+3r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-1+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{1}{3}\right\}$
- Each term must be 0
  $a_1(1+r)(2+3r) + 2a_0(1+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+1+r)(3ka_{k+1} + 3ra_{k+1} + 2a_k - 4a_{k-1} + 2a_{k+1}) = 0$
- Shift index using $k- >k+1$
  $(k+r+2)(3(k+1)\, a_{k+2} + 3ra_{k+2} + 2a_{k+1} - 4a_k + 2a_{k+2}) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5+3r}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}, 2a_1 + 2a_0 = 0\right]$
- Recursion relation for $r = \frac{1}{3}$
  $a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+6}$
- Solution for $r = \frac{1}{3}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+6}, 4a_1 + \frac{8a_0}{3} = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}, 2a_1 + 2a_0 = 0, b_{k+2} = \frac{2(-b_{k+1}+2b_k)}{3k+6}, 4b_1 +\right.$

1.503.3 Maple trace

```
Methods for second order ODEs:
```

1.503.4 Maple dsolve solution

Solving time : `0.034` (sec)
Leaf size : 38

```
dsolve(3*x^2*diff(diff(y(x),x),x)+2*x*(-2*x^2+x+1)*diff(y(x),x)+(-8*x^2+2*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x^{1/3} \mathrm{e}^{\frac{2x(x-1)}{3}} + c_2 \,\mathrm{HeunB}\left(-\frac{1}{3}, \frac{\sqrt{6}}{3}, -\frac{7}{3}, \frac{4\sqrt{6}}{9}, -\frac{\sqrt{6}\,x}{3}\right)$$

1.503.5 Mathematica DSolve solution

Solving time : `0.548` (sec)
Leaf size : 53

```
DSolve[{3*x^2*D[y[x],{x,2}]+2*x*(1+x-2*x^2)*D[y[x],x]+(2*x-8*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{\frac{2}{3}(x-1)x} \sqrt[3]{x} \left(c_2 \int_1^x \frac{e^{-\frac{2}{3}(K[1]-1)K[1]}}{K[1]^{4/3}} dK[1] + c_1\right)$$

1.504 problem 520

Internal problem ID [8642]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 520
Date solved : `Monday, October 21, 2024 at 05:18:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$12x^2(1+x)\,y''+x\left(3x^2+35x+11\right)y'-\left(-5x^2-10x+1\right)y=0$$

1.504.1 Solved as second order ode using Kovacic algorithm

Time used: 0.365 (sec)

Writing the ode as

$$\left(12x^3+12x^2\right)y''+\left(3x^3+35x^2+11x\right)y'+\left(5x^2+10x-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=12x^3+12x^2\\ B&=3x^3+35x^2+11x\\ C&=5x^2+10x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576 (x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9x^4 - 30x^3 - 197x^2 - 190x - 95$$
$$t = 576(x^2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576 (x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 953: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 576(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{64} - \frac{7}{64\,(1+x)^2} - \frac{1}{12\,(1+x)} - \frac{95}{576x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{95}{576}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{24} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{24} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{8} - \frac{1}{3x} - \frac{29}{24x^2} - \frac{193}{72x^3} - \frac{3017}{216x^4} - \frac{40009}{648x^5} - \frac{642029}{1944x^6} - \frac{10350493}{5832x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{8} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2} \\ &= Q + \frac{R}{576x^4 + 1152x^3 + 576x^2} \\ &= \left(\frac{1}{64}\right) + \left(\frac{-48x^3 - 206x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2}\right) \\ &= \frac{1}{64} + \frac{-48x^3 - 206x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -48. Dividing this by leading coefficient in t which is 576 gives $-\frac{1}{12}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{12}\right) - (0) \\ &= -\frac{1}{12} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{8} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{12}}{\frac{1}{8}} - 0\right) &&= -\frac{1}{3} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{12}}{\frac{1}{8}} - 0\right) &&= \frac{1}{3} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{8}$	$\frac{1}{8}$
0	2	0	$\frac{19}{24}$	$\frac{5}{24}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{8}$	$-\frac{1}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{3}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{8+8x} + \frac{5}{24x} + (-)\left(\frac{1}{8}\right) \\
&= \frac{1}{8+8x} + \frac{5}{24x} - \frac{1}{8} \\
&= \frac{1}{8+8x} + \frac{5}{24x} - \frac{1}{8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8+8x} + \frac{5}{24x} - \frac{1}{8}\right)(0) + \left(\left(-\frac{1}{8(1+x)^2} - \frac{5}{24x^2}\right) + \left(\frac{1}{8+8x} + \frac{5}{24x} - \frac{1}{8}\right)^2 - \left(\frac{9x^4 - 30x^3 -}{576}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{8+8x}+\frac{5}{24x}-\frac{1}{8}\right)dx} \\
&= (1+x)^{1/8}\, x^{5/24} \mathrm{e}^{-\frac{x}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3+35x^2+11x}{12x^3+12x^2}\, dx} \\
&= z_1 e^{-\frac{x}{8}-\frac{7\ln(1+x)}{8}-\frac{11\ln(x)}{24}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{8}}}{(1+x)^{7/8}\, x^{11/24}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-\frac{x}{4}}}{(1+x)^{3/4}\, x^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+35x^2+11x}{12x^3+12x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x}{4}-\frac{7\ln(1+x)}{4}-\frac{11\ln(x)}{12}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \mathrm{e}^{-\frac{x}{4}-\frac{7\ln(1+x)}{4}-\frac{11\ln(x)}{12}} (1+x)^{3/2} \sqrt{x}\, \mathrm{e}^{\frac{x}{2}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\frac{x}{4}}}{(1+x)^{3/4}\, x^{1/4}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{x}{4}}}{(1+x)^{3/4}\, x^{1/4}} \left(\int \mathrm{e}^{-\frac{x}{4}-\frac{7\ln(1+x)}{4}-\frac{11\ln(x)}{12}} (1+x)^{3/2} \sqrt{x}\, \mathrm{e}^{\frac{x}{2}} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.504.2 Maple step by step solution

Let's solve

$12x^2(1+x)\left(\frac{d}{dx}y'\right) + x(3x^2+35x+11)\,y' - (-5x^2-10x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(5x^2+10x-1)y}{12x^2(1+x)} - \frac{(3x^2+35x+11)y'}{12x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3x^2+35x+11)y'}{12x(1+x)} + \frac{(5x^2+10x-1)y}{12x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3x^2+35x+11}{12x(1+x)}, P_3(x) = \frac{5x^2+10x-1}{12x^2(1+x)}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{7}{4}$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $12x^2(1+x)\left(\frac{d}{dx}y'\right) + x(3x^2+35x+11)\,y' + (5x^2+10x-1)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(12u^3-24u^2+12u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (3u^3+26u^2-50u+21)\left(\frac{d}{du}y(u)\right) + (5u^2-6)\,y(u) = 0$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(3+4r)\,u^{-1+r} + (3a_1(1+r)\,(7+4r) - 2a_0(3+4r)\,(1+3r))\,u^r + (3a_2(2+r)\,(11+4r) - 2$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $3r(3+4r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{3}{4}\right\}$
- The coefficients of each power of u must be 0
  $[3a_1(1+r)\,(7+4r) - 2a_0(3+4r)\,(1+3r) = 0, 3a_2(2+r)\,(11+4r) - 2a_1(7+4r)\,(4+3r) + 2a$
- Solve for the dependent coefficient(s)
  $\left\{a_1 = \frac{2a_0(12r^2+13r+3)}{3(4r^2+11r+7)}, a_2 = \frac{2a_0(54r^3+135r^2+101r+24)}{9(4r^3+23r^2+41r+22)}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $12(-2a_k + a_{k-1} + a_{k+1})\,k^2 + (24(-2a_k + a_{k-1} + a_{k+1})\,r - 26a_k + 3a_{k-2} - 10a_{k-1} + 33a_{k+1})\,k +$
- Shift index using $k- >k + 2$

$12(-2a_{k+2}+a_{k+1}+a_{k+3})(k+2)^2+(24(-2a_{k+2}+a_{k+1}+a_{k+3})r-26a_{k+2}+3a_k-10a_{k+1}+3$

- Recursion relation that defines series solution to ODE

$a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+24kra_{k+1}-48kra_{k+2}+12r^2a_{k+1}-24r^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+3ra_k+38ra_{k+1}-122r}{3(4k^2+8kr+4r^2+27k+27r+45)}$

- Recursion relation for $r=0$

$a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a_k+26a_{k+1}-154a_{k+2}}{3(4k^2+27k+45)}$

- Solution for $r=0$

$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k, a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a_k+26a_{k+1}-154a_{k+2}}{3(4k^2+27k+45)}, a_1=\frac{2a_0}{7}, a_2\right.$

- Revert the change of variables $u=1+x$

$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k, a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a_k+26a_{k+1}-154a_{k+2}}{3(4k^2+27k+45)}, a_1=\frac{2a_0}{7},\right.$

- Recursion relation for $r=-\frac{3}{4}$

$a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+20ka_{k+1}-86ka_{k+2}+\frac{11}{4}a_k+\frac{17}{4}a_{k+1}-76a_{k+2}}{3(4k^2+21k+27)}$

- Solution for $r=-\frac{3}{4}$

$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-\frac{3}{4}}, a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+20ka_{k+1}-86ka_{k+2}+\frac{11}{4}a_k+\frac{17}{4}a_{k+1}-76a_{k+2}}{3(4k^2+21k+27)}, a_1=0, a_2\right.$

- Revert the change of variables $u=1+x$

$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k-\frac{3}{4}}, a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+20ka_{k+1}-86ka_{k+2}+\frac{11}{4}a_k+\frac{17}{4}a_{k+1}-76a_{k+2}}{3(4k^2+21k+27)}, a_1=0,\right.$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty}a_k(1+x)^k\right)+\left(\sum_{k=0}^{\infty}b_k(1+x)^{k-\frac{3}{4}}\right), a_{k+3}=-\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a}{3(4k^2+27k+45)}\right.$

1.504.3 Maple trace

```
Methods for second order ODEs:
```

1.504.4 Maple dsolve solution

Solving time : 0.046 (sec)
Leaf size : 43

```
dsolve(12*x^2*(1+x)*diff(diff(y(x),x),x)+x*(3*x^2+35*x+11)*diff(y(x),x)-(-5*x^2-10*x+1
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{x}{4}} \left(x^{7/12} \operatorname{HeunC}\left(\frac{1}{4}, \frac{7}{12}, -\frac{3}{4}, -\frac{1}{12}, \frac{1}{2}, -x\right) c_2 + \operatorname{HeunC}\left(\frac{1}{4}, -\frac{7}{12}, -\frac{3}{4}, -\frac{1}{12}, \frac{1}{2}, -x\right) c_1\right)}{(1+x)^{3/4} x^{1/4}}$$

1.504.5 Mathematica DSolve solution

Solving time : 0.703 (sec)
Leaf size : 61

```
DSolve[{12*x^2*(1+x)*D[y[x],{x,2}]+x*(11+35*x+3*x^2)*D[y[x],x]-(1-10*x-5*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{e^{-x/4}\left(c_2 \int_1^x \frac{e^{\frac{K[1]}{4}}}{K[1]^{5/12} \sqrt[4]{K[1]+1}} dK[1] + c_1\right)}{\sqrt[4]{x}(x+1)^{3/4}}$$

1.505 problem 521

Internal problem ID [8643]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 521
Date solved : `Monday, October 21, 2024 at 05:18:58 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + 3y' + 4y = 0$$

1.505.1 Solved as second order ode using Kovacic algorithm

Time used: 0.214 (sec)

Writing the ode as

$$y'' + 3y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = -7$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{7z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 955: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{7}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{7}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\, dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1 \left(e^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{7}\, x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{7}\,\tan\left(\frac{\sqrt{7}\,x}{2}\right)}{7}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{3x}{2}}\cos\left(\frac{\sqrt{7}\,x}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{3x}{2}}\cos\left(\frac{\sqrt{7}\,x}{2}\right)\left(\frac{2\sqrt{7}\,\tan\left(\frac{\sqrt{7}\,x}{2}\right)}{7}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

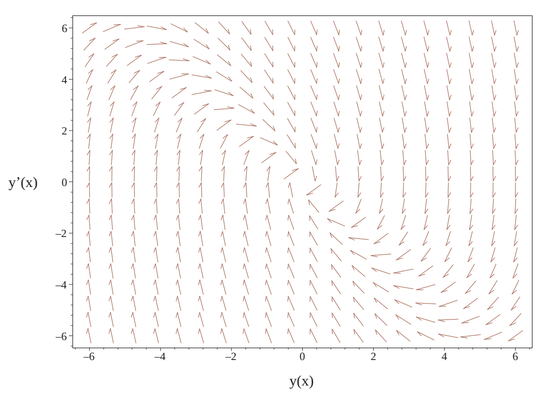

Figure 2: Slope field plot
$y'' + 3y' + 4y = 0$

1.505.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 3y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$
- Characteristic polynomial of ODE

  $r^2 + 3r + 4 = 0$
- Use quadratic formula to solve for r

  $r = \frac{(-3)\pm(\sqrt{-7})}{2}$
- Roots of the characteristic polynomial

  $r = \left(-\frac{3}{2} - \frac{I\sqrt{7}}{2}, -\frac{3}{2} + \frac{I\sqrt{7}}{2}\right)$
- 1st solution of the ODE

  $y_1(x) = e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{7}\,x}{2}\right)$
- 2nd solution of the ODE

  $y_2(x) = e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{7}\,x}{2}\right)$
- General solution of the ODE

  $y = C1 y_1(x) + C2 y_2(x)$
- Substitute in solutions

  $y = C1\, e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{7}\,x}{2}\right) + C2\, e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{7}\,x}{2}\right)$

1.505.3 Maple trace

```
Methods for second order ODEs:
```

1.505.4 Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 28

```
dsolve(diff(diff(y(x),x),x)+3*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{-\frac{3x}{2}} \left(c_1 \sin\left(\frac{\sqrt{7}\,x}{2}\right) + c_2 \cos\left(\frac{\sqrt{7}\,x}{2}\right)\right)$$

1.505.5 Mathematica DSolve solution

Solving time : 0.04 (sec)
Leaf size : 42

```
DSolve[{D[y[x],{x,2}]+3*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-3x/2}\left(c_2 \cos\left(\frac{\sqrt{7}x}{2}\right) + c_1 \sin\left(\frac{\sqrt{7}x}{2}\right)\right)$$

1.506 problem 522

Internal problem ID [8644]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 522
Date solved : `Monday, October 21, 2024 at 05:18:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$18x^2(1+x)\,y''+3x\left(x^2+11x+5\right)y'-\left(-5x^2-2x+1\right)y=0$$

1.506.1 Solved as second order ode using Kovacic algorithm

Time used: 0.360 (sec)

Writing the ode as

$$\left(18x^3+18x^2\right)y''+\left(3x^3+33x^2+15x\right)y'+\left(5x^2+2x-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=18x^3+18x^2\\B&=3x^3+33x^2+15x\\C&=5x^2+2x-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 18x^3 - 45x^2 - 18x - 27$$
$$t = 144\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 957: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{144} - \frac{7}{18\,(1+x)} - \frac{3}{16x^2} + \frac{1}{4x} - \frac{35}{144\,(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{12} - \frac{5}{6x} - \frac{53}{12x^2} - \frac{523}{12x^3} - \frac{6659}{12x^4} - \frac{94267}{12x^5} - \frac{1432421}{12x^6} - \frac{22802941}{12x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{12}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{12}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{144}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2} \\
&= Q + \frac{R}{144x^4 + 288x^3 + 144x^2} \\
&= \left(\frac{1}{144}\right) + \left(\frac{-20x^3 - 46x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2}\right) \\
&= \frac{1}{144} + \frac{-20x^3 - 46x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -20. Dividing this by leading coefficient in t which is 144 gives $-\frac{5}{36}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{5}{36}\right) - (0) \\
&= -\frac{5}{36}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{12} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{36}}{\frac{1}{12}} - 0\right) &= -\frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{36}}{\frac{1}{12}} - 0\right) &= \frac{5}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{12}$	$\frac{5}{12}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{12}$	$-\frac{5}{6}$	$\frac{5}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\ &= \frac{5}{6} - \left(\frac{5}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{7}{12\,(1+x)} + \frac{1}{4x} + (-)\left(\frac{1}{12}\right) \\ &= \frac{7}{12\,(1+x)} + \frac{1}{4x} - \frac{1}{12} \\ &= \frac{7}{12+12x} + \frac{1}{4x} - \frac{1}{12} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{12\,(1+x)} + \frac{1}{4x} - \frac{1}{12}\right)(0) + \left(\left(-\frac{7}{12\,(1+x)^2} - \frac{1}{4x^2}\right) + \left(\frac{7}{12\,(1+x)} + \frac{1}{4x} - \frac{1}{12}\right)^2 - \left(\frac{x^4 - 1}{}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{7}{12(1+x)}+\frac{1}{4x}-\frac{1}{12}\right)dx} \\
&= (1+x)^{7/12} x^{1/4} \mathrm{e}^{-\frac{x}{12}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3+33x^2+15x}{18x^3+18x^2}\, dx} \\
&= z_1 e^{-\frac{x}{12}-\frac{5\ln(1+x)}{12}-\frac{5\ln(x)}{12}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{12}}}{(1+x)^{5/12} x^{5/12}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(1+x)^{1/6} \mathrm{e}^{-\frac{x}{6}}}{x^{1/6}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+33x^2+15x}{18x^3+18x^2}\, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x}{6}-\frac{5\ln(1+x)}{6}-\frac{5\ln(x)}{6}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x}{6}-\frac{5\ln(1+x)}{6}-\frac{5\ln(x)}{6}} x^{1/3} \mathrm{e}^{\frac{x}{3}}}{(1+x)^{1/3}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(1+x)^{1/6} \mathrm{e}^{-\frac{x}{6}}}{x^{1/6}}\right) + c_2 \left(\frac{(1+x)^{1/6} \mathrm{e}^{-\frac{x}{6}}}{x^{1/6}} \left(\int \frac{\mathrm{e}^{-\frac{x}{6}-\frac{5\ln(1+x)}{6}-\frac{5\ln(x)}{6}} x^{1/3} \mathrm{e}^{\frac{x}{3}}}{(1+x)^{1/3}} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.506.2 Maple step by step solution

Let's solve

$18x^2(1+x)\left(\frac{d}{dx}y'\right) + 3x(x^2+11x+5)\, y' - (-5x^2-2x+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(5x^2+2x-1)y}{18x^2(1+x)} - \frac{(x^2+11x+5)y'}{6x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(x^2+11x+5)y'}{6x(1+x)} + \frac{(5x^2+2x-1)y}{18x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x^2+11x+5}{6x(1+x)}, P_3(x) = \frac{5x^2+2x-1}{18x^2(1+x)}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.((1+x)\cdot P_2(x))\right|_{x=-1} = \frac{5}{6}$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left.((1+x)^2\cdot P_3(x))\right|_{x=-1} = 0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $18x^2(1+x)\left(\frac{d}{dx}y'\right) + 3x(x^2+11x+5)\, y' + (5x^2+2x-1)\, y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(18u^3-36u^2+18u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (3u^3+24u^2-42u+15)\left(\frac{d}{du}y(u)\right) + (5u^2-8u+2)\, y(u) = 0$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

- □ Rewrite ODE with series expansions
  - ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

  - ○ Shift index using $k- >k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

  - ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

  - ○ Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

  - ○ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

  - ○ Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(-1+6r)\, u^{-1+r} + (3a_1(1+r)\,(5+6r) - 2a_0(1+3r)\,(-1+6r))\, u^r + (3a_2(2+r)\,(11+6r)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$3r(-1+6r) = 0$
- Values of r that satisfy the indicial equation
$r \in \left\{0, \frac{1}{6}\right\}$
- The coefficients of each power of u must be 0
$[3a_1(1+r)\,(5+6r) - 2a_0(1+3r)\,(-1+6r) = 0, 3a_2(2+r)\,(11+6r) - 2a_1(4+3r)\,(5+6r) + 2$
- Solve for the dependent coefficient(s)
$\left\{a_1 = \frac{2a_0(18r^2+3r-1)}{3(6r^2+11r+5)}, a_2 = \frac{2a_0(81r^3+126r^2+21r+4)}{9(6r^3+29r^2+45r+22)}\right\}$
- Each term in the series must be 0, giving the recursion relation
$18(-2a_k + a_{k-1} + a_{k+1})\, k^2 + 3(12(-2a_k + a_{k-1} + a_{k+1})\, r - 2a_k + a_{k-2} - 10a_{k-1} + 11a_{k+1})\, k + 1$
- Shift index using $k- >k+2$

$$18(-2a_{k+2}+a_{k+1}+a_{k+3})(k+2)^2+3(12(-2a_{k+2}+a_{k+1}+a_{k+3})r-2a_{k+2}+a_k-10a_{k+1}+11$$

- Recursion relation that defines series solution to ODE

$$a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+36kra_{k+1}-72kra_{k+2}+18r^2a_{k+1}-36r^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+3ra_k+42ra_{k+1}-150r}{3(6k^2+12kr+6r^2+35k+35r+51)}$$

- Recursion relation for $r=0$

$$a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k, a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)}, a_1=-\frac{2a_0}{15}, a\right.$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k, a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)}, a_1=-\frac{2a_0}{15}\right.$$

- Recursion relation for $r=\frac{1}{6}$

$$a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+48ka_{k+1}-162ka_{k+2}+\frac{11}{2}a_k+\frac{47}{2}a_{k+1}-180a_{k+2}}{3(6k^2+37k+57)}$$

- Solution for $r=\frac{1}{6}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+\frac{1}{6}}, a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+48ka_{k+1}-162ka_{k+2}+\frac{11}{2}a_k+\frac{47}{2}a_{k+1}-180a_{k+2}}{3(6k^2+37k+57)}, a_1=0, a\right.$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k+\frac{1}{6}}, a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+48ka_{k+1}-162ka_{k+2}+\frac{11}{2}a_k+\frac{47}{2}a_{k+1}-180a_{k+2}}{3(6k^2+37k+57)}, a_1=\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k(1+x)^k\right)+\left(\sum_{k=0}^{\infty}b_k(1+x)^{k+\frac{1}{6}}\right), a_{k+3}=-\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a}{3(6k^2+35k+51)}\right.$$

1.506.3 Maple trace

```
Methods for second order ODEs:
```

1.506.4 Maple dsolve solution

Solving time : `0.044` (sec)
Leaf size : 38

```
dsolve(18*x^2*(1+x)*diff(diff(y(x),x),x)+3*x*(x^2+11*x+5)*diff(y(x),x)-(-5*x^2-2*x+1)*
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{x}{6}} \left(\sqrt{x}\, \mathrm{HeunC}\left(\frac{1}{6}, \frac{1}{2}, -\frac{1}{6}, -\frac{5}{36}, \frac{1}{4}, -x\right) c_2 + \mathrm{HeunC}\left(\frac{1}{6}, -\frac{1}{2}, -\frac{1}{6}, -\frac{5}{36}, \frac{1}{4}, -x\right) c_1\right)}{x^{1/6}}$$

1.506.5 Mathematica DSolve solution

Solving time : `0.857` (sec)
Leaf size : 73

```
DSolve[{18*x^2*(1+x)*D[y[x],{x,2}]+3*x*(5+11*x+x^2)*D[y[x],x]-(1-2*x-5*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x/6}\left(c_2 \int_1^x \frac{e^{\frac{K[1]}{6}} \sqrt[3]{\frac{K[1]}{K[1]+1}}}{K[1]^{5/6}(K[1]+1)^{5/6}} dK[1] + c_1\right)}{\sqrt[6]{\frac{x}{x+1}}}$$

1.507 problem 523

Internal problem ID [8645]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 523
Date solved : `Monday, October 21, 2024 at 05:19:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + x(3+2x)\,y' - (1-x)\,y = 0$$

1.507.1 Solved as second order ode using Kovacic algorithm

Time used: 0.271 (sec)

Writing the ode as

$$2x^2y'' + \left(2x^2+3x\right)y' + (x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 2x^2+3x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4x^2 + 4x + 5}{16x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4x^2 + 4x + 5 \\
t &= 16x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{4x^2 + 4x + 5}{16x^2} \right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 959: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4} + \frac{1}{4x} + \frac{5}{16x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4x} + \frac{1}{4x^2} - \frac{1}{8x^3} + \frac{1}{16x^5} - \frac{3}{64x^6} - \frac{1}{128x^7} + \frac{11}{256x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{4x^2 + 4x + 5}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x + 5}{16x^2}\right) \\ &= \frac{1}{4} + \frac{4x + 5}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 4x + 5}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4x} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{4x} - \frac{1}{2} \\
&= -\frac{1}{4x} - \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{4x} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x} - \frac{1}{2}\right)^2 - \left(\frac{4x^2+4x+5}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{4x} - \frac{1}{2}\right) dx} \\
&= \frac{e^{-\frac{x}{2}}}{x^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^2+3x}{2x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{e^{-\frac{x}{2}}}{x^{3/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+3x}{2x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x-\frac{3\ln(x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\sqrt{x}\, e^{x} - \frac{\sqrt{\pi}\, \operatorname{erfi}\left(\sqrt{x}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-x}}{x} \right) + c_2 \left(\frac{e^{-x}}{x} \left(\sqrt{x}\, e^{x} - \frac{\sqrt{\pi}\, \operatorname{erfi}\left(\sqrt{x}\right)}{2} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.507.2 Maple step by step solution

Let's solve

$2x^2 \left(\frac{d}{dx} y'\right) + x(3+2x)\, y' - (1-x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(x-1)y}{2x^2} - \frac{(3+2x)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{(3+2x)y'}{2x} + \frac{(x-1)y}{2x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3+2x}{2x}, P_3(x) = \frac{x-1}{2x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{3}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{2}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d}{dx}y'\right) + x(3+2x)\,y' + (x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r-1) + a_{k-1}(2k+2r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+r)(-1+2r)=0$
- Values of r that satisfy the indicial equation
  $r\in\left\{-1,\frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $2(a_k(k+r+1)+a_{k-1})\left(k+r-\frac{1}{2}\right)=0$
- Shift index using $k->k+1$
  $2(a_{k+1}(k+2+r)+a_k)\left(k+\frac{1}{2}+r\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k}{k+2+r}$
- Recursion relation for $r=-1$
  $a_{k+1}=-\frac{a_k}{k+1}$
- Solution for $r=-1$
  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-1},a_{k+1}=-\frac{a_k}{k+1}\right]$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1}=-\frac{a_k}{k+\frac{5}{2}}$
- Solution for $r=\frac{1}{2}$
  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}},a_{k+1}=-\frac{a_k}{k+\frac{5}{2}}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right),a_{k+1}=-\frac{a_k}{k+1},b_{k+1}=-\frac{b_k}{k+\frac{5}{2}}\right]$

1.507.3 Maple trace

```
Methods for second order ODEs:
```

1.507.4 Maple dsolve solution

Solving time : 0.015 (sec)
Leaf size : 52

```
dsolve(2*x^2*diff(diff(y(x),x),x)+x*(3+2*x)*diff(y(x),x)-(1-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{3\left(2c_1(-x)^{3/2} + \mathrm{e}^{-x}\left(x c_1 \sqrt{\pi}\, \mathrm{erf}\left(\sqrt{-x}\right) - \frac{4c_2\sqrt{x}\,\sqrt{-x}}{3}\right)\right)}{4\sqrt{-x}\, x^{3/2}}$$

1.507.5 Mathematica DSolve solution

Solving time : 0.053 (sec)
Leaf size : 33

```
DSolve[{2*x^2*D[y[x],{x,2}]+x*(3+2*x)*D[y[x],x]-(1-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x}\left(c_2 x^{3/2} L_{-\frac{3}{2}}^{\frac{3}{2}}(x) + c_1\right)}{x}$$

1.508 problem 524

Internal problem ID [8646]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 524
Date solved : `Monday, October 21, 2024 at 05:19:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + x(5+x)\,y' - (2-3x)\,y = 0$$

1.508.1 Solved as second order ode using Kovacic algorithm

Time used: 0.296 (sec)

Writing the ode as

$$2x^2y'' + \left(x^2+5x\right)y' + (3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= x^2+5x \\ C &= 3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 14x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 14x + 21$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 14x + 21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 961: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} - \frac{7}{8x} + \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{7}{4x} - \frac{7}{2x^2} - \frac{49}{2x^3} - \frac{196}{x^4} - \frac{1715}{x^5} - \frac{31899}{2x^6} - \frac{309729}{2x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 14x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-14x + 21}{16x^2}\right) \\ &= \frac{1}{16} + \frac{-14x + 21}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -14. Dividing this by leading coefficient in t which is 16 gives $-\frac{7}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{8}\right) - (0) \\ &= -\frac{7}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{7}{8}}{\frac{1}{4}} - 0\right) &= -\frac{7}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{7}{8}}{\frac{1}{4}} - 0\right) &= \frac{7}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 14x + 21}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$-\frac{7}{4}$	$\frac{7}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{7}{4} - \left(\frac{7}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{7}{4x} + (-)\left(\frac{1}{4}\right) \\
&= \frac{7}{4x} - \frac{1}{4} \\
&= -\frac{x - 7}{4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{4x} - \frac{1}{4}\right)(0) + \left(\left(-\frac{7}{4x^2}\right) + \left(\frac{7}{4x} - \frac{1}{4}\right)^2 - \left(\frac{x^2 - 14x + 21}{16x^2}\right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{7}{4x} - \frac{1}{4}\right) dx} \\
&= x^{7/4} e^{-\frac{x}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 + 5x}{2x^2} \, dx} \\
&= z_1 e^{-\frac{x}{4} - \frac{5 \ln(x)}{4}} \\
&= z_1 \left(\frac{e^{-\frac{x}{4}}}{x^{5/4}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+5x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{2}-\frac{5\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{2\,\mathrm{e}^{\frac{x}{2}}}{5x^{5/2}} - \frac{2\,\mathrm{e}^{\frac{x}{2}}}{15x^{3/2}} - \frac{2\,\mathrm{e}^{\frac{x}{2}}}{15\sqrt{x}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right)}{15}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\left(-\frac{2\,\mathrm{e}^{\frac{x}{2}}}{5x^{5/2}} - \frac{2\,\mathrm{e}^{\frac{x}{2}}}{15x^{3/2}} - \frac{2\,\mathrm{e}^{\frac{x}{2}}}{15\sqrt{x}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right)}{15}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.508.2 Maple step by step solution

Let's solve

$2x^2\left(\frac{d}{dx}y'\right) + x(5+x)\,y' - (2-3x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(3x-2)y}{2x^2} - \frac{(5+x)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + \frac{(5+x)y'}{2x} + \frac{(3x-2)y}{2x^2} = 0$
- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{5+x}{2x}, P_3(x) = \frac{3x-2}{2x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{5}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -1$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $2x^2\left(\frac{d}{dx}y'\right) + x(5+x)\,y' + (3x-2)\,y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$
    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$
    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$
    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$
    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$
    Rewrite ODE with series expansions

$$a_0(2+r)(-1+2r)x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)(2k+2r-1) + a_{k-1}(k+r+2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-2, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k+r+2)\left(\left(k+r-\frac{1}{2}\right)a_k + \frac{a_{k-1}}{2}\right) = 0$
- Shift index using $k- >k+1$

  $2(k+r+3)\left(\left(k+\frac{1}{2}+r\right)a_{k+1} + \frac{a_k}{2}\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{2k+1+2r}$
- Recursion relation for $r = -2$

  $a_{k+1} = -\frac{a_k}{2k-3}$
- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{a_k}{2k-3}\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{2k+2}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{2k+2}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{2k-3}, b_{k+1} = -\frac{b_k}{2k+2}\right]$

1.508.3 Maple trace

```
Methods for second order ODEs:
```

1.508.4 Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 52

```
dsolve(2*x^2*diff(diff(y(x),x),x)+x*(5+x)*diff(y(x),x)-(2-3*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{i\sqrt{\pi}\,\sqrt{2}\,x^{5/2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right)\mathrm{e}^{-\frac{x}{2}}c_2 + c_1\,x^{5/2}\mathrm{e}^{-\frac{x}{2}} + 2c_2(x^2+x+3)}{x^2}$$

1.508.5 Mathematica DSolve solution

Solving time : `0.136` (sec)
Leaf size : 70

```
DSolve[{2*x^2*D[y[x],{x,2}]+x*(5+x)*D[y[x],x]-(2-3*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{15}\left(-\frac{2c_2(x^2+x+3)}{x^2} + 15c_1e^{-x/2}\sqrt{x} + \sqrt{2}c_2e^{-x/2}\sqrt{-x}\Gamma\left(\frac{1}{2},-\frac{x}{2}\right)\right)$$

1.509 problem 525

Internal problem ID [8647]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 525
Date solved : `Monday, October 21, 2024 at 05:19:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2y'' + x(1+x)\,y' - y = 0$$

1.509.1 Solved as second order ode using Kovacic algorithm

Time used: 0.444 (sec)

Writing the ode as

$$3x^2y'' + \left(x^2+x\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= x^2+x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 2x + 7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 2x + 7$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 2x + 7}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 963: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{7}{36x^2} + \frac{1}{18x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{1}{6x} + \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{1}{4x^4} + \frac{7}{4x^5} - \frac{7}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 2x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{2x + 7}{36x^2}\right) \\ &= \frac{1}{36} + \frac{2x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 2. Dividing this by leading coefficient in t which is 36 gives $\frac{1}{18}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{18}\right) - (0) \\ &= \frac{1}{18} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{18}}{\frac{1}{6}} - 0\right) &&= \frac{1}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{18}}{\frac{1}{6}} - 0\right) &&= -\frac{1}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 2x + 7}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{6} - \left(-\frac{1}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{6x} + (-)\left(\frac{1}{6}\right) \\
&= -\frac{1}{6x} - \frac{1}{6} \\
&= -\frac{1+x}{6x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6x} - \frac{1}{6}\right)(0) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} - \frac{1}{6}\right)^2 - \left(\frac{x^2+2x+7}{36x^2}\right) \right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int (-\frac{1}{6x} - \frac{1}{6}) dx} \\
&= \frac{e^{-\frac{x}{6}}}{x^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{3x^2}\, dx} \\
&= z_1 e^{-\frac{x}{6} - \frac{\ln(x)}{6}} \\
&= z_1 \left(\frac{e^{-\frac{x}{6}}}{x^{1/6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{x}{3}}}{x^{1/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{3x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \mathrm{e}^{-\frac{x}{3}-\frac{\ln(x)}{3}} x^{2/3} \mathrm{e}^{\frac{2x}{3}}\,dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\mathrm{e}^{-\frac{x}{3}}}{x^{1/3}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{x}{3}}}{x^{1/3}} \left(\int \mathrm{e}^{-\frac{x}{3}-\frac{\ln(x)}{3}} x^{2/3} \mathrm{e}^{\frac{2x}{3}}\,dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.509.2 Maple step by step solution

Let's solve

$3x^2\left(\frac{d}{dx}y'\right) + x(1+x)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{y}{3x^2} - \frac{(1+x)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(1+x)y'}{3x} - \frac{y}{3x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{1+x}{3x}, P_3(x) = -\frac{1}{3x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{3}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{3}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2\left(\frac{d}{dx}y'\right) + x(1+x)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k- > k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+3r)\,(-1+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r+1)\,(k+r-1) + a_{k-1}(k+r-1)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, -\frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

$$3(k+r-1)\left(\left(k+r+\frac{1}{3}\right)a_k+\frac{a_{k-1}}{3}\right)=0$$

- Shift index using $k->k+1$

$$3(k+r)\left(\left(k+\frac{4}{3}+r\right)a_{k+1}+\frac{a_k}{3}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{a_k}{3k+4+3r}$$

- Recursion relation for $r=1$

$$a_{k+1}=-\frac{a_k}{3k+7}$$

- Solution for $r=1$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+1}, a_{k+1}=-\frac{a_k}{3k+7}\right]$$

- Recursion relation for $r=-\frac{1}{3}$

$$a_{k+1}=-\frac{a_k}{3k+3}$$

- Solution for $r=-\frac{1}{3}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{3}}, a_{k+1}=-\frac{a_k}{3k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k-\frac{1}{3}}\right), a_{k+1}=-\frac{a_k}{3k+7}, b_{k+1}=-\frac{b_k}{3k+3}\right]$$

1.509.3 Maple trace

```
Methods for second order ODEs:
```

1.509.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 30

```
dsolve(3*x^2*diff(diff(y(x),x),x)+x*(1+x)*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{\mathrm{e}^{-\frac{x}{6}}\left(x^{1/6}\,\mathrm{WhittakerM}\left(-\frac{1}{6},\frac{2}{3},\frac{x}{3}\right)c_1+\mathrm{e}^{-\frac{x}{6}}c_2\right)}{x^{1/3}}$$

1.509.5 Mathematica DSolve solution

Solving time : 0.027 (sec)
Leaf size : 50

```
DSolve[{3*x^2*D[y[x],{x,2}]+x*(1+x)*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x/3}\left(c_2x^{2/3} - 3\sqrt[3]{3}c_1(-x)^{2/3}\Gamma\left(\frac{4}{3}, -\frac{x}{3}\right)\right)}{x}$$

1.510 problem 526

Internal problem ID [8648]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 526
Date solved : `Monday, October 21, 2024 at 05:19:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' - xy' + (1-2x)\,y = 0$$

1.510.1 Solved as second order ode using Kovacic algorithm

Time used: 0.189 (sec)

Writing the ode as

$$2x^2y'' - xy' + (1-2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -x \\ C &= 1-2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3 + 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3 + 16x$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3 + 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 965: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1-16x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+4\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= e^{\int \omega \,dx} \\
&= \mathrm{e}^{\int \frac{1+4\sqrt{x}}{4x}dx} \\
&= x^{1/4}\mathrm{e}^{2\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{2x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{4}} \\
&= z_1 \left(x^{1/4}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{2\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{x}\, \mathrm{e}^{2\sqrt{x}}\right) + c_2 \left(\sqrt{x}\, \mathrm{e}^{2\sqrt{x}} \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.510.2 Maple step by step solution

Let's solve

$2x^2\left(\frac{d}{dx}y'\right) - xy' + (1-2x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(2x-1)y}{2x^2} + \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'}{2x} - \frac{(2x-1)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = -\frac{2x-1}{2x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -\frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{2}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d}{dx}y'\right) - xy' + (1-2x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) - 2a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r-\frac{1}{2}\right)(k+r-1)\, a_k - 2a_{k-1} = 0$
- Shift index using $k- >k + 1$

  $2\left(k+\frac{1}{2}+r\right)(k+r)\, a_{k+1} - 2a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k}{(2k+1+2r)(k+r)}$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{2a_k}{(2k+3)(k+1)}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{2a_k}{(2k+3)(k+1)}\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{2a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$
- Solution for $r = \frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=\frac{2a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1}=\frac{2a_k}{(2k+3)(k+1)}, b_{k+1}=\frac{2b_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

1.510.3 Maple trace

```
Methods for second order ODEs:
```

1.510.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 25

```
dsolve(2*x^2*diff(diff(y(x),x),x)-x*diff(y(x),x)+(1-2*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\sqrt{x}\left(c_1 \sinh\left(2\sqrt{x}\right)+c_2 \cosh\left(2\sqrt{x}\right)\right)$$

1.510.5 Mathematica DSolve solution

Solving time : 0.074 (sec)
Leaf size : 41

```
DSolve[{2*x^2*D[y[x],{x,2}]-x*D[y[x],x]+(1-2*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{1}{2}e^{-2\sqrt{x}}\sqrt{x}\left(2c_1 e^{4\sqrt{x}}-c_2\right)$$

1.511 problem 527

Internal problem ID [8649]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 527
Date solved : `Monday, October 21, 2024 at 05:19:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2y'' + x(1+x)\,y' - (1+3x)\,y = 0$$

1.511.1 Solved as second order ode using Kovacic algorithm

Time used: 0.587 (sec)

Writing the ode as

$$3x^2y'' + \left(x^2+x\right)y' + (-3x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= x^2+x \\ C &= -3x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 38x + 7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 38x + 7$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 38x + 7}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 967: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{19}{18x} + \frac{7}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{19}{6x} - \frac{59}{2x^2} + \frac{1121}{2x^3} - \frac{53041}{4x^4} + \frac{1404613}{4x^5} - \frac{39845827}{4x^6} + \frac{1184064097}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 38x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{38x + 7}{36x^2}\right) \\ &= \frac{1}{36} + \frac{38x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 38. Dividing this by leading coefficient in t which is 36 gives $\frac{19}{18}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{19}{18}\right) - (0) \\ &= \frac{19}{18} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{19}{18}}{\frac{1}{6}} - 0\right) &= \frac{19}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{19}{18}}{\frac{1}{6}} - 0\right) &= -\frac{19}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 38x + 7}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{19}{6}$	$-\frac{19}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{19}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{19}{6} - \left(\frac{7}{6}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{7}{6x} + \left(\frac{1}{6}\right) \\ &= \frac{7}{6x} + \frac{1}{6} \\ &= \frac{7+x}{6x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{7}{6x} + \frac{1}{6}\right)(2x + a_1) + \left(\left(-\frac{7}{6x^2}\right) + \left(\frac{7}{6x} + \frac{1}{6}\right)^2 - \left(\frac{x^2+38x+7}{36x^2}\right) \right) = 0$$

$$\frac{(-a_1 + 20)\, x - 2a_0 + 7a_1}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 70, a_1 = 20\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 20x + 70$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2 + 20x + 70\right) e^{\int \left(\frac{7}{6x} + \frac{1}{6}\right) dx} \\ &= \left(x^2 + 20x + 70\right) e^{\frac{x}{6} + \frac{7\ln(x)}{6}} \\ &= \left(x^2 + 20x + 70\right) x^{7/6} e^{\frac{x}{6}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{3x^2}\,dx} \\
&= z_1 e^{-\frac{x}{6}-\frac{\ln(x)}{6}} \\
&= z_1 \left(\frac{e^{-\frac{x}{6}}}{x^{1/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x^2 + 20x + 70)\, x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{3x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(x^2+20x+70)^2\, x^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left((x^2 + 20x + 70)\, x\right) + c_2 \left((x^2 + 20x + 70)\, x \left(\int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(x^2+20x+70)^2\, x^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.511.2 Maple step by step solution

Let's solve

$3x^2\left(\frac{d}{dx}y'\right)+x(1+x)\,y'-(1+3x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{(1+3x)y}{3x^2}-\frac{(1+x)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(1+x)y'}{3x}-\frac{(1+3x)y}{3x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{1+x}{3x},P_3(x)=-\frac{1+3x}{3x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=\frac{1}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=-\frac{1}{3}$

  ◦ $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $3x^2\left(\frac{d}{dx}y'\right)+x(1+x)\,y'+(-3x-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+3r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r+1)\,(k+r-1) + a_{k-1}(k-4+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+3r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, -\frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3\left(k+r+\frac{1}{3}\right)(k+r-1)\, a_k + a_{k-1}(k-4+r) = 0$$

- Shift index using $k->k+1$

$$3\left(k+\frac{4}{3}+r\right)(k+r)\, a_{k+1} + a_k(k+r-3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r-3)}{(3k+4+3r)(k+r)}$$

- Recursion relation for $r=1$; series terminates at $k=2$

$$a_{k+1} = -\frac{a_k(k-2)}{(3k+7)(k+1)}$$

- Apply recursion relation for $k=0$

$$a_1 = \frac{2a_0}{7}$$

- Apply recursion relation for $k=1$

$$a_2 = \frac{a_1}{20}$$

- Express in terms of a_0

$$a_2 = \frac{a_0}{70}$$

- Terminating series solution of the ODE for $r=1$. Use reduction of order to find the second li

$$y = a_0 \cdot \left(1 + \frac{2}{7}x + \frac{1}{70}x^2\right)$$

- Recursion relation for $r = -\frac{1}{3}$

$$a_{k+1} = -\frac{a_k\left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}$$

- Solution for $r = -\frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+1} = -\frac{a_k \left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = a_0 \cdot \left(1 + \frac{2}{7}x + \frac{1}{70}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{3}}\right), b_{k+1} = -\frac{b_k \left(k-\frac{10}{3}\right)}{(3k+3)\left(k-\frac{1}{3}\right)}\right]$$

1.511.3 Maple trace

```
Methods for second order ODEs:
```

1.511.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 41

```
dsolve(3*x^2*diff(diff(y(x),x),x)+x*(1+x)*diff(y(x),x)-(1+3*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \,\mathrm{e}^{-\frac{x}{3}} \operatorname{hypergeom}\left(\left[3\right], \left[-\frac{1}{3}\right], \frac{x}{3}\right) + 70 c_1 \left(x^{4/3} + \frac{2x^{7/3}}{7} + \frac{x^{10/3}}{70}\right)}{x^{1/3}}$$

1.511.5 Mathematica DSolve solution

Solving time : `0.244` (sec)
Leaf size : 78

```
DSolve[{3*x^2*D[y[x],{x,2}]+x*(1+x)*D[y[x],x]-(1+3*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x\left(x^2 + 20x + 70\right) - \frac{c_2 x(x^2 + 20x + 70)\,\Gamma\left(\frac{2}{3}, \frac{x}{3}\right)}{1680\sqrt[3]{3}} + \frac{c_2 e^{-x/3}(x^3 + 19x^2 + 54x - 18)}{1680\sqrt[3]{x}}$$

1.512 problem 528

Internal problem ID [8650]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 528
Date solved : `Monday, October 21, 2024 at 05:19:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(3+x)\,y''+x(1+5x)\,y'+(1+x)\,y=0$$

1.512.1 Solved as second order ode using Kovacic algorithm

Time used: 0.306 (sec)

Writing the ode as

$$\left(2x^3+6x^2\right)y''+\left(5x^2+x\right)y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^3+6x^2\\ B&=5x^2+x\\ C&=1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 30x - 35}{16(x^2 + 3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2 - 30x - 35 \\ t &= 16(x^2 + 3x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 30x - 35}{16(x^2 + 3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 969: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{108\,(3+x)} - \frac{5}{108x} - \frac{35}{144x^2} + \frac{7}{36\,(3+x)^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(3+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 30x - 35}{16\,(x^2 + 3x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-3x^2-30x-35}{16\left(x^2+3x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-3	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\
&= \frac{1}{4}-\left(\frac{1}{4}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{6\left(3+x\right)} + \frac{5}{12x} + (-)\left(0\right) \\
&= -\frac{1}{6\left(3+x\right)} + \frac{5}{12x} \\
&= \frac{x+5}{4x\left(3+x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{6\left(3+x\right)} + \frac{5}{12x}\right)(0) + \left(\left(\frac{1}{6\left(3+x\right)^2} - \frac{5}{12x^2}\right) + \left(-\frac{1}{6\left(3+x\right)} + \frac{5}{12x}\right)^2 - \left(\frac{-3x^2-30x-3}{16\left(x^2+3x\right)^2}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{6(3+x)} + \frac{5}{12x}\right) dx} \\
&= \frac{x^{5/12}}{\left(3+x\right)^{1/6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{5x^2+x}{2x^3+6x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{12} - \frac{7\ln(3+x)}{6}} \\
&= z_1 \left(\frac{1}{x^{1/12}\left(3+x\right)^{7/6}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{(3+x)^{4/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2+x}{2x^3+6x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{6}-\frac{7\ln(3+x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{6}-\frac{7\ln(3+x)}{3}}(3+x)^{8/3}}{x^{2/3}}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/3}}{(3+x)^{4/3}}\right) + c_2\left(\frac{x^{1/3}}{(3+x)^{4/3}}\left(\int \frac{\mathrm{e}^{-\frac{\ln(x)}{6}-\frac{7\ln(3+x)}{3}}(3+x)^{8/3}}{x^{2/3}}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.512.2 Maple step by step solution

Let's solve

$2x^2(3+x)\left(\frac{d}{dx}y'\right) + x(1+5x)\,y' + (1+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(1+x)y}{2x^2(3+x)} - \frac{(1+5x)y'}{2x(3+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(1+5x)y'}{2x(3+x)} + \frac{(1+x)y}{2x^2(3+x)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{1+5x}{2x(3+x)}, P_3(x) = \frac{1+x}{2x^2(3+x)}\right]$
  - $(3+x) \cdot P_2(x)$ is analytic at $x = -3$

    $\left((3+x) \cdot P_2(x)\right)\Big|_{x=-3} = \frac{7}{3}$
  - $(3+x)^2 \cdot P_3(x)$ is analytic at $x = -3$

    $\left((3+x)^2 \cdot P_3(x)\right)\Big|_{x=-3} = 0$
  - $x = -3$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -3$
- Multiply by denominators

  $2x^2(3+x)\left(\frac{d}{dx}y'\right) + x(1+5x)\, y' + (1+x)\, y = 0$
- Change variables using $x = u - 3$ so that the regular singular point is at $u = 0$

  $(2u^3 - 12u^2 + 18u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (5u^2 - 29u + 42)\left(\frac{d}{du}y(u)\right) + (-2+u)\, y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- □ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k- > k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
  - Shift index using $k- > k + 1 - m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$6a_0r(4+3r)\,u^{-1+r} + (6a_1(1+r)\,(7+3r) - a_0(12r^2+17r+2))\,u^r + \left(\sum_{k=1}^{\infty} (6a_{k+1}(k+r+1)\,(3k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$6r(4+3r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{0, -\frac{4}{3}\right\}$$
- Each term must be 0
$$6a_1(1+r)\,(7+3r) - a_0(12r^2+17r+2) = 0$$
- Each term in the series must be 0, giving the recursion relation
$$2(-6a_k + a_{k-1} + 9a_{k+1})\,k^2 + (4(-6a_k + a_{k-1} + 9a_{k+1})\,r - 17a_k - a_{k-1} + 60a_{k+1})\,k + 2(-6a_k +$$
- Shift index using $k->k+1$
$$2(-6a_{k+1} + a_k + 9a_{k+2})\,(k+1)^2 + (4(-6a_{k+1} + a_k + 9a_{k+2})\,r - 17a_{k+1} - a_k + 60a_{k+2})\,(k+1) -$$
- Recursion relation that defines series solution to ODE
$$a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 4kra_k - 24kra_{k+1} + 2r^2a_k - 12r^2a_{k+1} + 3ka_k - 41ka_{k+1} + 3ra_k - 41ra_{k+1} + a_k - 31a_{k+1}}{6(3k^2+6kr+3r^2+16k+16r+20)}$$
- Recursion relation for $r = 0$
$$a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}$$
- Solution for $r = 0$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}, 42a_1 - 2a_0 = 0\right]$$
- Revert the change of variables $u = 3+x$
$$\left[y = \sum_{k=0}^{\infty} a_k(3+x)^k, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}, 42a_1 - 2a_0 = 0\right]$$
- Recursion relation for $r = -\frac{4}{3}$
$$a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} - \frac{7}{3}ka_k - 9ka_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}$$
- Solution for $r = -\frac{4}{3}$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} - \frac{7}{3}ka_k - 9ka_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}, -6a_1 - \frac{2a_0}{3} = 0\right]$$
- Revert the change of variables $u = 3+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (3+x)^{k-\frac{4}{3}}, a_{k+2} = -\frac{2k^2 a_k - 12k^2 a_{k+1} - \frac{7}{3} k a_k - 9k a_{k+1} + \frac{5}{9} a_k + \frac{7}{3} a_{k+1}}{6(3k^2+8k+4)}, -6a_1 - \frac{2a_0}{3} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (3+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (3+x)^{k-\frac{4}{3}}\right), a_{k+2} = -\frac{2k^2 a_k - 12k^2 a_{k+1} + 3k a_k - 41k a_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}, 42a$$

1.512.3 Maple trace

```
Methods for second order ODEs:
```

1.512.4 Maple dsolve solution

Solving time : `0.015` (sec)
Leaf size : 36

```
dsolve(2*x^2*(3+x)*diff(diff(y(x),x),x)+x*(1+5*x)*diff(y(x),x)+(1+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \sqrt{x}\, \text{hypergeom}\left(\left[1, \frac{3}{2}\right], \left[\frac{7}{6}\right], -\frac{x}{3}\right) + \frac{c_2 x^{1/3}}{(3+x)\left(1+\frac{x}{3}\right)^{1/3}}$$

1.512.5 Mathematica DSolve solution

Solving time : `0.113` (sec)
Leaf size : 50

```
DSolve[{2*x^2*(3+x)*D[y[x],{x,2}]+x*(1+5*x)*D[y[x],x]+(1+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[3]{x}\left(6\sqrt[3]{3}c_2\sqrt[6]{x}\, \text{Hypergeometric2F1}\left(-\frac{1}{3}, \frac{1}{6}, \frac{7}{6}, -\frac{x}{3}\right) + c_1\right)}{(x+3)^{4/3}}$$

1.513 problem 529

Internal problem ID [8651]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 529
Date solved : `Monday, October 21, 2024 at 05:19:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(4+x)\,y'' - x(1-3x)\,y' + y = 0$$

1.513.1 Solved as second order ode using Kovacic algorithm

Time used: 0.538 (sec)

Writing the ode as

$$x^2(4+x)\,y'' + \left(3x^2 - x\right)y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(4+x) \\ B &= 3x^2 - x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 6x - 7}{4\left(x^2 + 4x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^2 - 6x - 7$$
$$t = 4\left(x^2 + 4x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 6x - 7}{4\left(x^2 + 4x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 971: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 4x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -4$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2} + \frac{65}{64(4+x)^2} + \frac{5}{128(4+x)} - \frac{5}{128x}$$

For the pole at $x = -4$ let b be the coefficient of $\frac{1}{(4+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{8} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 6x - 7}{4(x^2 + 4x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{3x^2-6x-7}{4\left(x^2+4x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-4	2	0	$\frac{13}{8}$	$-\frac{5}{8}$
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=-\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= -\frac{1}{2}-\left(-\frac{1}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{5}{8\left(4+x\right)} + \frac{1}{8x} + (-)\left(0\right) \\
&= -\frac{5}{8\left(4+x\right)} + \frac{1}{8x} \\
&= -\frac{x-1}{2x\left(4+x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{5}{8\left(4+x\right)} + \frac{1}{8x}\right)(0) + \left(\left(\frac{5}{8\left(4+x\right)^2} - \frac{1}{8x^2}\right) + \left(-\frac{5}{8\left(4+x\right)} + \frac{1}{8x}\right)^2 - \left(\frac{3x^2-6x-7}{4\left(x^2+4x\right)^2}\right)\right) = 0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{8(4+x)} + \frac{1}{8x}\right) dx} \\
&= \frac{x^{1/8}}{\left(4+x\right)^{5/8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x^2 - x}{x^2(4+x)} \, dx} \\
&= z_1 e^{-\frac{13 \ln(4+x)}{8} + \frac{\ln(x)}{8}} \\
&= z_1 \left(\frac{x^{1/8}}{\left(4+x\right)^{13/8}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x^{1/4}}{(4+x)^{9/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2-x}{x^2(4+x)}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{13\ln(4+x)}{4}+\frac{\ln(x)}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-\frac{13\ln(4+x)}{4}+\frac{\ln(x)}{4}} (4+x)^{9/2}}{\sqrt{x}} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/4}}{(4+x)^{9/4}} \right) + c_2 \left(\frac{x^{1/4}}{(4+x)^{9/4}} \left(\int \frac{e^{-\frac{13\ln(4+x)}{4}+\frac{\ln(x)}{4}} (4+x)^{9/2}}{\sqrt{x}} dx \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.513.2 Maple step by step solution

Let's solve

$x^2(4+x)\left(\frac{d}{dx}y'\right) - x(1-3x)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x^2(4+x)} - \frac{(3x-1)y'}{x(4+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(3x-1)y'}{x(4+x)} + \frac{y}{x^2(4+x)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3x-1}{x(4+x)}, P_3(x) = \frac{1}{x^2(4+x)}\right]$

- $(4+x) \cdot P_2(x)$ is analytic at $x = -4$

  $\left((4+x) \cdot P_2(x)\right)\Big|_{x=-4} = \frac{13}{4}$

- $(4+x)^2 \cdot P_3(x)$ is analytic at $x = -4$

  $\left((4+x)^2 \cdot P_3(x)\right)\Big|_{x=-4} = 0$

- $x = -4$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -4$

- Multiply by denominators

  $x^2(4+x)\left(\frac{d}{dx}y'\right) + x(3x-1)y' + y = 0$

- Change variables using $x = u - 4$ so that the regular singular point is at $u = 0$

  $(u^3 - 8u^2 + 16u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (3u^2 - 25u + 52)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$4a_0 r(9+4r)\, u^{-1+r} + (4a_1(1+r)(13+4r) - a_0(8r^2+17r-1))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)(4k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$4r(9+4r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{0, -\frac{9}{4}\right\}$$
- Each term must be 0
$$4a_1(1+r)(13+4r) - a_0(8r^2+17r-1) = 0$$
- Each term in the series must be 0, giving the recursion relation
$$(-8a_k + a_{k-1} + 16a_{k+1})\, k^2 + (2(-8a_k + a_{k-1} + 16a_{k+1})\, r - 17a_k + 68a_{k+1})\, k + (-8a_k + a_{k-1} +$$
- Shift index using $k- >k+1$
$$(-8a_{k+1} + a_k + 16a_{k+2})(k+1)^2 + (2(-8a_{k+1} + a_k + 16a_{k+2})\, r - 17a_{k+1} + 68a_{k+2})(k+1) + (-$$
- Recursion relation that defines series solution to ODE
$$a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} + 2kra_k - 16kra_{k+1} + r^2 a_k - 8r^2 a_{k+1} + 2ka_k - 33ka_{k+1} + 2ra_k - 33ra_{k+1} - 24a_{k+1}}{4(4k^2 + 8kr + 4r^2 + 25k + 25r + 34)}$$
- Recursion relation for $r = 0$
$$a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} + 2ka_k - 33ka_{k+1} - 24a_{k+1}}{4(4k^2 + 25k + 34)}$$
- Solution for $r = 0$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} + 2ka_k - 33ka_{k+1} - 24a_{k+1}}{4(4k^2 + 25k + 34)}, 52a_1 + a_0 = 0\right]$$
- Revert the change of variables $u = 4 + x$
$$\left[y = \sum_{k=0}^{\infty} a_k (4+x)^k, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} + 2ka_k - 33ka_{k+1} - 24a_{k+1}}{4(4k^2 + 25k + 34)}, 52a_1 + a_0 = 0\right]$$
- Recursion relation for $r = -\frac{9}{4}$
$$a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - \frac{5}{2}ka_k + 3ka_{k+1} + \frac{9}{16}a_k + \frac{39}{4}a_{k+1}}{4(4k^2 + 7k - 2)}$$
- Solution for $r = -\frac{9}{4}$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{9}{4}}, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - \frac{5}{2}ka_k + 3ka_{k+1} + \frac{9}{16}a_k + \frac{39}{4}a_{k+1}}{4(4k^2 + 7k - 2)}, -20a_1 - \frac{5a_0}{4} = 0\right]$$
- Revert the change of variables $u = 4 + x$
$$\left[y = \sum_{k=0}^{\infty} a_k (4+x)^{k-\frac{9}{4}}, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - \frac{5}{2}ka_k + 3ka_{k+1} + \frac{9}{16}a_k + \frac{39}{4}a_{k+1}}{4(4k^2 + 7k - 2)}, -20a_1 - \frac{5a_0}{4} = 0\right]$$
- Combine solutions and rename parameters
$$\left[y = \left(\sum_{k=0}^{\infty} a_k (4+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (4+x)^{k-\frac{9}{4}}\right), a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} + 2ka_k - 33ka_{k+1} - 24a_{k+1}}{4(4k^2 + 25k + 34)}, 52a_1 + \right.$$

1.513.3 Maple trace

```
Methods for second order ODEs:
```

1.513.4 Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 27

```
dsolve(x^2*(4+x)*diff(diff(y(x),x),x)-x*(1-3*x)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1\, x^{1/4}}{(4+x)^{9/4}} + c_2 \,\text{hypergeom}\left([1,3]\,, \left[\frac{7}{4}\right], -\frac{x}{4}\right) x$$

1.513.5 Mathematica DSolve solution

Solving time : 0.149 (sec)
Leaf size : 89

```
DSolve[{x^2*(4+x)*D[y[x],{x,2}]-x*(1-3*x)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[4]{x}\left(-10c_2 \arctan\left(\sqrt[4]{\frac{x}{x+4}}\right) + 10c_2 \text{arctanh}\left(\sqrt[4]{\frac{x}{x+4}}\right) + c_2\sqrt[4]{x+4}x^{7/4} + 9c_2\sqrt[4]{x+4}x^{3/4} + 2c_1\right)}{2(x+4)^{9/4}}$$

1.514 problem 530

Internal problem ID [8652]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 530
Date solved : `Monday, October 21, 2024 at 05:19:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + 5xy' + (1 + x)\, y = 0$$

1.514.1 Solved as second order ode using Kovacic algorithm

Time used: 0.258 (sec)

Writing the ode as

$$2x^2y'' + 5xy' + (1 + x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 5x \\ C &= 1 + x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-3 - 8x}{16x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -3 - 8x \\
t &= 16x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-3 - 8x}{16x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 973: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} - \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{1}{2}(1-(1)) \\
&= 0
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\
&= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\
&= \frac{1}{2x}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}
$$

Since $d = 0$, then letting

$$
p = 1 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$
0 = 0
$$

And solving for p gives

$$
p = 1
$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta + \frac{p'}{p} \\
&= \frac{1}{2x}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0
$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1+8x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\,\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{2}\,\sqrt{-x}}{4x} dx} \\ &= x^{1/4} \mathrm{e}^{\sqrt{2}\,\sqrt{-x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x}{2x^2} \, dx} \\ &= z_1 e^{-\frac{5\ln(x)}{4}} \\ &= z_1 \left(\frac{1}{x^{5/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\sqrt{2}\,\sqrt{-x}}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{2x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(x)}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\sqrt{2}\,\sqrt{-x}\left(1 - \mathrm{e}^{-2\sqrt{2}\,\sqrt{-x}}\right)}{2\sqrt{x}} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\sqrt{2}\,\sqrt{-x}}}{x}\right) + c_2 \left(\frac{e^{\sqrt{2}\,\sqrt{-x}}}{x}\left(-\frac{\sqrt{2}\,\sqrt{-x}\left(1-e^{-2\sqrt{2}\,\sqrt{-x}}\right)}{2\sqrt{x}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.514.2 Maple step by step solution

Let's solve

$2x^2\left(\frac{d}{dx}y'\right) + 5xy' + (1+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$
- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(1+x)y}{2x^2} - \frac{5y'}{2x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{5y'}{2x} + \frac{(1+x)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{5}{2x}, P_3(x) = \frac{1+x}{2x^2}\right]$
- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{5}{2}$
- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{2}$
- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d}{dx}y'\right) + 5xy' + (1+x)\,y = 0$
- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + r) (1 + 2r) x^r + \left(\sum_{k=1}^{\infty} (a_k (k + r + 1) (2k + 2r + 1) + a_{k-1}) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + r) (1 + 2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-1, -\frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k + r + 1) \left(k + r + \frac{1}{2}\right) a_k + a_{k-1} = 0$
- Shift index using $k- >k + 1$

  $2(k + 2 + r) \left(k + \frac{3}{2} + r\right) a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+2+r)(2k+3+2r)}$
- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}\right]$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}$

- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}, b_{k+1} = -\frac{b_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

1.514.3 Maple trace

```
Methods for second order ODEs:
```

1.514.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 29

```
dsolve(2*x^2*diff(diff(y(x),x),x)+5*x*diff(y(x),x)+(1+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(\sqrt{x}\,\sqrt{2}\right) + c_2 \cos\left(\sqrt{x}\,\sqrt{2}\right)}{x}$$

1.514.5 Mathematica DSolve solution

Solving time : 0.104 (sec)
Leaf size : 60

```
DSolve[{2*x^2*D[y[x],{x,2}]+5*x*D[y[x],x]+(1+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{i\sqrt{2}\sqrt{x}} + i\sqrt{2}c_2 e^{-i\sqrt{2}\sqrt{x}}}{2x}$$

1.515 problem 531

Internal problem ID [8653]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 531
Date solved : `Monday, October 21, 2024 at 05:19:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2y'' + x(10-x)\,y' - (2+x)\,y = 0$$

1.515.1 Solved as second order ode using Kovacic algorithm

Time used: 0.353 (sec)

Writing the ode as

$$6x^2y'' + \left(-x^2+10x\right)y' + (-x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= -x^2+10x \\ C &= -x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 28}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x + 28$$
$$t = 144x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 28}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 975: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{144} + \frac{1}{36x} + \frac{7}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{12} + \frac{1}{6x} + \frac{1}{x^2} - \frac{2}{x^3} - \frac{2}{x^4} + \frac{28}{x^5} - \frac{56}{x^6} - \frac{272}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{12}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{12}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{144}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 28}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{1}{144}\right) + \left(\frac{4x + 28}{144x^2}\right) \\ &= \frac{1}{144} + \frac{4x + 28}{144x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 144 gives $\frac{1}{36}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{36}\right) - (0) \\ &= \frac{1}{36} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{12} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{36}}{\frac{1}{12}} - 0\right) &= \frac{1}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{36}}{\frac{1}{12}} - 0\right) &= -\frac{1}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 28}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{12}$	$\frac{1}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{6} - \left(-\frac{1}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{6x} + (-)\left(\frac{1}{12}\right) \\
&= -\frac{1}{6x} - \frac{1}{12} \\
&= -\frac{2 + x}{12x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6x} - \frac{1}{12}\right)(0) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} - \frac{1}{12}\right)^2 - \left(\frac{x^2 + 4x + 28}{144x^2}\right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{6x} - \frac{1}{12}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{12}}}{x^{1/6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2 + 10x}{6x^2} \, dx} \\
&= z_1 e^{\frac{x}{12} - \frac{5 \ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{12}}}{x^{5/6}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+10x}{6x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x}{6}-\frac{5\ln(x)}{3}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \mathrm{e}^{\frac{x}{6}-\frac{5\ln(x)}{3}} x^2 dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\int \mathrm{e}^{\frac{x}{6}-\frac{5\ln(x)}{3}} x^2 dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.515.2 Maple step by step solution

Let's solve

$6x^2\left(\frac{d}{dx}y'\right) + x(10-x)\,y' - (2+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(2+x)y}{6x^2} + \frac{(x-10)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x-10)y'}{6x} - \frac{(2+x)y}{6x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x-10}{6x}, P_3(x) = -\frac{2+x}{6x^2}\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$
  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{5}{3}$$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{3}$$
- $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

• Multiply by denominators
$$6x^2\left(\frac{d}{dx}y'\right) - x(x-10)\,y' + (-x-2)\,y = 0$$

• Assume series solution for y
$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions
- Convert $x^m \cdot y$ to series expansion for $m = 0..1$
  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
- Shift index using $k- > k - m$
  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$
  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$
- Shift index using $k- > k + 1 - m$
  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$
- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$
  Rewrite ODE with series expansions
  $$2a_0(1+r)\,(-1+3r)\,x^r + \left(\sum_{k=1}^{\infty} \left(2a_k(k+r+1)\,(3k+3r-1) - a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

• $a_0$cannot be 0 by assumption, giving the indicial equation

$2(1+r)(-1+3r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{-1, \frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

$6\left(k+r-\frac{1}{3}\right)(k+r+1)\, a_k - a_{k-1}(k+r) = 0$

- Shift index using $k- >k+1$

$6\left(k+\frac{2}{3}+r\right)(k+2+r)\, a_{k+1} - a_k(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r+1)}{2(3k+2+3r)(k+2+r)}$

- Recursion relation for $r = -1$

$a_{k+1} = \frac{a_k k}{2(3k-1)(k+1)}$

- Solution for $r = -1$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k k}{2(3k-1)(k+1)}\right]$

- Recursion relation for $r = \frac{1}{3}$

$a_{k+1} = \frac{a_k\left(k+\frac{4}{3}\right)}{2(3k+3)\left(k+\frac{7}{3}\right)}$

- Solution for $r = \frac{1}{3}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = \frac{a_k\left(k+\frac{4}{3}\right)}{2(3k+3)\left(k+\frac{7}{3}\right)}\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+1} = \frac{a_k k}{2(3k-1)(k+1)}, b_{k+1} = \frac{b_k\left(k+\frac{4}{3}\right)}{2(3k+3)\left(k+\frac{7}{3}\right)}\right]$

1.515.3 Maple trace

```
Methods for second order ODEs:
```

1.515.4 Maple dsolve solution

Solving time : 0.013 (sec)
Leaf size : 27

```
dsolve(6*x^2*diff(diff(y(x),x),x)+x*(10-x)*diff(y(x),x)-(2+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\, x^{5/6} + c_1 \operatorname{WhittakerM}\left(-\frac{1}{6}, \frac{2}{3}, \frac{x}{6}\right) \mathrm{e}^{\frac{x}{12}} x}{x^{11/6}}$$

1.515.5 Mathematica DSolve solution

Solving time : 0.053 (sec)
Leaf size : 38

```
DSolve[{6*x^2*D[y[x],{x,2}]+x*(10-x)*D[y[x],x]-(2+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 \sqrt[3]{x} L_{-\frac{4}{3}}^{\frac{4}{3}}\left(\frac{x}{6}\right) + \frac{6\sqrt[3]{6} c_1}{x}$$

1.516 problem 532

Internal problem ID [8654]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 532
Date solved : `Monday, October 21, 2024 at 05:19:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(3+4x)\,y''+x(11+4x)\,y'-(3+4x)\,y=0$$

1.516.1 Solved as second order ode using Kovacic algorithm

Time used: 0.358 (sec)

Writing the ode as

$$\left(4x^3+3x^2\right)y''+\left(4x^2+11x\right)y'+(-3-4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+3x^2 \\ B &= 4x^2+11x \\ C &= -3-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{48x^2 + 8x + 91}{4(4x^2 + 3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 48x^2 + 8x + 91$$
$$t = 4(4x^2 + 3x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{48x^2 + 8x + 91}{4(4x^2 + 3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 977: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(4x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{3}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{28}{9\left(x+\frac{3}{4}\right)^2} + \frac{176}{27\left(x+\frac{3}{4}\right)} - \frac{176}{27x} + \frac{91}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{91}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{6}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{6}\end{aligned}$$

For the pole at $x = -\frac{3}{4}$ let b be the coefficient of $\frac{1}{\left(x+\frac{3}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{28}{9}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{3}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{4}{3}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{48x^2 + 8x + 91}{4\left(4x^2 + 3x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{48x^2 + 8x + 91}{4\left(4x^2 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{13}{6}$	$-\frac{7}{6}$
$-\frac{3}{4}$	2	0	$\frac{7}{3}$	$-\frac{4}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{5}{2}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{7}{6x} - \frac{4}{3\left(x + \frac{3}{4}\right)} + (-)\,(0) \\
&= -\frac{7}{6x} - \frac{4}{3\left(x + \frac{3}{4}\right)} \\
&= \frac{-7 - 20x}{8x^2 + 6x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{7}{6x} - \frac{4}{3\left(x + \frac{3}{4}\right)}\right)(2x + a_1) + \left(\left(\frac{7}{6x^2} + \frac{4}{3\left(x + \frac{3}{4}\right)^2}\right) + \left(-\frac{7}{6x} - \frac{4}{3\left(x + \frac{3}{4}\right)}\right)^2 - \left(\frac{48x^2 + 8x}{4\left(4x^2 + 3\right.}\right.\right.
$$

$$
\frac{12a_1 x - 8x + 32a_0 -}{x\,(3 + 4x)}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{7}{48}, a_1 = \frac{2}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + \frac{2}{3}x + \frac{7}{48}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + \frac{2}{3}x + \frac{7}{48}\right) \mathrm{e}^{\int \left(-\frac{7}{6x} - \frac{4}{3\left(x + \frac{3}{4}\right)}\right) dx} \\
&= \left(x^2 + \frac{2}{3}x + \frac{7}{48}\right) \mathrm{e}^{-\frac{7\ln(x)}{6} - \frac{4\ln(3+4x)}{3}} \\
&= \frac{x^2 + \frac{2}{3}x + \frac{7}{48}}{x^{7/6}\,(3 + 4x)^{4/3}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^2+11x}{4x^3+3x^2}\,dx} \\
&= z_1 e^{-\frac{11\ln(x)}{6}+\frac{4\ln(3+4x)}{3}} \\
&= z_1\left(\frac{(3+4x)^{4/3}}{x^{11/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^2+11x}{4x^3+3x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{11\ln(x)}{3}+\frac{8\ln(3+4x)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{11\ln(x)}{3}+\frac{8\ln(3+4x)}{3}}x^6}{\left(x^2+\frac{2}{3}x+\frac{7}{48}\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3}\right) + c_2\left(\frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3}\left(\int \frac{\mathrm{e}^{-\frac{11\ln(x)}{3}+\frac{8\ln(3+4x)}{3}}x^6}{\left(x^2+\frac{2}{3}x+\frac{7}{48}\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.516.2 Maple step by step solution

Let's solve

$x^2(3+4x)\left(\frac{d}{dx}y'\right)+x(11+4x)\,y'-(3+4x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{y}{x^2}-\frac{(11+4x)y'}{x(3+4x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(11+4x)y'}{x(3+4x)}-\frac{y}{x^2}=0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{11+4x}{x(3+4x)},P_3(x)=-\frac{1}{x^2}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{11}{3}$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-1$
  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(3+4x)\left(\frac{d}{dx}y'\right)+x(11+4x)\,y'+(-3-4x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx} y'\right)$ to series expansion for $m = 2..3$

  $$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

  $$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 (3+r)\,(-1+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+3)\,(3k+3r-1) + 4a_{k-1} (k+r)\,(k-2+r)\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-3, \frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3(k+r+3)\left(k+r-\frac{1}{3}\right) a_k + 4a_{k-1}(k+r)\,(k-2+r) = 0$

- Shift index using $k- > k+1$

  $3(k+4+r)\left(k+\frac{2}{3}+r\right) a_{k+1} + 4a_k(k+r+1)\,(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k(k+r+1)(k+r-1)}{(k+4+r)(3k+2+3r)}$

- Recursion relation for $r = -3$; series terminates at $k = 2$

  $a_{k+1} = -\frac{4a_k(k-2)(k-4)}{(k+1)(3k-7)}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{32a_0}{7}$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{3a_1}{2}$

- Express in terms of a_0

  $a_2 = \frac{48a_0}{7}$

- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y = a_0 \cdot \left(\frac{48}{7}x^2 + \frac{32}{7}x + 1\right)$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+1} = -\frac{4a_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{\left(k+\frac{13}{3}\right)(3k+3)}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{4a_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{\left(k+\frac{13}{3}\right)(3k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(\frac{48}{7}x^2 + \frac{32}{7}x + 1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), b_{k+1} = -\frac{4b_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{\left(k+\frac{13}{3}\right)(3k+3)}\right]$

1.516.3 Maple trace

```
Methods for second order ODEs:
```

1.516.4 Maple dsolve solution

Solving time : `0.022` (sec)
Leaf size : 41

```
dsolve(x^2*(3+4*x)*diff(diff(y(x),x),x)+x*(11+4*x)*diff(y(x),x)-(3+4*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1(48x^2 + 32x + 7)}{x^3} + c_2 \,\mathrm{hypergeom}\left([3,5], \left[\frac{13}{3}\right], -\frac{4x}{3}\right) x^{1/3}(3+4x)^{11/3}$$

1.516.5 Mathematica DSolve solution

Solving time : `0.473` (sec)
Leaf size : 367

```
DSolve[{x^2*(3+4*x)*D[y[x],{x,2}]+x*(11+4*x)*D[y[x],x]-(3+4*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \\ \rightarrow \frac{12\sqrt[3]{2}\sqrt{3}c_2(48x^2+32x+7)\arctan\left(\frac{\sqrt{3}\sqrt[3]{4x+3}}{2\ 2^{2/3}\sqrt[3]{x}+\sqrt[3]{4x+3}}\right) + 384c_2(4x+3)^{2/3}x^{10/3} + 576c_2(4x+3)^{2/3}x}{}$$

1.517 problem 533

Internal problem ID [8655]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 533
Date solved : `Monday, October 21, 2024 at 05:19:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+3x)\,y''+x(4+11x)\,y'-(1-x)\,y=0$$

1.517.1 Solved as second order ode using Kovacic algorithm

Time used: 0.208 (sec)

Writing the ode as

$$\left(6x^3+4x^2\right)y''+\left(11x^2+4x\right)y'+(x-1)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^3+4x^2 \\ B &= 11x^2+4x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-35}{16\,(2+3x)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -35 \\
t &= 16(2+3x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{35}{16\,(2+3x)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 979: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2+3x)^2$. There is a pole at $x = -\frac{2}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144\left(x + \frac{2}{3}\right)^2}$$

For the pole at $x = -\frac{2}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{35}{16\left(2+3x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{35}{16\left(2+3x\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{2}{3}$	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{12}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{5}{12} - \left(\frac{5}{12}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{12\left(x+\frac{2}{3}\right)} + (-)\left(0\right) \\ &= \frac{5}{12\left(x+\frac{2}{3}\right)} \\ &= \frac{5}{8+12x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{12\left(x+\frac{2}{3}\right)}\right)(0) + \left(\left(-\frac{5}{12\left(x+\frac{2}{3}\right)^2}\right) + \left(\frac{5}{12\left(x+\frac{2}{3}\right)}\right)^2 - \left(-\frac{35}{16\,(2+3x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{5}{12\left(x+\frac{2}{3}\right)} dx} \\ &= (2+3x)^{5/12} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{11x^2+4x}{6x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2} - \frac{5\ln(2+3x)}{12}} \\ &= z_1\left(\frac{1}{\sqrt{x}\,(2+3x)^{5/12}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^2+4x}{6x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)-\frac{5\ln(2+3x)}{6}}}{(y_1)^2}\,dx \\
&= y_1 \left(2\,\mathrm{e}^{-\ln(x)-\frac{5\ln(2+3x)}{6}}\,x(2+3x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{\sqrt{x}}\right) + c_2\left(\frac{1}{\sqrt{x}}\left(2\,\mathrm{e}^{-\ln(x)-\frac{5\ln(2+3x)}{6}}\,x(2+3x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.517.2 Maple step by step solution

Let's solve

$2x^2(2+3x)\left(\frac{d}{dx}y'\right) + x(4+11x)\,y' - (1-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x-1)y}{2x^2(2+3x)} - \frac{(4+11x)y'}{2x(2+3x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(4+11x)y'}{2x(2+3x)} + \frac{(x-1)y}{2x^2(2+3x)} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{4+11x}{2x(2+3x)}, P_3(x) = \frac{x-1}{2x^2(2+3x)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2(2+3x)\left(\frac{d}{dx}y'\right) + x(4+11x)\,y' + (x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + a_{k-1}(2k+2r-1)\,(3k-2+3r\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(\left(\frac{3k}{2}+\frac{3r}{2}-1\right)a_{k-1}+a_k\left(k+r+\frac{1}{2}\right)\right)=0$
- Shift index using $k->k+1$

  $4\left(k+r+\frac{1}{2}\right)\left(\left(\frac{3k}{2}+\frac{1}{2}+\frac{3r}{2}\right)a_k+a_{k+1}\left(k+\frac{3}{2}+r\right)\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{(3k+3r+1)a_k}{2k+3+2r}$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+1}=-\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}$
- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}, a_{k+1}=-\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=-\frac{\left(3k+\frac{5}{2}\right)a_k}{2k+4}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{\left(3k+\frac{5}{2}\right)a_k}{2k+4}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}, b_{k+1}=-\frac{\left(3k+\frac{5}{2}\right)b_k}{2k+4}\right]$

1.517.3 Maple trace

```
Methods for second order ODEs:
```

1.517.4 Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 19

```
dsolve(2*x^2*(2+3*x)*diff(diff(y(x),x),x)+x*(4+11*x)*diff(y(x),x)-(1-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_2(2+3x)^{1/6}+c_1}{\sqrt{x}}$$

1.517.5 Mathematica DSolve solution

Solving time : 0.091 (sec)
Leaf size : 32

```
DSolve[{2*x^2*(2+3*x)*D[y[x],{x,2}]+x*(4+11*x)*D[y[x],x]-(1-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 \sqrt[6]{6x+4} + 2^{5/6} c_1}{\sqrt{x}}$$

1.518 problem 534

Internal problem ID [8656]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 534
Date solved : `Monday, October 21, 2024 at 05:19:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(2+x)\,y''+5x(1-x)\,y'-(2-8x)\,y=0$$

1.518.1 Solved as second order ode using Kovacic algorithm

Time used: 0.879 (sec)

Writing the ode as

$$x^2(2+x)\,y''+\left(-5x^2+5x\right)y'+(8x-2)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(2+x)\\ B &= -5x^2+5x\\ C &= 8x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3x^2 - 126x + 21}{4\left(x^2 + 2x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3x^2 - 126x + 21 \\
t &= 4\left(x^2 + 2x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3x^2 - 126x + 21}{4\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 981: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{147}{16\,(2+x)} + \frac{285}{16\,(2+x)^2} - \frac{147}{16x} + \frac{21}{16x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{285}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{15}{4}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 126x + 21}{4\,(x^2+2x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{3x^2-126x+21}{4\left(x^2+2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{19}{4}$	$-\frac{15}{4}$
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=-\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\ &= -\frac{1}{2}-\left(-\frac{9}{2}\right) \\ &= 4 \end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{15}{4\left(2+x\right)} - \frac{3}{4x} + (-)\left(0\right) \\
&= -\frac{15}{4\left(2+x\right)} - \frac{3}{4x} \\
&= -\frac{3(3x+1)}{2x\left(2+x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(-\frac{15}{4\left(2+x\right)} - \frac{3}{4x}\right)\left(4x^3 + 3x^2a_3 + 2a_2x + a_1\right) + \left(\left(\frac{15}{4\left(2+x\right)^2} + \frac{3}{4x^2}\right) + \left(-\right.\right.$$

$$\underline{3(4+a_3)\,x^3 + (8a_2 +}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{40}, a_1 = \frac{1}{5}, a_2 = \frac{3}{2}, a_3 = -4\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= \left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right) e^{\int \left(-\frac{15}{4(2+x)} - \frac{3}{4x}\right)dx} \\
&= \left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right) e^{-\frac{3\ln(x)}{4} - \frac{15\ln(2+x)}{4}} \\
&= \frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^{3/4}\left(2+x\right)^{15/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x^2+5x}{x^2(2+x)}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{4}+\frac{15\ln(2+x)}{4}} \\
&= z_1\left(\frac{(2+x)^{15/4}}{x^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{40x^4-160x^3+60x^2+8x+1}{40x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5x^2+5x}{x^2(2+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{2}+\frac{15\ln(2+x)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{10\sqrt{2+x}\,x^{5/2}\left(8x^5\sqrt{x(2+x)}+4200\ln\left(\frac{x+\sqrt{x(2+x)}}{x}\right)x^4-4200\ln\left(\frac{\sqrt{x(2+x)}-x}{x}\right)x^4+328x^4\sqrt{x(2\right.}{}\right.
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left(\frac{40x^4-160x^3+60x^2+8x+1}{40x^2}\right) \\
&\quad +c_2\left(\frac{40x^4-160x^3+60x^2+8x+1}{40x^2}\left(\frac{10\sqrt{2+x}\,x^{5/2}\left(8x^5\sqrt{x(2+x)}+4200\ln\left(\frac{x+\sqrt{x(2+x)}}{x}\right)x^4-420\right.}{}\right.\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.518.2 Maple step by step solution

Let's solve

$x^2(2+x)\left(\frac{d}{dx}y'\right)+5x(1-x)\,y'-(2-8x)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{2(4x-1)y}{x^2(2+x)}+\frac{5(x-1)y'}{x(2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{5(x-1)y'}{x(2+x)}+\frac{2(4x-1)y}{x^2(2+x)}=0$

□ Check to see if x_0 is a regular singular point

  ∘ Define functions

  $\left[P_2(x)=-\frac{5(x-1)}{x(2+x)},P_3(x)=\frac{2(4x-1)}{x^2(2+x)}\right]$

  ∘ $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((2+x)\cdot P_2(x)\right)\bigg|_{x=-2}=-\frac{15}{2}$

  ∘ $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((2+x)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

  ∘ $x=-2$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-2$

- Multiply by denominators

$x^2(2+x)\left(\frac{d}{dx}y'\right)-5x(x-1)\,y'+(8x-2)\,y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$(u^3-4u^2+4u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-5u^2+25u-30)\left(\frac{d}{du}y(u)\right)+(8u-18)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ∘ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  ∘ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$2a_0 r(-17+2r)\, u^{-1+r} + \left(2a_1(1+r)\,(-15+2r) - a_0(4r^2-29r+18)\right) u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r(-17+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{17}{2}\right\}$$

- Each term must be 0

$$2a_1(1+r)\,(-15+2r) - a_0(4r^2-29r+18) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + \left((-8a_k + 2a_{k-1} + 8a_{k+1})\, r + 29a_k - 8a_{k-1} - 26a_{k+1}\right) k + (-4a_k + a$$

- Shift index using $k->k+1$

$$(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + \left((-8a_{k+1} + 2a_k + 8a_{k+2})\, r + 29a_{k+1} - 8a_k - 26a_{k+2}\right)(k+1) -$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 2kra_k - 8kra_{k+1} + r^2 a_k - 4r^2 a_{k+1} - 6ka_k + 21ka_{k+1} - 6ra_k + 21ra_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2+4kr+2r^2-9k-9r-26)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 6ka_k + 21ka_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2-9k-26)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 6ka_k + 21ka_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2-9k-26)}, -30a_1 - 18a_0 = 0\right]$$

- Revert the change of variables $u = 2 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^k, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 6k a_k + 21k a_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2 - 9k - 26)}, -30a_1 - 18a_0 = 0\right]$$

- Recursion relation for $r = \frac{17}{2}$

$$a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 11k a_k - 47k a_{k+1} + \frac{117}{4} a_k - \frac{207}{2} a_{k+1}}{2(2k^2 + 25k + 42)}$$

- Solution for $r = \frac{17}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{17}{2}}, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 11k a_k - 47k a_{k+1} + \frac{117}{4} a_k - \frac{207}{2} a_{k+1}}{2(2k^2 + 25k + 42)}, 38a_1 - \frac{121 a_0}{2} = 0\right]$$

- Revert the change of variables $u = 2 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k+\frac{17}{2}}, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 11k a_k - 47k a_{k+1} + \frac{117}{4} a_k - \frac{207}{2} a_{k+1}}{2(2k^2 + 25k + 42)}, 38a_1 - \frac{121 a_0}{2} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (2+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (2+x)^{k+\frac{17}{2}}\right), a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 6k a_k + 21k a_{k+1} + 8a_k + 7a_{k+1}}{2(2k^2 - 9k - 26)}, -30\right.$$

1.518.3 Maple trace

```
Methods for second order ODEs:
```

1.518.4 Maple dsolve solution

Solving time : `0.028` (sec)
Leaf size : 113

```
dsolve(x^2*(2+x)*diff(diff(y(x),x),x)+5*x*(1-x)*diff(y(x),x)-(2-8*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1(40x^4 - 160x^3 + 60x^2 + 8x + 1)}{x^2} + \frac{4c_2(-2-x)^{3/4}\left(1050x^{3/2}\left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right)\operatorname{arcsinh}\left(\frac{\sqrt{2}\sqrt{x}}{2}\right) + \sqrt{2+x}\,x^2\left(x^5 + 41x^4 - \frac{6987}{4}\right.\right.}{105\,(2+x)^{3/4}\,x^{7/2}}$$

1.518.5 Mathematica DSolve solution

Solving time : `0.258` (sec)
Leaf size : 114

```
DSolve[{x^2*(2+x)*D[y[x],{x,2}]+5*x*(1-x)*D[y[x],x]-(2-8*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1050c_2(40x^4-160x^3+60x^2+8x+1)\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{x}{x+2}}}\right)+2c_1(40x^4-160x^3+60x^2+8x+1)+5c_2\sqrt{}}{80x^2}$$

1.519 problem 535

Internal problem ID [8657]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 535
Date solved : `Monday, October 21, 2024 at 05:19:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$8x^2\left(-x^2+1\right)y''+2x\left(-13x^2+1\right)y'+\left(-9x^2+1\right)y=0$$

1.519.1 Solved as second order ode using Kovacic algorithm

Time used: 0.374 (sec)

Writing the ode as

$$\left(-8x^4+8x^2\right)y''+\left(-26x^3+2x\right)y'+\left(-9x^2+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-8x^4+8x^2\\ B&=-26x^3+2x\\ C&=-9x^2+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7x^4 - 26x^2 - 15}{64\left(x^3 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -7x^4 - 26x^2 - 15$$
$$t = 64\left(x^3 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-7x^4 - 26x^2 - 15}{64\left(x^3 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 983: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 4 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4x - 4} - \frac{3}{16 (x + 1)^2} - \frac{3}{16 (x - 1)^2} - \frac{1}{4 (x + 1)} - \frac{15}{64x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{3}{8}
\end{aligned}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4}
\end{aligned}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-7x^4 - 26x^2 - 15}{64\left(x^3 - x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-7x^4 - 26x^2 - 15}{64\left(x^3 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{8}$	$\frac{3}{8}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{7}{8} - \left(\frac{7}{8}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} + (0) \\
&= \frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} \\
&= \frac{7x^2-3}{8x^3-8x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right)(0) + \left(\left(-\frac{3}{8x^2} - \frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2}\right) + \left(\frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right)dx} \\
&= (x+1)^{1/4}(x-1)^{1/4}x^{3/8}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-26x^3+2x}{-8x^4+8x^2}\, dx} \\
&= z_1 e^{-\frac{3\ln(x+1)}{4}-\frac{3\ln(x-1)}{4}-\frac{\ln(x)}{8}} \\
&= z_1 \left(\frac{1}{(x+1)^{3/4}(x-1)^{3/4}x^{1/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{1/4}(x^2-1)^{1/4}}{(x+1)^{3/4}(x-1)^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-26x^3+2x}{-8x^4+8x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x+1)}{2}-\frac{3\ln(x-1)}{2}-\frac{\ln(x)}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{3\ln(x+1)}{2}-\frac{3\ln(x-1)}{2}-\frac{\ln(x)}{4}}(x+1)^{3/2}(x-1)^{3/2}}{\sqrt{x}\,\sqrt{x^2-1}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/4}(x^2-1)^{1/4}}{(x+1)^{3/4}(x-1)^{3/4}}\right) + c_2\left(\frac{x^{1/4}(x^2-1)^{1/4}}{(x+1)^{3/4}(x-1)^{3/4}}\left(\int \frac{\mathrm{e}^{-\frac{3\ln(x+1)}{2}-\frac{3\ln(x-1)}{2}-\frac{\ln(x)}{4}}(x+1)^{3/2}(x-1)^{3/2}}{\sqrt{x}\,\sqrt{x^2-1}}dx
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.519.2 Maple step by step solution

Let's solve

$8x^2(-x^2+1)\left(\frac{d}{dx}y'\right)+2x(-13x^2+1)\,y'+(-9x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(9x^2-1)y}{8x^2(x^2-1)}-\frac{(13x^2-1)y'}{4x(x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(13x^2-1)y'}{4x(x^2-1)}+\frac{(9x^2-1)y}{8x^2(x^2-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{13x^2-1}{4x(x^2-1)},P_3(x)=\frac{9x^2-1}{8x^2(x^2-1)}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{3}{2}$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $8x^2(x^2-1)\left(\frac{d}{dx}y'\right)+2x(13x^2-1)\,y'+(9x^2-1)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(8u^4-32u^3+40u^2-16u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(26u^3-78u^2+76u-24)\left(\frac{d}{du}y(u)\right)+(9u^2-18u+8)$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..4$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-8a_0r(1+2r)\, u^{-1+r} + \left(-8a_1(1+r)\,(3+2r) + 4a_0(1+2r)\,(2+5r)\right) u^r + \left(-8a_2(2+r)\,(5+2r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$-8r(1+2r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{0, -\frac{1}{2}\right\}$$
- The coefficients of each power of u must be 0
$$[-8a_1(1+r)\,(3+2r) + 4a_0(1+2r)\,(2+5r) = 0, -8a_2(2+r)\,(5+2r) + 4a_1(3+2r)\,(7+5r) -$$
- Solve for the dependent coefficient(s)
$$\left\{a_1 = \frac{a_0(10r^2+9r+2)}{2(2r^2+5r+3)}, a_2 = \frac{a_0(34r^3+76r^2+41r+5)}{4(2r^3+11r^2+19r+10)}\right\}$$
- Each term in the series must be 0, giving the recursion relation
$$8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, k^2 + 2(8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, r + 18a_k - 7a_{k-2} + 9a_{k-1}$$
- Shift index using $k->k+2$
$$8(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\,(k+2)^2 + 2(8(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\, r + 18a_{k+2} - 7a_k + 9$$
- Recursion relation that defines series solution to ODE
$$a_{k+3} = \frac{8k^2a_k - 32k^2a_{k+1} + 40k^2a_{k+2} + 16kra_k - 64kra_{k+1} + 80kra_{k+2} + 8r^2a_k - 32r^2a_{k+1} + 40r^2a_{k+2} + 18ka_k - 110ka_{k+1} + 196ka_{k+}}{8(2k^2+4kr+2r^2+13k+13r+21)}$$
- Recursion relation for $r = 0$
$$a_{k+3} = \frac{8k^2a_k - 32k^2a_{k+1} + 40k^2a_{k+2} + 18ka_k - 110ka_{k+1} + 196ka_{k+2} + 9a_k - 96a_{k+1} + 240a_{k+2}}{8(2k^2+13k+21)}$$
- Solution for $r = 0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)}, a_1=\frac{a_0}{3}\right.$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)}, a_1=\right.$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}, a_1=\right.$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{1}{2}}, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}, a_1\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k(x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{1}{2}}\right), a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka}{8(2k^2+13k+21)}\right.$$

1.519.3 Maple trace

```
Methods for second order ODEs:
```

1.519.4 Maple dsolve solution

Solving time : `0.024` (sec)
Leaf size : 34

```
dsolve(8*x^2*(-x^2+1)*diff(diff(y(x),x),x)+2*x*(-13*x^2+1)*diff(y(x),x)+(-9*x^2+1)*y(x
       y(x),singsol=all)
```

$$y=\frac{x^{1/4}\left(\operatorname{LegendreQ}\left(-\frac{1}{8},\frac{1}{8},\sqrt{-x^2+1}\right)c_2\,x^{1/8}+c_1\right)}{\sqrt{x^2-1}}$$

1.519.5 Mathematica DSolve solution

Solving time : 0.126 (sec)
Leaf size : 47

```
DSolve[{8*x^2*(1-x^2)*D[y[x],{x,2}]+2*x*(1-13*x^2)*D[y[x],x]+(1-9*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[4]{x}\left(4c_2\sqrt[4]{x}\,\text{Hypergeometric2F1}\left(\frac{1}{8},\frac{1}{2},\frac{9}{8},x^2\right)+c_1\right)}{\sqrt{1-x^2}}$$

1.520 problem 536

Internal problem ID [8658]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 536
Date solved : `Monday, October 21, 2024 at 05:19:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2 \left(x^2+1\right) y''-2x\left(-x^2+2\right) y'+4y=0$$

1.520.1 Solved as second order ode using Kovacic algorithm

Time used: 0.375 (sec)

Writing the ode as

$$\left(x^4+x^2\right) y''+\left(2x^3-4x\right) y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 2x^3-4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 2}{(x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 2$$
$$t = (x^3 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 2}{(x^3 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 985: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{7i}{4(x-i)} - \frac{7i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2+2}{(x^3+x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}+\alpha_{c_3}^{-}\right) \\
&= 1-(1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} + (-)(0) \\
&= \frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} \\
&= \frac{x^2+2}{x^3+x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right)(0) + \left(\left(-\frac{2}{x^2} + \frac{1}{2(x-i)^2} + \frac{1}{2(x+i)^2}\right) + \left(\frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right) dx} \\
&= \frac{x^2}{\sqrt{x^2+1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3-4x}{x^4+x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+1\right)}{2}+2\ln(x)} \\
&= z_1 \left(\frac{x^2}{\left(x^2+1\right)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^4}{\left(x^2+1\right)^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-4x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln\left(x^2+1\right)+4\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{\left(3x^2+1\right)\left(x^2+1\right)^3 \mathrm{e}^{-3\ln\left(x^2+1\right)+4\ln(x)}}{3x^7}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^4}{\left(x^2+1\right)^2}\right) + c_2 \left(\frac{x^4}{\left(x^2+1\right)^2}\left(-\frac{\left(3x^2+1\right)\left(x^2+1\right)^3 \mathrm{e}^{-3\ln\left(x^2+1\right)+4\ln(x)}}{3x^7}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.520.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right) - 2x(-x^2+2)\,y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{4y}{x^2(x^2+1)} - \frac{2(x^2-2)\,y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2(x^2-2)\,y'}{x(x^2+1)} + \frac{4y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2(x^2-2)}{x(x^2+1)}, P_3(x) = \frac{4}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d}{dx}y'\right) + 2x(x^2-2)\,y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)\,(-4+r)\,x^r + a_1 r(-3+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)\,(k+r-4) + a_{k-2}(k-2+r)\,($$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 4\}$

- Each term must be 0

  $a_1 r(-3+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r-4) + a_{k-2}(k-2+r)) = 0$

- Shift index using $k- >k+2$

  $(k+r+1)\,(a_{k+2}(k-2+r) + a_k(k+r)) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r)}{k-2+r}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k(k+1)}{k-1}$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+1)}{k-1}, a_1 = 0\right]$$

- Recursion relation for $r = 4$

  $a_{k+2} = -\frac{a_k(k+4)}{k+2}$

- Solution for $r = 4$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+4}\right), a_{k+2}=-\frac{a_k(k+1)}{k-1}, a_1=0, b_{k+2}=-\frac{b_k(k+4)}{k+2}, b_1=0\right]$$

1.520.3 Maple trace

```
Methods for second order ODEs:
```

1.520.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 26

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-2*x*(-x^2+2)*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{x(c_1\,x^3+3c_2\,x^2+c_2)}{\left(x^2+1\right)^2}$$

1.520.5 Mathematica DSolve solution

Solving time : 0.078 (sec)
Leaf size : 35

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-2*x*(2-x^2)*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\frac{-3c_1x^4+3c_2x^3+c_2x}{3\left(x^2+1\right)^2}$$

1.521 problem 537

Internal problem ID [8659]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 537
Date solved : `Monday, October 21, 2024 at 05:19:18 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x\left(x^2+3\right)y''+\left(-x^2+2\right)y'-8xy=0$$

1.521.1 Solved as second order ode using Kovacic algorithm

Time used: 0.321 (sec)

Writing the ode as

$$\left(x^3+3x\right)y''+\left(-x^2+2\right)y'-8xy=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3+3x \\ B &= -x^2+2 \\ C &= -8x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 35x^4 + 74x^2 - 8 \\
t &= 4\left(x^3 + 3x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 987: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9x^2} + \frac{85}{144\left(x - i\sqrt{3}\right)^2} + \frac{85}{144\left(x + i\sqrt{3}\right)^2} - \frac{187i\sqrt{3}}{144\left(x - i\sqrt{3}\right)} + \frac{187i\sqrt{3}}{144\left(x + i\sqrt{3}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{85}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{12} \end{aligned}$$

For the pole at $x=-i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{3}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{85}{144}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{17}{12} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{5}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{35x^4+74x^2-8}{4\left(x^3+3x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{35x^4+74x^2-8}{4\left(x^3+3x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$
$i\sqrt{3}$	2	0	$\frac{17}{12}$	$-\frac{5}{12}$
$-i\sqrt{3}$	2	0	$\frac{17}{12}$	$-\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{7}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+} + \alpha_{c_3}^{+}\right)\\
&= \frac{7}{2} - \left(\frac{7}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3}\right) + (+)[\sqrt{r}]_\infty\\
&= \frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)} + (0)\\
&= \frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)}\\
&= \frac{2}{3x} + \frac{17x}{6x^2+18}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)}\right)(0) + \left(\left(-\frac{2}{3x^2} - \frac{17}{12\left(x-i\sqrt{3}\right)^2} - \frac{17}{12\left(x+i\sqrt{3}\right)^2}\right) + \left(\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{3x} + \frac{17}{12\left(x - i\sqrt{3}\right)} + \frac{17}{12\left(x + i\sqrt{3}\right)}\right) dx} \\
&= \left(x^2 + 3\right)^{17/12} x^{2/3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2+2}{x^3+3x} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{3} + \frac{5 \ln\left(x^2+3\right)}{12}} \\
&= z_1 \left(\frac{\left(x^2 + 3\right)^{5/12}}{x^{1/3}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{1/3} \left(x^2 + 3\right)^{11/6}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^3+3x} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{2 \ln(x)}{3} + \frac{5 \ln\left(x^2+3\right)}{6}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(-\frac{x^{1/3} \left(8x^4 + 44x^2 + 55\right) \mathrm{e}^{-\frac{2 \ln(x)}{3} + \frac{5 \ln\left(x^2+3\right)}{6}}}{55 \left(x^2 + 3\right)^{8/3}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{1/3} (x^2+3)^{11/6}\right) \\
&\quad + c_2 \left(x^{1/3} (x^2+3)^{11/6} \left(-\frac{x^{1/3}(8x^4+44x^2+55)\,\mathrm{e}^{-\frac{2\ln(x)}{3}+\frac{5\ln(x^2+3)}{6}}}{55\,(x^2+3)^{8/3}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.521.2 Maple step by step solution

Let's solve

$x(x^2+3)\left(\frac{d}{dx}y'\right)+(-x^2+2)\,y'-8xy=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{8y}{x^2+3}+\frac{(x^2-2)y'}{x(x^2+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{(x^2-2)y'}{x(x^2+3)}-\frac{8y}{x^2+3}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{x^2-2}{x(x^2+3)},P_3(x)=-\frac{8}{x^2+3}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{2}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x^2+3)\left(\frac{d}{dx}y'\right)+(-x^2+2)\,y'-8xy=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

$$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k->k-1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r) x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+3r) x^{-1+r} + a_1(1+r)(2+3r) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)(3k+2+3r) + a_{k-1}(k+r-$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$r(-1+3r) = 0$
- Values of r that satisfy the indicial equation
$r \in \left\{0, \frac{1}{3}\right\}$
- Each term must be 0
$a_1(1+r)(2+3r) = 0$
- Each term in the series must be 0, giving the recursion relation
$(k+r+1)\left(a_{k-1}(k-5+r) + 3\left(k+\frac{2}{3}+r\right) a_{k+1}\right) = 0$
- Shift index using $k->k+1$
$(k+r+2)\left(a_k(k+r-4) + 3\left(k+\frac{5}{3}+r\right) a_{k+2}\right) = 0$
- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k(k+r-4)}{3k+5+3r}$

- Recursion relation for $r = 0$; series terminates at $k = 4$

$a_{k+2} = -\frac{a_k(k-4)}{3k+5}$

- Solution for $r = 0$

$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k-4)}{3k+5}, 2a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{3}$

$a_{k+2} = -\frac{a_k\left(k-\frac{11}{3}\right)}{3k+6}$

- Solution for $r = \frac{1}{3}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k\left(k-\frac{11}{3}\right)}{3k+6}, 4a_1 = 0\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{a_k(k-4)}{3k+5}, 2a_1 = 0, b_{k+2} = -\frac{b_k\left(k-\frac{11}{3}\right)}{3k+6}, 4b_1 = 0\right]$

1.521.3 Maple trace

```
Methods for second order ODEs:
```

1.521.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 32

```
dsolve(x*(x^2+3)*diff(diff(y(x),x),x)+(-x^2+2)*diff(y(x),x)-8*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x^{1/3}\left(x^2+3\right)^{11/6} + \frac{c_2(8x^4+44x^2+55)}{8}$$

1.521.5 Mathematica DSolve solution

Solving time : 0.139 (sec)
Leaf size : 41

```
DSolve[{x*(3+x^2)*D[y[x],{x,2}]+(2-x^2)*D[y[x],x]-8*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(x^2+3\right)^{11/6} - \frac{1}{55} c_2 \left(8x^4+44x^2+55\right)$$

1.522 problem 538

Internal problem ID [8660]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 538
Date solved : `Monday, October 21, 2024 at 05:19:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(-x^2+1\right)y''+x\left(-19x^2+7\right)y'-\left(14x^2+1\right)y=0$$

1.522.1 Solved as second order ode using Kovacic algorithm

Time used: 0.347 (sec)

Writing the ode as

$$\left(-4x^4+4x^2\right)y''+\left(-19x^3+7x\right)y'+\left(-14x^2-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-4x^4+4x^2\\B&=-19x^3+7x\\C&=-14x^2-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-15x^4 - 42x^2 + 9}{64\left(x^3 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -15x^4 - 42x^2 + 9 \\ t &= 64\left(x^3 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-15x^4 - 42x^2 + 9}{64\left(x^3 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 989: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 4 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(x+1)^2} - \frac{3}{16\,(x-1)^2} + \frac{9}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8}
\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x=-1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4}\\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r=\frac{s}{t}=\frac{-15x^4-42x^2+9}{64\left(x^3-x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{8}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-15x^4-42x^2+9}{64\left(x^3-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{8}$	$-\frac{1}{8}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{8}$	$\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{3}{8} - \left(\frac{3}{8}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} + (-)\,(0) \\ &= -\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} \\ &= \frac{3x^2+1}{8x^3-8x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right)(0) + \left(\left(\frac{1}{8x^2} - \frac{1}{4\,(x-1)^2} - \frac{1}{4\,(x+1)^2}\right) + \left(-\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right) dx} \\ &= \frac{(x+1)^{1/4}\,(x-1)^{1/4}}{x^{1/8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-19x^3+7x}{-4x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x+1)}{4}-\frac{7\ln(x)}{8}-\frac{3\ln(x-1)}{4}} \\
&= z_1 \left(\frac{1}{(x+1)^{3/4}\, x^{7/8}\,(x-1)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x^2-1)^{1/4}}{(x+1)^{3/4}\, x\,(x-1)^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-19x^3+7x}{-4x^4+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x+1)}{2}-\frac{7\ln(x)}{4}-\frac{3\ln(x-1)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{e^{-\frac{3\ln(x+1)}{2}-\frac{7\ln(x)}{4}-\frac{3\ln(x-1)}{2}}(x+1)^{3/2}\,x^2(x-1)^{3/2}}{\sqrt{x^2-1}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x^2-1)^{1/4}}{(x+1)^{3/4}\, x\,(x-1)^{3/4}}\right) + c_2\left(\frac{(x^2-1)^{1/4}}{(x+1)^{3/4}\, x\,(x-1)^{3/4}}\left(\int \frac{e^{-\frac{3\ln(x+1)}{2}-\frac{7\ln(x)}{4}-\frac{3\ln(x-1)}{2}}(x+1)^{3/2}\,x^2(x-
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.522.2 Maple step by step solution

Let's solve

$4x^2(-x^2+1)\left(\frac{d}{dx}y'\right)+x(-19x^2+7)\,y'-(14x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(14x^2+1)y}{4x^2(x^2-1)}-\frac{(19x^2-7)y'}{4x(x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(19x^2-7)y'}{4x(x^2-1)}+\frac{(14x^2+1)y}{4x^2(x^2-1)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{19x^2-7}{4x(x^2-1)},P_3(x)=\frac{14x^2+1}{4x^2(x^2-1)}\right]$

- ○ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=\frac{3}{2}$

- ○ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$4x^2(x^2-1)\left(\frac{d}{dx}y'\right)+x(19x^2-7)\,y'+(14x^2+1)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(4u^4-16u^3+20u^2-8u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(19u^3-57u^2+50u-12)\left(\frac{d}{du}y(u)\right)+(14u^2-28u+15)$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ○ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..4$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-4a_0 r(1+2r)\, u^{-1+r} + \left(-4a_1(1+r)\,(3+2r) + 5a_0(4r^2+6r+3)\right) u^r + \left(-4a_2(2+r)\,(5+2r) +\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-4r(1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{1}{2}\right\}$
- The coefficients of each power of u must be 0
  $[-4a_1(1+r)\,(3+2r) + 5a_0(4r^2+6r+3) = 0, -4a_2(2+r)\,(5+2r) + 5a_1(4r^2+14r+13) - a_0$
- Solve for the dependent coefficient(s)
  $\left\{a_1 = \frac{5a_0(4r^2+6r+3)}{4(2r^2+5r+3)}, a_2 = \frac{a_0(272r^4+1352r^3+2464r^2+1948r+639)}{16(4r^4+28r^3+71r^2+77r+30)}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $4(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, k^2 + (8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, r + 30a_k - a_{k-2} - 9a_{k-1} -$
- Shift index using $k->k+2$
  $4(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\,(k+2)^2 + (8(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\, r + 30a_{k+2} - a_k - 9a_k$
- Recursion relation that defines series solution to ODE
  $a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 8kra_k - 32kra_{k+1} + 40kra_{k+2} + 4r^2a_k - 16r^2a_{k+1} + 20r^2a_{k+2} + 15ka_k - 73ka_{k+1} + 110ka_{k+2}}{4(2k^2+4kr+2r^2+13k+13r+21)}$
- Recursion relation for $r = 0$
  $a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 15ka_k - 73ka_{k+1} + 110ka_{k+2} + 14a_k - 85a_{k+1} + 155a_{k+2}}{4(2k^2+13k+21)}$
- Solution for $r = 0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+3}=\frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}+14a_k-85a_{k+1}+155a_{k+2}}{4(2k^2+13k+21)}, a_1=\frac{5a}{4}\right.$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+3}=\frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}+14a_k-85a_{k+1}+155a_{k+2}}{4(2k^2+13k+21)}, a_1=\right.$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+3}=\frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+2}}{4(2k^2+11k+15)}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+3}=\frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+2}}{4(2k^2+11k+15)}, a_1=\right.$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{1}{2}}, a_{k+3}=\frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+2}}{4(2k^2+11k+15)}, a\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k(x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{1}{2}}\right), a_{k+3}=\frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka}{4(2k^2+13k+21)}\right.$$

1.522.3 Maple trace

```
Methods for second order ODEs:
```

1.522.4 Maple dsolve solution

Solving time : `0.025` (sec)
Leaf size : 44

```
dsolve(4*x^2*(-x^2+1)*diff(diff(y(x),x),x)+x*(-19*x^2+7)*diff(y(x),x)-(14*x^2+1)*y(x)
       y(x),singsol=all)
```

$$y=\frac{c_1 \operatorname{LegendreP}\left(-\frac{3}{8},\frac{5}{8},\sqrt{-x^2+1}\right)+c_2 \operatorname{LegendreQ}\left(-\frac{3}{8},\frac{5}{8},\sqrt{-x^2+1}\right)}{x^{3/8}\sqrt{x^2-1}}$$

1.522.5 Mathematica DSolve solution

Solving time : 0.121 (sec)
Leaf size : 50

```
DSolve[{4*x^2*(1-x^2)*D[y[x],{x,2}]+x*(7-19*x^2)*D[y[x],x]-(1+14*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_2x^{5/4}\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{5}{8},\frac{13}{8},x^2\right)+5c_1}{5x\sqrt{1-x^2}}$$

1.523 problem 539

Internal problem ID [8661]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 539
Date solved : `Monday, October 21, 2024 at 05:19:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2\left(-x^2+2\right)y''+x\left(-11x^2+1\right)y'+\left(-5x^2+1\right)y=0$$

1.523.1 Solved as second order ode using Kovacic algorithm

Time used: 0.390 (sec)

Writing the ode as

$$\left(-3x^4+6x^2\right)y''+\left(-11x^3+x\right)y'+\left(-5x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -3x^4+6x^2 \\ B &= -11x^3+x \\ C &= -5x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5x^4 - 4x^2 - 35}{36 (x^3 - 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5x^4 - 4x^2 - 35$$
$$t = 36(x^3 - 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-5x^4 - 4x^2 - 35}{36 (x^3 - 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 991: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144x^2} - \frac{7}{64\left(x - \sqrt{2}\right)^2} - \frac{7}{64\left(x + \sqrt{2}\right)^2} + \frac{31\sqrt{2}}{384\left(x - \sqrt{2}\right)} - \frac{31\sqrt{2}}{384\left(x + \sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{5}{12} \end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8}\\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-5x^4 - 4x^2 - 35}{36\left(x^3 - 2x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6}\\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-5x^4 - 4x^2 - 35}{36\left(x^3 - 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$
$\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{5}{6} - \left(\frac{5}{6}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}} + (0) \\
&= \frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}} \\
&= \frac{5x^2-7}{6x^3-12x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}}\right)(0) + \left(\left(-\frac{7}{12x^2} - \frac{1}{8\left(x-\sqrt{2}\right)^2} - \frac{1}{8\left(x+\sqrt{2}\right)^2}\right) + \left(\frac{7}{12x} + \frac{}{8x}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}}\right) dx} \\
&= \left(x+\sqrt{2}\right)^{1/8} \left(x-\sqrt{2}\right)^{1/8} x^{7/12}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-11x^3+x}{-3x^4+6x^2}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(x^2-2\right)}{8}-\frac{\ln(x)}{12}} \\
&= z_1\left(\frac{1}{\left(x^2-2\right)^{7/8} x^{1/12}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x}}{\left(x^2-2\right)^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-11x^3+x}{-3x^4+6x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(x^2-2\right)}{4}-\frac{\ln(x)}{6}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{7\ln\left(x^2-2\right)}{4}-\frac{\ln(x)}{6}}\left(x^2-2\right)^{3/2}}{x}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{\left(x^2-2\right)^{3/4}}\right) + c_2\left(\frac{\sqrt{x}}{\left(x^2-2\right)^{3/4}}\left(\int \frac{\mathrm{e}^{-\frac{7\ln\left(x^2-2\right)}{4}-\frac{\ln(x)}{6}}\left(x^2-2\right)^{3/2}}{x}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.523.2 Maple step by step solution

Let's solve

$3x^2(-x^2+2)\left(\frac{d}{dx}y'\right)+x(-11x^2+1)\,y'+(-5x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(5x^2-1)y}{3x^2(x^2-2)}-\frac{(11x^2-1)y'}{3x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(11x^2-1)y'}{3x(x^2-2)}+\frac{(5x^2-1)y}{3x^2(x^2-2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{11x^2-1}{3x(x^2-2)},P_3(x)=\frac{5x^2-1}{3x^2(x^2-2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=\frac{1}{6}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=\frac{1}{6}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

$3x^2(x^2-2)\left(\frac{d}{dx}y'\right)+x(11x^2-1)\,y'+(5x^2-1)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+3r)\,(-1+2r)\, x^r - a_1(2+3r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (-a_k(3k+3r-1)\,(2k+2r-1) +\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-1+3r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$$

- Each term must be 0

$$-a_1(2+3r)\,(1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-6\left(k+r-\frac{1}{3}\right)\left(\frac{(-k-r+1)a_{k-2}}{2} + a_k\left(k+r-\frac{1}{2}\right)\right) = 0$$

- Shift index using $k- >k+2$

$$-6\left(k+\frac{5}{3}+r\right)\left(\frac{(-k-1-r)a_k}{2} + a_{k+2}\left(k+\frac{3}{2}+r\right)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{(k+r+1)a_k}{2k+3+2r}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = \frac{\left(k+\frac{4}{3}\right)a_k}{2k+\frac{11}{3}}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{\left(k+\frac{4}{3}\right)a_k}{2k+\frac{11}{3}}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}, a_1 = 0, b_{k+2} = \frac{\left(k+\frac{4}{3}\right)b_k}{2k+\frac{11}{3}}, b_1 = 0\right]$

1.523.3 Maple trace

```
Methods for second order ODEs:
```

1.523.4 Maple dsolve solution

Solving time : `0.020` (sec)
Leaf size : 35

```
dsolve(3*x^2*(-x^2+2)*diff(diff(y(x),x),x)+x*(-11*x^2+1)*diff(y(x),x)+(-5*x^2+1)*y(x)
       y(x),singsol=all)
```

$$y = \frac{c_1\sqrt{x}}{\left(-2x^2+4\right)^{3/4}} + c_2\, x^{1/3} \operatorname{hypergeom}\left(\left[\frac{2}{3}, 1\right], \left[\frac{11}{12}\right], \frac{x^2}{2}\right)$$

1.523.5 Mathematica DSolve solution

Solving time : `0.152` (sec)
Leaf size : 57

```
DSolve[{3*x^2*(2-x^2)*D[y[x],{x,2}]+x*(1-11*x^2)*D[y[x],x]+(1-5*x^2)*y[x]==0,{}},
      y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1\sqrt{x} - 3\ 2^{3/4} c_2 \sqrt[3]{x}\,\text{Hypergeometric2F1}\left(-\frac{1}{12}, \frac{1}{4}, \frac{11}{12}, \frac{x^2}{2}\right)}{\left(2-x^2\right)^{3/4}}$$

1.524 problem 540

Internal problem ID [8662]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 540
Date solved : `Monday, October 21, 2024 at 05:19:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(x^2+2)\,y'' - x(-7x^2+12)\,y' + (3x^2+7)\,y = 0$$

1.524.1 Solved as second order ode using Kovacic algorithm

Time used: 0.369 (sec)

Writing the ode as

$$(2x^4+4x^2)\,y'' + (7x^3-12x)\,y' + (3x^2+7)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+4x^2 \\ B &= 7x^3-12x \\ C &= 3x^2+7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^4 - 72x^2 + 128$$
$$t = 16\left(x^3 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 993: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 4 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + \frac{65}{64\left(x - i\sqrt{2}\right)^2} + \frac{65}{64\left(x + i\sqrt{2}\right)^2} + \frac{135i\sqrt{2}}{128\left(x - i\sqrt{2}\right)} - \frac{135i\sqrt{2}}{128\left(x + i\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{13}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{8}
\end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{8} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1
$i\sqrt{2}$	2	0	$\frac{13}{8}$	$-\frac{5}{8}$
$-i\sqrt{2}$	2	0	$\frac{13}{8}$	$-\frac{5}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{2}{x} - \frac{5}{8\left(x - i\sqrt{2}\right)} - \frac{5}{8\left(x + i\sqrt{2}\right)} + (0) \\
&= \frac{2}{x} - \frac{5}{8\left(x - i\sqrt{2}\right)} - \frac{5}{8\left(x + i\sqrt{2}\right)} \\
&= \frac{2}{x} - \frac{5x}{4x^2 + 8}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{2}{x} - \frac{5}{8\left(x - i\sqrt{2}\right)} - \frac{5}{8\left(x + i\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{2}{x^2} + \frac{5}{8\left(x - i\sqrt{2}\right)^2} + \frac{5}{8\left(x + i\sqrt{2}\right)^2}\right) + \left(\frac{2}{x} - \frac{}{8\left(\right.}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{x} - \frac{5}{8\left(x - i\sqrt{2}\right)} - \frac{5}{8\left(x + i\sqrt{2}\right)}\right) dx} \\
&= \frac{x^2}{\left(x^2+2\right)^{5/8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{7x^3 - 12x}{2x^4 + 4x^2} \, dx} \\
&= z_1 e^{-\frac{13 \ln\left(x^2+2\right)}{8} + \frac{3 \ln(x)}{2}} \\
&= z_1 \left(\frac{x^{3/2}}{\left(x^2+2\right)^{13/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{7/2}}{\left(x^2+2\right)^{9/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3 - 12x}{2x^4 + 4x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{13 \ln\left(x^2+2\right)}{4} + 3 \ln(x)}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{13 \ln\left(x^2+2\right)}{4} + 3 \ln(x)} \left(x^2+2\right)^{9/2}}{x^7} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{7/2}}{(x^2+2)^{9/4}}\right) + c_2 \left(\frac{x^{7/2}}{(x^2+2)^{9/4}} \left(\int \frac{e^{-\frac{13\ln(x^2+2)}{4}+3\ln(x)} (x^2+2)^{9/2}}{x^7} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.524.2 Maple step by step solution

Let's solve
$2x^2(x^2+2)\left(\frac{d}{dx}y'\right) - x(-7x^2+12)\,y' + (3x^2+7)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{(3x^2+7)y}{2x^2(x^2+2)} - \frac{(7x^2-12)y'}{2x(x^2+2)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + \frac{(7x^2-12)y'}{2x(x^2+2)} + \frac{(3x^2+7)y}{2x^2(x^2+2)} = 0$
- □ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{7x^2-12}{2x(x^2+2)}, P_3(x) = \frac{3x^2+7}{2x^2(x^2+2)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $(x \cdot P_2(x))\Big|_{x=0} = -3$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{7}{4}$
  - $x = 0$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $2x^2(x^2+2)\left(\frac{d}{dx}y'\right) + x(7x^2-12)\,y' + (3x^2+7)\,y = 0$
- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)\,(-7+2r)\,x^r + a_1(1+2r)\,(-5+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-7) + a\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+2r)\,(-7+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{2}, \frac{7}{2}\right\}$$

- Each term must be 0

  $$a_1(1+2r)\,(-5+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$4\left(k+r-\frac{1}{2}\right)\left(\frac{a_{k-2}(k+r-1)}{2} + a_k\left(k+r-\frac{7}{2}\right)\right) = 0$$

- Shift index using $k- >k + 2$

$$4\left(k+\frac{3}{2}+r\right)\left(\frac{a_k(k+r+1)}{2}+a_{k+2}\left(k-\frac{3}{2}+r\right)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a_k(k+r+1)}{2k-3+2r}$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}, a_1=0\right]$$

- Recursion relation for $r=\frac{7}{2}$

$$a_{k+2}=-\frac{a_k\left(k+\frac{9}{2}\right)}{2k+4}$$

- Solution for $r=\frac{7}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{7}{2}}, a_{k+2}=-\frac{a_k\left(k+\frac{9}{2}\right)}{2k+4}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{7}{2}}\right), a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}, a_1=0, b_{k+2}=-\frac{b_k\left(k+\frac{9}{2}\right)}{2k+4}, b_1=0\right]$$

1.524.3 Maple trace

```
Methods for second order ODEs:
```

1.524.4 Maple dsolve solution

Solving time : `0.020` (sec)
Leaf size : 35

```
dsolve(2*x^2*(x^2+2)*diff(diff(y(x),x),x)-x*(-7*x^2+12)*diff(y(x),x)+(3*x^2+7)*y(x) =
       y(x),singsol=all)
```

$$y=\frac{c_1\,x^{7/2}}{\left(2x^2+4\right)^{9/4}}+c_2\sqrt{x}\ \mathrm{hypergeom}\left(\left[\frac{3}{4},1\right],\left[-\frac{1}{2}\right],-\frac{x^2}{2}\right)$$

1.524.5 Mathematica DSolve solution

Solving time : 0.163 (sec)
Leaf size : 57

```
DSolve[{2*x^2*(2+x^2)*D[y[x],{x,2}]-x*(12-7*x^2)*D[y[x],x]+(7+3*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}\left(3c_1x^3 - 2\sqrt[4]{2}c_2\,\text{Hypergeometric2F1}\left(-\frac{3}{2}, -\frac{5}{4}, -\frac{1}{2}, -\frac{x^2}{2}\right)\right)}{3\left(x^2+2\right)^{9/4}}$$

1.525 problem 541

Internal problem ID [8663]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 541
Date solved : `Monday, October 21, 2024 at 05:19:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+2\right)y''+x\left(7x^2+4\right)y'-\left(-3x^2+1\right)y=0$$

1.525.1 Solved as second order ode using Kovacic algorithm

Time used: 0.349 (sec)

Writing the ode as

$$\left(2x^4+4x^2\right)y''+\left(7x^3+4x\right)y'+\left(3x^2-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+4x^2 \\ B &= 7x^3+4x \\ C &= 3x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 + 24}{16\left(x^2 + 2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 + 24$$
$$t = 16\left(x^2 + 2\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 + 24}{16\left(x^2 + 2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 995: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+2)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{15}{64\left(x - i\sqrt{2}\right)^2} - \frac{15}{64\left(x + i\sqrt{2}\right)^2} - \frac{9i\sqrt{2}}{128\left(x - i\sqrt{2}\right)} + \frac{9i\sqrt{2}}{128\left(x + i\sqrt{2}\right)}
$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}
$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 24}{16 \left(x^2 + 2\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 + 24}{16 \left(x^2 + 2\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$i\sqrt{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$
$-i\sqrt{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)} + (0) \\
&= \frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)} \\
&= \frac{3x}{4x^2 + 8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{3}{8\left(x - i\sqrt{2}\right)^2} - \frac{3}{8\left(x + i\sqrt{2}\right)^2}\right) + \left(\frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{}{8(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)}\right) dx} \\
&= \left(-x^2 - 2\right)^{3/8}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^3+4x}{2x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}-\frac{5\ln\left(x^2+2\right)}{8}} \\
&= z_1 \left(\frac{1}{\sqrt{x}\,\left(x^2+2\right)^{5/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(-1)^{3/8}}{\sqrt{x}\,\left(x^2+2\right)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3+4x}{2x^4+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)-\frac{5\ln\left(x^2+2\right)}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int -\mathrm{e}^{-\ln(x)-\frac{5\ln\left(x^2+2\right)}{4}} x\sqrt{x^2+2}\,(-1)^{1/4}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(-1)^{3/8}}{\sqrt{x}\,\left(x^2+2\right)^{1/4}}\right) + c_2 \left(\frac{(-1)^{3/8}}{\sqrt{x}\,\left(x^2+2\right)^{1/4}}\left(\int -\mathrm{e}^{-\ln(x)-\frac{5\ln\left(x^2+2\right)}{4}} x\sqrt{x^2+2}\,(-1)^{1/4}\,dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.525.2 Maple step by step solution

Let's solve

$2x^2(x^2+2)\left(\frac{d}{dx}y'\right)+x(7x^2+4)\,y'-(-3x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(3x^2-1)y}{2x^2(x^2+2)}-\frac{(7x^2+4)y'}{2x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(7x^2+4)y'}{2x(x^2+2)}+\frac{(3x^2-1)y}{2x^2(x^2+2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{7x^2+4}{2x(x^2+2)},P_3(x)=\frac{3x^2-1}{2x^2(x^2+2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=-\frac{1}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(x^2+2)\left(\frac{d}{dx}y'\right)+x(7x^2+4)\,y'+(3x^2-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(\frac{a_{k-2}(k+r-1)}{2} + a_k\left(k+r+\frac{1}{2}\right)\right) = 0$

- Shift index using $k- >k+2$

  $4\left(k+\frac{3}{2}+r\right)\left(\frac{a_k(k+r+1)}{2} + a_{k+2}\left(k+\frac{5}{2}+r\right)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+1)}{2k+5+2r}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+6}$
- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+6}, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0, b_{k+2} = -\frac{b_k\left(k+\frac{3}{2}\right)}{2k+6}, b_1 = 0\right]$$

1.525.3 Maple trace

```
Methods for second order ODEs:
```

1.525.4 Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 35

```
dsolve(2*x^2*(x^2+2)*diff(diff(y(x),x),x)+x*(7*x^2+4)*diff(y(x),x)-(-3*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \operatorname{LegendreQ}\left(-\frac{1}{4}, \frac{1}{4}, \frac{i\sqrt{2}\,x}{2}\right)(x^2+2)^{1/8} + c_1}{\sqrt{x}\,(x^2+2)^{1/4}}$$

1.525.5 Mathematica DSolve solution

Solving time : 0.109 (sec)
Leaf size : 68

```
DSolve[{2*x^2*(2+x^2)*D[y[x],{x,2}]+x*(4+7*x^2)*D[y[x],x]-(1-3*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 \sqrt[8]{x^2+2}\,\mathrm{Gamma}\left(\frac{3}{4}\right) Q_{-\frac{1}{4}}^{\frac{1}{4}}\left(\frac{ix}{\sqrt{2}}\right) + 2^{3/8} c_1}{\sqrt{x}\sqrt[4]{x^2+2}\,\mathrm{Gamma}\left(\frac{3}{4}\right)}$$

1.526 problem 542

Internal problem ID [8664]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 542
Date solved : `Monday, October 21, 2024 at 05:19:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(2x^2+1\right)y''+5x\left(6x^2+1\right)y'-\left(-40x^2+2\right)y=0$$

1.526.1 Solved as second order ode using Kovacic algorithm

Time used: 0.565 (sec)

Writing the ode as

$$\left(4x^4+2x^2\right)y''+\left(30x^3+5x\right)y'+\left(40x^2-2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^4+2x^2\\B&=30x^3+5x\\C&=40x^2-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{20x^4 + 12x^2 + 21}{16\left(2x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 20x^4 + 12x^2 + 21 \\ t &= 16\left(2x^3 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{20x^4 + 12x^2 + 21}{16\left(2x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 997: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16x^2} + \frac{5}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{5}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} + \frac{13i\sqrt{2}}{16\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{13i\sqrt{2}}{16\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{20x^4 + 12x^2 + 21}{16\left(2x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{20x^4 + 12x^2 + 21}{16\left(2x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{5}{4} - \left(\frac{5}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{4x} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)} + (0) \\
&= \frac{7}{4x} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)} \\
&= \frac{10x^2+7}{8x^3+4x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{7}{4x} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)}\right)(0) + \left(\left(-\frac{7}{4x^2} + \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)^2} + \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{7}{4x}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \mathrm{e}^{\int \left(\frac{7}{4x} - \frac{1}{4\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x + \frac{i\sqrt{2}}{2}\right)}\right) dx} \\
&= \frac{2^{3/4} x^{7/4}}{2 \left(2x^2+1\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \,dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{30x^3+5x}{4x^4+2x^2} \,dx} \\
&= z_1 e^{-\frac{5 \ln\left(x\left(2x^2+1\right)\right)}{4}} \\
&= z_1 \left(\frac{1}{\left(2x^3+x\right)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{3/4} x^{3/4}}{2 \left(2x^2+1\right)^{5/4} \left(2x^3+x\right)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \,dx}}{y_1^2} \,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{30x^3+5x}{4x^4+2x^2} \,dx}}{\left(y_1\right)^2} \,dx \\
&= y_1 \int \frac{e^{-\frac{5 \ln\left(2x^3+x\right)}{2}}}{\left(y_1\right)^2} \,dx \\
&= y_1 \left(\int \frac{\left(2x^2+1\right)^{5/2} \sqrt{2}}{\left(2x^3+x\right)^2 x^{3/2}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{2^{3/4} x^{3/4}}{2\left(2x^2+1\right)^{5/4}\left(2x^3+x\right)^{1/4}}\right) + c_2 \left(\frac{2^{3/4} x^{3/4}}{2\left(2x^2+1\right)^{5/4}\left(2x^3+x\right)^{1/4}}\left(\int \frac{\left(2x^2+1\right)^{5/2}\sqrt{2}}{\left(2x^3+x\right)^2 x^{3/2}} dx\right)\right)$$

Will add steps showing solving for IC soon.

1.526.2 Maple step by step solution

Let's solve

$2x^2(2x^2+1)\left(\frac{d}{dx}y'\right) + 5x(6x^2+1)\,y' - (-40x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(20x^2-1)y}{x^2(2x^2+1)} - \frac{5(6x^2+1)y'}{2x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{5(6x^2+1)y'}{2x(2x^2+1)} + \frac{(20x^2-1)y}{x^2(2x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{5(6x^2+1)}{2x(2x^2+1)}, P_3(x) = \frac{20x^2-1}{x^2(2x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -1$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2(2x^2+1)\left(\frac{d}{dx}y'\right) + 5x(6x^2+1)\,y' + (40x^2-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(-1+2r)\, x^r + a_1(3+r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(2k+2r-1) + 2a_{k-2}(k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-2, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $2\left(a_{k-2}(2k+1+2r) + a_k\left(k+r-\frac{1}{2}\right)\right)(k+r+2) = 0$

- Shift index using $k->k+2$

  $2\left(a_k(2k+2r+5) + a_{k+2}\left(k+\frac{3}{2}+r\right)\right)(k+r+4) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k(2k+2r+5)}{2k+3+2r}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{2a_k(2k+1)}{2k-1}$
- Solution for $r = -2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2a_k(2k+1)}{2k-1}, a_1 = 0\right]$$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{2a_k(2k+6)}{2k+4}$
- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{2a_k(2k+6)}{2k+4}, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{2a_k(2k+1)}{2k-1}, a_1 = 0, b_{k+2} = -\frac{2b_k(2k+6)}{2k+4}, b_1 = 0\right]$$

1.526.3 Maple trace

```
Methods for second order ODEs:
```

1.526.4 Maple dsolve solution

Solving time : `0.020` (sec)
Leaf size : 35

```
dsolve(2*x^2*(2*x^2+1)*diff(diff(y(x),x),x)+5*x*(6*x^2+1)*diff(y(x),x)-(-40*x^2+2)*y(x
       y(x),singsol=all)
```

$$y = \frac{c_1\sqrt{x}}{(2x^2+1)^{3/2}} + \frac{c_2 \,\text{hypergeom}\left(\left[\frac{1}{4}, 1\right], \left[-\frac{1}{4}\right], -2x^2\right)}{x^2}$$

1.526.5 Mathematica DSolve solution

Solving time : 0.159 (sec)
Leaf size : 52

```
DSolve[{2*x^2*(1+2*x^2)*D[y[x],{x,2}]+5*x*(1+6*x^2)*D[y[x],x]-(2-40*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{5c_1x^{5/2} - 2c_2 \operatorname{Hypergeometric2F1}\left(-\frac{5}{4}, -\frac{1}{2}, -\frac{1}{4}, -2x^2\right)}{5x^2\left(2x^2+1\right)^{3/2}}$$

1.527 problem 543

Internal problem ID [8665]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 543
Date solved : `Monday, October 21, 2024 at 05:19:26 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x\left(x^2+1\right)y''+\left(7x^2+4\right)y'+8xy=0$$

1.527.1 Solved as second order ode using Kovacic algorithm

Time used: 0.308 (sec)

Writing the ode as

$$\left(x^3+x\right)y''+\left(7x^2+4\right)y'+8xy=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3+x \\ B &= 7x^2+4 \\ C &= 8x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^4 + 14x^2 + 8 \\ t &= 4\left(x^3 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 999: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(x-i)^2} - \frac{3}{16(x+i)^2} + \frac{7i}{16(x-i)} - \frac{7i}{16(x+i)} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1
i	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-i$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} + (-)(0) \\
&= -\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} \\
&= -\frac{1}{x} + \frac{x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)(0) + \left(\left(\frac{1}{x^2} - \frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2}\right) + \left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)dx} \\
&= \frac{(x^2+1)^{1/4}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^2+4}{x^3+x}\,dx} \\
&= z_1 e^{-2\ln(x)-\frac{3\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{1}{x^2\left(x^2+1\right)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^3\sqrt{x^2+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2+4}{x^3+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x)-\frac{3\ln\left(x^2+1\right)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^5}{\sqrt{x^2+1}} - \frac{x^3}{3\sqrt{x^2+1}} + \frac{4x^7}{\sqrt{x^2+1}} + \frac{8x^9}{3\sqrt{x^2+1}} + \frac{x\sqrt{x^2+1}}{2} - \frac{\operatorname{arcsinh}(x)}{2} \right. \\
&\qquad \left. + \frac{x^3\sqrt{x^2+1}}{3} - \frac{4x^5\sqrt{x^2+1}}{3} - \frac{8x^7\sqrt{x^2+1}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^3\sqrt{x^2+1}}\right) + c_2 \left(\frac{1}{x^3\sqrt{x^2+1}} \left(\frac{x^5}{\sqrt{x^2+1}} - \frac{x^3}{3\sqrt{x^2+1}} + \frac{4x^7}{\sqrt{x^2+1}} + \frac{8x^9}{3\sqrt{x^2+1}} \right.\right. \\
&\qquad \left.\left. + \frac{x\sqrt{x^2+1}}{2} - \frac{\operatorname{arcsinh}(x)}{2} + \frac{x^3\sqrt{x^2+1}}{3} - \frac{4x^5\sqrt{x^2+1}}{3} - \frac{8x^7\sqrt{x^2+1}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.527.2 Maple step by step solution

Let's solve

$x(x^2+1)\left(\frac{d}{dx}y'\right)+(7x^2+4)\,y'+8xy=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{8y}{x^2+1}-\frac{(7x^2+4)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(7x^2+4)y'}{x(x^2+1)}+\frac{8y}{x^2+1}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{7x^2+4}{x(x^2+1)}, P_3(x)=\frac{8}{x^2+1}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=4$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x^2+1)\left(\frac{d}{dx}y'\right)+(7x^2+4)\,y'+8xy=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x\cdot y$ to series expansion

  $x\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+1}$

  ◦ Shift index using $k->k-1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx} y'\right)$ to series expansion for $m = 1..3$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(3+r)\, x^{-1+r} + a_1(1+r)\,(4+r)\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)\,(k+r+4) + a_{k-1}(k+r+3)\,(k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-3, 0\}$$

- Each term must be 0

$$a_1(1+r)\,(4+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r+1)\,(a_{k+1}(k+r+4) + a_{k-1}(k+r+3)) = 0$$

- Shift index using $k->k+1$

$$(k+r+2)\,(a_{k+2}(k+5+r) + a_k(k+r+4)) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+4)}{k+5+r}$$

- Recursion relation for $r = -3$

$$a_{k+2} = -\frac{a_k(k+1)}{k+2}$$

- Solution for $r = -3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a_k(k+1)}{k+2}, -2a_1 = 0\right]$$

- Recursion relation for $r = 0$

$a_{k+2} = -\frac{a_k(k+4)}{k+5}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{k+5}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k(k+1)}{k+2}, -2a_1 = 0, b_{k+2} = -\frac{b_k(k+4)}{k+5}, 4b_1 = 0\right]$$

1.527.3 Maple trace

```
Methods for second order ODEs:
```

1.527.4 Maple dsolve solution

Solving time : `0.016` (sec)
Leaf size : 32

```
dsolve(x*(x^2+1)*diff(diff(y(x),x),x)+(7*x^2+4)*diff(y(x),x)+8*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-\sqrt{x^2+1}\, c_2 x + \operatorname{arcsinh}(x)\, c_2 + c_1}{x^3\sqrt{x^2+1}}$$

1.527.5 Mathematica DSolve solution

Solving time : `0.121` (sec)
Leaf size : 55

```
DSolve[{x*(1+x^2)*D[y[x],{x,2}]+(4+7*x^2)*D[y[x],x]+8*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-c_2 \operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right) + c_2 x\sqrt{x^2+1} + 2c_1}{2x^3\sqrt{x^2+1}}$$

1.528 problem 544

Internal problem ID [8666]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 544
Date solved : `Monday, October 21, 2024 at 05:19:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(x^2+1)\,y''+x(8x^2+3)\,y'-(-4x^2+3)\,y=0$$

1.528.1 Solved as second order ode using Kovacic algorithm

Time used: 0.349 (sec)

Writing the ode as

$$(2x^4+2x^2)\,y''+(8x^3+3x)\,y'+(4x^2-3)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+2x^2 \\ B &= 8x^3+3x \\ C &= 4x^2-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{36x^2 + 21}{16\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 36x^2 + 21$$
$$t = 16\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{36x^2 + 21}{16\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1001: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16x^2} - \frac{15}{64(x-i)^2} - \frac{15}{64(x+i)^2} + \frac{27i}{64(x-i)} - \frac{27i}{64(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8} \end{aligned}$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{15}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{8} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{3}{8}
\end{aligned}
$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{36x^2+21}{16\left(x^3+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
i	2	0	$\frac{5}{8}$	$\frac{3}{8}$
$-i$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=0$ then

$$
\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-+\alpha_{c_3}^-\right) \\
&= 0-(0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{3}{4x} + \frac{3}{8\,(x-i)} + \frac{3}{8\,(x+i)} + (0) \\
&= -\frac{3}{4x} + \frac{3}{8\,(x-i)} + \frac{3}{8\,(x+i)} \\
&= -\frac{3}{4x\,(x^2+1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4x} + \frac{3}{8\,(x-i)} + \frac{3}{8\,(x+i)}\right)(0) + \left(\left(\frac{3}{4x^2} - \frac{3}{8\,(x-i)^2} - \frac{3}{8\,(x+i)^2}\right) + \left(-\frac{3}{4x} + \frac{3}{8\,(x-i)} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)}\right) dx} \\
&= \frac{\left(x^2+1\right)^{3/8}}{x^{3/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x^3+3x}{2x^4+2x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4}-\frac{5\ln\left(x^2+1\right)}{8}} \\
&= z_1\left(\frac{1}{x^{3/4}\left(x^2+1\right)^{5/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^{3/2}\left(x^2+1\right)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x^3+3x}{2x^4+2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}-\frac{5\ln\left(x^2+1\right)}{4}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \mathrm{e}^{-\frac{3\ln(x)}{2}-\frac{5\ln\left(x^2+1\right)}{4}} x^3\sqrt{x^2+1}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^{3/2}\left(x^2+1\right)^{1/4}}\right) + c_2\left(\frac{1}{x^{3/2}\left(x^2+1\right)^{1/4}}\left(\int \mathrm{e}^{-\frac{3\ln(x)}{2}-\frac{5\ln\left(x^2+1\right)}{4}} x^3\sqrt{x^2+1}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.528.2 Maple step by step solution

Let's solve

$2x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(8x^2+3)\,y'-(-4x^2+3)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(4x^2-3)y}{2x^2(x^2+1)}-\frac{(8x^2+3)y'}{2x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(8x^2+3)y'}{2x(x^2+1)}+\frac{(4x^2-3)y}{2x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ∘ Define functions

$\left[P_2(x)=\frac{8x^2+3}{2x(x^2+1)},P_3(x)=\frac{4x^2-3}{2x^2(x^2+1)}\right]$

- ∘ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=\frac{3}{2}$

- ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{3}{2}$

- ∘ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$2x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(8x^2+3)\,y'+(4x^2-3)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ∘ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ∘ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ∘ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m=2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2r+3)\,(-1+r)\, x^r + a_1(5+2r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+3)\,(k+r-1) + 2a_{k-2}(k+r)\,(k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2r+3)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, -\frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(5+2r)\, r = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(\left(k+r+\frac{3}{2}\right) a_k + a_{k-2}(k+r)\right)(k+r-1) = 0$$

- Shift index using $k->k+2$

$$2\left(\left(k+\frac{7}{2}+r\right) a_{k+2} + a_k(k+r+2)\right)(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2a_k(k+r+2)}{2k+7+2r}$$

- Recursion relation for $r=1$

$$a_{k+2} = -\frac{2a_k(k+3)}{2k+9}$$

- Solution for $r=1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_k(k+3)}{2k+9}, a_1 = 0\right]$$

- Recursion relation for $r=-\frac{3}{2}$

$a_{k+2} = -\frac{2a_k(k+\frac{1}{2})}{2k+4}$

- Solution for $r = -\frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{2a_k(k+\frac{1}{2})}{2k+4}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2a_k(k+3)}{2k+9}, a_1 = 0, b_{k+2} = -\frac{2b_k(k+\frac{1}{2})}{2k+4}, b_1 = 0\right]$

1.528.3 Maple trace

```
Methods for second order ODEs:
```

1.528.4 Maple dsolve solution

Solving time : `0.023` (sec)
Leaf size : 31

```
dsolve(2*x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(8*x^2+3)*diff(y(x),x)-(-4*x^2+3)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x \,\mathrm{hypergeom}\left(\left[1, \frac{3}{2}\right], \left[\frac{9}{4}\right], -x^2\right) + \frac{c_2}{x^{3/2}(x^2+1)^{1/4}}$$

1.528.5 Mathematica DSolve solution

Solving time : `0.133` (sec)
Leaf size : 60

```
DSolve[{2*x^2*(1+x^2)*D[y[x],{x,2}]+x*(3+8*x^2)*D[y[x],x]-(3-4*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{c_2 \,\mathrm{Hypergeometric2F1}\left(\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, -x^2\right)}{x\sqrt[4]{x^2+1}} + \frac{c_1}{x^{3/2}\sqrt[4]{x^2+1}} + \frac{c_2}{x}$$

1.529 problem 545

Internal problem ID [8667]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 545
Date solved : `Monday, October 21, 2024 at 05:19:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' + 3x\left(x^2+3\right)y' - \left(-5x^2+1\right)y = 0$$

1.529.1 Solved as second order ode using Kovacic algorithm

Time used: 0.319 (sec)

Writing the ode as

$$9x^2y'' + \left(3x^3+9x\right)y' + \left(5x^2-1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= 3x^3+9x \\ C &= 5x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^2 - 5}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 8x^2 - 5$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^2 - 5}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1003: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{36} - \frac{2}{9} - \frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \qquad (8) \end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} - \frac{2}{3x} - \frac{7}{4x^3} - \frac{7}{x^5} - \frac{595}{16x^7} - \frac{889}{4x^9} - \frac{45647}{32x^{11}} - \frac{76811}{8x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 8x^2 - 5}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{x^2}{36} - \frac{2}{9}\right) + \left(-\frac{5}{36x^2}\right) \\ &= \frac{x^2}{36} - \frac{2}{9} - \frac{5}{36x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{2}{9}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{2}{9}\right) - (0) \\ &= -\frac{2}{9} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{6} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{2}{9}}{\frac{1}{6}} - 1\right) &&= -\frac{7}{6} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{2}{9}}{\frac{1}{6}} - 1\right) &&= \frac{1}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 8x^2 - 5}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{6}$	$-\frac{7}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{6} - \left(\frac{1}{6}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{6x} + (-)\left(\frac{x}{6}\right) \\
&= \frac{1}{6x} - \frac{x}{6} \\
&= \frac{1}{6x} - \frac{x}{6}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{6x} - \frac{x}{6}\right)(0) + \left(\left(-\frac{1}{6x^2} - \frac{1}{6}\right) + \left(\frac{1}{6x} - \frac{x}{6}\right)^2 - \left(\frac{x^4 - 8x^2 - 5}{36x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{6x} - \frac{x}{6}\right) dx} \\
&= x^{1/6} \mathrm{e}^{-\frac{x^2}{12}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x^3 + 9x}{9x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{12} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x^2}{12}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{x^2}{6}}}{x^{1/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+9x}{9x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{6}-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \mathrm{e}^{-\frac{x^2}{6}-\ln(x)} x^{2/3} \mathrm{e}^{\frac{x^2}{3}}\, dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\mathrm{e}^{-\frac{x^2}{6}}}{x^{1/3}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{x^2}{6}}}{x^{1/3}} \left(\int \mathrm{e}^{-\frac{x^2}{6}-\ln(x)} x^{2/3} \mathrm{e}^{\frac{x^2}{3}}\, dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.529.2 Maple step by step solution

Let's solve

$9x^2\left(\frac{d}{dx}y'\right) + 3x(x^2+3)\, y' - (-5x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(5x^2-1)y}{9x^2} - \frac{(x^2+3)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(x^2+3)y'}{3x} + \frac{(5x^2-1)y}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $$\left[P_2(x) = \frac{x^2+3}{3x}, P_3(x) = \frac{5x^2-1}{9x^2}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\bigg|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{9}$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$9x^2\left(\frac{d}{dx}y'\right) + 3x(x^2+3)\,y' + (5x^2-1)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(1+3r)(-1+3r)x^r + a_1(4+3r)(2+3r)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+1)(3k+3r-1) + a_{k-2}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$(1+3r)(-1+3r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$$
- Each term must be 0
$$a_1(4+3r)(2+3r) = 0$$
- Solve for the dependent coefficient(s)
$$a_1 = 0$$
- Each term in the series must be 0, giving the recursion relation
$$(3k+3r-1)(3a_k k + 3a_k r + a_k + a_{k-2}) = 0$$
- Shift index using $k->k+2$
$$(3k+3r+5)(3a_{k+2}(k+2) + 3a_{k+2}r + a_{k+2} + a_k) = 0$$
- Recursion relation that defines series solution to ODE
$$a_{k+2} = -\frac{a_k}{3k+7+3r}$$
- Recursion relation for $r = -\frac{1}{3}$
$$a_{k+2} = -\frac{a_k}{3k+6}$$
- Solution for $r = -\frac{1}{3}$
$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = -\frac{a_k}{3k+6}, a_1 = 0\right]$$
- Recursion relation for $r = \frac{1}{3}$
$$a_{k+2} = -\frac{a_k}{3k+8}$$
- Solution for $r = \frac{1}{3}$
$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k}{3k+8}, a_1 = 0\right]$$
- Combine solutions and rename parameters
$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{a_k}{3k+6}, a_1 = 0, b_{k+2} = -\frac{b_k}{3k+8}, b_1 = 0\right]$$

1.529.3 Maple trace

```
Methods for second order ODEs:
```

1.529.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 37

```
dsolve(9*x^2*diff(diff(y(x),x),x)+3*x*(x^2+3)*diff(y(x),x)-(-5*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{x^2}{12}} \left(x^{1/3} \operatorname{WhittakerM}\left(\frac{1}{3}, \frac{1}{6}, \frac{x^2}{6}\right) c_1 + e^{-\frac{x^2}{12}} c_2 x\right)}{x^{4/3}}$$

1.529.5 Mathematica DSolve solution

Solving time : `0.117` (sec)
Leaf size : 61

```
DSolve[{9*x^2*D[y[x],{x,2}]+3*x*(3+x^2)*D[y[x],x]-(1-5*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{x^2}{6}} \left(2c_1 x^{4/3} + \sqrt[3]{6} c_2 (-x^2)^{2/3} \Gamma\left(\frac{1}{3}, -\frac{x^2}{6}\right)\right)}{2x^{5/3}}$$

1.530 problem 546

Internal problem ID [8668]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 546
Date solved : `Monday, October 21, 2024 at 05:19:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2y'' + x\left(6x^2 + 1\right) y' + \left(9x^2 + 1\right) y = 0$$

1.530.1 Solved as second order ode using Kovacic algorithm

Time used: 0.431 (sec)

Writing the ode as

$$6x^2y'' + \left(6x^3 + x\right) y' + \left(9x^2 + 1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= 6x^3 + x \\ C &= 9x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{36x^4 - 132x^2 - 35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 36x^4 - 132x^2 - 35$$
$$t = 144x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{36x^4 - 132x^2 - 35}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1005: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - \frac{11}{12} - \frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{11}{12x} - \frac{13}{12x^3} - \frac{143}{72x^5} - \frac{130}{27x^7} - \frac{17017}{1296x^9} - \frac{597961}{15552x^{11}} - \frac{11016863}{93312x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{36x^4 - 132x^2 - 35}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{x^2}{4} - \frac{11}{12}\right) + \left(-\frac{35}{144x^2}\right) \\ &= \frac{x^2}{4} - \frac{11}{12} - \frac{35}{144x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{11}{12}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{11}{12}\right) - (0) \\ &= -\frac{11}{12}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{11}{12}}{\frac{1}{2}} - 1\right) &&= -\frac{17}{12} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{11}{12}}{\frac{1}{2}} - 1\right) &&= \frac{5}{12}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{36x^4 - 132x^2 - 35}{144x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	$-\frac{17}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{12}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{5}{12} - \left(\frac{5}{12}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{5}{12x} + (-)\left(\frac{x}{2}\right) \\
&= \frac{5}{12x} - \frac{x}{2} \\
&= \frac{5}{12x} - \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{5}{12x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{5}{12x^2} - \frac{1}{2}\right) + \left(\frac{5}{12x} - \frac{x}{2}\right)^2 - \left(\frac{36x^4 - 132x^2 - 35}{144x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{5}{12x} - \frac{x}{2}\right) dx} \\
&= x^{5/12} \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{6x^3 + x}{6x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{4} - \frac{\ln(x)}{12}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x^2}{4}}}{x^{1/12}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^{1/3} \mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6x^3+x}{6x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}-\frac{\ln(x)}{6}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}-\frac{\ln(x)}{6}} \mathrm{e}^{x^2}}{x^{2/3}} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{1/3} \mathrm{e}^{-\frac{x^2}{2}} \right) + c_2 \left(x^{1/3} \mathrm{e}^{-\frac{x^2}{2}} \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}-\frac{\ln(x)}{6}} \mathrm{e}^{x^2}}{x^{2/3}} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.530.2 Maple step by step solution

Let's solve

$6x^2 \left(\frac{d}{dx} y'\right) + x(6x^2+1)\, y' + (9x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(9x^2+1)y}{6x^2} - \frac{(6x^2+1)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{(6x^2+1)y'}{6x} + \frac{(9x^2+1)y}{6x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{6x^2+1}{6x}, P_3(x) = \frac{9x^2+1}{6x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{6}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{6}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2\left(\frac{d}{dx}y'\right) + x(6x^2+1)\,y' + (9x^2+1)\,y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(-1+3r)(-1+2r)x^r + a_1(2+3r)(1+2r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r-1)(2k+2r-1)+3a$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+3r)(-1+2r)=0$
- Values of r that satisfy the indicial equation
  $r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$
- Each term must be 0
  $a_1(2+3r)(1+2r)=0$
- Solve for the dependent coefficient(s)
  $a_1=0$
- Each term in the series must be 0, giving the recursion relation
  $6\left(k+r-\frac{1}{2}\right)\left(\left(k+r-\frac{1}{3}\right)a_k+a_{k-2}\right)=0$
- Shift index using $k->k+2$
  $6\left(k+\frac{3}{2}+r\right)\left(\left(k+\frac{5}{3}+r\right)a_{k+2}+a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{3a_k}{3k+5+3r}$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+2}=-\frac{3a_k}{3k+\frac{13}{2}}$
- Solution for $r=\frac{1}{2}$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{3a_k}{3k+\frac{13}{2}}, a_1=0\right]$
- Recursion relation for $r=\frac{1}{3}$
  $a_{k+2}=-\frac{3a_k}{3k+6}$
- Solution for $r=\frac{1}{3}$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2}=-\frac{3a_k}{3k+6}, a_1=0\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2}=-\frac{3a_k}{3k+\frac{13}{2}}, a_1=0, b_{k+2}=-\frac{3b_k}{3k+6}, b_1=0\right]$

1.530.3 Maple trace

```
Methods for second order ODEs:
```

1.530.4 Maple dsolve solution

Solving time : 0.013 (sec)
Leaf size : 36

```
dsolve(6*x^2*diff(diff(y(x),x),x)+x*(6*x^2+1)*diff(y(x),x)+(9*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{x^2}{4}} \left(x^{11/12} e^{-\frac{x^2}{4}} c_2 + \text{WhittakerM}\left(\frac{11}{24}, \frac{1}{24}, \frac{x^2}{2}\right) c_1\right)}{x^{7/12}}$$

1.530.5 Mathematica DSolve solution

Solving time : 0.126 (sec)
Leaf size : 61

```
DSolve[{6*x^2*D[y[x],{x,2}]+x*(1+6*x^2)*D[y[x],x]+(1+9*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{x^2}{2}} \left(2c_1 x^{11/6} + \sqrt[12]{2} c_2 (-x^2)^{11/12} \Gamma\left(\frac{1}{12}, -\frac{x^2}{2}\right)\right)}{2x^{3/2}}$$

1.531 problem 547

Internal problem ID [8669]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 547
Date solved : `Monday, October 21, 2024 at 05:19:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2\left(x^2+1\right)y''+3x\left(13x^2+3\right)y'-\left(-25x^2+1\right)y=0$$

1.531.1 Solved as second order ode using Kovacic algorithm

Time used: 0.355 (sec)

Writing the ode as

$$\left(9x^4+9x^2\right)y''+\left(39x^3+9x\right)y'+\left(25x^2-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=9x^4+9x^2\\ B&=39x^3+9x\\ C&=25x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9x^4 + 6x^2 - 5}{36\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -9x^4 + 6x^2 - 5$$
$$t = 36\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-9x^4 + 6x^2 - 5}{36\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1007: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 4 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36x^2} - \frac{5}{36(x-i)^2} - \frac{5}{36(x+i)^2} - \frac{i}{12(x-i)} + \frac{i}{12x + 12i}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{6}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{6}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-9x^4 + 6x^2 - 5}{36\left(x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-9x^4 + 6x^2 - 5}{36\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$
i	2	0	$\frac{5}{6}$	$\frac{1}{6}$
$-i$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i} + (-)\,(0) \\
&= \frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i} \\
&= \frac{1}{6x} + \frac{x}{3x^2+3}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i}\right)(0) + \left(\left(-\frac{1}{6x^2} - \frac{1}{6\,(x-i)^2} - \frac{1}{6\,(x+i)^2}\right) + \left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i}\right) dx} \\
&= \left(x^2+1\right)^{1/6} (-x)^{1/6}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{39x^3+9x}{9x^4+9x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln\left(x^2+1\right)}{6}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\left(x^2+1\right)^{5/6}\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(-x)^{1/6}}{\left(x^2+1\right)^{2/3}\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{39x^3+9x}{9x^4+9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(x^2+1\right)}{3}-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{5\ln\left(x^2+1\right)}{3}-\ln(x)}\left(x^2+1\right)^{4/3} x}{(-x)^{1/3}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(-x)^{1/6}}{\left(x^2+1\right)^{2/3}\sqrt{x}}\right) + c_2 \left(\frac{(-x)^{1/6}}{\left(x^2+1\right)^{2/3}\sqrt{x}}\left(\int \frac{\mathrm{e}^{-\frac{5\ln\left(x^2+1\right)}{3}-\ln(x)}\left(x^2+1\right)^{4/3} x}{(-x)^{1/3}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.531.2 Maple step by step solution

Let's solve

$9x^2(x^2+1)\left(\frac{d}{dx}y'\right)+3x(13x^2+3)\,y'-(-25x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(25x^2-1)y}{9x^2(x^2+1)}-\frac{(13x^2+3)y'}{3x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(13x^2+3)y'}{3x(x^2+1)}+\frac{(25x^2-1)y}{9x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{13x^2+3}{3x(x^2+1)},P_3(x)=\frac{25x^2-1}{9x^2(x^2+1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$(x\cdot P_2(x))\bigg|_{x=0}=1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2\cdot P_3(x))\bigg|_{x=0}=-\frac{1}{9}$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$9x^2(x^2+1)\left(\frac{d}{dx}y'\right)+3x(13x^2+3)\,y'+(25x^2-1)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m=2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+3r)\,(-1+3r)\, x^r + a_1(4+3r)\,(2+3r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+1)\,(3k+3r-1) + a_{k-2}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+3r)\,(-1+3r)=0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$
- Each term must be 0
  $a_1(4+3r)\,(2+3r)=0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $9\left(\left(k+r-\frac{1}{3}\right) a_{k-2} + a_k\left(k+r+\frac{1}{3}\right)\right)\left(k+r-\frac{1}{3}\right)=0$
- Shift index using $k- >k+2$
  $9\left(\left(k+\frac{5}{3}+r\right) a_k + a_{k+2}\left(k+\frac{7}{3}+r\right)\right)\left(k+\frac{5}{3}+r\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{(3k+3r+5)a_k}{3k+7+3r}$
- Recursion relation for $r=-\frac{1}{3}$
  $a_{k+2} = -\frac{(3k+4)a_k}{3k+6}$
- Solution for $r=-\frac{1}{3}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = -\frac{(3k+4)a_k}{3k+6}, a_1 = 0\right]$
- Recursion relation for $r=\frac{1}{3}$

$a_{k+2} = -\frac{(3k+6)a_k}{3k+8}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{(3k+6)a_k}{3k+8}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{(3k+4)a_k}{3k+6}, a_1 = 0, b_{k+2} = -\frac{(3k+6)b_k}{3k+8}, b_1 = 0\right]$

1.531.3 Maple trace

```
Methods for second order ODEs:
```

1.531.4 Maple dsolve solution

Solving time : `0.029` (sec)
Leaf size : 33

```
dsolve(9*x^2*(x^2+1)*diff(diff(y(x),x),x)+3*x*(13*x^2+3)*diff(y(x),x)-(-25*x^2+1)*y(x
       y(x),singsol=all)
```

$$y = \frac{c_1}{(x^2+1)^{2/3} x^{1/3}} + c_2 x^{1/3} \operatorname{hypergeom}\left([1,1], \left[\frac{4}{3}\right], -x^2\right)$$

1.531.5 Mathematica DSolve solution

Solving time : `0.173` (sec)
Leaf size : 124

```
DSolve[{9*x^2*(1+x^2)*D[y[x],{x,2}]+3*x*(3+13*x^2)*D[y[x],x]-(1-25*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$y(x)$

$$\rightarrow \frac{2\sqrt{3}c_2 \arctan\left(\frac{\sqrt{3}x^{2/3}}{x^{2/3}+2\sqrt[3]{x^2+1}}\right) - 2c_2 \log\left(\sqrt[3]{x^2+1} - x^{2/3}\right) + c_2 \log\left(x^{4/3} + (x^2+1)^{2/3} + \sqrt[3]{x^2+1}x^{2/3}\right.}{4\sqrt[3]{x}\,(x^2+1)^{2/3}}$$

1.532 problem 548

Internal problem ID [8670]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 548
Date solved : `Monday, October 21, 2024 at 05:19:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+1\right)y''+4x\left(6x^2+1\right)y'-\left(-25x^2+1\right)y=0$$

1.532.1 Solved as second order ode using Kovacic algorithm

Time used: 0.313 (sec)

Writing the ode as

$$\left(4x^4+4x^2\right)y''+\left(24x^3+4x\right)y'+\left(25x^2-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 24x^3+4x \\ C &= 25x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-x^2 - 6}{4(x^2+1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -x^2 - 6 \\
t &= 4(x^2+1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-x^2 - 6}{4(x^2+1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1009: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\,(x-i)^2} + \frac{5}{16\,(x+i)^2} + \frac{7i}{16\,(x-i)} - \frac{7i}{16\,(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 6}{4\,(x^2+1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^2-6}{4\left(x^2+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{1}{2}-\left(-\frac{1}{2}\right)\\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(x - i\right)} - \frac{1}{4\left(x + i\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x - i\right)} - \frac{1}{4\left(x + i\right)} \\
&= -\frac{x}{2x^{2} + 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^{2} - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(x - i\right)} - \frac{1}{4\left(x + i\right)}\right)(1) + \left(\left(\frac{1}{4\left(x - i\right)^{2}} + \frac{1}{4\left(x + i\right)^{2}}\right) + \left(-\frac{1}{4\left(x - i\right)} - \frac{1}{4\left(x + i\right)}\right)^{2} - \left(-\frac{\cdots}{4(\cdots}\right.\right.
$$

$$
\frac{(x^{2} + 1 \cdots}{(-x + i)^{2} (\cdots}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\
&= (x) \frac{1}{\left(\left(-x + i\right)\left(x + i\right)\right)^{1/4}} \\
&= \frac{x}{\left(-x^{2} - 1\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{24x^3+4x}{4x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln\left(x^2+1\right)}{4}-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\left(x^2+1\right)^{5/4}\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{x}\,\sqrt{2}}{\left(x^2+1\right)^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{24x^3+4x}{4x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(x^2+1\right)}{2}-\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(i\left(-\frac{\left(x^2+1\right)^{3/2}}{x}+x\sqrt{x^2+1}+\operatorname{arcsinh}(x)\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{x}\,\sqrt{2}}{\left(x^2+1\right)^{3/2}}\right) \\
&\quad + c_2\left(\frac{\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{x}\,\sqrt{2}}{\left(x^2+1\right)^{3/2}}\left(i\left(-\frac{\left(x^2+1\right)^{3/2}}{x}+x\sqrt{x^2+1}+\operatorname{arcsinh}(x)\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.532.2 Maple step by step solution

Let's solve

$4x^2(x^2+1)\left(\frac{d}{dx}y'\right)+4x(6x^2+1)\,y'-(-25x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(25x^2-1)y}{4x^2(x^2+1)}-\frac{(6x^2+1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(6x^2+1)y'}{x(x^2+1)}+\frac{(25x^2-1)y}{4x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{6x^2+1}{x(x^2+1)},P_3(x)=\frac{25x^2-1}{4x^2(x^2+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=-\frac{1}{4}$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $4x^2(x^2+1)\left(\frac{d}{dx}y'\right)+4x(6x^2+1)\,y'+(25x^2-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\,x^r + a_1(3+2r)\,(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + a_{k-2}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\left(k+r+\frac{1}{2}\right) a_{k-2} + a_k\left(k+r-\frac{1}{2}\right)\right)\left(k+r+\frac{1}{2}\right) = 0$

- Shift index using $k- >k+2$

  $4\left(\left(k+\frac{5}{2}+r\right) a_k + a_{k+2}\left(k+\frac{3}{2}+r\right)\right)\left(k+\frac{5}{2}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{(2k+2r+5)a_k}{2k+3+2r}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{(2k+4)a_k}{2k+2}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{(2k+4)a_k}{2k+2}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+2} = -\frac{(2k+6)a_k}{2k+4}$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{(2k+6)a_k}{2k+4}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{(2k+4)a_k}{2k+2}, a_1 = 0, b_{k+2} = -\frac{(2k+6)b_k}{2k+4}, b_1 = 0\right]$$

1.532.3 Maple trace

```
Methods for second order ODEs:
```

1.532.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 34

```
dsolve(4*x^2*(x^2+1)*diff(diff(y(x),x),x)+4*x*(6*x^2+1)*diff(y(x),x)-(-25*x^2+1)*y(x)
       y(x),singsol=all)
```

$$y = \frac{-\sqrt{x^2+1}\, c_2 + x(c_2 \operatorname{arcsinh}(x) + c_1)}{(x^2+1)^{3/2} \sqrt{x}}$$

1.532.5 Mathematica DSolve solution

Solving time : `0.122` (sec)
Leaf size : 54

```
DSolve[{4*x^2*(1+x^2)*D[y[x],{x,2}]+4*x*(1+6*x^2)*D[y[x],x]-(1-25*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x \operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right) - c_2\sqrt{x^2+1} + c_1 x}{\sqrt{x}\,(x^2+1)^{3/2}}$$

1.533 problem 549

Internal problem ID [8671]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 549
Date solved : `Monday, October 21, 2024 at 05:19:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$8x^2\left(2x^2+1\right)y''+2x\left(34x^2+5\right)y'-\left(-30x^2+1\right)y=0$$

1.533.1 Solved as second order ode using Kovacic algorithm

Time used: 0.520 (sec)

Writing the ode as

$$\left(16x^4+8x^2\right)y''+\left(68x^3+10x\right)y'+\left(30x^2-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^4+8x^2 \\ B &= 68x^3+10x \\ C &= 30x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{132x^4 + 148x^2 - 7}{64\,(2x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 132x^4 + 148x^2 - 7 \\ t &= 64\left(2x^3 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{132x^4 + 148x^2 - 7}{64\,(2x^3 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1011: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(2x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2} - \frac{3}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{3}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{i\sqrt{2}}{2\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{i\sqrt{2}}{2x + i\sqrt{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x=-\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{132x^4+148x^2-7}{64\left(2x^3+x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{33}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{11}{8} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{3}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{132x^4+148x^2-7}{64\left(2x^3+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{11}{8}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= \frac{11}{8} - \left(\frac{11}{8}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}} + (0) \\
&= \frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}} \\
&= \frac{22x^2+7}{16x^3+8x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}}\right)(0) + \left(\left(-\frac{7}{8x^2} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{7}{8x} + \frac{}{4x}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{7}{8x} + \frac{1}{4x - 2i\sqrt{2}} + \frac{1}{4x + 2i\sqrt{2}}\right) dx} \\
&= 2^{1/4} \left(2x^2 + 1\right)^{1/4} x^{7/8}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{68x^3 + 10x}{16x^4 + 8x^2} \, dx} \\
&= z_1 e^{-\frac{3 \ln\left(2x^2+1\right)}{4} - \frac{5 \ln(x)}{8}} \\
&= z_1 \left(\frac{1}{\left(2x^2 + 1\right)^{3/4} x^{5/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{1/4} 2^{1/4}}{\sqrt{2x^2 + 1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{68x^3 + 10x}{16x^4 + 8x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{3 \ln\left(2x^2+1\right)}{2} - \frac{5 \ln(x)}{4}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{3 \ln\left(2x^2+1\right)}{2} - \frac{5 \ln(x)}{4}} \left(2x^2 + 1\right) \sqrt{2}}{2\sqrt{x}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/4} 2^{1/4}}{\sqrt{2x^2+1}}\right) + c_2\left(\frac{x^{1/4} 2^{1/4}}{\sqrt{2x^2+1}}\left(\int \frac{e^{-\frac{3\ln\left(2x^2+1\right)}{2}-\frac{5\ln(x)}{4}}\left(2x^2+1\right)\sqrt{2}}{2\sqrt{x}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.533.2 Maple step by step solution

Let's solve

$8x^2(2x^2+1)\left(\frac{d}{dx}y'\right)+2x(34x^2+5)\,y'-(-30x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(30x^2-1)y}{8x^2(2x^2+1)}-\frac{(34x^2+5)y'}{4x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(34x^2+5)y'}{4x(2x^2+1)}+\frac{(30x^2-1)y}{8x^2(2x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{34x^2+5}{4x(2x^2+1)}, P_3(x)=\frac{30x^2-1}{8x^2(2x^2+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{5}{4}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{8}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $8x^2(2x^2+1)\left(\frac{d}{dx}y'\right)+2x(34x^2+5)\,y'+(30x^2-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- ○ Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- ○ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- ○ Shift index using $k- >k+2-m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-1+4r)\, x^r + a_1(3+2r)\,(3+4r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(4k+4r-1) + 2a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)\,(-1+4r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$$

- Each term must be 0

  $$a_1(3+2r)\,(3+4r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$8\left(k+r+\frac{1}{2}\right)\left(\left(2k+2r-\frac{5}{2}\right)a_{k-2} + a_k\left(k+r-\frac{1}{4}\right)\right) = 0$$

- Shift index using $k- >k+2$

  $$8\left(k+\frac{5}{2}+r\right)\left(\left(2k+\frac{3}{2}+2r\right)a_k + a_{k+2}\left(k+\frac{7}{4}+r\right)\right) = 0$$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{2(4k+4r+3)a_k}{4k+7+4r}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{2(4k+1)a_k}{4k+5}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{2(4k+1)a_k}{4k+5}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{4}$

  $a_{k+2} = -\frac{2(4k+4)a_k}{4k+8}$

- Solution for $r = \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{2(4k+4)a_k}{4k+8}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2} = -\frac{2(4k+1)a_k}{4k+5}, a_1 = 0, b_{k+2} = -\frac{2(4k+4)b_k}{4k+8}, b_1 = 0\right]$

1.533.3 Maple trace

```
Methods for second order ODEs:
```

1.533.4 Maple dsolve solution

Solving time : `0.020` (sec)
Leaf size : 46

```
dsolve(8*x^2*(2*x^2+1)*diff(diff(y(x),x),x)+2*x*(34*x^2+5)*diff(y(x),x)-(-30*x^2+1)*y(
       y(x),singsol=all)
```

$$y = \frac{c_1 \operatorname{LegendreP}\left(\frac{3}{8}, \frac{3}{8}, \sqrt{2x^2+1}\right) + c_2 \operatorname{LegendreQ}\left(\frac{3}{8}, \frac{3}{8}, \sqrt{2x^2+1}\right)}{\sqrt{2x^2+1}\, x^{1/8}}$$

1.533.5 Mathematica DSolve solution

Solving time : `0.15` (sec)
Leaf size : 54

```
DSolve[{8*x^2*(1+2*x^2)*D[y[x],{x,2}]+2*x*(5+34*x^2)*D[y[x],x]-(1-30*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3c_1x^{3/4} - 4c_2 \,\text{Hypergeometric2F1}\left(-\frac{3}{8}, \frac{1}{2}, \frac{5}{8}, -2x^2\right)}{3\sqrt{x}\sqrt{2x^2+1}}$$

1.534 problem 550

Internal problem ID [8672]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 550
Date solved : `Monday, October 21, 2024 at 05:19:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(1+x)\,y''-x(1-3x)\,y'+y=0$$

1.534.1 Solved as second order ode using Kovacic algorithm

Time used: 0.189 (sec)

Writing the ode as

$$\left(2x^3+2x^2\right)y''+\left(3x^2-x\right)y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+2x^2\\B&=3x^2-x\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-3}{16x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -3 \\
t &= 16x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{3}{16x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1013: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} + (-)(0) \\ &= \frac{1}{4x} \\ &= \frac{1}{4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x}\right)(0) + \left(\left(-\frac{1}{4x^2}\right) + \left(\frac{1}{4x}\right)^2 - \left(-\frac{3}{16x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{4x} dx} \\ &= x^{1/4} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3x^2 - x}{2x^3 + 2x^2} \, dx} \\ &= z_1 e^{-\ln(1+x) + \frac{\ln(x)}{4}} \\ &= z_1 \left(\frac{x^{1/4}}{1+x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{1+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2-x}{2x^3+2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(1+x)+\frac{\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(2\,\mathrm{e}^{-2\ln(1+x)+\frac{\ln(x)}{2}}(1+x)^2\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{1+x}\right) + c_2 \left(\frac{\sqrt{x}}{1+x}\left(2\,\mathrm{e}^{-2\ln(1+x)+\frac{\ln(x)}{2}}(1+x)^2\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.534.2 Maple step by step solution

Let's solve

$2x^2(1+x)\left(\frac{d}{dx}y'\right) - x(1-3x)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{2x^2(1+x)} - \frac{(3x-1)y'}{2x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3x-1)y'}{2x(1+x)} + \frac{y}{2x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{3x-1}{2x(1+x)}, P_3(x) = \frac{1}{2x^2(1+x)}\right]$

- ○ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left.\left((1+x)\cdot P_2(x)\right)\right|_{x=-1} = 2$

- ○ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $2x^2(1+x)\left(\frac{d}{dx}y'\right) + x(3x-1)\,y' + y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(2u^3 - 4u^2 + 2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (3u^2 - 7u + 4)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(1+r)\,u^{-1+r} + (2a_1(1+r)\,(2+r) - a_0(1+r)\,(-1+4r))\,u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+r+1)\,(k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $2a_1(1+r)\,(2+r) - a_0(1+r)\,(-1+4r) = 0$

- Each term in the series must be 0, giving the recursion relation

$$(-4a_k + 2a_{k-1} + 2a_{k+1})\,k^2 + ((-8a_k + 4a_{k-1} + 4a_{k+1})\,r - 3a_k - 3a_{k-1} + 6a_{k+1})\,k + (-4a_k + 2a$$

- Shift index using $k- >k+1$

$$(-4a_{k+1} + 2a_k + 2a_{k+2})\,(k+1)^2 + ((-8a_{k+1} + 4a_k + 4a_{k+2})\,r - 3a_{k+1} - 3a_k + 6a_{k+2})\,(k+1) +$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + 4kra_k - 8kra_{k+1} + 2r^2a_k - 4r^2a_{k+1} + ka_k - 11ka_{k+1} + ra_k - 11ra_{k+1} - 6a_{k+1}}{2(k^2 + 2kr + r^2 + 5k + 5r + 6)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}$$

- Solution for $r = -1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}, 0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-1}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2 + 5k + 6)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2 + 5k + 6)}, 4a_1 + a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2 + 5k + 6)}, 4a_1 + a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^k\right), a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}, 0 = 0,\right.$$

1.534.3 Maple trace

```
Methods for second order ODEs:
```

1.534.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 19

```
dsolve(2*x^2*(1+x)*diff(diff(y(x),x),x)-x*(1-3*x)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\sqrt{x} + c_1 x}{1 + x}$$

1.534.5 Mathematica DSolve solution

Solving time : 0.056 (sec)
Leaf size : 25

```
DSolve[{2*x^2*(1+x)*D[y[x],{x,2}]-x*(1-3*x)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1\sqrt{x} + 2c_2 x}{x + 1}$$

1.535 problem 551

Internal problem ID [8673]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 551
Date solved : `Monday, October 21, 2024 at 05:19:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2\left(2x^2+1\right)y''+x\left(50x^2+1\right)y'+\left(30x^2+1\right)y=0$$

1.535.1 Solved as second order ode using Kovacic algorithm

Time used: 0.201 (sec)

Writing the ode as

$$\left(12x^4+6x^2\right)y''+\left(50x^3+x\right)y'+\left(30x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 12x^4+6x^2 \\ B &= 50x^3+x \\ C &= 30x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-35}{144x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -35 \\
t &= 144x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{35}{144x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1015: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{35}{144x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{35}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{12}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{5}{12} - \left(\frac{5}{12}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{5}{12x} + (-)\,(0) \\
&= \frac{5}{12x} \\
&= \frac{5}{12x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{5}{12x}\right)(0) + \left(\left(-\frac{5}{12x^2}\right) + \left(\frac{5}{12x}\right)^2 - \left(-\frac{35}{144x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{5}{12x} dx} \\ &= x^{5/12}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{50x^3+x}{12x^4+6x^2} \, dx} \\ &= z_1 e^{-\ln(2x^2+1) - \frac{\ln(x)}{12}} \\ &= z_1 \left(\frac{1}{(2x^2+1)\, x^{1/12}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{2x^2+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{50x^3+x}{12x^4+6x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(2x^2+1)-\frac{\ln(x)}{6}}}{(y_1)^2}\, dx \\
&= y_1 \left(6x^{1/3} \mathrm{e}^{-2\ln(2x^2+1)-\frac{\ln(x)}{6}} \left(2x^2+1\right)^2\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/3}}{2x^2+1}\right) + c_2 \left(\frac{x^{1/3}}{2x^2+1} \left(6x^{1/3} \mathrm{e}^{-2\ln(2x^2+1)-\frac{\ln(x)}{6}} \left(2x^2+1\right)^2\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.535.2 Maple step by step solution

Let's solve

$6x^2(2x^2+1)\left(\frac{d}{dx}y'\right) + x(50x^2+1)\,y' + (30x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(30x^2+1)y}{6x^2(2x^2+1)} - \frac{(50x^2+1)y'}{6x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(50x^2+1)y'}{6x(2x^2+1)} + \frac{(30x^2+1)y}{6x^2(2x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{50x^2+1}{6x(2x^2+1)}, P_3(x) = \frac{30x^2+1}{6x^2(2x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{6}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{6}$
- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2(2x^2+1)\left(\frac{d}{dx}y'\right) + x(50x^2+1)\,y' + (30x^2+1)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)\,(-1+2r)\,x^r + a_1(2+3r)\,(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)\,(2k+2r-1) + 2a\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(2+3r)(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(3k+3r-1)(2k+2r-1)(a_k+2a_{k-2}) = 0$
- Shift index using $k->k+2$

  $(3k+3r+5)(2k+2r+3)(a_{k+2}+2a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -2a_k$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -2a_k$
- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -2a_k, a_1 = 0\right]$$
- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -2a_k$
- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -2a_k, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -2a_k, a_1 = 0, b_{k+2} = -2b_k, b_1 = 0\right]$$

1.535.3 Maple trace

```
Methods for second order ODEs:
```

1.535.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 24

```
dsolve(6*x^2*(2*x^2+1)*diff(diff(y(x),x),x)+x*(50*x^2+1)*diff(y(x),x)+(30*x^2+1)*y(x)
       y(x),singsol=all)
```

$$y = \frac{x^{1/3}\left(c_1\, x^{1/6} + c_2\right)}{2x^2 + 1}$$

1.535.5 Mathematica DSolve solution

Solving time : 0.073 (sec)
Leaf size : 32

```
DSolve[{6*x^2*(1+2*x^2)*D[y[x],{x,2}]+x*(1+50*x^2)*D[y[x],x]+(1+30*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[3]{x}\left(6c_2\sqrt[6]{x} + c_1\right)}{2x^2 + 1}$$

1.536 problem 552

Internal problem ID [8674]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 552
Date solved : `Monday, October 21, 2024 at 05:19:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$28x^2(1-3x)\,y''-7x(5+9x)\,y'+7(2+9x)\,y=0$$

1.536.1 Solved as second order ode using Kovacic algorithm

Time used: 0.195 (sec)

Writing the ode as

$$\left(-84x^3+28x^2\right)y''+\left(-63x^2-35x\right)y'+(63x+14)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-84x^3+28x^2\\B&=-63x^2-35x\\C&=63x+14\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{33}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 33 \\ t &= 64x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{33}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1017: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{33}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{33}{64x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{33}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{3}{8} - \left(-\frac{3}{8}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{8x} + (-)(0) \\ &= -\frac{3}{8x} \\ &= -\frac{3}{8x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}
(0) + 2\left(-\frac{3}{8x}\right)(0) + \left(\left(\frac{3}{8x^2}\right) + \left(-\frac{3}{8x}\right)^2 - \left(\frac{33}{64x^2}\right)\right) = 0 \\
0 = 0
\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{3}{8x} dx} \\
&= \frac{1}{x^{3/8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-63x^2 - 35x}{-84x^3 + 28x^2} \, dx} \\
&= z_1 e^{-\ln(-1+3x) + \frac{5\ln(x)}{8}} \\
&= z_1 \left(\frac{x^{5/8}}{-1 + 3x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}}{-1 + 3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-63x^2-35x}{-84x^3+28x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(-1+3x)+\frac{5\ln(x)}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{4\sqrt{x}\,\mathrm{e}^{-2\ln(-1+3x)+\frac{5\ln(x)}{4}}(-1+3x)^2}{7}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/4}}{-1+3x}\right) + c_2 \left(\frac{x^{1/4}}{-1+3x}\left(\frac{4\sqrt{x}\,\mathrm{e}^{-2\ln(-1+3x)+\frac{5\ln(x)}{4}}(-1+3x)^2}{7}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.536.2 Maple step by step solution

Let's solve

$28x^2(1-3x)\left(\frac{d}{dx}y'\right) - 7x(5+9x)\,y' + 7(2+9x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(2+9x)y}{4x^2(-1+3x)} - \frac{(5+9x)y'}{4x(-1+3x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(5+9x)y'}{4x(-1+3x)} - \frac{(2+9x)y}{4x^2(-1+3x)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{5+9x}{4x(-1+3x)}, P_3(x) = -\frac{2+9x}{4x^2(-1+3x)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{5}{4}$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{2}$$

- $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$4x^2(-1+3x)\left(\frac{d}{dx}y'\right) + x(5+9x)\,y' + (-9x-2)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+4r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(4k+4r-1)\,(k+r-2) + 3a_{k-1}(4k+4r-1)\,(k+r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-1+4r)(-2+r)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{2, \frac{1}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-4(k+r-2)(a_k - 3a_{k-1})\left(k+r-\frac{1}{4}\right)=0$$

- Shift index using $k->k+1$

$$-4(k+r-1)(a_{k+1} - 3a_k)\left(k+\frac{3}{4}+r\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = 3a_k$$

- Recursion relation for $r=2$

$$a_{k+1} = 3a_k$$

- Solution for $r=2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = 3a_k\right]$$

- Recursion relation for $r=\frac{1}{4}$

$$a_{k+1} = 3a_k$$

- Solution for $r=\frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = 3a_k\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+1} = 3a_k, b_{k+1} = 3b_k\right]$$

1.536.3 Maple trace

```
Methods for second order ODEs:
```

1.536.4 Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 23

```
dsolve(28*x^2*(1-3*x)*diff(diff(y(x),x),x)-7*x*(5+9*x)*diff(y(x),x)+7*(2+9*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1\, x^2 + c_2\, x^{1/4}}{-1+3x}$$

1.536.5 Mathematica DSolve solution

Solving time : 0.072 (sec)
Leaf size : 30

```
DSolve[{28*x^2*(1-3*x)*D[y[x],{x,2}]-7*x*(5+9*x)*D[y[x],x]+7*(2+9*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_2x^2 + 7c_1\sqrt[4]{x}}{7 - 21x}$$

1.537 problem 553

Internal problem ID [8675]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 553
Date solved : `Monday, October 21, 2024 at 05:19:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$8x^2\left(-x^2+2\right)y''+2x\left(-21x^2+10\right)y'-\left(35x^2+2\right)y=0$$

1.537.1 Solved as second order ode using Kovacic algorithm

Time used: 0.197 (sec)

Writing the ode as

$$\left(-8x^4+16x^2\right)y''+\left(-42x^3+20x\right)y'+\left(-35x^2-2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -8x^4+16x^2 \\ B &= -42x^3+20x \\ C &= -35x^2-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -7$$
$$t = 64x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{7}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1019: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{7}{64x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{7}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{8} - \left(\frac{1}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{8x} + (-)\,(0) \\
&= \frac{1}{8x} \\
&= \frac{1}{8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8x}\right)(0) + \left(\left(-\frac{1}{8x^2}\right) + \left(\frac{1}{8x}\right)^2 - \left(-\frac{7}{64x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{8x} dx} \\ &= x^{1/8} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-42x^3+20x}{-8x^4+16x^2} \, dx} \\ &= z_1 e^{-\frac{5 \ln(x)}{8} - \ln(x^2-2)} \\ &= z_1 \left(\frac{1}{x^{5/8}\left(x^2-2\right)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}\,(x^2-2)}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-42x^3+20x}{-8x^4+16x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{4} - 2\ln(x^2-2)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{4x^2 \mathrm{e}^{-\frac{5\ln(x)}{4} - 2\ln(x^2-2)} (x^2-2)^2}{3} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\sqrt{x}\,(x^2-2)} \right) + c_2 \left(\frac{1}{\sqrt{x}\,(x^2-2)} \left(\frac{4x^2 \mathrm{e}^{-\frac{5\ln(x)}{4} - 2\ln(x^2-2)} (x^2-2)^2}{3} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.537.2 Maple step by step solution

Let's solve

$8x^2(-x^2+2)\left(\frac{d}{dx}y'\right) + 2x(-21x^2+10)\,y' - (35x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(35x^2+2)y}{8x^2(x^2-2)} - \frac{(21x^2-10)y'}{4x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(21x^2-10)y'}{4x(x^2-2)} + \frac{(35x^2+2)y}{8x^2(x^2-2)} = 0$

- □ Check to see if x_0 is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{21x^2-10}{4x(x^2-2)}, P_3(x) = \frac{35x^2+2}{8x^2(x^2-2)}\right]$

  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{4}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{8}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $8x^2(x^2-2)\left(\frac{d}{dx}y'\right) + 2x(21x^2-10)\,y' + (35x^2+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0(1+2r)\,(-1+4r)\,x^r - 2a_1(3+2r)\,(3+4r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (-2a_k(2k+2r+1)\,(4k+4r-1)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2(1+2r)\,(-1+4r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$

- Each term must be 0

  $-2a_1(3+2r)(3+4r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $-(2k+2r+1)(4k+4r-1)(2a_k - a_{k-2}) = 0$

- Shift index using $k->k+2$

  $-(2k+2r+5)(4k+4r+7)(2a_{k+2} - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{2}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{a_k}{2}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{4}$

  $a_{k+2} = \frac{a_k}{2}$

- Solution for $r = \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2} = \frac{a_k}{2}, a_1 = 0, b_{k+2} = \frac{b_k}{2}, b_1 = 0\right]$

1.537.3 Maple trace

```
Methods for second order ODEs:
```

1.537.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 22

```
dsolve(8*x^2*(-x^2+2)*diff(diff(y(x),x),x)+2*x*(-21*x^2+10)*diff(y(x),x)-(35*x^2+2)*y(
       y(x),singsol=all)
```

$$y = \frac{c_2\, x^{3/4} + c_1}{\sqrt{x}\, (x^2 - 2)}$$

1.537.5 Mathematica DSolve solution

Solving time : 0.085 (sec)
Leaf size : 34

```
DSolve[{8*x^2*(2-x^2)*D[y[x],{x,2}]+2*x*(10-21*x^2)*D[y[x],x]-(2+35*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\frac{3c_1}{\sqrt{x}} + 4c_2\sqrt[4]{x}}{6 - 3x^2}$$

1.538 problem 554

Internal problem ID [8676]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 554
Date solved : `Monday, October 21, 2024 at 05:19:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+3x+1\right)y''-4x\left(-3x^2-3x+1\right)y'+3\left(x^2-x+1\right)y=0$$

1.538.1 Solved as second order ode using Kovacic algorithm

Time used: 0.134 (sec)

Writing the ode as

$$\left(4x^4+12x^3+4x^2\right)y''+\left(12x^3+12x^2-4x\right)y'+\left(3x^2-3x+3\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^4+12x^3+4x^2\\B&=12x^3+12x^2-4x\\C&=3x^2-3x+3\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1021: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{12x^3+12x^2-4x}{4x^4+12x^3+4x^2}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \ln(x^2+3x+1)} \\ &= z_1 \left(\frac{\sqrt{x}}{x^2+3x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{x^2+3x+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{12x^3+12x^2-4x}{4x^4+12x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x) - 2\ln(x^2+3x+1)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{x^2 + 3x + 1}\right) + c_2 \left(\frac{\sqrt{x}}{x^2 + 3x + 1}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.538.2 Maple step by step solution

Let's solve

$4x^2(x^2 + 3x + 1)\left(\frac{d}{dx}y'\right) - 4x(-3x^2 - 3x + 1)\,y' + 3(x^2 - x + 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{3(x^2 - x + 1)y}{4x^2(x^2 + 3x + 1)} - \frac{(3x^2 + 3x - 1)y'}{x(x^2 + 3x + 1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3x^2 + 3x - 1)y'}{x(x^2 + 3x + 1)} + \frac{3(x^2 - x + 1)y}{4x^2(x^2 + 3x + 1)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{3x^2 + 3x - 1}{x(x^2 + 3x + 1)}, P_3(x) = \frac{3(x^2 - x + 1)}{4x^2(x^2 + 3x + 1)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2 + 3x + 1)\left(\frac{d}{dx}y'\right) + 4x(3x^2 + 3x - 1)\,y' + (3x^2 - 3x + 3)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)(-3+2r)\, x^r + \left(a_1(1+2r)(-1+2r) + 3a_0(1+2r)(-1+2r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+2r)(-3+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$
- Each term must be 0
  $a_1(1+2r)(-1+2r) + 3a_0(1+2r)(-1+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = -3a_0$
- Each term in the series must be 0, giving the recursion relation
  $(2k+2r-1)(2k+2r-3)(a_k + 3a_{k-1} + a_{k-2}) = 0$
- Shift index using $k- >k + 2$
  $(2k+2r+3)(2k+2r+1)(a_{k+2} + 3a_{k+1} + a_k) = 0$

- Recursion relation that defines series solution to ODE
  $a_{k+2} = -3a_{k+1} - a_k$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = -3a_{k+1} - a_k$
- Solution for $r = \frac{1}{2}$
  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$$
- Recursion relation for $r = \frac{3}{2}$
  $a_{k+2} = -3a_{k+1} - a_k$
- Solution for $r = \frac{3}{2}$
  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$$
- Combine solutions and rename parameters
  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0, b_{k+2} = -3b_{k+1} - b_k, b_1 =\right.$$

1.538.3 Maple trace

```
Methods for second order ODEs:
```

1.538.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 23

```
dsolve(4*x^2*(x^2+3*x+1)*diff(diff(y(x),x),x)-4*x*(-3*x^2-3*x+1)*diff(y(x),x)+3*(x^2-x
      y(x),singsol=all)
```

$$y = \frac{\sqrt{x}\,(c_2 x + c_1)}{x^2 + 3x + 1}$$

1.538.5 Mathematica DSolve solution

Solving time : 0.091 (sec)
Leaf size : 28

```
DSolve[{4*x^2*(1+3*x+x^2)*D[y[x],{x,2}]-4*x*(1-3*x-3*x^2)*D[y[x],x]+3*(1-x+x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}(c_2x + c_1)}{x^2 + 3x + 1}$$

1.539 problem 555

Internal problem ID [8677]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 555
Date solved : `Monday, October 21, 2024 at 05:19:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2(1+x)^2 y'' - x\left(-11x^2-10x+1\right) y' + \left(5x^2+1\right) y = 0$$

1.539.1 Solved as second order ode using Kovacic algorithm

Time used: 0.194 (sec)

Writing the ode as

$$3x^2(1+x)^2 y'' + \left(11x^3+10x^2-x\right) y' + \left(5x^2+1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2(1+x)^2 \\ B &= 11x^3+10x^2-x \\ C &= 5x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{5}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1023: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{36x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{6} - \left(\frac{1}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{6x} + (-)(0) \\ &= \frac{1}{6x} \\ &= \frac{1}{6x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{6x}\right)(0) + \left(\left(-\frac{1}{6x^2}\right) + \left(\frac{1}{6x}\right)^2 - \left(-\frac{5}{36x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{6x} dx} \\ &= x^{1/6} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{11x^3+10x^2-x}{3x^2(1+x)^2} \, dx} \\ &= z_1 e^{-2\ln(1+x)+\frac{\ln(x)}{6}} \\ &= z_1 \left(\frac{x^{1/6}}{(1+x)^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{(1+x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{11x^3+10x^2-x}{3x^2(1+x)^2} dx}}{(y_1)^2} dx$$

$$= y_1 \int \frac{e^{-4\ln(1+x)+\frac{\ln(x)}{3}}}{(y_1)^2} dx$$

$$= y_1 \left(\frac{3x^{1/3}\mathrm{e}^{-4\ln(1+x)+\frac{\ln(x)}{3}}(1+x)^4}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{x^{1/3}}{(1+x)^2}\right) + c_2 \left(\frac{x^{1/3}}{(1+x)^2} \left(\frac{3x^{1/3}\mathrm{e}^{-4\ln(1+x)+\frac{\ln(x)}{3}}(1+x)^4}{2}\right)\right)$$

Will add steps showing solving for IC soon.

1.539.2 Maple step by step solution

Let's solve

$3x^2(1+x)^2 \left(\frac{d}{dx}y'\right) - x(-11x^2 - 10x + 1)\, y' + (5x^2 + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(5x^2+1)y}{3x^2(1+x)^2} - \frac{y'(11x-1)}{3(1+x)x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'(11x-1)}{3(1+x)x} + \frac{(5x^2+1)y}{3x^2(1+x)^2} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{11x-1}{3x(1+x)}, P_3(x) = \frac{5x^2+1}{3x^2(1+x)^2}\right]$

- ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left.((1+x) \cdot P_2(x))\right|_{x=-1} = 4$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 2$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $3x^2(1+x)^2 \left(\frac{d}{dx}y'\right) + x(1+x)(11x-1)\, y' + (5x^2+1)\, y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(3u^4 - 6u^3 + 3u^2)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (11u^3 - 23u^2 + 12u)\left(\frac{d}{du}y(u)\right) + (5u^2 - 10u + 6)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- >k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 2..4$

    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $3a_0(2+r)(1+r)\, u^r + (3a_1(3+r)(2+r) - a_0(2+r)(5+6r))\, u^{1+r} + \left(\sum_{k=2}^{\infty} (3a_k(k+r+2)(k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $3(2+r)(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$
- Each term must be 0

  $3a_1(3+r)(2+r) - a_0(2+r)(5+6r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0(5+6r)}{3(3+r)}$
- Each term in the series must be 0, giving the recursion relation

  $3(a_k + a_{k-2} - 2a_{k-1})k^2 + (6(a_k + a_{k-2} - 2a_{k-1})r + 9a_k - 4a_{k-2} - 5a_{k-1})k + 3(a_k + a_{k-2} - 2a_{}$
- Shift index using $k- >k+2$

  $3(a_{k+2} + a_k - 2a_{k+1})(k+2)^2 + (6(a_{k+2} + a_k - 2a_{k+1})r + 9a_{k+2} - 4a_k - 5a_{k+1})(k+2) + 3(a_{k+}$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 6kra_k - 12kra_{k+1} + 3r^2a_k - 6r^2a_{k+1} + 8ka_k - 29ka_{k+1} + 8ra_k - 29ra_{k+1} + 5a_k - 33a_{k+1}}{3(k^2 + 2kr + r^2 + 7k + 7r + 12)}$
- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2 + 3k + 2)}$
- Solution for $r = -2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-2}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2 + 3k + 2)}, a_1 = -\frac{7a_0}{3}\right]$
- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-2}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2 + 3k + 2)}, a_1 = -\frac{7a_0}{3}\right]$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 2ka_k - 17ka_{k+1} - 10a_{k+1}}{3(k^2 + 5k + 6)}$
- Solution for $r = -1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 2ka_k - 17ka_{k+1} - 10a_{k+1}}{3(k^2 + 5k + 6)}, a_1 = -\frac{a_0}{6}\right]$
- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-1}, a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} + 2ka_k - 17ka_{k+1} - 10a_{k+1}}{3(k^2 + 5k + 6)}, a_1 = -\frac{a_0}{6}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k-1}\right), a_{k+2} = -\frac{3k^2a_k - 6k^2a_{k+1} - 4ka_k - 5ka_{k+1} + a_k + a_{k+1}}{3(k^2 + 3k + 2)}, a_1 =\right.$

1.539.3 Maple trace

```
Methods for second order ODEs:
```

1.539.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 19

```
dsolve(3*x^2*(1+x)^2*diff(diff(y(x),x),x)-x*(-11*x^2-10*x+1)*diff(y(x),x)+(5*x^2+1)*y(
       y(x),singsol=all)
```

$$y = \frac{c_2\, x^{1/3} + c_1 x}{\left(1+x\right)^{2}}$$

1.539.5 Mathematica DSolve solution

Solving time : 0.061 (sec)
Leaf size : 29

```
DSolve[{3*x^2*(1+x)^2*D[y[x],{x,2}]-x*(1-10*x-11*x^2)*D[y[x],x]+(1+5*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 \sqrt[3]{x} + 3c_2 x}{2(x+1)^2}$$

1.540 problem 556

Internal problem ID [8678]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 556
Date solved : `Monday, October 21, 2024 at 05:19:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+2x+3\right)y''-x\left(-15x^2-14x+3\right)y'+\left(7x^2+3\right)y=0$$

1.540.1 Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$\left(4x^4+8x^3+12x^2\right)y''+\left(15x^3+14x^2-3x\right)y'+\left(7x^2+3\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=4x^4+8x^3+12x^2\\ B&=15x^3+14x^2-3x \\ C&=7x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -7$$
$$t = 64x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{7}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1025: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{7}{64x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{7}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{8} - \left(\frac{1}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{8x} + (-)(0) \\
&= \frac{1}{8x} \\
&= \frac{1}{8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8x}\right)(0) + \left(\left(-\frac{1}{8x^2}\right) + \left(\frac{1}{8x}\right)^2 - \left(-\frac{7}{64x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1}{8x} dx} \\
&= x^{1/8}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{15x^3+14x^2-3x}{4x^4+8x^3+12x^2} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{8} - \ln(x^2+2x+3)} \\
&= z_1 \left(\frac{x^{1/8}}{x^2+2x+3}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}}{x^2+2x+3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{15x^3+14x^2-3x}{4x^4+8x^3+12x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{4}-2\ln(x^2+2x+3)}}{(y_1)^2} dx \\
&= y_1 \left(\frac{4\sqrt{x}\,\mathrm{e}^{\frac{\ln(x)}{4}-2\ln(x^2+2x+3)}(x^2+2x+3)^2}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/4}}{x^2+2x+3}\right) + c_2 \left(\frac{x^{1/4}}{x^2+2x+3} \left(\frac{4\sqrt{x}\,\mathrm{e}^{\frac{\ln(x)}{4}-2\ln(x^2+2x+3)}(x^2+2x+3)^2}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.540.2 Maple step by step solution

Let's solve

$4x^2(x^2+2x+3)\left(\frac{d}{dx}y'\right) - x(-15x^2-14x+3)\,y' + (7x^2+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(7x^2+3)y}{4x^2(x^2+2x+3)} - \frac{(15x^2+14x-3)y'}{4x(x^2+2x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(15x^2+14x-3)y'}{4x(x^2+2x+3)} + \frac{(7x^2+3)y}{4x^2(x^2+2x+3)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{15x^2+14x-3}{4x(x^2+2x+3)}, P_3(x) = \frac{7x^2+3}{4x^2(x^2+2x+3)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = -\frac{1}{4}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+2x+3)\left(\frac{d}{dx}y'\right) + x(15x^2+14x-3)\,y' + (7x^2+3)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $3a_0(-1+4r)\,(-1+r)\,x^r + \left(3a_1(3+4r)\,r + 2a_0 r(3+4r)\right)x^{1+r} + \left(\sum_{k=2}^{\infty} (3a_k(4k+4r-1)\,(k+r\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$3(-1+4r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{4}\right\}$

- Each term must be 0

  $3a_1(3+4r)r + 2a_0r(3+4r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -\frac{2a_0}{3}$

- Each term in the series must be 0, giving the recursion relation

  $(4k+4r-1)(k+r-1)(3a_k + 2a_{k-1} + a_{k-2}) = 0$

- Shift index using $k->k+2$

  $(4k+4r+7)(k+r+1)(3a_{k+2} + 2a_{k+1} + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}, a_1 = -\frac{2a_0}{3}\right]$

- Recursion relation for $r = \frac{1}{4}$

  $a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$

- Solution for $r = \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}, a_1 = -\frac{2a_0}{3}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}, a_1 = -\frac{2a_0}{3}, b_{k+2} = -\frac{2b_{k+1}}{3} - \frac{b_k}{3}, b_1 = -\right.$

1.540.3 Maple trace

```
Methods for second order ODEs:
```

1.540.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 24

```
dsolve(4*x^2*(x^2+2*x+3)*diff(diff(y(x),x),x)-x*(-15*x^2-14*x+3)*diff(y(x),x)+(7*x^2+3
       y(x),singsol=all)
```

$$y = \frac{c_2 x^{1/4} + c_1 x}{x^2 + 2x + 3}$$

1.540.5 Mathematica DSolve solution

Solving time : 0.095 (sec)
Leaf size : 33

```
DSolve[{4*x^2*(3+2*x+x^2)*D[y[x],{x,2}]-x*(3-14*x-15*x^2)*D[y[x],x]+(3+7*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3c_1 \sqrt[4]{x} + 4c_2 x}{3x^2 + 6x + 9}$$

1.541 problem 557

Internal problem ID [8679]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 557
Date solved : `Monday, October 21, 2024 at 05:19:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2-2x+1\right)y''-x(3+x)\,y'+(4+x)\,y=0$$

1.541.1 Solved as second order ode using Kovacic algorithm

Time used: 0.343 (sec)

Writing the ode as

$$x^2(x-1)^2\,y''+\left(-x^2-3x\right)y'+(4+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2(x-1)^2\\ B&=-x^2-3x\\ C&=4+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 10x - 1}{4x^2 (x - 1)^4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 7x^2 + 10x - 1$$
$$t = 4x^2(x - 1)^4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 10x - 1}{4x^2 (x - 1)^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1027: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2(x-1)^4$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4} + \frac{3}{2x} - \frac{2}{(x-1)^3} + \frac{7}{4(x-1)^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r=-\frac{1}{4x^{2}}-\frac{3}{2\left(x-1\right)}+\frac{4}{\left(x-1\right)^{4}}+\frac{3}{2x}-\frac{2}{\left(x-1\right)^{3}}+\frac{7}{4\left(x-1\right)^{2}}$$

There is pole in r at $x=1$ of order 4, hence $v=2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c=1$ gives

$$[\sqrt{r}]_{c}\approx\frac{2}{\left(x-1\right)^{2}}-\frac{1}{2\left(x-1\right)}+\frac{21}{32}-\frac{9x}{32}+\frac{53(x-1)^{2}}{256}-\frac{149(x-1)^{3}}{1024}+\dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v=2$ the above becomes

$$[\sqrt{r}]_{c}=\frac{2}{\left(x-1\right)^{2}} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-1)^{2}}$ is

$$a=2$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_{c}$ found in eq. (3B). Here c is current pole which is $c=1$. This term becomes $\frac{1}{(x-1)^{3}}$. The coefficient of this term in the sum $[\sqrt{r}]_{c}$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -2. Therefore

$$\begin{aligned}b&=(-2)-(0)\\&=-2\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_{c}&=\frac{2}{\left(x-1\right)^{2}}\\ \alpha_{c}^{+}&=\frac{1}{2}\left(\frac{b}{a}+v\right)&&=\frac{1}{2}\left(\frac{-2}{2}+2\right)&&=\frac{1}{2}\\ \alpha_{c}^{-}&=\frac{1}{2}\left(-\frac{b}{a}+v\right)&&=\frac{1}{2}\left(-\frac{-2}{2}+2\right)&&=\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}[\sqrt{r}]_{\infty}&=0\\ \alpha_{\infty}^{+}&=0\\ \alpha_{\infty}^{-}&=1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2 + 10x - 1}{4x^2 (x - 1)^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	4	$\frac{2}{(x-1)^2}$	$\frac{1}{2}$	$\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x - 2} + (-)(0) \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x - 2} \\ &= \frac{2x^2 + x + 1}{2x (x - 1)^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{4}{(x-1)^3} - \frac{1}{2(x-1)^2}\right) + \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right) dx} \\
&= \sqrt{x}\,\sqrt{x-1}\,\mathrm{e}^{-\frac{2}{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2(x-1)^2} \, dx} \\
&= z_1 e^{\frac{3\ln(x)}{2} - \frac{2}{x-1} - \frac{3\ln(x-1)}{2}} \\
&= z_1 \left(\frac{x^{3/2}\mathrm{e}^{-\frac{2}{x-1}}}{(x-1)^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}}{(x-1)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2(x-1)^2}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{3\ln(x)-\frac{4}{x-1}-3\ln(x-1)}}{(y_1)^2}\,dx$$

$$= y_1 \left(\mathrm{e}^{-4}\,\mathrm{Ei}_1\left(-\frac{4}{x-1}-4\right)\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}}{(x-1)^{3/2}}\right) + c_2 \left(\frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}}{(x-1)^{3/2}}\left(\mathrm{e}^{-4}\,\mathrm{Ei}_1\left(-\frac{4}{x-1}-4\right)\right)\right)$$

Will add steps showing solving for IC soon.

1.541.2 Maple step by step solution

Let's solve

$x^2(x^2-2x+1)\left(\frac{d}{dx}y'\right) - x(3+x)\,y' + (4+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4+x)y}{x^2(x^2-2x+1)} + \frac{(3+x)y'}{x(x^2-2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(3+x)y'}{x(x^2-2x+1)} + \frac{(4+x)y}{x^2(x^2-2x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{3+x}{x(x^2-2x+1)}, P_3(x) = \frac{4+x}{x^2(x^2-2x+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2-2x+1)\left(\frac{d}{dx}y'\right) - x(3+x)\,y' + (4+x)\,y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ◦ Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- ◦ Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- ◦ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- ◦ Shift index using $k->k+2-m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-2+r)^2\,x^r + \left(a_1(-1+r)^2 - a_0(1+2r)\,(-1+r)\right)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(2k -\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term must be 0
  $$a_1(-1+r)^2 - a_0(1+2r)(-1+r) = 0$$
- Solve for the dependent coefficient(s)
  $$a_1 = \frac{a_0(1+2r)}{-1+r}$$
- Each term in the series must be 0, giving the recursion relation
  $$(k+r-2)\left((a_k + a_{k-2} - 2a_{k-1})k + (a_k + a_{k-2} - 2a_{k-1})r - 2a_k - 3a_{k-2} + a_{k-1}\right) = 0$$
- Shift index using $k- >k+2$
  $$(k+r)\left((a_{k+2} + a_k - 2a_{k+1})(k+2) + (a_{k+2} + a_k - 2a_{k+1})r - 2a_{k+2} - 3a_k + a_{k+1}\right) = 0$$
- Recursion relation that defines series solution to ODE
  $$a_{k+2} = -\frac{ka_k - 2ka_{k+1} + ra_k - 2ra_{k+1} - a_k - 3a_{k+1}}{k+r}$$
- Recursion relation for $r=2$
  $$a_{k+2} = -\frac{ka_k - 2ka_{k+1} + a_k - 7a_{k+1}}{k+2}$$
- Solution for $r=2$
  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{ka_k - 2ka_{k+1} + a_k - 7a_{k+1}}{k+2}, a_1 = 5a_0\right]$$

1.541.3 Maple trace

```
Methods for second order ODEs:
```

1.541.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 45

```
dsolve(x^2*(x^2-2*x+1)*diff(diff(y(x),x),x)-x*(3+x)*diff(y(x),x)+(4+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^2\left(c_2\,\mathrm{e}^{-\frac{4x}{x-1}}\,\mathrm{Ei}_1\left(-\frac{4x}{x-1}\right) + \mathrm{e}^{-\frac{4}{x-1}}c_1\right)}{x-1}$$

1.541.5 Mathematica DSolve solution

Solving time : 0.293 (sec)
Leaf size : 54

```
DSolve[{x^2*(1-2*x+x^2)*D[y[x],{x,2}]-x*(3+x)*D[y[x],x]+(4+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{4x}{x-1}}\sqrt{1-x}x^2\left(c_2 \operatorname{ExpIntegralEi}\left(\frac{4x}{x-1}\right)+e^4 c_1\right)}{(x-1)^{3/2}}$$

1.542 problem 558

Internal problem ID [8680]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 558
Date solved : `Monday, October 21, 2024 at 05:19:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''+5x^2y'+(1+x)\,y=0$$

1.542.1 Solved as second order ode using Kovacic algorithm

Time used: 0.241 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y''+5x^2y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2 \\ B &= 5x^2 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 - 24x - 16$$
$$t = 16\left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1029: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{8x} + \frac{5}{16\left(2+x\right)^2} + \frac{1}{16+8x} - \frac{1}{4x^2}
$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-3x^2 - 24x - 16}{16\left(x^2+2x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-3x^2-24x-16}{16\left(x^2+2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= \frac{1}{4}-\left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4\,(2+x)} + \frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{4\,(2+x)} + \frac{1}{2x} \\
&= \frac{x+4}{4x\,(2+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\,(2+x)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{4\,(2+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{4\,(2+x)} + \frac{1}{2x}\right)^2 - \left(\frac{-3x^2-24x-16}{16\,(x^2+2x)^2}\right)
$$

$$
0 =
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{4(2+x)} + \frac{1}{2x}\right) dx} \\
&= \frac{\sqrt{x}}{(2+x)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x^2}{2x^3+4x^2}\, dx} \\
&= z_1 e^{-\frac{5\ln(2+x)}{4}} \\
&= z_1 \left(\frac{1}{(2+x)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(2+x)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2}{2x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(2+x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(2\sqrt{2+x} - 2\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{(2+x)^{3/2}}\right) + c_2\left(\frac{\sqrt{x}}{(2+x)^{3/2}}\left(2\sqrt{2+x} - 2\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.542.2 Maple step by step solution

Let's solve

$2x^2(2+x)\left(\frac{d}{dx}y'\right) + 5x^2y' + (1+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(1+x)y}{2x^2(2+x)} - \frac{5y'}{2(2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + \frac{5y'}{2(2+x)} + \frac{(1+x)y}{2x^2(2+x)} = 0$
- ☐ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{5}{2(2+x)}, P_3(x) = \frac{1+x}{2x^2(2+x)}\right]$
  - $(2+x)\cdot P_2(x)$ is analytic at $x=-2$
    $\left((2+x)\cdot P_2(x)\right)\Big|_{x=-2} = \frac{5}{2}$
  - $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$
    $\left((2+x)^2\cdot P_3(x)\right)\Big|_{x=-2} = 0$
  - $x=-2$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = -2$
- Multiply by denominators
  $2x^2(2+x)\left(\frac{d}{dx}y'\right) + 5x^2y' + (1+x)\,y = 0$
- Change variables using $x = u-2$ so that the regular singular point is at $u=0$
  $(2u^3 - 8u^2 + 8u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (5u^2 - 20u + 20)\left(\frac{d}{du}y(u)\right) + (-1+u)\,y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot y(u)$ to series expansion for $m = 0..1$
    $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k->k-m$
    $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$
    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$
    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)$ to series expansion for $m =1..3$

  $u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$
- Shift index using $k->k +2-m$

  $u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$

  Rewrite ODE with series expansions

  $4 a_{0} r \left(3+2 r \right) u^{-1+r}+\left(4 a_{1} \left(1+r \right) \left(5+2 r \right)-a_{0} \left(8 r^{2}+12 r +1\right)\right) u^{r}+\left(\sum_{k =1}^{\infty}\left(4 a_{k +1} \left(k +r +1\right) \left(2 k \right.\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $4 r \left(3+2 r \right)=0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{3}{2}\right\}$
- Each term must be 0

  $4 a_{1} \left(1+r \right) \left(5+2 r \right)-a_{0} \left(8 r^{2}+12 r +1\right)=0$
- Each term in the series must be 0, giving the recursion relation

  $2 \left(-4 a_{k}+a_{k -1}+4 a_{k +1}\right) k^{2}+\left(4 \left(-4 a_{k}+a_{k -1}+4 a_{k +1}\right) r -12 a_{k}-a_{k -1}+28 a_{k +1}\right) k +2 \left(-4 a_{k}+\right.$
- Shift index using $k->k +1$

  $2 \left(-4 a_{k +1}+a_{k}+4 a_{k +2}\right) \left(k +1\right)^{2}+\left(4 \left(-4 a_{k +1}+a_{k}+4 a_{k +2}\right) r -12 a_{k +1}-a_{k}+28 a_{k +2}\right) \left(k +1\right)$
- Recursion relation that defines series solution to ODE

  $a_{k +2}=-\frac{2 k^{2} a_{k}-8 k^{2} a_{k +1}+4 k r a_{k}-16 k r a_{k +1}+2 r^{2} a_{k}-8 r^{2} a_{k +1}+3 k a_{k}-28 k a_{k +1}+3 r a_{k}-28 r a_{k +1}+a_{k}-21 a_{k +1}}{4 \left(2 k^{2}+4 k r +2 r^{2}+11 k +11 r +14\right)}$
- Recursion relation for $r =0$

  $a_{k +2}=-\frac{2 k^{2} a_{k}-8 k^{2} a_{k +1}+3 k a_{k}-28 k a_{k +1}+a_{k}-21 a_{k +1}}{4 \left(2 k^{2}+11 k +14\right)}$
- Solution for $r =0$

  $\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k}, a_{k +2}=-\frac{2 k^{2} a_{k}-8 k^{2} a_{k +1}+3 k a_{k}-28 k a_{k +1}+a_{k}-21 a_{k +1}}{4 \left(2 k^{2}+11 k +14\right)}, 20 a_{1}-a_{0}=0\right]$
- Revert the change of variables $u =2+x$

  $\left[y =\sum_{k =0}^{\infty}a_{k} \left(2+x \right)^{k}, a_{k +2}=-\frac{2 k^{2} a_{k}-8 k^{2} a_{k +1}+3 k a_{k}-28 k a_{k +1}+a_{k}-21 a_{k +1}}{4 \left(2 k^{2}+11 k +14\right)}, 20 a_{1}-a_{0}=0\right]$
- Recursion relation for $r =-\frac{3}{2}$

  $a_{k +2}=-\frac{2 k^{2} a_{k}-8 k^{2} a_{k +1}-3 k a_{k}-4 k a_{k +1}+a_{k}+3 a_{k +1}}{4 \left(2 k^{2}+5 k +2\right)}$
- Solution for $r =-\frac{3}{2}$

  $\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k -\frac{3}{2}}, a_{k +2}=-\frac{2 k^{2} a_{k}-8 k^{2} a_{k +1}-3 k a_{k}-4 k a_{k +1}+a_{k}+3 a_{k +1}}{4 \left(2 k^{2}+5 k +2\right)}, -4 a_{1}-a_{0}=0\right]$

- Revert the change of variables $u = 2+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (2+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (2+x)^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a$$

1.542.3 Maple trace

```
Methods for second order ODEs:
```

1.542.4 Maple dsolve solution

Solving time : `0.012` (sec)
Leaf size : 39

```
dsolve(2*x^2*(2+x)*diff(diff(y(x),x),x)+5*x^2*diff(y(x),x)+(1+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\left(\sqrt{2}\,\sqrt{2+x}\,c_2 - 2\,\mathrm{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right) c_2 + c_1\right)\sqrt{x}}{(2+x)^{3/2}}$$

1.542.5 Mathematica DSolve solution

Solving time : `0.105` (sec)
Leaf size : 55

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}]+5*x^2*D[y[x],x]+(1+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}\left(-2\sqrt{2}c_2\mathrm{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right) + 2c_2\sqrt{x+2} + c_1\right)}{(x+2)^{3/2}}$$

1.543 problem 559

Internal problem ID [8681]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 559
Date solved : `Monday, October 21, 2024 at 05:19:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-x^2+2\right) y''-2x\left(2x^2+1\right) y'+\left(-2x^2+2\right) y=0$$

1.543.1 Solved as second order ode using Kovacic algorithm

Time used: 0.411 (sec)

Writing the ode as

$$\left(-x^4+2x^2\right) y''+\left(-4x^3-2x\right) y'+\left(-2x^2+2\right) y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+2x^2 \\ B &= -4x^3-2x \\ C &= -2x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 1}{(x^3 - 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^2 - 1$$
$$t = (x^3 - 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 1}{(x^3 - 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1031: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\left(x - \sqrt{2}\right)^2} + \frac{5}{16\left(x + \sqrt{2}\right)^2} - \frac{3\sqrt{2}}{32\left(x - \sqrt{2}\right)} + \frac{3\sqrt{2}}{32\left(x + \sqrt{2}\right)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - \sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^2 - 1}{\left(x^3 - 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\sqrt{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\sqrt{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)} + (0) \\ &= \frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)} \\ &= -\frac{1}{x^3-2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)} \right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{4\left(x-\sqrt{2}\right)^2} + \frac{1}{4\left(x+\sqrt{2}\right)^2} \right) + \left(\frac{1}{2x} - \frac{}{4(} \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)} \right) dx} \\ &= \frac{\sqrt{x}}{\left(x-\sqrt{2}\right)^{1/4} \left(x+\sqrt{2}\right)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^3-2x}{-x^4+2x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln\left(x^2-2\right)}{4}+\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\sqrt{x}}{\left(x^2-2\right)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{\left(x^2-2\right)^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3-2x}{-x^4+2x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(x^2-2\right)}{2}+\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\sqrt{x^2-2}+\sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{\left(x^2-2\right)^{3/2}}\right) + c_2 \left(\frac{x}{\left(x^2-2\right)^{3/2}} \left(\sqrt{x^2-2}+\sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.543.2 Maple step by step solution

Let's solve

$x^2(-x^2+2)\left(\frac{d}{dx}y'\right)-2x(2x^2+1)\,y'+(-2x^2+2)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{2(x^2-1)y}{x^2(x^2-2)}-\frac{2(2x^2+1)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{2(2x^2+1)y'}{x(x^2-2)}+\frac{2(x^2-1)y}{x^2(x^2-2)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2(2x^2+1)}{x(x^2-2)},P_3(x)=\frac{2(x^2-1)}{x^2(x^2-2)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=-1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=1$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $x^2(x^2-2)\left(\frac{d}{dx}y'\right)+2x(2x^2+1)\,y'+(2x^2-2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx} y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0(-1+r)^2\, x^r - 2a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(-2a_k(k+r-1)^2 + a_{k-2}(k+r)\,(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 1$$

- Each term must be 0

$$-2a_1 r^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-2a_k(k+r-1)^2 + a_{k-2}(k+r)\,(k+r-1) = 0$$

- Shift index using $k- > k+2$

$$-2a_{k+2}(k+r+1)^2 + a_k(k+r+2)\,(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a_k(k+r+2)}{2(k+r+1)}$$

- Recursion relation for $r = 1$

$$a_{k+2} = \frac{a_k(k+3)}{2(k+2)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k(k+3)}{2(k+2)}, a_1 = 0\right]$$

1.543.3 Maple trace

```
Methods for second order ODEs:
```

1.543.4 Maple dsolve solution

Solving time : 0.014 (sec)
Leaf size : 42

```
dsolve(x^2*(-x^2+2)*diff(diff(y(x),x),x)-2*x*(2*x^2+1)*diff(y(x),x)+(-2*x^2+2)*y(x) =
        y(x),singsol=all)
```

$$y = \frac{x\left(\sqrt{2}\, c_2\sqrt{x^2-2}+2\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right) c_2+c_1\right)}{\left(x^2-2\right)^{3/2}}$$

1.543.5 Mathematica DSolve solution

Solving time : 0.143 (sec)
Leaf size : 58

```
DSolve[{x^2*(2-x^2)*D[y[x],{x,2}]-2*x*(1+2*x^2)*D[y[x],x]+(2-2*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x\left(-\sqrt{2}c_2\text{arctanh}\left(\sqrt{1-\frac{x^2}{2}}\right)+c_2\sqrt{2-x^2}+c_1\right)}{\left(2-x^2\right)^{3/2}}$$

1.544 problem 560

Internal problem ID [8682]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 560
Date solved : `Monday, October 21, 2024 at 05:19:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(5-x)\,y' + (9-4x)\,y = 0$$

1.544.1 Solved as second order ode using Kovacic algorithm

Time used: 0.461 (sec)

Writing the ode as

$$x^2y'' + \left(x^2-5x\right)y' + (9-4x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2-5x \\ C &= 9-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 6x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1033: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4x^2} + \frac{3}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{2x} - \frac{5}{2x^2} + \frac{15}{2x^3} - \frac{115}{4x^4} + \frac{495}{4x^5} - \frac{2285}{4x^6} + \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{6x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 6. Dividing this by leading coefficient in t which is 4 gives $\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{3}{2}\right) - (0) \\
&= \frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\
&= \frac{3}{2} - \left(\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2x} + \frac{1}{2} \\
&= \frac{1+x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2+6x-1}{4x^2}\right)\right) = 0$$
$$\frac{1-a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1+x) \, e^{\int \left(\frac{1}{2x} + \frac{1}{2}\right) dx} \\
&= (1+x) \, e^{\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= (1+x) \sqrt{x} \, e^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2-5x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\frac{5\ln(x)}{2}} \\
&= z_1 \left(x^{5/2} \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^3(1+x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-5x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\operatorname{Ei}_1(x) - \frac{\mathrm{e}^{-x}}{-1-x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^3(1+x)\right) + c_2\left(x^3(1+x)\left(-\operatorname{Ei}_1(x) - \frac{\mathrm{e}^{-x}}{-1-x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.544.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x(5-x)\,y' + (9-4x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(-9+4x)y}{x^2} - \frac{(-5+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(-5+x)y'}{x} - \frac{(-9+4x)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{-5+x}{x}, P_3(x) = -\frac{-9+4x}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -5$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 9$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x(-5+x)\,y' + (9-4x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r-3)^2 + a_{k-1}(k-5+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 3$$

- Each term in the series must be 0, giving the recursion relation

$$a_k (k+r-3)^2 + a_{k-1}(k-5+r) = 0$$

- Shift index using $k->k+1$

$$a_{k+1}(k-2+r)^2 + a_k (k+r-4) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k (k+r-4)}{(k-2+r)^2}$$

- Recursion relation for $r = 3$; series terminates at $k = 1$

$$a_{k+1} = -\frac{a_k (k-1)}{(k+1)^2}$$

- Apply recursion relation for $k = 0$

$$a_1 = a_0$$

- Terminating series solution of the ODE for $r = 3$. Use reduction of order to find the second li

$$y = a_0 \cdot (1+x)$$

1.544.3 Maple trace

```
Methods for second order ODEs:
```

1.544.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 27

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(5-x)*diff(y(x),x)+(9-4*x)*y(x) = 0,
       y(x),singsol=all)
```

$$
y = x^{3}\left(-c_{2} \mathrm{e}^{-x}+\left(\mathrm{Ei}_{1}\left(x\right) c_{2}+c_{1}\right)\left(1+x\right)\right)
$$

1.544.5 Mathematica DSolve solution

Solving time : 0.099 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]-x*(5-x)*D[y[x],x]+(9-4*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to e^{-x} x^{3}\left(c_{2} e^{x}(x+1) \operatorname{ExpIntegralEi}(-x)+c_{1} e^{x}(x+1)+c_{2}\right)
$$

1.545 problem 561

Internal problem ID [8683]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 561
Date solved : `Monday, October 21, 2024 at 05:19:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+x+1\right)y''+12x^2(1+x)\,y'+\left(3x^2+3x+1\right)y=0$$

1.545.1 Solved as second order ode using Kovacic algorithm

Time used: 0.803 (sec)

Writing the ode as

$$\left(4x^4+4x^3+4x^2\right)y''+\left(12x^3+12x^2\right)y'+\left(3x^2+3x+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^4+4x^3+4x^2\\ B&=12x^3+12x^2\\ C&=3x^2+3x+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int\frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 4x - 1}{4\left(x^3 + x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 - 4x - 1$$
$$t = 4\left(x^3 + x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 4x - 1}{4\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1035: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{2x} - \frac{1}{4x^2} + \frac{-\frac{3}{8} - \frac{i\sqrt{3}}{8}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{3}{8} + \frac{i\sqrt{3}}{8}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{\frac{1}{4} - \frac{5i\sqrt{3}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{4} + \frac{5i\sqrt{3}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{8} - \frac{i\sqrt{3}}{8}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{-2 - 2i\sqrt{3}}}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{-2 - 2i\sqrt{3}}}{4} \end{aligned}$$

For the pole at $x=-\frac{1}{2}-\frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{3}{8}+\frac{i\sqrt{3}}{8}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}+\frac{\sqrt{-2+2i\sqrt{3}}}{4} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}-\frac{\sqrt{-2+2i\sqrt{3}}}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{2x^2-4x-1}{4\left(x^3+x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{-2-2i\sqrt{3}}}{4}$	$\frac{1}{2}-\frac{\sqrt{-2-2i\sqrt{3}}}{4}$
$-\frac{1}{2}-\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{-2+2i\sqrt{3}}}{4}$	$\frac{1}{2}-\frac{\sqrt{-2+2i\sqrt{3}}}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}+\alpha_{c_3}^{-}\right) \\
&= 1-(1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} + (-)(0) \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \\
&= \frac{2x^2 + 1}{2x\,(x^2 + x + 1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} \right) -$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2-2i\sqrt{3}}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-2+2i\sqrt{3}}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right) dx} \\
&= \left(x^2 + x + 1\right)^{1/4} \sqrt{x}\,\sqrt{2}\,e^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{12x^3+12x^2}{4x^4+4x^3+4x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+x+1\right)}{4}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}}{\left(x^2+x+1\right)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\sqrt{x}\,\sqrt{2}}{\sqrt{x^2+x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{12x^3+12x^2}{4x^4+4x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+x+1\right)}{2}-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{3\ln\left(x^2+x+1\right)}{2}-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\left(x^2+x+1\right)\mathrm{e}^{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{2x}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)} \sqrt{x}\, \sqrt{2}}{\sqrt{x^2+x+1}}\right) \\
&\quad + c_2 \left(\frac{e^{-\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)} \sqrt{x}\, \sqrt{2}}{\sqrt{x^2+x+1}} \left(\int \frac{e^{-\frac{3\ln\left(x^2+x+1\right)}{2} - \sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)} \left(x^2+x+1\right) e^{2\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{2x} dx
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.545.2 Maple step by step solution

Let's solve

$4x^2(x^2+x+1)\left(\frac{d}{dx}y'\right) + 12x^2(1+x)\, y' + (3x^2+3x+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(3x^2+3x+1)y}{4x^2(x^2+x+1)} - \frac{3(1+x)y'}{x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{3(1+x)y'}{x^2+x+1} + \frac{(3x^2+3x+1)y}{4x^2(x^2+x+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3(1+x)}{x^2+x+1}, P_3(x) = \frac{3x^2+3x+1}{4x^2(x^2+x+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+x+1)\left(\frac{d}{dx}y'\right) + 12x^2(1+x)\, y' + (3x^2+3x+1)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 2..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

  - Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k- >k + 2 - m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)^2 x^r + \left(a_1(1+2r)^2 + a_0(3+2r)\,(1+2r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-1}(2k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{2}$$

- Each term must be 0

$$a_1(1+2r)^2 + a_0(3+2r)\,(1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{(3+2r)a_0}{1+2r}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{1}{2}\right)\left(\left(a_k + a_{k-2} + a_{k-1}\right)k + \left(a_k + a_{k-2} + a_{k-1}\right)r - \frac{a_k}{2} - \frac{3a_{k-2}}{2} + \frac{a_{k-1}}{2}\right) = 0$$

- Shift index using $k->k+2$

  $4\left(k+\frac{3}{2}+r\right)\left(\left(a_{k+2}+a_{k}+a_{k+1}\right)(k+2)+\left(a_{k+2}+a_{k}+a_{k+1}\right)r-\frac{a_{k+2}}{2}-\frac{3a_{k}}{2}+\frac{a_{k+1}}{2}\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{2ka_{k}+2ka_{k+1}+2ra_{k}+2ra_{k+1}+a_{k}+5a_{k+1}}{2k+2r+3}$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-\frac{2ka_{k}+2ka_{k+1}+2a_{k}+6a_{k+1}}{2k+4}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_{k}x^{k+\frac{1}{2}}, a_{k+2}=-\frac{2ka_{k}+2ka_{k+1}+2a_{k}+6a_{k+1}}{2k+4}, a_{1}=-2a_{0}\right]$

1.545.3 Maple trace

```
Methods for second order ODEs:
```

1.545.4 Maple dsolve solution

Solving time : `0.280` (sec)
Leaf size : 143

```
dsolve(4*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+12*x^2*(1+x)*diff(y(x),x)+(3*x^2+3*x+1)*y(
       y(x),singsol=all)
```

$$y=\frac{\sqrt{i\sqrt{3}-2x-1}\,\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}\sqrt{x}\left(c_{1}\left(\frac{-2ix-i+\sqrt{3}}{\sqrt{3}+2ix+i}\right)^{\frac{1}{4}-\frac{i\sqrt{3}}{4}}+c_{2}\left(\frac{-2ix-i+\sqrt{3}}{\sqrt{3}+2ix+i}\right)^{\frac{3}{4}+\frac{i\sqrt{3}}{4}}\mathrm{hypergeom}\left(\left[1,\frac{1}{2}+\right.\right.\right.}{\left(x^{2}+x+1\right)^{3/4}}$$

1.545.5 Mathematica DSolve solution

Solving time : `1.341` (sec)

Leaf size : 93

```
DSolve[{4*x^2*(1+x+x^2)*D[y[x],{x,2}]+12*x^2*(1+x)*D[y[x],x]+(1+3*x+3*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}e^{-\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}\left(c_2\int_1^x \frac{e^{\sqrt{3}\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}}{K[1]\sqrt{K[1]^2+K[1]+1}}dK[1]+c_1\right)}{\sqrt{x^2+x+1}}$$

1.546 problem 562

Internal problem ID [8684]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 562
Date solved : `Monday, October 21, 2024 at 05:19:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+x+1)\,y''-x(-2x^2-4x+1)\,y'+y=0$$

1.546.1 Solved as second order ode using Kovacic algorithm

Time used: 0.912 (sec)

Writing the ode as

$$x^2(x^2+x+1)\,y''+(2x^3+4x^2-x)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(x^2+x+1) \\ B &= 2x^3+4x^2-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{10x^2 - 8x - 1}{4\left(x^3 + x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 10x^2 - 8x - 1$$
$$t = 4\left(x^3 + x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{10x^2 - 8x - 1}{4\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1037: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{29}{24} - \frac{7i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{29}{24} + \frac{7i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{\frac{3}{4} - \frac{41i\sqrt{3}}{36}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{3}{4} + \frac{41i\sqrt{3}}{36}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} - \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{24} - \frac{7i\sqrt{3}}{24}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{-138 - 42i\sqrt{3}}}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{-138 - 42i\sqrt{3}}}{12}\end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{24} + \frac{7i\sqrt{3}}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{-138+42i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{10x^2 - 8x - 1}{4\left(x^3 + x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{-138-42i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}$
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{-138+42i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} + (-)(0) \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \\
&= \frac{2x^2 - 2x + 1}{2x\,(x^2 + x + 1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} - \frac{\sqrt{-138+42i}}{12}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right.}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right) dx} \\
&= \sqrt{x}\,\left(x^2 + x + 1\right)^{1/4} \sqrt{2}\, \mathrm{e}^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+4x^2-x}{x^2(x^2+x+1)}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}-\frac{3\ln(x^2+x+1)}{4}-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} \\
&= z_1 \left(\frac{\sqrt{x}\,\mathrm{e}^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{(x^2+x+1)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x\,\mathrm{e}^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{\sqrt{x^2+x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+4x^2-x}{x^2(x^2+x+1)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{3\ln(x^2+x+1)}{2}-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\ln(x)-\frac{3\ln(x^2+x+1)}{2}-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}(x^2+x+1)\,\mathrm{e}^{\frac{14\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{2x^2}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x \,\mathrm{e}^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \sqrt{2}}{\sqrt{x^2+x+1}} \right) \\
&\quad + c_2 \left(\frac{x \,\mathrm{e}^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \sqrt{2}}{\sqrt{x^2+x+1}} \left(\int \frac{\mathrm{e}^{\ln(x)-\frac{3\ln\left(x^2+x+1\right)}{2}-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} (x^2+x+1)\,\mathrm{e}^{\frac{14\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{2x^2} \right.\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.546.2 Maple step by step solution

Let's solve

$x^2(x^2+x+1)\left(\frac{d}{dx}y'\right) - x(-2x^2-4x+1)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x^2(x^2+x+1)} - \frac{(2x^2+4x-1)\,y'}{x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(2x^2+4x-1)\,y'}{x(x^2+x+1)} + \frac{y}{x^2(x^2+x+1)} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{2x^2+4x-1}{x(x^2+x+1)}, P_3(x) = \frac{1}{x^2(x^2+x+1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 1$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+x+1)\left(\frac{d}{dx}y'\right) + x(2x^2+4x-1)\,y' + y = 0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m=1..3$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m=2..4$

    $x^m \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2 x^r+\left(a_1 r^2+a_0 r(3+r)\right) x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r-1)^2+a_{k-1}(k+r-1)\,(k+2+r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2=0$

- Values of r that satisfy the indicial equation

  $r=1$

- Each term must be 0

  $a_1 r^2+a_0 r(3+r)=0$

- Solve for the dependent coefficient(s)

  $a_1=-\frac{(3+r)a_0}{r}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left((a_k+a_{k-2}+a_{k-1})\, k+(a_k+a_{k-2}+a_{k-1})\, r-a_k-2a_{k-2}+2a_{k-1}\right)=0$

- Shift index using $k->k+2$

  $(k+r+1)\left((a_{k+2}+a_k+a_{k+1})\,(k+2)+(a_{k+2}+a_k+a_{k+1})\, r-a_{k+2}-2a_k+2a_{k+1}\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{ka_k+ka_{k+1}+ra_k+ra_{k+1}+4a_{k+1}}{k+r+1}$

- Recursion relation for $r=1$

  $a_{k+2}=-\frac{ka_k+ka_{k+1}+a_k+5a_{k+1}}{k+2}$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{k a_k + k a_{k+1} + a_k + 5 a_{k+1}}{k+2}, a_1 = -4 a_0\right]$$

1.546.3 Maple trace

```
Methods for second order ODEs:
```

1.546.4 Maple dsolve solution

Solving time : `0.166` (sec)
Leaf size : 147

```
dsolve(x^2*(x^2+x+1)*diff(diff(y(x),x),x)-x*(-2*x^2-4*x+1)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} x \left(c_2 \left(2x+1+i\sqrt{3}\right)^{\frac{3}{4}+\frac{7i\sqrt{3}}{12}} \left(i\sqrt{3}-2x-1\right)^{-\frac{1}{4}-\frac{7i\sqrt{3}}{12}} \operatorname{hypergeom}\left(\left[1, \frac{1}{2}+\frac{7i\sqrt{3}}{6}\right], \left[\frac{3}{2}+\right.\right.\right.}{\left(x^2+x+1\right)^{3/4}}$$

1.546.5 Mathematica DSolve solution

Solving time : `1.31` (sec)
Leaf size : 90

```
DSolve[{x^2*(1+x+x^2)*D[y[x],{x,2}]-x*(1-4*x-2*x^2)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x e^{-\frac{7\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}} \left(c_2 \int_1^x \frac{e^{\frac{7\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}{\sqrt{3}}}}{K[1]\sqrt{K[1]^2+K[1]+1}} dK[1] + c_1\right)}{\sqrt{x^2+x+1}}$$

1.547 problem 563

Internal problem ID [8685]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 563
Date solved : `Monday, October 21, 2024 at 05:19:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' + 3x\left(-2x^2+3x+5\right)y' + \left(-14x^2+12x+1\right)y = 0$$

1.547.1 Solved as second order ode using Kovacic algorithm

Time used: 0.535 (sec)

Writing the ode as

$$9x^2y'' + \left(-6x^3+9x^2+15x\right)y' + \left(-14x^2+12x+1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= -6x^3+9x^2+15x \\ C &= -14x^2+12x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^4 - 12x^3 + 33x^2 - 18x - 9$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1039: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{9} - \frac{x}{3} + \frac{11}{12} - \frac{1}{4x^2} - \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{3} - \frac{1}{2} + \frac{1}{x} + \frac{3}{4x^2} - \frac{3}{4x^3} - \frac{27}{8x^4} - \frac{117}{32x^5} + \frac{405}{64x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{3}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= -\frac{1}{2} + \frac{x}{3} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4} - \frac{1}{3}x + \frac{1}{9}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{9}x^2 - \frac{1}{3}x + \frac{11}{12}\right) + \left(\frac{-18x - 9}{36x^2}\right) \\ &= \frac{x^2}{9} - \frac{x}{3} + \frac{11}{12} + \frac{-18x - 9}{36x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $\frac{11}{12}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{11}{12}\right) - \left(\frac{1}{4}\right) \\ &= \frac{2}{3} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= -\frac{1}{2} + \frac{x}{3} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$-\frac{1}{2} + \frac{x}{3}$	$\frac{1}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \left(-\frac{1}{2} + \frac{x}{3}\right) \\
&= \frac{1}{2x} - \frac{1}{2} + \frac{x}{3} \\
&= \frac{1}{2x} - \frac{1}{2} + \frac{x}{3}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{3}\right) + \left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right)^2 - \left(\frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}\right)\right) = 0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{\frac{x(x-3)}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-6x^3 + 9x^2 + 15x}{9x^2} \, dx} \\
&= z_1 e^{\frac{x^2}{6} - \frac{x}{2} - \frac{5 \ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x(x-3)}{6}}}{x^{5/6}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^{\frac{x(x-3)}{3}}}{x^{1/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x^3+9x^2+15x}{9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{3}-x-\frac{5\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int e^{\frac{x^2}{3}-x-\frac{5\ln(x)}{3}} x^{2/3} e^{-\frac{2x(x-3)}{3}}\,dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\frac{x(x-3)}{3}}}{x^{1/3}}\right) + c_2 \left(\frac{e^{\frac{x(x-3)}{3}}}{x^{1/3}} \left(\int e^{\frac{x^2}{3}-x-\frac{5\ln(x)}{3}} x^{2/3} e^{-\frac{2x(x-3)}{3}}\,dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.547.2 Maple step by step solution

Let's solve
$9x^2\left(\frac{d}{dx}y'\right) + 3x(-2x^2+3x+5)\,y' + (-14x^2+12x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = \frac{(14x^2-12x-1)y}{9x^2} + \frac{(2x^2-3x-5)y'}{3x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{(2x^2-3x-5)y'}{3x} - \frac{(14x^2-12x-1)y}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$$\left[P_2(x) = -\frac{2x^2-3x-5}{3x}, P_3(x) = -\frac{14x^2-12x-1}{9x^2}\right]$$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{5}{3}$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{9}$$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$9x^2\left(\frac{d}{dx}y'\right) - 3x(2x^2 - 3x - 5)\, y' + (-14x^2 + 12x + 1)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

- ○ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$$

- ○ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+3r)^2 x^r + \left(a_1(4+3r)^2 + 3a_0(4+3r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+1)^2 + 3a_{k-1}(3k+3r+1\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = -\frac{1}{3}$
- Each term must be 0

  $a_1(4+3r)^2 + 3a_0(4+3r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = -\frac{3a_0}{4+3r}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r+1)^2 + (3k+3r+1)(-2a_{k-2} + 3a_{k-1}) = 0$
- Shift index using $k->k+2$

  $a_{k+2}(3k+3r+7)^2 + (3k+3r+7)(-2a_k + 3a_{k+1}) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+3r+7}$
- Recursion relation for $r = -\frac{1}{3}$

  $a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+6}$
- Solution for $r = -\frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+6}, a_1 = -a_0\right]$

1.547.3 Maple trace

```
Methods for second order ODEs:
```

1.547.4 Maple dsolve solution

Solving time : 0.047 (sec)
Leaf size : 32

```
dsolve(9*x^2*diff(diff(y(x),x),x)+3*x*(-2*x^2+3*x+5)*diff(y(x),x)+(-14*x^2+12*x+1)*y(x
       y(x),singsol=all)
```

$$y = \frac{e^{\frac{x(x-3)}{3}} \left(c_2 \left(\int \frac{e^{-\frac{x(x-3)}{3}}}{x} dx\right) + c_1\right)}{x^{1/3}}$$

1.547.5 Mathematica DSolve solution

Solving time : 0.521 (sec)
Leaf size : 52

```
DSolve[{9*x^2*D[y[x],{x,2}]+3*x*(5+3*x-2*x^2)*D[y[x],x]+(1+12*x-14*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{\frac{1}{3}(x-3)x} \left(c_2 \int_1^x \frac{e^{K[1]-\frac{K[1]^2}{3}}}{K[1]} dK[1] + c_1\right)}{\sqrt[3]{x}}$$

1.548 problem 564

Internal problem ID [8686]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 564
Date solved : `Monday, October 21, 2024 at 05:19:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+2x)\,y''+x\left(3x^2+14x+5\right)y'+\left(12x^2+18x+4\right)y=0$$

1.548.1 Solved as second order ode using Kovacic algorithm

Time used: 0.580 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(3x^3+14x^2+5x\right)y'+\left(12x^2+18x+4\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+x^2\\B&=3x^3+14x^2+5x\\C&=12x^2+18x+4\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{4(2x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9x^4 - 12x^3 - 16x^2 - 4x - 1$$
$$t = 4(2x^2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{4(2x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1041: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{16} - \frac{1}{4x^2} - \frac{15}{64\left(x + \frac{1}{2}\right)^2} - \frac{21}{16\left(x + \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{3}{8} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$[\sqrt{r}]_\infty = \sum_{i=0}^{v} a_i x^i$$
$$= \sum_{i=0}^{0} a_i x^i \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{4} - \frac{7}{8x} - \frac{19}{48x^2} - \frac{151}{288x^3} - \frac{139}{192x^4} - \frac{11383}{10368x^5} - \frac{38729}{20736x^6} - \frac{1212655}{373248x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$[\sqrt{r}]_\infty = \sum_{i=0}^{0} a_i x^i$$
$$= \frac{3}{4} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t}\\
&= \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2}\\
&= Q + \frac{R}{16x^4 + 16x^3 + 4x^2}\\
&= \left(\frac{9}{16}\right) + \left(\frac{-21x^3 - \frac{73}{4}x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2}\right)\\
&= \frac{9}{16} + \frac{-21x^3 - \frac{73}{4}x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -21. Dividing this by leading coefficient in t which is 16 gives $-\frac{21}{16}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{21}{16}\right) - (0)\\
&= -\frac{21}{16}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{4}\\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{21}{16}}{\frac{3}{4}} - 0\right) &= -\frac{7}{8}\\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{21}{16}}{\frac{3}{4}} - 0\right) &= \frac{7}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{4\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{3}{4}$	$-\frac{7}{8}$	$\frac{7}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\ &= \frac{7}{8} - \left(\frac{7}{8}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} + (-)\left(\frac{3}{4}\right) \\ &= \frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4} \\ &= \frac{-3x^2+2x+1}{4x^2+2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{3}{8\left(x+\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4}\right)^2 - \left(\frac{9x^4 - 12x}{4}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4}\right) dx} \\
&= \sqrt{x}\,(1+2x)^{3/8}\,\mathrm{e}^{-\frac{3x}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3+14x^2+5x}{2x^3+x^2} \, dx} \\
&= z_1 e^{-\frac{3x}{4} - \frac{5\ln(x)}{2} - \frac{5\ln(1+2x)}{8}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{3x}{4}}}{x^{5/2}\,(1+2x)^{5/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-\frac{3x}{2}}}{x^2\,(1+2x)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+14x^2+5x}{2x^3+x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{3x}{2} - 5\ln(x) - \frac{5\ln(1+2x)}{4}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \mathrm{e}^{-\frac{3x}{2} - 5\ln(x) - \frac{5\ln(1+2x)}{4}} x^4 \sqrt{1+2x}\,\mathrm{e}^{3x} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{x^2 (1+2x)^{1/4}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{x^2 (1+2x)^{1/4}} \left(\int \mathrm{e}^{-\frac{3x}{2}-5\ln(x)-\frac{5\ln(1+2x)}{4}} x^4 \sqrt{1+2x}\,\mathrm{e}^{3x} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.548.2 Maple step by step solution

Let's solve

$x^2(1+2x)\left(\frac{d}{dx}y'\right) + x(3x^2+14x+5)\,y' + (12x^2+18x+4)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2(6x^2+9x+2)y}{x^2(1+2x)} - \frac{(3x^2+14x+5)y'}{x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3x^2+14x+5)y'}{x(1+2x)} + \frac{2(6x^2+9x+2)y}{x^2(1+2x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3x^2+14x+5}{x(1+2x)}, P_3(x) = \frac{2(6x^2+9x+2)}{x^2(1+2x)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 5$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(1+2x)\left(\frac{d}{dx}y'\right) + x(3x^2+14x+5)\,y' + (12x^2+18x+4)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^{m}\cdot y$ to series expansion for $m = 0..2$

  $x^{m}\cdot y = \sum_{k=0}^{\infty} a_{k} x^{k+r+m}$
- Shift index using $k->k-m$

  $x^{m}\cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^{m}\cdot y'$ to series expansion for $m = 1..3$

  $x^{m}\cdot y' = \sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^{m}\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

  $x^{m}\cdot\left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^{m}\cdot\left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(2+r)^{2} x^{r} + \left(a_{1}(3+r)^{2} + 2a_{0}(3+r)^{2}\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_{k}(k+r+2)^{2} + 2a_{k-1}(k+r+2)^{2} + 3a\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(2+r)^{2} = 0$
- Values of r that satisfy the indicial equation

  $r = -2$
- Each term must be 0

  $a_{1}(3+r)^{2} + 2a_{0}(3+r)^{2} = 0$
- Solve for the dependent coefficient(s)

  $a_{1} = -2a_{0}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r+2)\left((2k+2r+4)\, a_{k-1} + (k+r+2)\, a_{k} + 3a_{k-2}\right) = 0$
- Shift index using $k->k+2$

  $(k+r+4)\left((2k+8+2r)\, a_{k+1} + (k+r+4)\, a_{k+2} + 3a_{k}\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2k a_{k+1} + 2r a_{k+1} + 3a_{k} + 8a_{k+1}}{k+r+4}$
- Recursion relation for $r = -2$

$a_{k+2} = -\frac{2ka_{k+1}+3a_k+4a_{k+1}}{k+2}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2ka_{k+1}+3a_k+4a_{k+1}}{k+2}, a_1 = -2a_0\right]$

1.548.3 Maple trace

```
Methods for second order ODEs:
```

1.548.4 Maple dsolve solution

Solving time : `0.046` (sec)
Leaf size : 53

```
dsolve(x^2*(1+2*x)*diff(diff(y(x),x),x)+x*(3*x^2+14*x+5)*diff(y(x),x)+(12*x^2+18*x+4)*
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{3x}{2}}\left((1+2x)^{1/4}\operatorname{HeunC}\left(-\frac{3}{4}, \frac{1}{4}, 0, \frac{21}{32}, -\frac{5}{32}, 1+2x\right)c_2 + \operatorname{HeunC}\left(-\frac{3}{4}, -\frac{1}{4}, 0, \frac{21}{32}, -\frac{5}{32}, 1+2x\right)c_1\right)}{(1+2x)^{1/4}\,x^2}$$

1.548.5 Mathematica DSolve solution

Solving time : `0.663` (sec)
Leaf size : 61

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]+x*(5+14*x+3*x^2)*D[y[x],x]+(4+18*x+12*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-3x/2}\left(c_2\int_1^x \frac{e^{\frac{3K[1]}{2}}}{K[1](2K[1]+1)^{3/4}}dK[1] + c_1\right)}{x^2\sqrt[4]{2x+1}}$$

1.549 problem 565

Internal problem ID [8687]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 565
Date solved : `Monday, October 21, 2024 at 05:19:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$16x^2y'' + 4x\left(2x^2 + x + 6\right)y' + \left(18x^2 + 5x + 1\right)y = 0$$

1.549.1 Solved as second order ode using Kovacic algorithm

Time used: 0.517 (sec)

Writing the ode as

$$16x^2y'' + \left(8x^3 + 4x^2 + 24x\right)y' + \left(18x^2 + 5x + 1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^2 \\ B &= 8x^3 + 4x^2 + 24x \\ C &= 18x^2 + 5x + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4x^4 + 4x^3 - 31x^2 - 8x - 16 \\
t &= 64x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1043: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} + \frac{x}{16} - \frac{31}{64} - \frac{1}{4x^2} - \frac{1}{8x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} + \frac{1}{8} - \frac{1}{x} + \frac{1}{4x^2} - \frac{21}{8x^3} + \frac{37}{16x^4} - \frac{377}{32x^5} + \frac{1137}{64x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{1}{8} + \frac{x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64} + \frac{1}{16}x + \frac{1}{16}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{64}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2} \\ &= Q + \frac{R}{64x^2} \\ &= \left(\frac{1}{16}x^2 + \frac{1}{16}x - \frac{31}{64}\right) + \left(\frac{-8x - 16}{64x^2}\right) \\ &= \frac{x^2}{16} + \frac{x}{16} - \frac{31}{64} + \frac{-8x - 16}{64x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{31}{64}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{31}{64}\right) - \left(\frac{1}{64}\right) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{8} + \frac{x}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{1}{8} + \frac{x}{4}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{1}{8} + \frac{x}{4}\right) \\
&= \frac{1}{2x} - \frac{1}{8} - \frac{x}{4} \\
&= \frac{1}{2x} - \frac{1}{8} - \frac{x}{4}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right)^2 - \left(\frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-\frac{x(x+1)}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{8x^3 + 4x^2 + 24x}{16x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{8} - \frac{x}{8} - \frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x(x+1)}{8}}}{x^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{x(x+1)}{4}}}{x^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8x^3+4x^2+24x}{16x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{4}-\frac{x}{4}-\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \mathrm{e}^{-\frac{x^2}{4}-\frac{x}{4}-\frac{3\ln(x)}{2}} \sqrt{x}\,\mathrm{e}^{\frac{x(x+1)}{2}}\,dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\mathrm{e}^{-\frac{x(x+1)}{4}}}{x^{1/4}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{x(x+1)}{4}}}{x^{1/4}} \left(\int \mathrm{e}^{-\frac{x^2}{4}-\frac{x}{4}-\frac{3\ln(x)}{2}} \sqrt{x}\,\mathrm{e}^{\frac{x(x+1)}{2}}\,dx\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.549.2 Maple step by step solution

Let's solve

$16x^2\left(\frac{d}{dx}y'\right) + 4x(2x^2+x+6)\,y' + (18x^2+5x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(18x^2+5x+1)y}{16x^2} - \frac{(2x^2+x+6)y'}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(2x^2+x+6)y'}{4x} + \frac{(18x^2+5x+1)y}{16x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2x^2+x+6}{4x}, P_3(x) = \frac{18x^2+5x+1}{16x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $$\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{3}{2}$$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{16}$$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $$16x^2\left(\frac{d}{dx}y'\right) + 4x(2x^2 + x + 6)\,y' + (18x^2 + 5x + 1)\,y = 0$$
- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$$
  - Shift index using $k- >k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$$

    Rewrite ODE with series expansions

$$a_0(1+4r)^2 x^r + \left(a_1(5+4r)^2 + a_0(5+4r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r+1)^2 + a_{k-1}(4k+4r+1) \right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+4r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = -\frac{1}{4}$
- Each term must be 0
  $a_1(5+4r)^2 + a_0(5+4r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = -\frac{a_0}{5+4r}$
- Each term in the series must be 0, giving the recursion relation
  $a_k(4k+4r+1)^2 + (4k+4r+1)(2a_{k-2} + a_{k-1}) = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(4k+4r+9)^2 + (4k+4r+9)(2a_k + a_{k+1}) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+4r+9}$
- Recursion relation for $r = -\frac{1}{4}$
  $a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+8}$
- Solution for $r = -\frac{1}{4}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}, a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+8}, a_1 = -\frac{a_0}{4}\right]$

1.549.3 Maple trace

```
Methods for second order ODEs:
```

1.549.4 Maple dsolve solution

Solving time : 0.047 (sec)
Leaf size : 32

```
dsolve(16*x^2*diff(diff(y(x),x),x)+4*x*(2*x^2+x+6)*diff(y(x),x)+(18*x^2+5*x+1)*y(x) =
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{x(x+1)}{4}} \left(c_2 \left(\int \frac{e^{\frac{x(x+1)}{4}}}{x} dx\right) + c_1\right)}{x^{1/4}}$$

1.549.5 Mathematica DSolve solution

Solving time : 0.487 (sec)
Leaf size : 51

```
DSolve[{16*x^2*D[y[x],{x,2}]+4*x*(6+x+2*x^2)*D[y[x],x]+(1+5*x+18*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{1}{4}x(x+1)} \left(c_2 \int_1^x \frac{e^{\frac{1}{4}K[1](K[1]+1)}}{K[1]} dK[1] + c_1\right)}{\sqrt[4]{x}}$$

1.550 problem 566

Internal problem ID [8688]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 566
Date solved : `Monday, October 21, 2024 at 05:20:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2(1+x)\,y''+3x\left(-x^2+11x+5\right)y'+\left(-7x^2+16x+1\right)y=0$$

1.550.1 Solved as second order ode using Kovacic algorithm

Time used: 0.384 (sec)

Writing the ode as

$$\left(9x^3+9x^2\right)y''+\left(-3x^3+33x^2+15x\right)y'+\left(-7x^2+16x+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=9x^3+9x^2\\ B&=-3x^3+33x^2+15x\\ C&=-7x^2+16x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 + 6x^3 + 3x^2 - 18x - 9$$
$$t = 36\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1045: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 4 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{36} - \frac{1}{4x^2} + \frac{7}{36(1+x)^2} + \frac{1}{9+9x}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{6} + \frac{1}{3x} - \frac{5}{6x^2} + \frac{5}{6x^3} - \frac{7}{3x^4} + \frac{41}{6x^5} - \frac{149}{6x^6} + \frac{277}{3x^7} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{6}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{6}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t}\\
&= \frac{x^4+6x^3+3x^2-18x-9}{36x^4+72x^3+36x^2}\\
&= Q + \frac{R}{36x^4+72x^3+36x^2}\\
&= \left(\frac{1}{36}\right) + \left(\frac{4x^3+2x^2-18x-9}{36x^4+72x^3+36x^2}\right)\\
&= \frac{1}{36} + \frac{4x^3+2x^2-18x-9}{36x^4+72x^3+36x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is 4. Dividing this by leading coefficient in t which is 36 gives $\frac{1}{9}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{1}{9}\right) - (0)\\
&= \frac{1}{9}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{6}\\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{9}}{\frac{1}{6}} - 0\right) &= \frac{1}{3}\\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{9}}{\frac{1}{6}} - 0\right) &= -\frac{1}{3}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4+6x^3+3x^2-18x-9}{36\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to

determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{+} = \frac{1}{3}$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{6\,(1+x)} + \frac{1}{2x} + \left(\frac{1}{6}\right) \\
&= -\frac{1}{6\,(1+x)} + \frac{1}{2x} + \frac{1}{6} \\
&= -\frac{1}{6+6x} + \frac{1}{2x} + \frac{1}{6}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6\,(1+x)} + \frac{1}{2x} + \frac{1}{6}\right)(0) + \left(\left(\frac{1}{6\,(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{6\,(1+x)} + \frac{1}{2x} + \frac{1}{6}\right)^2 - \left(\frac{x^4 + 6x^3 +}{36\,(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{6(1+x)}+\frac{1}{2x}+\frac{1}{6}\right)dx} \\
&= \frac{\sqrt{x}\, \mathrm{e}^{\frac{x}{6}}}{(1+x)^{1/6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x^3+33x^2+15x}{9x^3+9x^2}\, dx} \\
&= z_1 e^{\frac{x}{6}-\frac{7\ln(1+x)}{6}-\frac{5\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{6}}}{(1+x)^{7/6}\, x^{5/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{\frac{x}{3}}}{(1+x)^{4/3}\, x^{1/3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+33x^2+15x}{9x^3+9x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x}{3}-\frac{7\ln(1+x)}{3}-\frac{5\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \mathrm{e}^{\frac{x}{3}-\frac{7\ln(1+x)}{3}-\frac{5\ln(x)}{3}} (1+x)^{8/3}\, x^{2/3} \mathrm{e}^{-\frac{2x}{3}}\, dx\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{e^{\frac{x}{3}}}{(1+x)^{4/3} x^{1/3}}\right) + c_2 \left(\frac{e^{\frac{x}{3}}}{(1+x)^{4/3} x^{1/3}} \left(\int e^{\frac{x}{3}-\frac{7\ln(1+x)}{3}-\frac{5\ln(x)}{3}} (1+x)^{8/3} x^{2/3} e^{-\frac{2x}{3}} dx\right)\right)$$

Will add steps showing solving for IC soon.

1.550.2 Maple step by step solution

Let's solve

$9x^2(1+x)\left(\frac{d}{dx}y'\right) + 3x(-x^2+11x+5)\,y' + (-7x^2+16x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$
- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(7x^2-16x-1)y}{9x^2(1+x)} + \frac{(x^2-11x-5)y'}{3x(1+x)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x^2-11x-5)y'}{3x(1+x)} - \frac{(7x^2-16x-1)y}{9x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{x^2-11x-5}{3x(1+x)}, P_3(x) = -\frac{7x^2-16x-1}{9x^2(1+x)}\right]$
- ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{7}{3}$
- ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$
- ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $9x^2(1+x)\left(\frac{d}{dx}y'\right) - 3x(x^2-11x-5)\,y' + (-7x^2+16x+1)\,y = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(9u^3-18u^2+9u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-3u^3+42u^2-60u+21)\left(\frac{d}{du}y(u)\right) + (-7u^2+30u-22)\,y(u) =$
- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r) u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r) u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du} y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du} y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r) u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0 r(4+3r) u^{-1+r} + \left(3a_1(1+r)(7+3r) - 2a_0(9r^2+21r+11)\right) u^r + \left(3a_2(2+r)(10+3r) - 2a\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $3r(4+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{4}{3}\right\}$
- The coefficients of each power of u must be 0
  $[3a_1(1+r)(7+3r) - 2a_0(9r^2+21r+11) = 0, 3a_2(2+r)(10+3r) - 2a_1(9r^2+39r+41) + 3a$
- Solve for the dependent coefficient(s)
  $\left\{a_1 = \frac{2a_0(9r^2+21r+11)}{3(3r^2+10r+7)}, a_2 = \frac{a_0(243r^4+1593r^3+3699r^2+3567r+1174)}{9(9r^4+78r^3+241r^2+312r+140)}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $9(-2a_k + a_{k-1} + a_{k+1}) k^2 + 3(6(-2a_k + a_{k-1} + a_{k+1}) r - 14a_k - a_{k-2} + 5a_{k-1} + 10a_{k+1}) k + 9($
- Shift index using $k- >k+2$

$$9(-2a_{k+2}+a_{k+1}+a_{k+3})\,(k+2)^2+3(6(-2a_{k+2}+a_{k+1}+a_{k+3})\,r-14a_{k+2}-a_k+5a_{k+1}+10a_{k+}$$

- Recursion relation that defines series solution to ODE

$$a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}+18kra_{k+1}-36kra_{k+2}+9r^2a_{k+1}-18r^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-3ra_k+51ra_{k+1}-114ra_{}}{3(3k^2+6kr+3r^2+22k+22r+39)}$$

- Recursion relation for $r=0$

$$a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k, a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}, a_1=\frac{22a_0}{21}, a_2=\right.$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k, a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}, a_1=\frac{22a_0}{21},\right.$$

- Recursion relation for $r=-\frac{4}{3}$

$$a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}$$

- Solution for $r=-\frac{4}{3}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-\frac{4}{3}}, a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}, a_1=\frac{2a_0}{3}, a_2=\right.$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k-\frac{4}{3}}, a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}, a_1=\frac{2a_0}{3},\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k(1+x)^k\right)+\left(\sum_{k=0}^{\infty}b_k(1+x)^{k-\frac{4}{3}}\right), a_{k+3}=-\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_{}}{3(3k^2+22k+39)}\right.$$

1.550.3 Maple trace

```
Methods for second order ODEs:
```

1.550.4 Maple dsolve solution

Solving time : 0.045 (sec)
Leaf size : 36

```
dsolve(9*x^2*(1+x)*diff(diff(y(x),x),x)+3*x*(-x^2+11*x+5)*diff(y(x),x)+(-7*x^2+16*x+1)
       y(x),singsol=all)
```

$$y = \frac{\frac{c_1 \operatorname{HeunC}\left(-\frac{1}{3},-\frac{4}{3},0,-\frac{1}{9},\frac{11}{18},1+x\right)}{(1+x)^{4/3}} + c_2 \operatorname{HeunC}\left(-\frac{1}{3},\frac{4}{3},0,-\frac{1}{9},\frac{11}{18},1+x\right)}{x^{1/3}}$$

1.550.5 Mathematica DSolve solution

Solving time : 0.206 (sec)
Leaf size : 50

```
DSolve[{9*x^2*(1+x)*D[y[x],{x,2}]+3*x*(5+11*x-x^2)*D[y[x],x]+(1+16*x-7*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{x/3}\left(c_1 - \sqrt[3]{3e}c_2\Gamma\left(\frac{1}{3},\frac{x+1}{3}\right)\right)}{\sqrt[3]{x}(x+1)^{4/3}}$$

1.551 problem 567

Internal problem ID [8689]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 567
Date solved : `Monday, October 21, 2024 at 05:20:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$36x^2(1-2x)\,y''+24x(1-9x)\,y'+(1-70x)\,y=0$$

1.551.1 Solved as second order ode using Kovacic algorithm

Time used: 0.357 (sec)

Writing the ode as

$$\left(-72x^3+36x^2\right)y''+\left(-216x^2+24x\right)y'+(1-70x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-72x^3+36x^2\\B&=-216x^2+24x\\C&=1-70x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-32x^2 + 48x - 9}{36 \left(2x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -32x^2 + 48x - 9 \\ t &= 36\left(2x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-32x^2 + 48x - 9}{36 \left(2x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1047: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{1}{3x} + \frac{7}{36\left(x - \frac{1}{2}\right)^2} - \frac{1}{3\left(x - \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-32x^2 + 48x - 9}{36\left(2x^2 - x\right)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-32x^2 + 48x - 9}{36\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)} + (-)(0) \\ &= \frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)} \\ &= \frac{-3+4x}{12x^2-6x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{6\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{-32x^2+48x-9}{36\left(2x^2-x\right)^2}\right)\right.$$

$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\,dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)}\right)dx} \\ &= \frac{\sqrt{x}}{(-1+2x)^{1/6}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-216x^2+24x}{-72x^3+36x^2}\,dx} \\ &= z_1 e^{-\frac{7\ln(-1+2x)}{6} - \frac{\ln(x)}{3}} \\ &= z_1\left(\frac{1}{(-1+2x)^{7/6}\,x^{1/3}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/6}}{(-1+2x)^{4/3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-216x^2+24x}{-72x^3+36x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(-1+2x)}{3}-\frac{2\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \Bigg(3(-1+2x)^{1/3} \\
&\quad -\ln\left((-1+2x)^{1/3}+1\right)+\frac{\ln\left((-1+2x)^{2/3}-(-1+2x)^{1/3}+1\right)}{2}-\sqrt{3}\arctan\left(\frac{\left(-1+2(-1+2x)^{1/3}\right.}{3}\right.
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/6}}{(-1+2x)^{4/3}}\right) \\
&\quad + c_2\left(\frac{x^{1/6}}{(-1+2x)^{4/3}}\left(3(-1+2x)^{1/3}-\ln\left((-1+2x)^{1/3}+1\right)+\frac{\ln\left((-1+2x)^{2/3}-(-1+2x)^{1/3}+1\right)}{2}-\sqrt{\right.\right.
\end{aligned}$$

Will add steps showing solving for IC soon.

1.551.2 Maple step by step solution

Let's solve

$36x^2(1-2x)\left(\frac{d}{dx}y'\right)+24x(1-9x)\,y'+(1-70x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$
- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(70x-1)y}{36x^2(-1+2x)}-\frac{2(-1+9x)y'}{3x(-1+2x)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{2(-1+9x)y'}{3x(-1+2x)}+\frac{(70x-1)y}{36x^2(-1+2x)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{2(-1+9x)}{3x(-1+2x)},P_3(x)=\frac{70x-1}{36x^2(-1+2x)}\right]$
- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=\frac{2}{3}$
- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=\frac{1}{36}$
- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $36x^2(-1+2x)\left(\frac{d}{dx}y'\right)+24x(-1+9x)\,y'+(70x-1)\,y=0$
- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
- ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
- ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+6r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(6k+6r-1)^2 + 2a_{k-1}(6k+1+6r)\,(6k+6r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-1+6r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{6}$$

- Each term in the series must be 0, giving the recursion relation

$$-36\left(k+r-\frac{1}{6}\right)\left(\left(-2k-2r-\frac{1}{3}\right) a_{k-1} + a_k\left(k+r-\frac{1}{6}\right)\right) = 0$$

- Shift index using $k->k+1$

$$-36\left(k+\frac{5}{6}+r\right)\left(\left(-2k-\frac{7}{3}-2r\right) a_k + a_{k+1}\left(k+\frac{5}{6}+r\right)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2(6k+6r+7)a_k}{6k+6r+5}$$

- Recursion relation for $r = \frac{1}{6}$

$$a_{k+1} = \frac{2(6k+8)a_k}{6k+6}$$

- Solution for $r = \frac{1}{6}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{6}}, a_{k+1} = \frac{2(6k+8)a_k}{6k+6}\right]$$

1.551.3 Maple trace

```
Methods for second order ODEs:
```

1.551.4 Maple dsolve solution

Solving time : 0.036 (sec)
Leaf size : 93

```
dsolve(36*x^2*(1-2*x)*diff(diff(y(x),x),x)+24*x*(1-9*x)*diff(y(x),x)+(1-70*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^{1/6}\left(2\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,(-1+2x)^{1/3}}{-2+(-1+2x)^{1/3}}\right)c_2 - 2\ln\left((-1+2x)^{1/3}+1\right)c_2 + \ln\left((-1+2x)^{2/3} - (-1+2x)^{1/3} + \right.}{3\,(-1+2x)^{4/3}}$$

1.551.5 Mathematica DSolve solution

Solving time : 0.183 (sec)
Leaf size : 111

```
DSolve[{36*x^2*(1-2*x)*D[y[x],{x,2}]+24*x*(1-9*x)*D[y[x],x]+(1-70*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[6]{x}\left(-2\sqrt{3}c_2\arctan\left(\frac{2\sqrt[3]{1-2x}+1}{\sqrt{3}}\right) + 6c_2\sqrt[3]{1-2x} + 2c_2\log\left(\sqrt[3]{1-2x}-1\right) - c_2\log\left((1-2x)^{2/3} + \sqrt[3]{1}\right.\right.}{2(1-2x)^{4/3}}$$

1.552 problem 568

Internal problem ID [8690]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 568
Date solved : `Monday, October 21, 2024 at 05:20:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\, y'' - x(3-x)\, y' + 4y = 0$$

1.552.1 Solved as second order ode using Kovacic algorithm

Time used: 0.263 (sec)

Writing the ode as

$$x^2(1+x)\, y'' + \left(x^2-3x\right) y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= x^2-3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 10x - 1$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1049: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x} + \frac{2}{(1+x)^2} + \frac{2}{1+x} - \frac{1}{4x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4(x^2 + x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2-10x-1}{4\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{1+x} + \frac{1}{2x} + (-)(0) \\
&= -\frac{1}{1+x} + \frac{1}{2x} \\
&= -\frac{x-1}{2x(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{1+x} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{1+x} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2 - 10x - 1}{4(x^2+x)^2}\right)\right) = 0
$$

$$
\frac{1 + a_0}{x(1+x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-1)\, e^{\int \left(-\frac{1}{1+x} + \frac{1}{2x}\right) dx} \\
&= (x-1)\, e^{-\ln(1+x) + \frac{\ln(x)}{2}} \\
&= \frac{(x-1)\sqrt{x}}{1+x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2-3x}{x^2(1+x)}\,dx} \\
&= z_1 e^{-2\ln(1+x)+\frac{3\ln(x)}{2}} \\
&= z_1\left(\frac{x^{3/2}}{(1+x)^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2(x-1)}{(1+x)^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-3x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(1+x)+3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{4}{x-1}+\ln(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2(x-1)}{(1+x)^3}\right) + c_2\left(\frac{x^2(x-1)}{(1+x)^3}\left(-\frac{4}{x-1}+\ln(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.552.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right)-x(3-x)\,y'+4y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{4y}{x^2(1+x)}-\frac{(x-3)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(x-3)y'}{x(1+x)}+\frac{4y}{x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{x-3}{x(1+x)},P_3(x)=\frac{4}{x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=4$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\left(\frac{d}{dx}y'\right)+x(x-3)\,y'+4y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(u^2-5u+4)\left(\frac{d}{du}y(u)\right)+4y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty}a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)$ to series expansion for $m =1..3$

$$u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- Shift index using $k->k +2-m$

$$u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$a_{0} r \left(3+r \right) u^{-1+r}+\left(a_{1} \left(1+r \right) \left(4+r \right)-a_{0} \left(2 r^{2}+3 r -4\right)\right) u^{r}+\left(\sum_{k =1}^{\infty}\left(a_{k +1} \left(k +1+r \right) \left(k +4+r \right)\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation
$$r \left(3+r \right)=0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{-3, 0\right\}$$
- Each term must be 0
$$a_{1} \left(1+r \right) \left(4+r \right)-a_{0} \left(2 r^{2}+3 r -4\right)=0$$
- Each term in the series must be 0, giving the recursion relation
$$a_{k +1} \left(k +1+r \right) \left(k +4+r \right)-a_{k} \left(2 k^{2}+4 k r +2 r^{2}+3 k +3 r -4\right)+a_{k -1} \left(k +r -1\right)^{2}=0$$
- Shift index using $k->k +1$
$$a_{k +2} \left(k +2+r \right) \left(k +5+r \right)-a_{k +1} \left(2 \left(k +1\right)^{2}+4 \left(k +1\right) r +2 r^{2}+3 k -1+3 r \right)+a_{k} \left(k +r \right)^{2}=$$
- Recursion relation that defines series solution to ODE
$$a_{k +2}=-\frac{k^{2} a_{k}-2 k^{2} a_{k +1}+2 k r a_{k}-4 k r a_{k +1}+r^{2} a_{k}-2 r^{2} a_{k +1}-7 k a_{k +1}-7 r a_{k +1}-a_{k +1}}{\left(k +2+r \right) \left(k +5+r \right)}$$
- Recursion relation for $r =-3$
$$a_{k +2}=-\frac{k^{2} a_{k}-2 k^{2} a_{k +1}-6 k a_{k}+5 k a_{k +1}+9 a_{k}+2 a_{k +1}}{\left(k -1\right) \left(k +2\right)}$$
- Series not valid for $r =-3$, division by 0 in the recursion relation at $k =1$
$$a_{k +2}=-\frac{k^{2} a_{k}-2 k^{2} a_{k +1}-6 k a_{k}+5 k a_{k +1}+9 a_{k}+2 a_{k +1}}{\left(k -1\right) \left(k +2\right)}$$
- Recursion relation for $r =0$
$$a_{k +2}=-\frac{k^{2} a_{k}-2 k^{2} a_{k +1}-7 k a_{k +1}-a_{k +1}}{\left(k +2\right) \left(k +5\right)}$$
- Solution for $r =0$
$$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k}, a_{k +2}=-\frac{k^{2} a_{k}-2 k^{2} a_{k +1}-7 k a_{k +1}-a_{k +1}}{\left(k +2\right) \left(k +5\right)}, 4 a_{1}+4 a_{0}=0\right]$$
- Revert the change of variables $u =1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 7k a_{k+1} - a_{k+1}}{(k+2)(k+5)}, 4a_1 + 4a_0 = 0\right]$$

1.552.3 Maple trace

```
Methods for second order ODEs:
```

1.552.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 30

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)-x*(3-x)*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(c_2(x-1)\ln(x) + c_1 x - c_1 - 4c_2)\, x^2}{(1+x)^3}$$

1.552.5 Mathematica DSolve solution

Solving time : `0.091` (sec)
Leaf size : 33

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-x*(3-x)*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^2(c_1(x-1) + c_2(x-1)\log(x) - 4c_2)}{(x+1)^3}$$

1.553 problem 569

Internal problem ID [8691]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 569
Date solved : `Monday, October 21, 2024 at 05:20:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-2x)\, y'' - x(5-4x)\, y' + (9-4x)\, y = 0$$

1.553.1 Solved as second order ode using Kovacic algorithm

Time used: 0.283 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right) y'' + \left(4x^2-5x\right) y' + (9-4x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= 4x^2-5x \\ C &= 9-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x - 1}{4(2x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 8x - 1$$
$$t = 4(2x^2 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x - 1}{4(2x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1051: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{1}{4x^2} + \frac{3}{4\left(x - \frac{1}{2}\right)^2} - \frac{1}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8x-1}{4\left(2x^{2}-x\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-}\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{1}{2\left(x-\frac{1}{2}\right)} + (0) \\
&= \frac{1}{2x} - \frac{1}{2\left(x-\frac{1}{2}\right)} \\
&= -\frac{1}{2x\left(-1+2x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2\left(x - \frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{2\left(x - \frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{1}{2\left(x - \frac{1}{2}\right)}\right)^2 - \left(\frac{8x - 1}{4\left(2x^2 - x\right)^2}\right)\right) = 0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2\left(x - \frac{1}{2}\right)}\right) dx} \\
&= \frac{\sqrt{x}}{\sqrt{-1 + 2x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4x^2 - 5x}{-2x^3 + x^2} \, dx} \\
&= z_1 e^{\frac{5 \ln(x)}{2} - \frac{3 \ln(-1 + 2x)}{2}} \\
&= z_1 \left(\frac{x^{5/2}}{(-1 + 2x)^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^3}{(-1 + 2x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{4x^2-5x}{-2x^3+x^2}\, dx}}{(y_1)^2}\, dx$$
$$= y_1 \int \frac{e^{5\ln(x)-3\ln(-1+2x)}}{(y_1)^2}\, dx$$
$$= y_1(2x - \ln(x))$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{x^3}{(-1+2x)^2}\right) + c_2\left(\frac{x^3}{(-1+2x)^2}(2x-\ln(x))\right)$$

Will add steps showing solving for IC soon.

1.553.2 Maple step by step solution

Let's solve

$x^2(1-2x)\left(\frac{d}{dx}y'\right) - x(5-4x)\,y' + (9-4x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(-9+4x)y}{x^2(-1+2x)} + \frac{(-5+4x)y'}{x(-1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(-5+4x)y'}{x(-1+2x)} + \frac{(-9+4x)y}{x^2(-1+2x)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{-5+4x}{x(-1+2x)}, P_3(x) = \frac{-9+4x}{x^2(-1+2x)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -5$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 9$$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(-1+2x)\left(\frac{d}{dx}y'\right) - x(-5+4x)\,y' + (-9+4x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ○ Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(-3+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-3)^2 + 2a_{k-1}(k+r-2)\,(k+r-3)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-3+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 3$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r-3)^2 + 2a_{k-1}(k+r-2)(k+r-3) = 0$

- Shift index using $k- >k+1$

  $-a_{k+1}(k+r-2)^2 + 2a_k(k+r-1)(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r-1)}{k+r-2}$

- Recursion relation for $r = 3$

  $a_{k+1} = \frac{2a_k(k+2)}{k+1}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{2a_k(k+2)}{k+1}\right]$

1.553.3 Maple trace

```
Methods for second order ODEs:
```

1.553.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 26

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)-x*(5-4*x)*diff(y(x),x)+(9-4*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^3(2c_2x - c_2\ln(x) + c_1)}{(-1+2x)^2}$$

1.553.5 Mathematica DSolve solution

Solving time : `0.075` (sec)
Leaf size : 29

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]-x*(5-4*x)*D[y[x],x]+(9-4*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^3(-2c_2x + c_2\log(x) + c_1)}{(1-2x)^2}$$

1.554 problem 570

Internal problem ID [8692]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 570
Date solved : `Monday, October 21, 2024 at 05:20:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''+x^2y'+(1-x)\,y=0$$

1.554.1 Solved as second order ode using Kovacic algorithm

Time used: 0.238 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y''+x^2y'+(1-x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2 \\ B &= x^2 \\ C &= 1-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 8x - 16}{16 (x^2 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 + 8x - 16$$
$$t = 16(x^2 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 8x - 16}{16 (x^2 + 2x)^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1053: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{3}{16\,(2+x)^2} + \frac{3}{8x} - \frac{1}{4x^2} - \frac{3}{8\,(2+x)}
$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{5x^2 + 8x - 16}{16\,(x^2 + 2x)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{5x^2+8x-16}{16\left(x^2+2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^++\alpha_{c_2}^+\right)\\
&= \frac{5}{4}-\left(\frac{5}{4}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4\,(2+x)} + \frac{1}{2x} + (0) \\
&= \frac{3}{4\,(2+x)} + \frac{1}{2x} \\
&= \frac{5x+4}{4x\,(2+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{gathered}
(0) + 2\left(\frac{3}{4\,(2+x)} + \frac{1}{2x}\right)(0) + \left(\left(-\frac{3}{4\,(2+x)^2} - \frac{1}{2x^2}\right) + \left(\frac{3}{4\,(2+x)} + \frac{1}{2x}\right)^2 - \left(\frac{5x^2+8x-16}{16\,(x^2+2x)^2}\right)\right) = 0 \\
0 = 0
\end{gathered}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{4(2+x)} + \frac{1}{2x}\right) dx} \\
&= (2+x)^{3/4} \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2}{2x^3+4x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(2+x)}{4}} \\
&= z_1 \left(\frac{1}{(2+x)^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \sqrt{2+x}\,\sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{2x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(2+x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{1}{\sqrt{2+x}} - \frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{2+x}\,\sqrt{x}\right) + c_2\left(\sqrt{2+x}\,\sqrt{x}\left(\frac{1}{\sqrt{2+x}} - \frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.554.2 Maple step by step solution

Let's solve

$2x^2(2+x)\left(\frac{d}{dx}y'\right) + x^2y' + (1-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(x-1)y}{2x^2(2+x)} - \frac{y'}{2(2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + \frac{y'}{2(2+x)} - \frac{(x-1)y}{2x^2(2+x)} = 0$

☐ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{1}{2(2+x)}, P_3(x) = -\frac{x-1}{2x^2(2+x)}\right]$
  - $(2+x) \cdot P_2(x)$ is analytic at $x = -2$
    $\left((2+x) \cdot P_2(x)\right)\Big|_{x=-2} = \frac{1}{2}$
  - $(2+x)^2 \cdot P_3(x)$ is analytic at $x = -2$
    $\left((2+x)^2 \cdot P_3(x)\right)\Big|_{x=-2} = 0$
  - $x = -2$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = -2$

- Multiply by denominators
  $2x^2(2+x)\left(\frac{d}{dx}y'\right) + x^2y' + (1-x)\,y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$
  $(2u^3 - 8u^2 + 8u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (u^2 - 4u + 4)\left(\frac{d}{du}y(u)\right) + (3-u)\,y(u) = 0$

- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$
    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k- >k - m$
    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$4a_0 r(-1+2r)\,u^{-1+r} + \left(4a_1(1+r)(1+2r) - a_0(8r^2-4r-3)\right)u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)(2k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term must be 0

  $$4a_1(1+r)(1+2r) - a_0(8r^2-4r-3) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$2(-4a_k + a_{k-1} + 4a_{k+1})\,k^2 + \left(4(-4a_k + a_{k-1} + 4a_{k+1})\,r + 4a_k - 5a_{k-1} + 12a_{k+1}\right)k + 2(-4a_k +$$

- Shift index using $k- >k+1$

  $$2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + \left(4(-4a_{k+1} + a_k + 4a_{k+2})\,r + 4a_{k+1} - 5a_k + 12a_{k+2}\right)(k+1) -$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} - ka_k - 12ka_{k+1} - ra_k - 12ra_{k+1} - a_k - a_{k+1}}{4(2k^2+4kr+2r^2+7k+7r+6)}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 + 3a_0 = 0\right]$$

- Revert the change of variables $u = 2 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(2+x)^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 + 3a_0 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + ka_k - 20ka_{k+1} - a_k - 9a_{k+1}}{4(2k^2+9k+10)}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + ka_k - 20ka_{k+1} - a_k - 9a_{k+1}}{4(2k^2+9k+10)}, 12a_1 + 3a_0 = 0\right]$$

- Revert the change of variables $u = 2 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2+x)^{k+\frac{1}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + k a_k - 20 k a_{k+1} - a_k - 9 a_{k+1}}{4(2k^2+9k+10)}, 12a_1 + 3a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (2+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (2+x)^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - k a_k - 12 k a_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 +$$

1.554.3 Maple trace

```
Methods for second order ODEs:
```

1.554.4 Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 50

```
dsolve(2*x^2*(2+x)*diff(diff(y(x),x),x)+x^2*diff(y(x),x)+(1-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \sqrt{x(2+x)} + \frac{c_2 \left((2+x) \operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right) - \sqrt{2+x}\,\sqrt{2}\right)\sqrt{x}}{\sqrt{2+x}}$$

1.554.5 Mathematica DSolve solution

Solving time : `0.14` (sec)
Leaf size : 65

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}]+x^2*D[y[x],x]+(1-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}\left(2\left(c_1\sqrt{x+2} + c_2\right) - \sqrt{2}c_2\sqrt{x+2}\operatorname{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right)\right)}{2\sqrt[4]{2}}$$

1.555 problem 571

Internal problem ID [8693]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 571
Date solved : `Monday, October 21, 2024 at 05:20:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(1+x)\,y'' - x(6-x)\,y' + (8-x)\,y = 0$$

1.555.1 Solved as second order ode using Kovacic algorithm

Time used: 0.243 (sec)

Writing the ode as

$$\left(2x^3+2x^2\right)y'' + \left(x^2-6x\right)y' + (8-x)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+2x^2 \\ B &= x^2-6x \\ C &= 8-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 - 20x - 4}{16\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 - 20x - 4$$
$$t = 16\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 - 20x - 4}{16\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1055: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16(1+x)^2} + \frac{3}{4(1+x)} - \frac{1}{4x^2} - \frac{3}{4x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 - 20x - 4}{16(x^2 + x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4}\\ \alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{5x^2-20x-4}{16\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=-\frac{1}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right)\\ &= -\frac{1}{4}-\left(-\frac{1}{4}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{4\,(1+x)} + \frac{1}{2x} + (-)\,(0) \\
&= -\frac{3}{4\,(1+x)} + \frac{1}{2x} \\
&= -\frac{x-2}{4x\,(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4\,(1+x)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{3}{4\,(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{3}{4\,(1+x)} + \frac{1}{2x}\right)^2 - \left(\frac{5x^2-20x-4}{16\,(x^2+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{3}{4(1+x)} + \frac{1}{2x}\right) dx} \\
&= \frac{\sqrt{x}}{(1+x)^{3/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2-6x}{2x^3+2x^2}\, dx} \\
&= z_1 e^{\frac{3\ln(x)}{2} - \frac{7\ln(1+x)}{4}} \\
&= z_1 \left(\frac{x^{3/2}}{(1+x)^{7/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2}{(1+x)^{5/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-6x}{2x^3+2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{3\ln(x) - \frac{7\ln(1+x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2(1+x)^{3/2}}{3} + 2\sqrt{1+x} + \ln\left(\sqrt{1+x}-1\right) - \ln\left(1+\sqrt{1+x}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2}{(1+x)^{5/2}}\right) \\
&\quad + c_2 \left(\frac{x^2}{(1+x)^{5/2}} \left(\frac{2(1+x)^{3/2}}{3} + 2\sqrt{1+x} + \ln\left(\sqrt{1+x}-1\right) - \ln\left(1+\sqrt{1+x}\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.555.2 Maple step by step solution

Let's solve

$2x^2(1+x)\left(\frac{d}{dx}y'\right)-x(6-x)\,y'+(8-x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{(x-8)y}{2x^2(1+x)}-\frac{(x-6)y'}{2x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(x-6)y'}{2x(1+x)}-\frac{(x-8)y}{2x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x-6}{2x(1+x)},P_3(x)=-\frac{x-8}{2x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=\frac{7}{2}$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $2x^2(1+x)\left(\frac{d}{dx}y'\right)+x(x-6)\,y'+(8-x)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(2u^3-4u^2+2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(u^2-8u+7)\left(\frac{d}{du}y(u)\right)+(9-u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_ku^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_ku^{k+r+m}$

  ◦ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(5+2r)\, u^{-1+r} + \left(a_1(1+r)\,(7+2r) - a_0(4r^2+4r-9)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+7+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(5+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{5}{2}\right\}$$

- Each term must be 0

$$a_1(1+r)\,(7+2r) - a_0(4r^2+4r-9) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-4a_k + 2a_{k-1} + 2a_{k+1})\, k^2 + \left((-8a_k + 4a_{k-1} + 4a_{k+1})\, r - 4a_k - 5a_{k-1} + 9a_{k+1}\right) k + (-4a_k + 2a$$

- Shift index using $k- >k+1$

$$(-4a_{k+1} + 2a_k + 2a_{k+2})\,(k+1)^2 + \left((-8a_{k+1} + 4a_k + 4a_{k+2})\, r - 4a_{k+1} - 5a_k + 9a_{k+2}\right)(k+1) +$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} + 4kr a_k - 8kr a_{k+1} + 2r^2 a_k - 4r^2 a_{k+1} - k a_k - 12k a_{k+1} - r a_k - 12r a_{k+1} - a_k + a_{k+1}}{2k^2 + 4kr + 2r^2 + 13k + 13r + 18}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - k a_k - 12k a_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - k a_k - 12k a_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}, 7a_1 + 9a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - k a_k - 12k a_{k+1} - a_k + a_{k+1}}{2k^2+13k+18}, 7a_1 + 9a_0 = 0\right]$$

- Recursion relation for $r = -\frac{5}{2}$

$$a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - 11k a_k + 8k a_{k+1} + 14a_k + 6a_{k+1}}{2k^2+3k-2}$$

- Solution for $r = -\frac{5}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{5}{2}}, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - 11k a_k + 8k a_{k+1} + 14a_k + 6a_{k+1}}{2k^2+3k-2}, -3a_1 - 6a_0 = 0\right]$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-\frac{5}{2}}, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - 11k a_k + 8k a_{k+1} + 14a_k + 6a_{k+1}}{2k^2+3k-2}, -3a_1 - 6a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k-\frac{5}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - k a_k - 12k a_{k+1} - a_k + a_{k+1}}{2k^2+13k+18}, 7a_1 +\right.$$

1.555.3 Maple trace

```
Methods for second order ODEs:
```

1.555.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 50

```
dsolve(2*x^2*(1+x)*diff(diff(y(x),x),x)-x*(6-x)*diff(y(x),x)+(8-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{2x^2 \left(-\frac{3\ln\left(1+\sqrt{1+x}\right)c_2}{2} + \frac{3\ln\left(\sqrt{1+x}-1\right)c_2}{2} + (x+4)\, c_2 \sqrt{1+x} + \frac{3c_1}{2}\right)}{3\,(1+x)^{5/2}}$$

1.555.5 Mathematica DSolve solution

Solving time : `0.109` (sec)
Leaf size : 50

```
DSolve[{2*x^2*(1+x)*D[y[x],{x,2}]-x*(6-x)*D[y[x],x]+(8-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^2\left(-6c_2\text{arctanh}\left(\sqrt{x+1}\right)+2c_2\sqrt{x+1}(x+4)+3c_1\right)}{3(x+1)^{5/2}}$$

1.556 problem 572

Internal problem ID [8694]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 572
Date solved : `Monday, October 21, 2024 at 05:20:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+2x)\,y''+x(5+9x)\,y'+(4+3x)\,y=0$$

1.556.1 Solved as second order ode using Kovacic algorithm

Time used: 0.258 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(9x^2+5x\right)y'+(4+3x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+x^2\\ B&=9x^2+5x\\ C&=4+3x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 21x^2 + 6x - 1 \\ t &= 4\left(2x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1057: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\left(x + \frac{1}{2}\right)^2} - \frac{5}{2\left(x + \frac{1}{2}\right)} - \frac{1}{4x^2} + \frac{5}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{7}{4} - \left(\frac{7}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \frac{5}{4\left(x + \frac{1}{2}\right)} + (0) \\
&= \frac{1}{2x} + \frac{5}{4\left(x + \frac{1}{2}\right)} \\
&= \frac{1 + 7x}{4x^2 + 2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} + \frac{5}{4\left(x + \frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{5}{4\left(x + \frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{5}{4\left(x + \frac{1}{2}\right)}\right)^2 - \left(\frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}\right)\right)
$$
$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{5}{4\left(x + \frac{1}{2}\right)}\right) dx} \\
&= (1 + 2x)^{5/4} \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{9x^2 + 5x}{2x^3 + x^2} \, dx} \\
&= z_1 e^{\frac{\ln(1 + 2x)}{4} - \frac{5 \ln(x)}{2}} \\
&= z_1 \left(\frac{(1 + 2x)^{1/4}}{x^{5/2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{(1+2x)^{3/2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{9x^2+5x}{2x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{\ln(1+2x)}{2}-5\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln\left(\sqrt{1+2x}-1\right) + \frac{2}{3\,(1+2x)^{3/2}} + \frac{2}{\sqrt{1+2x}} - \ln\left(\sqrt{1+2x}+1\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(1+2x)^{3/2}}{x^2}\right) \\
&\quad + c_2 \left(\frac{(1+2x)^{3/2}}{x^2} \left(\ln\left(\sqrt{1+2x}-1\right) + \frac{2}{3\,(1+2x)^{3/2}} + \frac{2}{\sqrt{1+2x}} - \ln\left(\sqrt{1+2x}+1\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.556.2 Maple step by step solution

Let's solve

$x^2(1+2x)\left(\frac{d}{dx}y'\right)+x(5+9x)\,y'+(4+3x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(4+3x)y}{x^2(1+2x)}-\frac{(5+9x)y'}{x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(5+9x)y'}{x(1+2x)}+\frac{(4+3x)y}{x^2(1+2x)}=0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{5+9x}{x(1+2x)},P_3(x)=\frac{4+3x}{x^2(1+2x)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=5$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=4$
  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(1+2x)\left(\frac{d}{dx}y'\right)+x(5+9x)\,y'+(4+3x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)^2 + a_{k-1}(k+r+2)\,(2k-1+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -2$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r+2)\left(a_k(k+r+2) + a_{k-1}(2k-1+2r)\right) = 0$$

- Shift index using $k->k+1$

$$(k+r+3)\left(a_{k+1}(k+r+3) + a_k(2k+2r+1)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(2k+2r+1)}{k+r+3}$$

- Recursion relation for $r = -2$

$$a_{k+1} = -\frac{a_k(2k-3)}{k+1}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{a_k(2k-3)}{k+1}\right]$$

1.556.3 Maple trace

```
Methods for second order ODEs:
```

1.556.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 73

```
dsolve(x^2*(1+2*x)*diff(diff(y(x),x),x)+x*(5+9*x)*diff(y(x),x)+(4+3*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\left(x+\frac{1}{2}\right)^{2} c_{2} \ln\left(\sqrt{1+2x}-1\right)-\left(x+\frac{1}{2}\right)^{2} c_{2} \ln\left(\sqrt{1+2x}+1\right)+c_{2}\left(x+\frac{2}{3}\right)\sqrt{1+2x}+4\left(x+\frac{1}{2}\right)^{2} c_{1}}{\sqrt{1+2x}\, x^{2}}$$

1.556.5 Mathematica DSolve solution

Solving time : 0.133 (sec)
Leaf size : 56

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]+x*(5+9*x)*D[y[x],x]+(4+3*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_2\left(-3(2x+1)^{3/2}\text{arctanh}\left(\sqrt{2x+1}\right)+6x+4\right)+3c_1(2x+1)^{3/2}}{3x^2}$$

1.557 problem 573

Internal problem ID [8695]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 573
Date solved : `Monday, October 21, 2024 at 05:20:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-2x)\,y''-x(5+4x)\,y'+(9+4x)\,y=0$$

1.557.1 Solved as second order ode using Kovacic algorithm

Time used: 0.289 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(-4x^2-5x\right)y'+(9+4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= -4x^2-5x \\ C &= 9+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{32x^2 + 56x - 1}{4(2x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 32x^2 + 56x - 1$$
$$t = 4(2x^2 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{32x^2 + 56x - 1}{4(2x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1059: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4\left(x - \frac{1}{2}\right)^2} - \frac{13}{x - \frac{1}{2}} + \frac{13}{x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{32x^2 + 56x - 1}{4\left(2x^2 - x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{32x^2 + 56x - 1}{4\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= -1 - (-2) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)} + (-)\left(0\right) \\ &= \frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)} \\ &= \frac{-1-8x}{4x^2-2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right)(1) + \left(\left(-\frac{1}{2x^2} + \frac{5}{2\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{32x^2+56x-1}{4\left(2x^2-x\right)^2}\right)\right)$$

$$\frac{-1+8a_0}{x\left(-1+2x\right)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{8}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x+\frac{1}{8}\right) \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right) dx} \\ &= \left(x+\frac{1}{8}\right) \mathrm{e}^{\frac{\ln(x)}{2} - \frac{5\ln(-1+2x)}{2}} \\ &= \frac{\left(x+\frac{1}{8}\right)\sqrt{x}}{\left(-1+2x\right)^{5/2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^2-5x}{-2x^3+x^2}\,dx} \\
&= z_1 e^{\frac{5\ln(x)}{2}-\frac{7\ln(-1+2x)}{2}} \\
&= z_1\left(\frac{x^{5/2}}{(-1+2x)^{7/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^3\left(x+\frac{1}{8}\right)}{(-1+2x)^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-5x}{-2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(x)-7\ln(-1+2x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{32x^3}{3}-44x^2+\frac{203x}{2}-64\ln(x)-\frac{3125}{16(1+8x)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^3\left(x+\frac{1}{8}\right)}{(-1+2x)^6}\right)+c_2\left(\frac{x^3\left(x+\frac{1}{8}\right)}{(-1+2x)^6}\left(\frac{32x^3}{3}-44x^2+\frac{203x}{2}-64\ln(x)-\frac{3125}{16(1+8x)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.557.2 Maple step by step solution

Let's solve

$x^2(1-2x)\left(\frac{d}{dx}y'\right)-x(5+4x)\,y'+(9+4x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{(9+4x)y}{x^2(-1+2x)}-\frac{(5+4x)y'}{x(-1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(5+4x)y'}{x(-1+2x)}-\frac{(9+4x)y}{x^2(-1+2x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{5+4x}{x(-1+2x)},P_3(x)=-\frac{9+4x}{x^2(-1+2x)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=-5$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=9$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(-1+2x)\left(\frac{d}{dx}y'\right)+x(5+4x)\,y'+(-4x-9)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-3)^2 + 2a_{k-1}(k+1+r)\,(k-2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 3$$

- Each term in the series must be 0, giving the recursion relation

$$-a_k(k+r-3)^2 + 2a_{k-1}(k+1+r)\,(k-2+r) = 0$$

- Shift index using $k- >k+1$

$$-a_{k+1}(k-2+r)^2 + 2a_k(k+r+2)\,(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k(k+r+2)(k+r-1)}{(k-2+r)^2}$$

- Recursion relation for $r = 3$

$$a_{k+1} = \frac{2a_k(k+5)(k+2)}{(k+1)^2}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{2a_k(k+5)(k+2)}{(k+1)^2}\right]$$

1.557.3 Maple trace

```
Methods for second order ODEs:
```

1.557.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 54

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)-x*(5+4*x)*diff(y(x),x)+(9+4*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\left(-6c_2\left(x+\frac{1}{8}\right)\ln\left(x\right)+c_2\,x^4-4c_2\,x^3+9c_2\,x^2+\left(8c_1+\frac{609c_2}{512}\right)x+c_1-\frac{9375c_2}{4096}\right)x^3}{\left(-1+2x\right)^6}$$

1.557.5 Mathematica DSolve solution

Solving time : 0.157 (sec)
Leaf size : 63

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]-x*(5+4*x)*D[y[x],x]+(9+4*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\frac{x^3(c_2(4096x^4-16384x^3+36864x^2+4872x-9375)-48c_1(8x+1)-3072c_2(8x+1)\log(x))}{384(1-2x)^6}$$

1.558 problem 574

Internal problem ID [8696]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 574
Date solved : `Monday, October 21, 2024 at 05:20:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-x)\,y''+x(7+x)\,y'+(9-x)\,y=0$$

1.558.1 Solved as second order ode using Kovacic algorithm

Time used: 0.297 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(x^2+7x\right)y'+(9-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^3+x^2\\B&=x^2+7x\\C&=9-x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 82x - 1}{4(x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 82x - 1$$
$$t = 4(x^2 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 82x - 1}{4(x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1061: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{20}{x} - \frac{1}{4x^2} - \frac{20}{-1+x} + \frac{20}{(-1+x)^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 5\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -4\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 82x - 1}{4(x^2 - x)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 82x - 1}{4\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{7}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} - \frac{4}{-1+x} + (-)(0) \\
&= \frac{1}{2x} - \frac{4}{-1+x} \\
&= -\frac{1+7x}{2x(-1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(12x^2 + 6xa_3 + 2a_2) + 2\left(\frac{1}{2x} - \frac{4}{-1+x}\right)(4x^3 + 3x^2 a_3 + 2a_2 x + a_1) + \left(\left(-\frac{1}{2x^2} + \frac{4}{(-1+x)^2}\right) + \left(\frac{1}{2x}\right.\right.
$$

$$
\frac{(a_3 - 16)\, x^3 + (4a_2 - 9a_3)}{x\,(}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1, a_1 = 16, a_2 = 36, a_3 = 16\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^4 + 16x^3 + 36x^2 + 16x + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^4 + 16x^3 + 36x^2 + 16x + 1)\, e^{\int \left(\frac{1}{2x} - \frac{4}{-1+x}\right) dx} \\
&= (x^4 + 16x^3 + 36x^2 + 16x + 1)\, e^{-4\ln(-1+x) + \frac{\ln(x)}{2}} \\
&= \frac{(x^4 + 16x^3 + 36x^2 + 16x + 1)\sqrt{x}}{(-1+x)^4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+7x}{-x^3+x^2}\,dx} \\
&= z_1 e^{4\ln(-1+x)-\frac{7\ln(x)}{2}} \\
&= z_1 \left(\frac{(-1+x)^4}{x^{7/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^4+16x^3+36x^2+16x+1}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+7x}{-x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{8\ln(-1+x)-7\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{20\left(-2x^3-\frac{15}{2}x^2-\frac{14}{3}x-\frac{5}{12}\right)}{x^4+16x^3+36x^2+16x+1}+\ln(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^4+16x^3+36x^2+16x+1}{x^3}\right) \\
&\quad + c_2 \left(\frac{x^4+16x^3+36x^2+16x+1}{x^3}\left(-\frac{20\left(-2x^3-\frac{15}{2}x^2-\frac{14}{3}x-\frac{5}{12}\right)}{x^4+16x^3+36x^2+16x+1}+\ln(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.558.2 Maple step by step solution

Let's solve

$x^2(1-x)\left(\frac{d}{dx}y'\right)+x(7+x)\,y'+(9-x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(-9+x)y}{x^2(-1+x)}+\frac{(7+x)y'}{x(-1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{(7+x)y'}{x(-1+x)}+\frac{(-9+x)y}{x^2(-1+x)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{7+x}{x(-1+x)},P_3(x)=\frac{-9+x}{x^2(-1+x)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=7$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=9$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(-1+x)\left(\frac{d}{dx}y'\right)-x(7+x)\,y'+(-9+x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ○ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 (3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k (k+r+3)^2 + a_{k-1}(k-2+r)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -3$$

- Each term in the series must be 0, giving the recursion relation

$$-a_k (k+r+3)^2 + a_{k-1}(k-2+r)^2 = 0$$

- Shift index using $k->k+1$

$$-a_{k+1}(k+4+r)^2 + a_k (k+r-1)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k (k+r-1)^2}{(k+4+r)^2}$$

- Recursion relation for $r = -3$; series terminates at $k = 4$

$$a_{k+1} = \frac{a_k (k-4)^2}{(k+1)^2}$$

- Apply recursion relation for $k = 0$

$$a_1 = 16 a_0$$

- Apply recursion relation for $k = 1$

$$a_2 = \frac{9 a_1}{4}$$

- Express in terms of a_0

$$a_2 = 36 a_0$$

- Apply recursion relation for $k = 2$

$$a_3 = \frac{4 a_2}{9}$$

- Express in terms of a_0

$a_3 = 16a_0$

- Apply recursion relation for $k = 3$

  $a_4 = \frac{a_3}{16}$
- Express in terms of a_0

  $a_4 = a_0$
- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y = a_0 \cdot (x^4 + 16x^3 + 36x^2 + 16x + 1)$

1.558.3 Maple trace

```
Methods for second order ODEs:
```

1.558.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 72

```
dsolve(x^2*(1-x)*diff(diff(y(x),x),x)+x*(7+x)*diff(y(x),x)+(9-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{3c_2(x^4+16x^3+36x^2+16x+1)\ln(x)+c_1x^4+(16c_1+120c_2)x^3+(36c_1+450c_2)x^2+(16c_1+280c_2}{x^3}$$

1.558.5 Mathematica DSolve solution

Solving time : `0.146` (sec)
Leaf size : 78

```
DSolve[{x^2*(1-x)*D[y[x],{x,2}]+x*(7+x)*D[y[x],x]+(9-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{5c_2(24x^3+90x^2+56x+5)+3c_1(x^4+16x^3+36x^2+16x+1)+3c_2(x^4+16x^3+36x^2+16x+1)\log}{3x^3}$$

1.559 problem 575

Internal problem ID [8697]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 575
Date solved : `Monday, October 21, 2024 at 05:20:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(-x^2+1)\,y' + (x^2+1)\,y = 0$$

1.559.1 Solved as second order ode using Kovacic algorithm

Time used: 0.284 (sec)

Writing the ode as

$$x^2y'' + (x^3-x)\,y' + (x^2+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 - x \\ C &= x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 4x^2 - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 4x^2 - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1063: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{1}{x} - \frac{5}{4x^3} - \frac{5}{2x^5} - \frac{105}{16x^7} - \frac{155}{8x^9} - \frac{1965}{32x^{11}} - \frac{3265}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 4x^2 - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{x^2}{4} - 1\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{4} - 1 - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) &= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 4x^2 - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{x}{2}\right) \\
&= \frac{1}{2x} - \frac{x}{2} \\
&= \frac{1}{2x} - \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{2}\right) + \left(\frac{1}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{x}{2}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3 - x}{x^2}\, dx} \\
&= z_1 e^{-\frac{x^2}{4} + \frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\, \mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.559.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x(-x^2+1)\,y' + (x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+1)y}{x^2} - \frac{(x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$\frac{d}{dx}y' + \frac{(x^2-1)y'}{x} + \frac{(x^2+1)y}{x^2} = 0$$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $$\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2\left(\frac{d}{dx}y'\right) + x(x^2-1)\,y' + (x^2+1)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 1$
- Each term must be 0
  $a_1 r^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_k(k+r-1) + a_{k-2}\right) = 0$
- Shift index using $k- > k+2$
  $(k+r+1)\left(a_{k+2}(k+r+1) + a_k\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{k+r+1}$
- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{a_k}{k+2}$
- Solution for $r = 1$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$

1.559.3 Maple trace

```
Methods for second order ODEs:
```

1.559.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-x^2+1)*diff(y(x),x)+(x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x\,e^{-\frac{x^2}{2}} \left(c_1 + \mathrm{Ei}_1\left(-\frac{x^2}{2}\right) c_2\right)$$

1.559.5 Mathematica DSolve solution

Solving time : 0.021 (sec)
Leaf size : 35

```
DSolve[{x^2*D[y[x],{x,2}]-x*(1-x^2)*D[y[x],x]+(1+x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-\frac{x^2}{2}}x\left(c_1 \operatorname{ExpIntegralEi}\left(\frac{x^2}{2}\right) + 2c_2\right)$$

1.560 problem 576

Internal problem ID [8698]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 576
Date solved : `Monday, October 21, 2024 at 05:20:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+1)\,y''-3x(-x^2+1)\,y'+4y=0$$

1.560.1 Solved as second order ode using Kovacic algorithm

Time used: 0.370 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(3x^3-3x)\,y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 3x^3-3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3x^4 - 10x^2 - 1 \\
t &= 4\left(x^3 + x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1065: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 4 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{i}{4x - 4i} - \frac{i}{4(x+i)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} + (-)(0) \\
&= \frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} \\
&= \frac{1}{2x} - \frac{x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{2(x-i)^2} + \frac{1}{2(x+i)^2}\right) + \left(\frac{1}{2x} - \frac{1}{2(x-i)} - 2(
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right) dx} \\
&= \frac{\sqrt{x}}{\sqrt{x^2+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3-3x}{x^4+x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+1\right)}{2}+\frac{3\ln(x)}{2}} \\
&= z_1\left(\frac{x^{3/2}}{\left(x^2+1\right)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2}{\left(x^2+1\right)^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3-3x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln\left(x^2+1\right)+3\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{x^2}{2}+\ln\left(x\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2}{\left(x^2+1\right)^2}\right) + c_2\left(\frac{x^2}{\left(x^2+1\right)^2}\left(\frac{x^2}{2}+\ln\left(x\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.560.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)-3x(-x^2+1)y'+4y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{4y}{x^2(x^2+1)}-\frac{3(x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{3(x^2-1)y'}{x(x^2+1)}+\frac{4y}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{3(x^2-1)}{x(x^2+1)}, P_3(x)=\frac{4}{x^2(x^2+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$(x\cdot P_2(x))\Big|_{x=0}=-3$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2\cdot P_3(x))\Big|_{x=0}=4$

- ○ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)+3x(x^2-1)y'+4y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)x^{k+r}$

- Convert $x^{m} \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^{m} \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_{k} (k+r) (k+r-1) x^{k+r-2+m}$
- Shift index using $k- >k+2-m$

  $x^{m} \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r) (k+1-m+r) x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-2+r)^{2} x^{r} + a_{1}(-1+r)^{2} x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_{k}(k+r-2)^{2} + a_{k-2}(k+r-2)(k+r)\right) x^{k+r}\right) = 0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^{2} = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term must be 0

  $a_{1}(-1+r)^{2} = 0$
- Solve for the dependent coefficient(s)

  $a_{1} = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left(a_{k}(k+r-2) + a_{k-2}(k+r)\right) = 0$
- Shift index using $k- >k+2$

  $(k+r)\left(a_{k+2}(k+r) + a_{k}(k+r+2)\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_{k}(k+r+2)}{k+r}$
- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_{k}(k+4)}{k+2}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_{k} x^{k+2}, a_{k+2} = -\frac{a_{k}(k+4)}{k+2}, a_{1} = 0\right]$

1.560.3 Maple trace

```
Methods for second order ODEs:
```

1.560.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 27

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-3*x*(-x^2+1)*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^2\left(c_1 + c_2\left(\frac{x^2}{2} + \ln\left(x\right)\right)\right)}{\left(x^2+1\right)^2}$$

1.560.5 Mathematica DSolve solution

Solving time : 0.081 (sec)
Leaf size : 36

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-3*x*(1-x^2)*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^2(c_2x^2 + 2c_2\log(x) + 2c_1)}{2\left(x^2+1\right)^2}$$

1.561 problem 577

Internal problem ID [8699]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 577
Date solved : `Monday, October 21, 2024 at 05:20:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 2x^3y' + \left(3x^2 + 1\right)y = 0$$

1.561.1 Solved as second order ode using Kovacic algorithm

Time used: 0.283 (sec)

Writing the ode as

$$4x^2y'' + 2x^3y' + \left(3x^2 + 1\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 2x^3 \\ C &= 3x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^2 - 4}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 8x^2 - 4$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^2 - 4}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1067: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} - \frac{1}{2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{1}{x} - \frac{5}{2x^3} - \frac{10}{x^5} - \frac{105}{2x^7} - \frac{310}{x^9} - \frac{1965}{x^{11}} - \frac{13060}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^4 - 8x^2 - 4}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{x^2}{16} - \frac{1}{2}\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{16} - \frac{1}{2} - \frac{1}{4x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{4}\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &= -\frac{3}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 8x^2 - 4}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{4}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right)\\
&= \frac{1}{2} - \left(\frac{1}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{x}{4}\right) \\
&= \frac{1}{2x} - \frac{x}{4} \\
&= \frac{1}{2x} - \frac{x}{4}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{x}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{x}{4}\right)^2 - \left(\frac{x^4 - 8x^2 - 4}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{x}{4}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3}{4x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{8}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{4}}\sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3}{4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{4}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{4}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{x^2}{4}}\sqrt{x}\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{4}}\sqrt{x}\left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{4}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.561.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) + 2x^3y' + (3x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(3x^2+1)y}{4x^2} - \frac{y'x}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{y'x}{2} + \frac{(3x^2+1)y}{4x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{x}{2}, P_3(x) = \frac{3x^2+1}{4x^2}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{4}$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- • Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 2x^3y' + (3x^2+1)\,y = 0$
- • Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x^3 \cdot y'$ to series expansion

    $x^3 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+2}$
  - ◦ Shift index using $k- >k-2$

    $x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2}(k-2+r)\,x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

    Rewrite ODE with series expansions

$$a_0(-1+2r)^2 x^r + a_1(1+2r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$
- Each term must be 0

  $a_1(1+2r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1) = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(2k+2r+3)^2 + a_k(2k+2r+3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{2k+2r+3}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k}{2k+4}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{2k+4}, a_1 = 0\right]$

1.561.3 Maple trace

```
Methods for second order ODEs:
```

1.561.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 25

```
dsolve(4*x^2*diff(diff(y(x),x),x)+2*x^3*diff(y(x),x)+(3*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-\frac{x^2}{4}} \sqrt{x} \left(c_1 + \mathrm{Ei}_1\left(-\frac{x^2}{4}\right) c_2\right)$$

1.561.5 Mathematica DSolve solution

Solving time : 0.093 (sec)
Leaf size : 39

```
DSolve[{4*x^2*D[y[x],{x,2}]+2*x^3*D[y[x],x]+(1+3*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-\frac{x^2}{4}}\sqrt{x}\left(c_2 \operatorname{ExpIntegralEi}\left(\frac{x^2}{4}\right)+2c_1\right)$$

1.562 problem 578

Internal problem ID [8700]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 578
Date solved : `Monday, October 21, 2024 at 05:20:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''-x\left(-2x^2+1\right)y'+y=0$$

1.562.1 Solved as second order ode using Kovacic algorithm

Time used: 0.283 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(2x^3-x\right)y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^4+x^2\\B&=2x^3-x\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2 - 1 \\ t &= 4\left(x^3 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1069: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(x-i)^2} - \frac{3}{16(x+i)^2} - \frac{5i}{16(x-i)} + \frac{5i}{16(x+i)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^2 - 1}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-i$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} + (-)\,(0) \\
&= \frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} \\
&= \frac{1}{2x} + \frac{x}{2x^2+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4\,(x-i)^2} - \frac{1}{4\,(x+i)^2}\right) + \left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right) dx} \\
&= \sqrt{x}\,\left(x^2+1\right)^{1/4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3-x}{x^4+x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}-\frac{3\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{\sqrt{x}}{\left(x^2+1\right)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{\sqrt{x^2+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{3\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{\sqrt{x^2+1}}\right) + c_2\left(\frac{x}{\sqrt{x^2+1}}\left(-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.562.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)-x(-2x^2+1)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{y}{x^2(x^2+1)}-\frac{(2x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(2x^2-1)y'}{x(x^2+1)}+\frac{y}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{2x^2-1}{x(x^2+1)},P_3(x)=\frac{1}{x^2(x^2+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=1$

- ○ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(2x^2-1)\,y'+y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)^2\,x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k-2+r)\,(k+r-1)\right)x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 1$$

- Each term must be 0

  $$a_1 r^2 = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-1)\,(a_k(k+r-1) + a_{k-2}(k-2+r)) = 0$$

- Shift index using $k- >k+2$

  $$(k+r+1)\,(a_{k+2}(k+r+1) + a_k(k+r)) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k(k+r)}{k+r+1}$$

- Recursion relation for $r = 1$

  $$a_{k+2} = -\frac{a_k(k+1)}{k+2}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+1)}{k+2}, a_1 = 0\right]$$

1.562.3 Maple trace

```
Methods for second order ODEs:
```

1.562.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 25

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-2*x^2+1)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x\left(c_2 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)+c_1\right)}{\sqrt{x^2+1}}$$

1.562.5 Mathematica DSolve solution

Solving time : 0.095 (sec)
Leaf size : 33

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(1-2*x^2)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x\left(c_1 - c_2\text{arctanh}\left(\sqrt{x^2+1}\right)\right)}{\sqrt{x^2+1}}$$

1.563 problem 579

Internal problem ID [8701]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 579
Date solved : `Monday, October 21, 2024 at 05:20:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+2\right) y''+7x^3y'+\left(3x^2+1\right) y=0$$

1.563.1 Solved as second order ode using Kovacic algorithm

Time used: 0.352 (sec)

Writing the ode as

$$\left(2x^4+4x^2\right) y''+7x^3y'+\left(3x^2+1\right) y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+4x^2 \\ B &= 7x^3 \\ C &= 3x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^4 - 16}{16\left(x^3 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^4 - 16$$
$$t = 16\left(x^3 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^4 - 16}{16\left(x^3 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1071: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{7}{64\left(x - i\sqrt{2}\right)^2} - \frac{7}{64\left(x + i\sqrt{2}\right)^2} - \frac{9i\sqrt{2}}{128\left(x - i\sqrt{2}\right)} + \frac{9i\sqrt{2}}{128\left(x + i\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-3x^4 - 16}{16\left(x^3 + 2x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-3x^4 - 16}{16\left(x^3 + 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$i\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-i\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{3}{4} - \left(\frac{3}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}} + (0) \\ &= \frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}} \\ &= \frac{1}{2x} + \frac{x}{4x^2 + 8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{8\left(x - i\sqrt{2}\right)^2} - \frac{1}{8\left(x + i\sqrt{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{}{8x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}}\right) dx} \\ &= \left(x^2 + 2\right)^{1/8} \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^3}{2x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(x^2+2\right)}{8}} \\
&= z_1 \left(\frac{1}{\left(x^2+2\right)^{7/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x}}{\left(x^2+2\right)^{3/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3}{2x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(x^2+2\right)}{4}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{1}{\left(x^2+2\right)^{1/4} x}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{\left(x^2+2\right)^{3/4}}\right) + c_2 \left(\frac{\sqrt{x}}{\left(x^2+2\right)^{3/4}}\left(\int \frac{1}{\left(x^2+2\right)^{1/4} x}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.563.2 Maple step by step solution

Let's solve

$2x^2(x^2+2)\left(\frac{d}{dx}y'\right)+7x^3y'+(3x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(3x^2+1)y}{2x^2(x^2+2)}-\frac{7xy'}{2(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{7xy'}{2(x^2+2)}+\frac{(3x^2+1)y}{2x^2(x^2+2)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{7x}{2(x^2+2)},P_3(x)=\frac{3x^2+1}{2x^2(x^2+2)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=0$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=\frac{1}{4}$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $2x^2(x^2+2)\left(\frac{d}{dx}y'\right)+7x^3y'+(3x^2+1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^3\cdot y'$ to series expansion

$$x^3 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+2}$$

- Shift index using $k->k-2$

$$x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2}(k-2+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)^2 x^r + a_1(1+2r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1)\,(k+r-1)\right) x$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{2}$$

- Each term must be 0

$$a_1(1+2r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{1}{2}\right)\left(\frac{a_{k-2}(k+r-1)}{2} + a_k\left(k+r-\frac{1}{2}\right)\right) = 0$$

- Shift index using $k->k+2$

$$4\left(k+\frac{3}{2}+r\right)\left(\frac{a_k(k+r+1)}{2} + a_{k+2}\left(k+\frac{3}{2}+r\right)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+1)}{2k+2r+3}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+4}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+4}, a_1 = 0\right]$$

1.563.3 Maple trace

```
Methods for second order ODEs:
```

1.563.4 Maple dsolve solution

Solving time : `0.051` (sec)
Leaf size : 81

```
dsolve(2*x^2*(x^2+2)*diff(diff(y(x),x),x)+7*x^3*diff(y(x),x)+(3*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\sqrt{x}\left(2^{3/4}c_1 + 2\arctan\left(\frac{\sqrt{2}\left(2x^2+4\right)^{1/4}}{2}\right)c_2 + \ln\left(-\sqrt{2}\left(2x^2+4\right)^{1/4}+2\right)c_2 - \ln\left(\sqrt{2}\left(2x^2+4\right)^{1/4}+2\right)c_2\right)}{2\left(x^2+2\right)^{3/4}}$$

1.563.5 Mathematica DSolve solution

Solving time : `0.156` (sec)
Leaf size : 77

```
DSolve[{2*x^2*(2+x^2)*D[y[x],{x,2}]+7*x^3*D[y[x],x]+(1+3*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}\left(2^{3/4}c_2\arctan\left(\frac{\sqrt[4]{x^2+2}}{\sqrt[4]{2}}\right) - 2^{3/4}c_2\operatorname{arctanh}\left(\frac{\sqrt[4]{x^2+2}}{\sqrt[4]{2}}\right) + 2c_1\right)}{2\left(x^2+2\right)^{3/4}}$$

1.564 problem 580

Internal problem ID [8702]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 580
Date solved : `Monday, October 21, 2024 at 05:20:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+1)\,y''-x(-4x^2+1)\,y'+(2x^2+1)\,y=0$$

1.564.1 Solved as second order ode using Kovacic algorithm

Time used: 0.316 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(4x^3-x)\,y'+(2x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 4x^3-x \\ C &= 2x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-6x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -6x^2 - 1$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-6x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1073: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{5}{16\,(x-i)^2} + \frac{5}{16\,(x+i)^2} + \frac{3i}{16\,(x-i)} - \frac{3i}{16\,(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-6x^2-1}{4\left(x^3+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}+\alpha_{c_3}^{-}\right) \\
&= 0-(0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{1}{4(x - i)} - \frac{1}{4(x + i)} + (0) \\
&= \frac{1}{2x} - \frac{1}{4(x - i)} - \frac{1}{4(x + i)} \\
&= \frac{1}{2x^3 + 2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{4(x - i)} - \frac{1}{4(x + i)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{4(x - i)^2} + \frac{1}{4(x + i)^2}\right) + \left(\frac{1}{2x} - \frac{1}{4(x - i)} - \frac{1}{4(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} - \frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\
&= \frac{\sqrt{x}}{(x^2 + 1)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^3-x}{x^4+x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}-\frac{5\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{\sqrt{x}}{\left(x^2+1\right)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{\left(x^2+1\right)^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^3-x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{5\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{\left(x^2+1\right)^{3/2}}\right) + c_2 \left(\frac{x}{\left(x^2+1\right)^{3/2}} \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.564.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right) - x(-4x^2+1)\,y' + (2x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = -\frac{(2x^2+1)y}{x^2(x^2+1)} - \frac{(4x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(4x^2-1)y'}{x(x^2+1)} + \frac{(2x^2+1)y}{x^2(x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x) = \frac{4x^2-1}{x(x^2+1)}, P_3(x) = \frac{2x^2+1}{x^2(x^2+1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x=0$

$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x=0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$

- ○ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2(x^2+1)\left(\frac{d}{dx}y'\right) + x(4x^2-1)\,y' + (2x^2+1)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k->k-m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)^2\, x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r)\,(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 1$
- Each term must be 0

  $a_1 r^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r-1) + a_{k-2}(k+r)) = 0$
- Shift index using $k->k+2$

  $(k+r+1)\,(a_{k+2}(k+r+1) + a_k(k+r+2)) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+2)}{k+r+1}$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k(k+3)}{k+2}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+3)}{k+2}, a_1 = 0\right]$

1.564.3 Maple trace

```
Methods for second order ODEs:
```

1.564.4 Maple dsolve solution

Solving time : 0.011 (sec)
Leaf size : 35

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-4*x^2+1)*diff(y(x),x)+(2*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\left(-c_2 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) + c_2\sqrt{x^2+1} + c_1\right)x}{\left(x^2+1\right)^{3/2}}$$

1.564.5 Mathematica DSolve solution

Solving time : 0.11 (sec)
Leaf size : 45

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(1-4*x^2)*D[y[x],x]+(1+2*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x\left(-c_2 \operatorname{arctanh}\left(\sqrt{x^2+1}\right) + c_2\sqrt{x^2+1} + c_1\right)}{\left(x^2+1\right)^{3/2}}$$

1.565 problem 581

Internal problem ID [8703]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 581
Date solved : `Monday, October 21, 2024 at 05:20:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+4\right)y''+3x\left(3x^2+8\right)y'+\left(-9x^2+1\right)y=0$$

1.565.1 Solved as second order ode using Kovacic algorithm

Time used: 0.378 (sec)

Writing the ode as

$$\left(4x^4+16x^2\right)y''+\left(9x^3+24x\right)y'+\left(-9x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+16x^2 \\ B &= 9x^3+24x \\ C &= -9x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{153x^4 + 704x^2 - 256}{64\left(x^3 + 4x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 153x^4 + 704x^2 - 256$$
$$t = 64\left(x^3 + 4x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{153x^4 + 704x^2 - 256}{64\left(x^3 + 4x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1075: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 + 4x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2i$ of order 2. There is a pole at $x = -2i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{39}{256\,(x-2i)^2} - \frac{39}{256\,(x+2i)^2} - \frac{377i}{512\,(x-2i)} + \frac{377i}{512\,(x+2i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 2i$ let b be the coefficient of $\frac{1}{(x-2i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{39}{256}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{16} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{16} \end{aligned}$$

For the pole at $x = -2i$ let b be the coefficient of $\frac{1}{(x+2i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{39}{256}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{16} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{16}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{153x^4 + 704x^2 - 256}{64\left(x^3 + 4x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{153}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{8} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{8}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{153x^4 + 704x^2 - 256}{64\left(x^3 + 4x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$2i$	2	0	$\frac{13}{16}$	$\frac{3}{16}$
$-2i$	2	0	$\frac{13}{16}$	$\frac{3}{16}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{17}{8}$	$-\frac{9}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{17}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+} + \alpha_{c_3}^{+}\right) \\ &= \frac{17}{8} - \left(\frac{17}{8}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)} + (0) \\ &= \frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)} \\ &= \frac{1}{2x} + \frac{13x}{8x^2+32} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{13}{16\,(x-2i)^2} - \frac{13}{16\,(x+2i)^2}\right) + \left(\frac{1}{2x} + \frac{1}{16\,(x\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{13}{16(x-2i)} + \frac{13}{16(x+2i)}\right) dx} \\ &= \left(x^2+4\right)^{13/16}\sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{9x^3+24x}{4x^4+16x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x^2+4)}{16}-\frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{(x^2+4)^{3/16}\, x^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x^2+4)^{5/8}}{x^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{9x^3+24x}{4x^4+16x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x^2+4)}{8}-\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{3\ln(x^2+4)}{8}-\frac{3\ln(x)}{2}}\sqrt{x}}{(x^2+4)^{5/4}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+4)^{5/8}}{x^{1/4}}\right) + c_2 \left(\frac{(x^2+4)^{5/8}}{x^{1/4}}\left(\int \frac{\mathrm{e}^{-\frac{3\ln(x^2+4)}{8}-\frac{3\ln(x)}{2}}\sqrt{x}}{(x^2+4)^{5/4}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.565.2 Maple step by step solution

Let's solve

$4x^2(x^2+4)\left(\frac{d}{dx}y'\right)+3x(3x^2+8)\,y'+(-9x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{(9x^2-1)y}{4x^2(x^2+4)}-\frac{3(3x^2+8)y'}{4x(x^2+4)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{3(3x^2+8)y'}{4x(x^2+4)}-\frac{(9x^2-1)y}{4x^2(x^2+4)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3(3x^2+8)}{4x(x^2+4)},P_3(x)=-\frac{9x^2-1}{4x^2(x^2+4)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=\frac{3}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=\frac{1}{16}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+4)\left(\frac{d}{dx}y'\right)+3x(3x^2+8)\,y'+(-9x^2+1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^{m}\cdot y' = \sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{m}\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=2..4$

$$x^{m}\cdot\left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^{m}\cdot\left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_{0}(1+4r)^{2} x^{r} + a_{1}(5+4r)^{2} x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_{k}(4k+4r+1)^{2} + a_{k-2}(4k+4r+1)\,(k-3+r)\right) x^{k+}\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$(1+4r)^{2}=0$$

- Values of r that satisfy the indicial equation

$$r=-\frac{1}{4}$$

- Each term must be 0

$$a_{1}(5+4r)^{2}=0$$

- Solve for the dependent coefficient(s)

$$a_{1}=0$$

- Each term in the series must be 0, giving the recursion relation

$$16\left(\frac{a_{k-2}(k-3+r)}{4}+a_{k}\left(k+r+\frac{1}{4}\right)\right)\left(k+r+\frac{1}{4}\right)=0$$

- Shift index using $k->k+2$

$$16\left(\frac{a_{k}(k+r-1)}{4}+a_{k+2}\left(k+\frac{9}{4}+r\right)\right)\left(k+\frac{9}{4}+r\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a_{k}(k+r-1)}{4k+4r+9}$$

- Recursion relation for $r=-\frac{1}{4}$

$$a_{k+2}=-\frac{a_{k}\left(k-\frac{5}{4}\right)}{4k+8}$$

- Solution for $r=-\frac{1}{4}$

$$\left[y=\sum_{k=0}^{\infty} a_{k} x^{k-\frac{1}{4}}, a_{k+2}=-\frac{a_{k}\left(k-\frac{5}{4}\right)}{4k+8}, a_{1}=0\right]$$

1.565.3 Maple trace

```
Methods for second order ODEs:
```

1.565.4 Maple dsolve solution

Solving time : 0.032 (sec)
Leaf size : 66

```
dsolve(4*x^2*(x^2+4)*diff(diff(y(x),x),x)+3*x*(3*x^2+8)*diff(y(x),x)+(-9*x^2+1)*y(x) =
       y(x),singsol=all)
```

$$y = \frac{\left(x^2 \operatorname{hypergeom}\left(\left[1,1,\frac{13}{8}\right],[2,2],-\frac{x^2}{4}\right)-\frac{32\gamma}{5}+\frac{64\ln(2)}{5}-\frac{64\ln(x)}{5}-\frac{32\Psi\left(\frac{5}{8}\right)}{5}\right)(x^2+4)^{5/8}c_2 2^{3/4}+c_1(x^2+4}{x^{1/4}}$$

1.565.5 Mathematica DSolve solution

Solving time : 0.404 (sec)
Leaf size : 198

```
DSolve[{4*x^2*(4+x^2)*D[y[x],{x,2}]+3*x*(8+3*x^2)*D[y[x],x]+(1-9*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2\left(5\ 2^{3/4}(x^2+4)^{5/8}\arctan\left(\frac{\sqrt[8]{x^2+4}}{\sqrt[4]{2}}\right)+5\sqrt[4]{2}(x^2+4)^{5/8}\arctan\left(\frac{\sqrt{2}-\sqrt[4]{x^2+4}}{2^{3/4}\sqrt[8]{x^2+4}}\right)-5\ 2^{3/4}(x^2+4)^{5/8}\right.}{80\sqrt[4]{x}}$$

1.566 problem 582

Internal problem ID [8704]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 582
Date solved : `Monday, October 21, 2024 at 05:20:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2\left(x^2+3\right)y''+x\left(11x^2+3\right)y'+\left(5x^2+1\right)y=0$$

1.566.1 Solved as second order ode using Kovacic algorithm

Time used: 0.355 (sec)

Writing the ode as

$$\left(3x^4+9x^2\right)y''+\left(11x^3+3x\right)y'+\left(5x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^4+9x^2 \\ B &= 11x^3+3x \\ C &= 5x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5x^4 + 18x^2 - 81$$
$$t = 36\left(x^3 + 3x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1077: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36\left(x - i\sqrt{3}\right)^2} - \frac{5}{36\left(x + i\sqrt{3}\right)^2} - \frac{7i\sqrt{3}}{108\left(x - i\sqrt{3}\right)} + \frac{7i\sqrt{3}}{108\left(x + i\sqrt{3}\right)} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{6} \end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$i\sqrt{3}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$
$-i\sqrt{3}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{5}{6} - \left(\frac{5}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}} + (0) \\ &= \frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}} \\ &= \frac{1}{2x} + \frac{x}{3x^2+9} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{6\left(x-i\sqrt{3}\right)^2} - \frac{1}{6\left(x+i\sqrt{3}\right)^2}\right) + \left(\frac{1}{2x} + \frac{1}{6x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}}\right) dx} \\ &= \sqrt{x}\left(x^2+3\right)^{1/6} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^3+3x}{3x^4+9x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln\left(x^2+3\right)}{6}-\frac{\ln(x)}{6}} \\
&= z_1\left(\frac{1}{\left(x^2+3\right)^{5/6} x^{1/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{1/3}}{\left(x^2+3\right)^{2/3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+3x}{3x^4+9x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(x^2+3\right)}{3}-\frac{\ln(x)}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{5\ln\left(x^2+3\right)}{3}-\frac{\ln(x)}{3}}\left(x^2+3\right)^{4/3}}{x^{2/3}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{1/3}}{\left(x^2+3\right)^{2/3}}\right) + c_2\left(\frac{x^{1/3}}{\left(x^2+3\right)^{2/3}}\left(\int \frac{\mathrm{e}^{-\frac{5\ln\left(x^2+3\right)}{3}-\frac{\ln(x)}{3}}\left(x^2+3\right)^{4/3}}{x^{2/3}}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.566.2 Maple step by step solution

Let's solve

$3x^2(x^2+3)\left(\frac{d}{dx}y'\right)+x(11x^2+3)\,y'+(5x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(5x^2+1)y}{3x^2(x^2+3)}-\frac{(11x^2+3)y'}{3x(x^2+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(11x^2+3)y'}{3x(x^2+3)}+\frac{(5x^2+1)y}{3x^2(x^2+3)}=0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x)=\frac{11x^2+3}{3x(x^2+3)},P_3(x)=\frac{5x^2+1}{3x^2(x^2+3)}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=\frac{1}{3}$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=\frac{1}{9}$

- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $3x^2(x^2+3)\left(\frac{d}{dx}y'\right)+x(11x^2+3)\,y'+(5x^2+1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+3r)^2\, x^r + a_1(2+3r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 + a_{k-2}(3k+3r-1)\,(k+r-1)\right) x$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+3r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{3}$$

- Each term must be 0

$$a_1(2+3r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$9\left(\frac{a_{k-2}(k+r-1)}{3} + a_k\left(k+r-\frac{1}{3}\right)\right)\left(k+r-\frac{1}{3}\right) = 0$$

- Shift index using $k->k+2$

$$9\left(\frac{a_k(k+r+1)}{3} + a_{k+2}\left(k+\frac{5}{3}+r\right)\right)\left(k+\frac{5}{3}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+1)}{3k+3r+5}$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+2} = -\frac{a_k\left(k+\frac{4}{3}\right)}{3k+6}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k\left(k+\frac{4}{3}\right)}{3k+6}, a_1 = 0\right]$$

1.566.3 Maple trace

```
Methods for second order ODEs:
```

1.566.4 Maple dsolve solution

Solving time : `0.050` (sec)
Leaf size : 102

```
dsolve(3*x^2*(x^2+3)*diff(diff(y(x),x),x)+x*(11*x^2+3)*diff(y(x),x)+(5*x^2+1)*y(x) = (
       y(x),singsol=all)
```

$$y = \frac{x^{1/3}\left(2\sqrt{3}\,\arctan\left(\frac{\left(9x^2+27\right)^{1/3}\sqrt{3}}{6+\left(9x^2+27\right)^{1/3}}\right)c_2 + 2\ln\left(3-\left(9x^2+27\right)^{1/3}\right)c_2 - \ln\left(\left(9x^2+27\right)^{2/3} + 3\left(9x^2+27\right)^{1/3}\right.\right.}{9\left(x^2+3\right)^{2/3}}$$

1.566.5 Mathematica DSolve solution

Solving time : `0.089` (sec)
Leaf size : 94

```
DSolve[{3*x^2*(3+x^2)*D[y[x],x]+x*(3+11*x^2)*D[y[x],x]+(1+5*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1 \exp\left(\frac{1}{3}\text{RootSum}\left[3\#1^3+11\#1^2+9\#1+3\&, \frac{3\#1^2\log(x-\#1)-4\#1\log(x-\#1)+9\log(x-\#1)}{9\#1^2+22\#1+9}\&\right]\right)}{\sqrt[3]{x}}$$

$$y(x) \to 0$$

1.567 problem 583

Internal problem ID [8705]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 583
Date solved : `Monday, October 21, 2024 at 05:20:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' - 3x(-2x^2+7)\,y' + (2x^2+25)\,y = 0$$

1.567.1 Solved as second order ode using Kovacic algorithm

Time used: 0.316 (sec)

Writing the ode as

$$9x^2y'' + (6x^3-21x)\,y' + (2x^2+25)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= 6x^3 - 21x \\ C &= 2x^2 + 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 24x^2 - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 - 24x^2 - 9 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 24x^2 - 9}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1079: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{9} - \frac{2}{3} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{3} - \frac{1}{x} - \frac{15}{8x^3} - \frac{45}{8x^5} - \frac{2835}{128x^7} - \frac{12555}{128x^9} - \frac{477495}{1024x^{11}} - \frac{2380185}{1024x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{3}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{3} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{9}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 24x^2 - 9}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{x^2}{9} - \frac{2}{3}\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{9} - \frac{2}{3} - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{2}{3}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{2}{3}\right) - (0) \\ &= -\frac{2}{3} \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{3} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 - 24x^2 - 9}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{3}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{x}{3}\right) \\
&= \frac{1}{2x} - \frac{x}{3} \\
&= \frac{1}{2x} - \frac{x}{3}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{x}{3}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{3}\right) + \left(\frac{1}{2x} - \frac{x}{3}\right)^2 - \left(\frac{4x^4 - 24x^2 - 9}{36x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{x}{3}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{6x^3 - 21x}{9x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{6} + \frac{7 \ln(x)}{6}} \\
&= z_1 \left(x^{7/6} \mathrm{e}^{-\frac{x^2}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^{5/3} \mathrm{e}^{-\frac{x^2}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6x^3-21x}{9x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{3}+\frac{7\ln(x)}{3}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{3}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{5/3} \mathrm{e}^{-\frac{x^2}{3}}\right) + c_2 \left(x^{5/3} \mathrm{e}^{-\frac{x^2}{3}} \left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{3}\right)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.567.2 Maple step by step solution

Let's solve

$9x^2 \left(\frac{d}{dx} y'\right) - 3x(-2x^2 + 7) y' + (2x^2 + 25) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(2x^2+25)y}{9x^2} - \frac{(2x^2-7)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(2x^2-7)y'}{3x} + \frac{(2x^2+25)y}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{2x^2-7}{3x}, P_3(x) = \frac{2x^2+25}{9x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -\frac{7}{3}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{25}{9}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2\left(\frac{d}{dx}y'\right) + 3x(2x^2-7)\,y' + (2x^2+25)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-5+3r)^2 x^r + a_1(-2+3r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-5)^2 + 2a_{k-2}(3k+3r-5)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-5+3r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = \frac{5}{3}$
- Each term must be 0
  $a_1(-2+3r)^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(3k+3r-5)^2 + 2a_{k-2}(3k+3r-5) = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(3k+3r+1)^2 + 2a_k(3k+3r+1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2a_k}{3k+3r+1}$
- Recursion relation for $r = \frac{5}{3}$
  $a_{k+2} = -\frac{2a_k}{3k+6}$
- Solution for $r = \frac{5}{3}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{3}}, a_{k+2} = -\frac{2a_k}{3k+6}, a_1 = 0\right]$

1.567.3 Maple trace

```
Methods for second order ODEs:
```

1.567.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 25

```
dsolve(9*x^2*diff(diff(y(x),x),x)-3*x*(-2*x^2+7)*diff(y(x),x)+(2*x^2+25)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^{5/3} e^{-\frac{x^2}{3}} \left(c_1 + \text{Ei}_1\left(-\frac{x^2}{3}\right) c_2\right)$$

1.567.5 Mathematica DSolve solution

Solving time : 0.097 (sec)
Leaf size : 39

```
DSolve[{9*x^2*D[y[x],{x,2}]-3*x*(7-2*x^2)*D[y[x],x]+(25+2*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-\frac{x^2}{3}}x^{5/3}\left(c_2 \operatorname{ExpIntegralEi}\left(\frac{x^2}{3}\right) + 2c_1\right)$$

1.568 problem 584

Internal problem ID [8706]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 584
Date solved : `Monday, October 21, 2024 at 05:20:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2 y'' - x(-x^2+1)\, y' + (x^2+1)\, y = 0$$

1.568.1 Solved as second order ode using Kovacic algorithm

Time used: 0.286 (sec)

Writing the ode as

$$x^2 y'' + (x^3 - x)\, y' + (x^2+1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 - x \\ C &= x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 4x^2 - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 4x^2 - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1081: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{1}{x} - \frac{5}{4x^3} - \frac{5}{2x^5} - \frac{105}{16x^7} - \frac{155}{8x^9} - \frac{1965}{32x^{11}} - \frac{3265}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 4x^2 - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{x^2}{4} - 1\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{4} - 1 - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) &&= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) &&= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 4x^2 - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{x}{2}\right) \\
&= \frac{1}{2x} - \frac{x}{2} \\
&= \frac{1}{2x} - \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{2}\right) + \left(\frac{1}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{x}{2}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^3 - x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{4} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2 \left(x\,\mathrm{e}^{-\frac{x^2}{2}} \left(-\frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.568.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x(-x^2+1)\,y' + (x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+1)y}{x^2} - \frac{(x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(x^2-1)y'}{x} + \frac{(x^2+1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x(x^2-1)\, y' + (x^2+1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ○ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 1$
- Each term must be 0

  $a_1 r^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_k(k+r-1) + a_{k-2}\right) = 0$
- Shift index using $k- > k+2$

  $(k+r+1)\left(a_{k+2}(k+r+1) + a_k\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k+r+1}$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{k+2}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$

1.568.3 Maple trace

```
Methods for second order ODEs:
```

1.568.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-x^2+1)*diff(y(x),x)+(x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x \,\mathrm{e}^{-\frac{x^2}{2}} \left(c_1 + \mathrm{Ei}_1\left(-\frac{x^2}{2}\right) c_2\right)$$

1.568.5 Mathematica DSolve solution

Solving time : 0.019 (sec)
Leaf size : 35

```
DSolve[{x^2*D[y[x],{x,2}]-x*(1-x^2)*D[y[x],x]+(1+x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-\frac{x^2}{2}}x\left(c_1 \operatorname{ExpIntegralEi}\left(\frac{x^2}{2}\right) + 2c_2\right)$$

1.569 problem 585

Internal problem ID [8707]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 585
Date solved : `Monday, October 21, 2024 at 05:20:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-2x)\,y''+3xy'+(1+4x)\,y=0$$

1.569.1 Solved as second order ode using Kovacic algorithm

Time used: 0.265 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right)y''+3xy'+(1+4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= 3x \\ C &= 1+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 32x^2 + 16x - 1$$
$$t = 4\left(2x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1083: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4\left(x - \frac{1}{2}\right)^2} - \frac{3}{x - \frac{1}{2}} + \frac{3}{x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)} + (-)(0) \\
&= \frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)} \\
&= \frac{-1-4x}{4x^2-2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{3}{2\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{32x^2+16x-1}{4\left(2x^2-x\right)^2}\right)\right)
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)}\right) dx} \\
&= \frac{\sqrt{x}}{(-1+2x)^{3/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{-2x^3+x^2}\, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2} + \frac{3\ln(-1+2x)}{2}} \\
&= z_1 \left(\frac{(-1+2x)^{3/2}}{x^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{-2x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3\ln(x)+3\ln(-1+2x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{8x^3}{3} + \frac{1}{2} + 6x - 6x^2 - \ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{8x^3}{3} + \frac{1}{2} + 6x - 6x^2 - \ln(x)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.569.2 Maple step by step solution

Let's solve

$x^2(1-2x)\left(\frac{d}{dx}y'\right) + 3xy' + (1+4x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(1+4x)y}{x^2(-1+2x)} + \frac{3y'}{x(-1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{3y'}{x(-1+2x)} - \frac{(1+4x)y}{x^2(-1+2x)} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x)=-\frac{3}{x(-1+2x)}, P_3(x)=-\frac{1+4x}{x^2(-1+2x)}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=3$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=1$

- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(-1+2x)\left(\frac{d}{dx}y'\right)-3xy'+(-1-4x)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=2..3$

  $x^m\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(1+r)^2\, x^r+\left(\sum_{k=1}^{\infty}\left(-a_k(k+r+1)^2+2a_{k-1}(k+r)\,(k-3+r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = -1$
- Each term in the series must be 0, giving the recursion relation
  $-a_k(k+r+1)^2 + 2a_{k-1}(k+r)(k-3+r) = 0$
- Shift index using $k- >k+1$
  $-a_{k+1}(k+2+r)^2 + 2a_k(k+r+1)(k+r-2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{2a_k(k+r+1)(k+r-2)}{(k+2+r)^2}$
- Recursion relation for $r = -1$; series terminates at $k = 3$
  $a_{k+1} = \frac{2a_k k(k-3)}{(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = 0$
- Apply recursion relation for $k = 1$
  $a_2 = -a_1$
- Express in terms of a_0
  $a_2 = 0$
- Apply recursion relation for $k = 2$
  $a_3 = -\frac{4a_2}{9}$
- Express in terms of a_0
  $a_3 = 0$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second
  $y = a_0 \cdot 0$

1.569.3 Maple trace

```
Methods for second order ODEs:
```

1.569.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 31

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)+3*x*diff(y(x),x)+(1+4*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{3 \ln \left(x \right) c_2 + \left(-8 x^3 + 18 x^2 - 18 x \right) c_2 + c_1}{x}$$

1.569.5 Mathematica DSolve solution

Solving time : `0.071` (sec)
Leaf size : 36

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]+3*x*D[y[x],x]+(1+4*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{2}{3} c_2 \left(4 x^2 - 9 x + 9\right) + \frac{c_1}{x} + \frac{c_2 \log(x)}{x}$$

1.570 problem 586

Internal problem ID [8708]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 586
Date solved : `Monday, October 21, 2024 at 05:20:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(1+x)\,y''+(1-x)\,y'+y=0$$

1.570.1 Solved as second order ode using Kovacic algorithm

Time used: 0.259 (sec)

Writing the ode as

$$\left(x^2+x\right)y''+(1-x)\,y'+y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+x \\ B &= 1-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 10x - 1$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1085: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{1+x} - \frac{2}{x} - \frac{1}{4x^2} + \frac{2}{(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^2-10x-1}{4\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{1+x} + \frac{1}{2x} + (-)(0) \\
&= -\frac{1}{1+x} + \frac{1}{2x} \\
&= -\frac{x-1}{2x(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{1+x} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{1+x} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2-10x-1}{4(x^2+x)^2}\right)\right) = 0
$$

$$
\frac{1+a_0}{x(1+x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-1)\, e^{\int \left(-\frac{1}{1+x} + \frac{1}{2x}\right) dx} \\
&= (x-1)\, e^{-\ln(1+x) + \frac{\ln(x)}{2}} \\
&= \frac{(x-1)\sqrt{x}}{1+x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-x}{x^2+x}\,dx} \\
&= z_1 e^{\ln(1+x)-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1+x}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x - 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-x}{x^2+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(1+x)-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\ln(x) - \frac{4}{x-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x-1) + c_2\left(x - 1\left(\ln(x) - \frac{4}{x-1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.570.2 Maple step by step solution

Let's solve

$x(1+x)\left(\frac{d}{dx}y'\right)+(1-x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{y}{x(1+x)}+\frac{(x-1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{(x-1)y'}{x(1+x)}+\frac{y}{x(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{x-1}{x(1+x)},P_3(x)=\frac{1}{x(1+x)}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=-2$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  - $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$x(1+x)\left(\frac{d}{dx}y'\right)+(1-x)\,y'+y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2-u)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(k-2+r) + a_k(k+r-1)^2\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(-3+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

$-a_{k+1}(k+1+r)(k-2+r) + a_k(k+r-1)^2 = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r-1)^2}{(k+1+r)(k-2+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$a_{k+1} = \frac{a_k(k-1)^2}{(k+1)(k-2)}$

- Apply recursion relation for $k = 0$

$a_1 = -\frac{a_0}{2}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$y(u) = a_0 \cdot \left(1 - \frac{u}{2}\right)$

- Revert the change of variables $u = 1 + x$

$\left[y = a_0\left(-\frac{x}{2} + \frac{1}{2}\right)\right]$

- Recursion relation for $r = 3$

$a_{k+1} = \frac{a_k(k+2)^2}{(k+4)(k+1)}$

- Solution for $r = 3$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = \frac{a_k(k+2)^2}{(k+4)(k+1)}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y=\sum_{k=0}^{\infty} a_k (1+x)^{k+3}, a_{k+1}=\frac{a_k (k+2)^2}{(k+4)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y=a_0\left(-\frac{x}{2}+\frac{1}{2}\right)+\left(\sum_{k=0}^{\infty} b_k (1+x)^{k+3}\right), b_{k+1}=\frac{b_k (k+2)^2}{(k+4)(k+1)}\right]$$

1.570.3 Maple trace

```
Methods for second order ODEs:
```

1.570.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 20

```
dsolve(x*(1+x)*diff(diff(y(x),x),x)+(1-x)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y=c_2(x-1)\ln(x)-4c_2+c_1(x-1)$$

1.570.5 Mathematica DSolve solution

Solving time : `0.074` (sec)
Leaf size : 23

```
DSolve[{x*(1+x)*D[y[x],{x,2}]+(1-x)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_1(x-1)+c_2((x-1)\log(x)-4)$$

1.571 problem 587

Internal problem ID [8709]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 587
Date solved : `Monday, October 21, 2024 at 05:20:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-x)\,y''-x(3-5x)\,y'+(4-5x)\,y=0$$

1.571.1 Solved as second order ode using Kovacic algorithm

Time used: 0.257 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(5x^2-3x\right)y'+(4-5x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^3+x^2\\ B&=5x^2-3x\\ C&=4-5x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 15x^2 - 6x - 1 \\
t &= 4\left(x^2 - x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1087: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{-1+x} + \frac{2}{(-1+x)^2} - \frac{1}{4x^2} - \frac{2}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 6x - 1}{4(x^2 - x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{15x^2-6x-1}{4\left(x^2-x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{+}\right) \\
&= \frac{5}{2}-\left(\frac{5}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \frac{2}{-1+x} + (0) \\
&= \frac{1}{2x} + \frac{2}{-1+x} \\
&= \frac{-1+5x}{2x\left(-1+x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x} + \frac{2}{-1+x}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{2}{\left(-1+x\right)^2}\right) + \left(\frac{1}{2x} + \frac{2}{-1+x}\right)^2 - \left(\frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{2}{-1+x}\right) dx} \\
&= \sqrt{x}\left(-1+x\right)^2
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{5x^2 - 3x}{-x^3 + x^2} \, dx} \\
&= z_1 e^{\frac{3\ln(x)}{2} + \ln(-1+x)} \\
&= z_1 \left(x^{3/2}(-1+x)\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2(-1+x)^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2-3x}{-x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{3\ln(x)+2\ln(-1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln(x) - \frac{1}{3(-1+x)^3} - \frac{1}{-1+x} + \frac{1}{2(-1+x)^2} - \ln(-1+x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2(-1+x)^3\right) \\
&\quad + c_2 \left(x^2(-1+x)^3 \left(\ln(x) - \frac{1}{3(-1+x)^3} - \frac{1}{-1+x} + \frac{1}{2(-1+x)^2} - \ln(-1+x)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.571.2 Maple step by step solution

Let's solve

$x^2(1-x)\left(\frac{d}{dx}y'\right) - x(3-5x)\,y' + (4-5x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(-4+5x)y}{x^2(-1+x)} + \frac{(5x-3)y'}{x(-1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{(5x-3)y'}{x(-1+x)} + \frac{(-4+5x)y}{x^2(-1+x)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{5x-3}{x(-1+x)}, P_3(x) = \frac{-4+5x}{x^2(-1+x)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -3$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 4$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2(-1+x)\left(\frac{d}{dx}y'\right) - x(5x-3)\,y' + (-4+5x)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$-a_0(-2+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-2)^2 + a_{k-1}(k+r-2)(k-6+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 2$
- Each term in the series must be 0, giving the recursion relation
  $-a_k(k+r-2)^2 + a_{k-1}(k+r-2)(k-6+r) = 0$
- Shift index using $k- >k+1$
  $-a_{k+1}(k+r-1)^2 + a_k(k+r-1)(k+r-5) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+r-5)}{k+r-1}$
- Recursion relation for $r = 2$; series terminates at $k = 3$
  $a_{k+1} = \frac{a_k(k-3)}{k+1}$
- Apply recursion relation for $k = 0$
  $a_1 = -3a_0$
- Apply recursion relation for $k = 1$
  $a_2 = -a_1$
- Express in terms of a_0
  $a_2 = 3a_0$
- Apply recursion relation for $k = 2$
  $a_3 = -\frac{a_2}{3}$
- Express in terms of a_0
  $a_3 = -a_0$
- Terminating series solution of the ODE for $r = 2$. Use reduction of order to find the second li
  $y = a_0 \cdot (-x^3 + 3x^2 - 3x + 1)$

1.571.3 Maple trace

```
Methods for second order ODEs:
```

1.571.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 47

```
dsolve(x^2*(1-x)*diff(diff(y(x),x),x)-x*(3-5*x)*diff(y(x),x)+(4-5*x)*y(x) = 0,
       y(x),singsol=all)
```

$$
y = x^2\left(c_1(-1+x)^3 + c_2\left(-(-1+x)^3\ln\left(-1+x\right) + (-1+x)^3\ln\left(x\right) - x^2 + \frac{5x}{2} - \frac{11}{6}\right)\right)
$$

1.571.5 Mathematica DSolve solution

Solving time : 0.112 (sec)
Leaf size : 76

```
DSolve[{x^2*(1-x)*D[y[x],{x,2}]-x*(3-5*x)*D[y[x],x]+(4-5*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to -\frac{1}{6}x^2\left(6c_1x^3 - 18c_1x^2 - 6c_2x^2 + 18c_1x + 15c_2x - 6c_2(x-1)^3\log(x-1) + 6c_2(x-1)^3\log(x) - 6c_1 - 11c_2\right)
$$

1.572 problem 588

Internal problem ID [8710]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 588
Date solved : `Monday, October 21, 2024 at 05:20:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''-x\left(9x^2+1\right)y'+\left(25x^2+1\right)y=0$$

1.572.1 Solved as second order ode using Kovacic algorithm

Time used: 0.352 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(-9x^3-x\right)y'+\left(25x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^4+x^2\\ B&=-9x^3-x\\ C&=25x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 - 98x^2 - 1}{4(x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^4 - 98x^2 - 1$$
$$t = 4(x^3 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 - 98x^2 - 1}{4(x^3 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1089: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{6}{(x-i)^2} + \frac{6}{(x+i)^2} + \frac{6i}{x-i} - \frac{6i}{x+i}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-x^4 - 98x^2 - 1}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^4 - 98x^2 - 1}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	3	-2
$-i$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right)\\
&= \frac{1}{2} - \left(-\frac{7}{2}\right)\\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty\\
&= \frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i} + (-)\,(0)\\
&= \frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i}\\
&= \frac{1}{2x} - \frac{4x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i}\right)\left(4x^3 + 3x^2a_3 + 2a_2x + a_1\right) + \left(\left(-\frac{1}{2x^2} + \frac{2}{(x-i)^2} + \frac{}{(x-}\right.\right.$$

$$\overline{(x^4a_3 + 4(4-}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0, a_2 = -4, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 4x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^4 - 4x^2 + 1\right) \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i}\right) dx} \\ &= \left(x^4 - 4x^2 + 1\right) \mathrm{e}^{-2\ln\left(x^2+1\right) + \frac{\ln(x)}{2}} \\ &= \frac{\left(x^4 - 4x^2 + 1\right)\sqrt{x}}{\left(x^2+1\right)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-9x^3-x}{x^4+x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} + 2\ln\left(x^2+1\right)} \\ &= z_1\left(\sqrt{x}\left(x^2+1\right)^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^5 - 4x^3 + x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-9x^3-x}{x^4+x^2}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x) + 4\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\, dx \\ &= y_1\left(\ln(x) + \frac{-6x^2+3}{x^4 - 4x^2 + 1}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^5 - 4x^3 + x\right) + c_2 \left(x^5 - 4x^3 + x \left(\ln\left(x\right) + \frac{-6x^2 + 3}{x^4 - 4x^2 + 1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.572.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right) - x(9x^2+1)\,y' + (25x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(25x^2+1)y}{x^2(x^2+1)} + \frac{(9x^2+1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(9x^2+1)y'}{x(x^2+1)} + \frac{(25x^2+1)y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{9x^2+1}{x(x^2+1)}, P_3(x) = \frac{25x^2+1}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.\left(x \cdot P_2(x)\right)\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.\left(x^2 \cdot P_3(x)\right)\right|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d}{dx}y'\right) - x(9x^2+1)\,y' + (25x^2+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2\, x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k-7+r)^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 1$
- Each term must be 0

  $a_1 r^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)^2 + a_{k-2}(k-7+r)^2 = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}(k+1+r)^2 + a_k(k+r-5)^2 = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r-5)^2}{(k+1+r)^2}$

- Recursion relation for $r = 1$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k (k-4)^2}{(k+2)^2}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k (k-4)^2}{(k+2)^2}, a_1 = 0\right]$

1.572.3 Maple trace

```
Methods for second order ODEs:
```

1.572.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 41

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(9*x^2+1)*diff(y(x),x)+(25*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x\left(c_2\left(x^4 - 4x^2 + 1\right)\ln(x) + c_1 x^4 + (-4c_1 - 6c_2)\, x^2 + c_1 + 3c_2\right)$$

1.572.5 Mathematica DSolve solution

Solving time : 0.127 (sec)
Leaf size : 43

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(1+9*x^2)*D[y[x],x]+(1+25*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1\left(x^5 - 4x^3 + x\right) + c_2 x\left(-6x^2 + \left(x^4 - 4x^2 + 1\right)\log(x) + 3\right)$$

1.573 problem 589

Internal problem ID [8711]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 589
Date solved : `Monday, October 21, 2024 at 05:20:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2y'' + 3x\left(-x^2+1\right)y' + \left(7x^2+1\right)y = 0$$

1.573.1 Solved as second order ode using Kovacic algorithm

Time used: 1.158 (sec)

Writing the ode as

$$9x^2y'' + \left(-3x^3+3x\right)y' + \left(7x^2+1\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= -3x^3+3x \\ C &= 7x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 36x^2 - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 36x^2 - 9$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 36x^2 - 9}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1091: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{x^2}{36} - 1 - \frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} - \frac{3}{x} - \frac{111}{4x^3} - \frac{999}{2x^5} - \frac{180819}{16x^7} - \frac{2292705}{8x^9} - \frac{249239511}{32x^{11}} - \frac{3548540907}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 36x^2 - 9}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{x^2}{36} - 1\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{36} - 1 - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{6} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{6}} - 1\right) &&= -\frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{6}} - 1\right) &&= \frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 36x^2 - 9}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{6}$	$-\frac{7}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{1}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{x}{6}\right) \\
&= \frac{1}{2x} - \frac{x}{6} \\
&= \frac{1}{2x} - \frac{x}{6}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{1}{2x} - \frac{x}{6}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2} - \frac{1}{6}\right) + \left(\frac{1}{2x} - \frac{x}{6}\right)^2 - \left(\frac{x^4 - 36x^2 - 9}{36x^2}\right)\right) = 0
$$

$$
\frac{x^2 a_1 + 2(6 + a_0)\,x + 3a_1}{3x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -6, a_1 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 6
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 6\right) e^{\int \left(\frac{1}{2x} - \frac{x}{6}\right) dx} \\
&= \left(x^2 - 6\right) e^{-\frac{x^2}{12} + \frac{\ln(x)}{2}} \\
&= \left(x^2 - 6\right) \sqrt{x}\, e^{-\frac{x^2}{12}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x^3+3x}{9x^2}\,dx} \\
&= z_1 e^{\frac{x^2}{12}-\frac{\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x^2}{12}}}{x^{1/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{1/3}\left(x^2-6\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+3x}{9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{6}-\frac{\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{6}-\frac{\ln(x)}{3}}}{x^{2/3}\left(x^2-6\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{1/3}\left(x^2-6\right)\right) + c_2\left(x^{1/3}\left(x^2-6\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{6}-\frac{\ln(x)}{3}}}{x^{2/3}\left(x^2-6\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.573.2 Maple step by step solution

Let's solve

$9x^2\left(\frac{d}{dx}y'\right)+3x(-x^2+1)\,y'+(7x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(7x^2+1)y}{9x^2}+\frac{(x^2-1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{(x^2-1)y'}{3x}+\frac{(7x^2+1)y}{9x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{x^2-1}{3x},P_3(x)=\frac{7x^2+1}{9x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{3}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{9}$

- ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$9x^2\left(\frac{d}{dx}y'\right)-3x(x^2-1)\,y'+(7x^2+1)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+3r)^2\, x^r + a_1(2+3r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 - a_{k-2}(3k-13+3r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+3r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{3}$$

- Each term must be 0

$$a_1(2+3r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(3k+3r-1)^2 + (-3k+13-3r)\, a_{k-2} = 0$$

- Shift index using $k->k+2$

$$a_{k+2}(3k+5+3r)^2 + a_k(-3k-3r+7) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a_k(3k+3r-7)}{(3k+5+3r)^2}$$

- Recursion relation for $r = \frac{1}{3}$; series terminates at $k = 2$

$$a_{k+2} = \frac{a_k(3k-6)}{(3k+6)^2}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{a_k(3k-6)}{(3k+6)^2}, a_1 = 0\right]$$

1.573.3 Maple trace

```
Methods for second order ODEs:
```

1.573.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 19

```
dsolve(9*x^2*diff(diff(y(x),x),x)+3*x*(-x^2+1)*diff(y(x),x)+(7*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{x^{1/3}(x^2 - 6)(c_1 - c_2)}{6}$$

1.573.5 Mathematica DSolve solution

Solving time : 0.448 (sec)
Leaf size : 53

```
DSolve[{9*x^2*D[y[x],{x,2}]+3*x*(1-x^2)*D[y[x],x]+(1+7*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{72}\sqrt[3]{x}\left(c_2(x^2 - 6)\,\text{ExpIntegralEi}\left(\frac{x^2}{6}\right) + 72c_1(x^2 - 6) - 6c_2e^{\frac{x^2}{6}}\right)$$

1.574 problem 590

Internal problem ID [8712]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 590
Date solved : `Monday, October 21, 2024 at 05:20:28 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x\left(x^{2}+1\right) y''+\left(-x^{2}+1\right) y'-8 x y=0$$

1.574.1 Solved as second order ode using Kovacic algorithm

Time used: 0.389 (sec)

Writing the ode as

$$\left(x^{3}+x\right) y''+\left(-x^{2}+1\right) y'-8 x y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{3}+x \\ B &= -x^{2}+1 \\ C &= -8 x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2 A} d x}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35x^4 + 22x^2 - 1$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1093: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} - \frac{15i}{4(x-i)} + \frac{15i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{7}{2} - \left(\frac{7}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (0) \\
&= \frac{1}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(x+i)} \\
&= \frac{1}{2x} + \frac{3x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(x+i)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{3}{2\,(x-i)^2} - \frac{3}{2\,(x+i)^2}\right) + \left(\frac{1}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \left(x^2+1\right)^{3/2}\sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+1}{x^3+x}\,dx} \\
&= z_1 e^{\frac{\ln\left(x^2+1\right)}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\sqrt{x^2+1}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2+1\right)^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+1}{x^3+x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\ln\left(x^2+1\right)-\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{\ln\left(x^2+1\right)}{2}+\frac{1}{2x^2+2}+\frac{1}{4\left(x^2+1\right)^2}+\ln\left(x\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2+1\right)^2\right) + c_2\left(\left(x^2+1\right)^2\left(-\frac{\ln\left(x^2+1\right)}{2}+\frac{1}{2x^2+2}+\frac{1}{4\left(x^2+1\right)^2}+\ln\left(x\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.574.2 Maple step by step solution

Let's solve

$x(x^2+1)\left(\frac{d}{dx}y'\right)+(-x^2+1)\,y'-8xy=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$
- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{8y}{x^2+1}+\frac{(x^2-1)y'}{x(x^2+1)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{(x^2-1)y'}{x(x^2+1)}-\frac{8y}{x^2+1}=0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{x^2-1}{x(x^2+1)},P_3(x)=-\frac{8}{x^2+1}\right]$
- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=1$
- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=0$
- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x(x^2+1)\left(\frac{d}{dx}y'\right)+(-x^2+1)\,y'-8xy=0$
- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x\cdot y$ to series expansion

  $x\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+1}$
- ◦ Shift index using $k->k-1$

  $x\cdot y=\sum_{k=1}^{\infty}a_{k-1}x^{k+r}$
- ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + a_1 (1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 + a_{k-1}(k+r+1)\,(k-5+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term must be 0

$$a_1 (1+r)^2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$\left((a_{k-1} + a_{k+1})\, k - 5a_{k-1} + a_{k+1}\right)(k+1) = 0$$

- Shift index using $k- > k+1$

$$\left((a_k + a_{k+2})\,(k+1) - 5a_k + a_{k+2}\right)(k+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k (k-4)}{k+2}$$

- Recursion relation for $r = 0$; series terminates at $k = 4$

$$a_{k+2} = -\frac{a_k (k-4)}{k+2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k (k-4)}{k+2}, a_1 = 0\right]$$

1.574.3 Maple trace

```
Methods for second order ODEs:
```

1.574.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 48

```
dsolve(x*(x^2+1)*diff(diff(y(x),x),x)+(-x^2+1)*diff(y(x),x)-8*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 (x^2+1)^2 + c_2 \left(-\frac{(x^2+1)^2 \ln(x^2+1)}{2} + (x^2+1)^2 \ln(x) + \frac{x^2}{2} + \frac{3}{4}\right)$$

1.574.5 Mathematica DSolve solution

Solving time : `0.115` (sec)
Leaf size : 55

```
DSolve[{x*(1+x^2)*D[y[x],{x,2}]+(1-x^2)*D[y[x],x]-8*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 (x^2+1)^2 + \frac{1}{4} c_2 \left(2x^2 + 4(x^2+1)^2 \log(x) - 2(x^2+1)^2 \log(x^2+1) + 3\right)$$

1.575 problem 591

Internal problem ID [8713]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 591
Date solved : `Monday, October 21, 2024 at 05:20:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 2x\left(-x^2+4\right)y' + \left(7x^2+1\right)y = 0$$

1.575.1 Solved as second order ode using Kovacic algorithm

Time used: 0.736 (sec)

Writing the ode as

$$4x^2y'' + \left(-2x^3+8x\right)y' + \left(7x^2+1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -2x^3+8x \\ C &= 7x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^4 - 40x^2 - 4}{16x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^4 - 40x^2 - 4 \\
t &= 16x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^4 - 40x^2 - 4}{16x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1095: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} - \frac{5}{2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{5}{x} - \frac{101}{2x^3} - \frac{1010}{x^5} - \frac{50601}{2x^7} - \frac{710030}{x^9} - \frac{21351501}{x^{11}} - \frac{672670100}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 40x^2 - 4}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{x^2}{16} - \frac{5}{2}\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{16} - \frac{5}{2} - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{4}} - 1\right) &&= -\frac{11}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{4}} - 1\right) &&= \frac{9}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 40x^2 - 4}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{4}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{9}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{9}{2} - \left(\frac{1}{2}\right) \\
&= 4
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{x}{4}\right) \\
&= \frac{1}{2x} - \frac{x}{4} \\
&= \frac{1}{2x} - \frac{x}{4}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(12x^2 + 6xa_3 + 2a_2) + 2\left(\frac{1}{2x} - \frac{x}{4}\right)(4x^3 + 3a_3x^2 + 2a_2x + a_1) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{x}{4}\right)^2 - \left(\frac{x^4 \cdots}{\cdots}\right.\right.
$$

$$
\frac{x^4 a_3 + 2(16 + a_2)\, x^3 + 3(a_1 + 6a_3)\, x^2 + 4(a_0 + 2a \cdots}{2x}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 32, a_1 = 0, a_2 = -16, a_3 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^4 - 16x^2 + 32
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 - 16x^2 + 32\right) e^{\int \left(\frac{1}{2x} - \frac{x}{4}\right) dx} \\
&= \left(x^4 - 16x^2 + 32\right) e^{-\frac{x^2}{8} + \frac{\ln(x)}{2}} \\
&= \left(x^4 - 16x^2 + 32\right) \sqrt{x}\, e^{-\frac{x^2}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^3+8x}{4x^2}\,dx} \\
&= z_1 e^{\frac{x^2}{8}-\ln(x)} \\
&= z_1\left(\frac{\mathrm{e}^{\frac{x^2}{8}}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^4-16x^2+32}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+8x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{4}-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{\frac{x^2}{4}-2\ln(x)}x}{(x^4-16x^2+32)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^4-16x^2+32}{\sqrt{x}}\right) + c_2\left(\frac{x^4-16x^2+32}{\sqrt{x}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{4}-2\ln(x)}x}{(x^4-16x^2+32)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.575.2 Maple step by step solution

Let's solve

$4x^2 \left(\frac{d}{dx}y'\right) + 2x(-x^2+4)\, y' + (7x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = -\frac{(7x^2+1)y}{4x^2} + \frac{(x^2-4)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{(x^2-4)y'}{2x} + \frac{(7x^2+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = -\frac{x^2-4}{2x}, P_3(x) = \frac{7x^2+1}{4x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left.(x \cdot P_2(x))\right|_{x=0} = 2$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{4}$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$4x^2 \left(\frac{d}{dx}y'\right) - 2(x^2-4)\, xy' + (7x^2+1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)^2\, x^r + a_1(3+2r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)^2 - a_{k-2}(2k-11+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -\frac{1}{2}$$

- Each term must be 0

$$a_1(3+2r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(2k+2r+1)^2 + (-2k+11-2r)\, a_{k-2} = 0$$

- Shift index using $k->k+2$

$$a_{k+2}(2k+5+2r)^2 + a_k(-2k-2r+7) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a_k(2k+2r-7)}{(2k+5+2r)^2}$$

- Recursion relation for $r = -\frac{1}{2}$; series terminates at $k = 4$

$$a_{k+2} = \frac{a_k(2k-8)}{(2k+4)^2}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{a_k(2k-8)}{(2k+4)^2}, a_1 = 0\right]$$

1.575.3 Maple trace

```
Methods for second order ODEs:
```

1.575.4 Maple dsolve solution

Solving time : `0.017` (sec)
Leaf size : 24

```
dsolve(4*x^2*diff(diff(y(x),x),x)+2*x*(-x^2+4)*diff(y(x),x)+(7*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(x^4 - 16x^2 + 32)(c_1 + 2c_2)}{32\sqrt{x}}$$

1.575.5 Mathematica DSolve solution

Solving time : `0.189` (sec)
Leaf size : 68

```
DSolve[{4*x^2*D[y[x],{x,2}]+2*x*(4-x^2)*D[y[x],x]+(1+7*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2(x^4 - 16x^2 + 32)\,\text{ExpIntegralEi}\left(\frac{x^2}{4}\right) - 4c_2e^{\frac{x^2}{4}}(x^2 - 12) + 2048c_1(x^4 - 16x^2 + 32)}{2048\sqrt{x}}$$

1.576 problem 592

Internal problem ID [8714]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 592
Date solved : `Monday, October 21, 2024 at 05:20:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y''+8x^2y'+(1+x)\,y=0$$

1.576.1 Solved as second order ode using Kovacic algorithm

Time used: 0.179 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+8x^2y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=4x^3+4x^2\\ B&=8x^2\\ C&=1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1097: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0 \\ 0 = 0\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^2}{4x^3+4x^2} \, dx} \\ &= z_1 e^{-\ln(1+x)} \\ &= z_1 \left(\frac{1}{1+x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{1+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{8x^2}{4x^3+4x^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-2\ln(1+x)}}{(y_1)^2}\,dx$$
$$= y_1(\ln(x))$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{\sqrt{x}}{1+x}\right) + c_2\left(\frac{\sqrt{x}}{1+x}(\ln(x))\right)$$

Will add steps showing solving for IC soon.

1.576.2 Maple step by step solution

Let's solve

$4x^2(1+x)\left(\frac{d}{dx}y'\right) + 8x^2y' + (1+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{4x^2} - \frac{2y'}{1+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2y'}{1+x} + \frac{y}{4x^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2}{1+x}, P_3(x) = \frac{1}{4x^2}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = 2$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

- $x = -1$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0 = -1$
- Multiply by denominators
  $4x^2(1+x)\left(\frac{d}{dx}y'\right) + 8x^2y' + (1+x)\,y = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$
  $(4u^3 - 8u^2 + 4u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (8u^2 - 16u + 8)\left(\frac{d}{du}y(u)\right) + uy(u) = 0$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$
- □ Rewrite ODE with series expansions
  - Convert $u \cdot y(u)$ to series expansion
    $$u \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+1}$$
  - Shift index using $k- >k-1$
    $$u \cdot y(u) = \sum_{k=1}^{\infty} a_{k-1} u^{k+r}$$
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$
    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$
  - Shift index using $k- >k+1-m$
    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$
    $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$
  - Shift index using $k- >k+2-m$
    $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions
  $$4a_0r(1+r)\,u^{-1+r} + (4a_1(1+r)(2+r) - 8a_0r(1+r))\,u^r + \left(\sum_{k=1}^{\infty}\left(4a_{k+1}(k+r+1)(k+2+r)\right.\right.$$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r(1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 0\}$

- Each term must be 0

  $4a_1(1+r)(2+r) - 8a_0 r(1+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k-1}(2k-1+2r)^2 - 8(k+r+1)\left(\left(-\frac{k}{2} - \frac{r}{2} - 1\right)a_{k+1} + a_k(k+r)\right) = 0$
- Shift index using $k- > k+1$

  $a_k(2k+2r+1)^2 - 8(k+2+r)\left(\left(-\frac{k}{2} - \frac{3}{2} - \frac{r}{2}\right)a_{k+2} + a_{k+1}(k+r+1)\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 8kra_k - 16kra_{k+1} + 4r^2a_k - 8r^2a_{k+1} + 4ka_k - 24ka_{k+1} + 4ra_k - 24ra_{k+1} + a_k - 16a_{k+1}}{4(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+1)(k+2)}$
- Solution for $r = -1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+1)(k+2)}, 0 = 0\right]$
- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-1}, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+1)(k+2)}, 0 = 0\right]$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+2)(k+3)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+2)(k+3)}, 8a_1 = 0\right]$
- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+2)(k+3)}, 8a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^k\right), a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+1)(k+2)}, 0 = 0, b_{k+2}\right.$

1.576.3 Maple trace

```
Methods for second order ODEs:
```

1.576.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 19

```
dsolve(4*x^2*(1+x)*diff(diff(y(x),x),x)+8*x^2*diff(y(x),x)+(1+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\sqrt{x}\,(c_2 \ln(x) + c_1)}{1 + x}$$

1.576.5 Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 24

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+8*x^2*D[y[x],x]+(1+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}(c_2 \log(x) + c_1)}{x + 1}$$

1.577 problem 593

Internal problem ID [8715]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 593
Date solved : `Monday, October 21, 2024 at 05:20:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2(3+x)\,y''+3x(3+7x)\,y'+(3+4x)\,y=0$$

1.577.1 Solved as second order ode using Kovacic algorithm

Time used: 0.246 (sec)

Writing the ode as

$$\left(9x^3+27x^2\right)y''+\left(21x^2+9x\right)y'+(3+4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^3+27x^2 \\ B &= 21x^2+9x \\ C &= 3+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1099: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{21x^2+9x}{9x^3+27x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{6} - \ln(3+x)} \\ &= z_1 \left(\frac{1}{x^{1/6}\,(3+x)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{3+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{21x^2+9x}{9x^3+27x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{3}-2\ln(3+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln(x) + \frac{x^2}{9+3x} + \frac{2x}{3+x} + \frac{3}{3+x} - \frac{2\ln(3+x)\, x}{3} - 2\ln(3+x) \right. \\
&\qquad \left. + \frac{2\ln(3+x)(3+x)}{3} - \frac{x}{3} - 2\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/3}}{3+x}\right) \\
&\quad + c_2 \left(\frac{x^{1/3}}{3+x} \left(\ln(x) + \frac{x^2}{9+3x} + \frac{2x}{3+x} + \frac{3}{3+x} - \frac{2\ln(3+x)\, x}{3} - 2\ln(3+x) + \frac{2\ln(3+x)(3+x)}{3} - \frac{x}{3} - 2\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.577.2 Maple step by step solution

Let's solve

$9x^2(3+x)\left(\frac{d}{dx}y'\right) + 3x(3+7x)\, y' + (3+4x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(3+4x)y}{9x^2(3+x)} - \frac{(3+7x)y'}{3x(3+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3+7x)y'}{3x(3+x)} + \frac{(3+4x)y}{9x^2(3+x)} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{3+7x}{3x(3+x)}, P_3(x) = \frac{3+4x}{9x^2(3+x)}\right]$

- ○ $(3+x)\cdot P_2(x)$ is analytic at $x=-3$

  $$\left((3+x)\cdot P_2(x)\right)\bigg|_{x=-3}=2$$

- ○ $(3+x)^2\cdot P_3(x)$ is analytic at $x=-3$

  $$\left((3+x)^2\cdot P_3(x)\right)\bigg|_{x=-3}=0$$

- ○ $x=-3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $$9x^2(3+x)\left(\frac{d}{dx}y'\right)+3x(3+7x)\,y'+(3+4x)\,y=0$$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $$(9u^3-54u^2+81u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(21u^2-117u+162)\left(\frac{d}{du}y(u)\right)+(-9+4u)\,y(u)=0$$

- Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$$

- ○ Shift index using $k->k-m$

  $$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$$

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

- ○ Shift index using $k->k+1-m$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- ○ Convert $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..3$

  $$u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- ○ Shift index using $k->k+2-m$

  $$u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$81a_0r(1+r)\,u^{-1+r}+\left(81a_1(1+r)\,(2+r)-9a_0(1+r)\,(1+6r)\right)u^r+\left(\sum_{k=1}^{\infty}\left(81a_{k+1}(k+r+1)\,(k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $81r(1+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{-1,0\}$
- Each term must be 0
  $81a_1(1+r)\,(2+r)-9a_0(1+r)\,(1+6r)=0$
- Each term in the series must be 0, giving the recursion relation
  $81a_{k+1}(k+r+1)\,(k+2+r)-54\left(k+r+\frac{1}{6}\right)a_k(k+r+1)+a_{k-1}(3k-1+3r)^2=0$
- Shift index using $k->k+1$
  $81a_{k+2}(k+2+r)\,(k+3+r)-54\left(k+\frac{7}{6}+r\right)a_{k+1}(k+2+r)+a_k(3k+3r+2)^2=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+18kra_k-108kra_{k+1}+9r^2a_k-54r^2a_{k+1}+12ka_k-171ka_{k+1}+12ra_k-171ra_{k+1}+4a_k-126a_{k+1}}{81(k+2+r)(k+3+r)}$
- Recursion relation for $r=-1$
  $a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)}$
- Solution for $r=-1$
  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-1},a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)},0=0\right]$
- Revert the change of variables $u=3+x$
  $\left[y=\sum_{k=0}^{\infty}a_k(3+x)^{k-1},a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)},0=0\right]$
- Recursion relation for $r=0$
  $a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)},162a_1-9a_0=0\right]$
- Revert the change of variables $u=3+x$
  $\left[y=\sum_{k=0}^{\infty}a_k(3+x)^k,a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)},162a_1-9a_0=0\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty}a_k(3+x)^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k(3+x)^k\right),a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)},0=\right.$

1.577.3 Maple trace

```
Methods for second order ODEs:
```

1.577.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 19

```
dsolve(9*x^2*(3+x)*diff(diff(y(x),x),x)+3*x*(3+7*x)*diff(y(x),x)+(3+4*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^{1/3}(\ln(x)\, c_2 + c_1)}{3 + x}$$

1.577.5 Mathematica DSolve solution

Solving time : 0.059 (sec)
Leaf size : 24

```
DSolve[{9*x^2*(3+x)*D[y[x],{x,2}]+3*x*(3+7*x)*D[y[x],x]+(3+4*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[3]{x}(c_2 \log(x) + c_1)}{x + 3}$$

1.578 problem 594

Internal problem ID [8716]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 594
Date solved : `Monday, October 21, 2024 at 05:20:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(-x^2+2)\,y''-x(3x^2+2)\,y'+(-x^2+2)\,y=0$$

1.578.1 Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$(-x^4+2x^2)\,y''+(-3x^3-2x)\,y'+(-x^2+2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+2x^2 \\ B &= -3x^3-2x \\ C &= -x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1101: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$\begin{aligned}
d&=\alpha_\infty^- -\left(\alpha_{c_1}^-\right)\\
&=\frac{1}{2}-\left(\frac{1}{2}\right)\\
&=0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&=\frac{1}{2x}+(-)\,(0)\\
&=\frac{1}{2x}\\
&=\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-3x^3-2x}{-x^4+2x^2} \, dx} \\ &= z_1 e^{-\ln(x^2-2)+\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\sqrt{x}}{x^2-2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{x^2-2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3-2x}{-x^4+2x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x^2-2)+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{x}{x^2-2}\right) + c_2\left(\frac{x}{x^2-2}(\ln(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.578.2 Maple step by step solution

Let's solve
$x^2(-x^2+2)\left(\frac{d}{dx}y'\right) - x(3x^2+2)\,y' + (-x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{y}{x^2} - \frac{(3x^2+2)y'}{x(x^2-2)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + \frac{(3x^2+2)y'}{x(x^2-2)} + \frac{y}{x^2} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{3x^2+2}{x(x^2-2)}, P_3(x) = \frac{1}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$$

◦ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$x^2(x^2-2)\left(\frac{d}{dx}y'\right) + x(3x^2+2)\, y' + (x^2-2)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

◦ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

◦ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

◦ Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0(-1+r)^2 x^r - 2a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(-2a_k(k+r-1)^2 + a_{k-2}(k+r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$r = 1$

- Each term must be 0
  $-2a_1r^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2\left(a_k - \frac{a_{k-2}}{2}\right)(k+r-1)^2 = 0$
- Shift index using $k->k+2$
  $-2\left(a_{k+2} - \frac{a_k}{2}\right)(k+r+1)^2 = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_k}{2}$
- Recursion relation for $r = 1$
  $a_{k+2} = \frac{a_k}{2}$
- Solution for $r = 1$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$

1.578.3 Maple trace

```
Methods for second order ODEs:
```

1.578.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 19

```
dsolve(x^2*(-x^2+2)*diff(diff(y(x),x),x)-x*(3*x^2+2)*diff(y(x),x)+(-x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x(c_1 + c_2 \ln(x))}{x^2 - 2}$$

1.578.5 Mathematica DSolve solution

Solving time : 0.058 (sec)
Leaf size : 23

```
DSolve[{x^2*(2-x^2)*D[y[x],{x,2}]-x*(2+3*x^2)*D[y[x],x]+(2-x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{x(c_2 \log(x) + c_1)}{x^2 - 2}$$

1.579 problem 595

Internal problem ID [8717]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 595
Date solved : `Monday, October 21, 2024 at 05:20:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$16x^2(x^2+1)\,y''+8x(9x^2+1)\,y'+(49x^2+1)\,y=0$$

1.579.1 Solved as second order ode using Kovacic algorithm

Time used: 0.294 (sec)

Writing the ode as

$$(16x^4+16x^2)\,y''+(72x^3+8x)\,y'+(49x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^4+16x^2 \\ B &= 72x^3+8x \\ C &= 49x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1103: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0 \\
0 = 0
\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{72x^3 + 8x}{16x^4 + 16x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{4} - \ln(x^2+1)} \\
&= z_1 \left(\frac{1}{x^{1/4} (x^2 + 1)}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}}{x^2 + 1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{72x^3+8x}{16x^4+16x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2}-2\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\ln(x) - \ln(x^2+1)\,x^2 - \ln(x^2+1) + \frac{x^4}{2x^2+2} + \frac{x^2}{x^2+1} + \frac{1}{2x^2+2} \right. \\
&\qquad \left. + \ln(x^2+1)\,(x^2+1) - \frac{x^2}{2} - 1\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/4}}{x^2+1}\right) \\
&\quad + c_2 \left(\frac{x^{1/4}}{x^2+1}\left(\ln(x) - \ln(x^2+1)\,x^2 - \ln(x^2+1) + \frac{x^4}{2x^2+2} + \frac{x^2}{x^2+1} + \frac{1}{2x^2+2} + \ln(x^2+1)\,(x^2+1) - \frac{x^2}{2}
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.579.2 Maple step by step solution

Let's solve

$16x^2(x^2+1)\left(\frac{d}{dx}y'\right) + 8x(9x^2+1)\,y' + (49x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(49x^2+1)y}{16x^2(x^2+1)} - \frac{(9x^2+1)y'}{2x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(9x^2+1)y'}{2x(x^2+1)} + \frac{(49x^2+1)y}{16x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$$\left[P_2(x) = \frac{9x^2+1}{2x(x^2+1)}, P_3(x) = \frac{49x^2+1}{16x^2(x^2+1)}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{2}$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{16}$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$16x^2(x^2+1)\left(\frac{d}{dx}y'\right) + 8x(9x^2+1)\,y' + (49x^2+1)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0(-1+4r)^2 x^r + a_1(3+4r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r-1)^2 + a_{k-2}(4k+4r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+4r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = \frac{1}{4}$
- Each term must be 0

  $a_1(3+4r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(4k+4r-1)^2 (a_k + a_{k-2}) = 0$
- Shift index using $k- >k+2$

  $(4k+4r+7)^2 (a_{k+2} + a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -a_k$
- Recursion relation for $r = \frac{1}{4}$

  $a_{k+2} = -a_k$
- Solution for $r = \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -a_k, a_1 = 0\right]$

1.579.3 Maple trace

```
Methods for second order ODEs:
```

1.579.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 21

```
dsolve(16*x^2*(x^2+1)*diff(diff(y(x),x),x)+8*x*(9*x^2+1)*diff(y(x),x)+(49*x^2+1)*y(x)
       y(x),singsol=all)
```

$$y = \frac{x^{1/4}(\ln(x) c_2 + c_1)}{x^2+1}$$

1.579.5 Mathematica DSolve solution

Solving time : 0.065 (sec)
Leaf size : 26

```
DSolve[{16*x^2*(1+x^2)*D[y[x],{x,2}]+8*x*(1+9*x^2)*D[y[x],x]+(1+49*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[4]{x}(c_2 \log(x) + c_1)}{x^2 + 1}$$

1.580 problem 596

Internal problem ID [8718]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 596
Date solved : `Monday, October 21, 2024 at 05:20:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(4+3x)\,y''-x(4-3x)\,y'+4y=0$$

1.580.1 Solved as second order ode using Kovacic algorithm

Time used: 0.194 (sec)

Writing the ode as

$$\left(3x^3+4x^2\right)y''+\left(3x^2-4x\right)y'+4y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^3+4x^2 \\ B &= 3x^2-4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1105: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^2-4x}{3x^3+4x^2}\, dx} \\ &= z_1 e^{-\ln(4+3x)+\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{\sqrt{x}}{4+3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{4+3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2-4x}{3x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(4+3x)+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{4+3x}\right) + c_2 \left(\frac{x}{4+3x}(\ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.580.2 Maple step by step solution

Let's solve

$x^2(4+3x)\left(\frac{d}{dx}y'\right) - x(4-3x)\, y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{4y}{x^2(4+3x)} - \frac{(3x-4)y'}{x(4+3x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3x-4)y'}{x(4+3x)} + \frac{4y}{x^2(4+3x)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{3x-4}{x(4+3x)}, P_3(x) = \frac{4}{x^2(4+3x)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(4+3x)\left(\frac{d}{dx}y'\right) + x(3x-4)\,y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $4a_0(-1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(4a_k(k+r-1)^2 + 3a_{k-1}(k+r-1)^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)^2\,(4a_k + 3a_{k-1}) = 0$

- Shift index using $k- >k+1$

  $(k+r)^2\,(4a_{k+1} + 3a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{3a_k}{4}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{3a_k}{4}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{3a_k}{4}\right]$

1.580.3 Maple trace

```
Methods for second order ODEs:
```

1.580.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 19

```
dsolve(x^2*(4+3*x)*diff(diff(y(x),x),x)-x*(4-3*x)*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x(c_2 \ln(x) + c_1)}{4 + 3x}$$

1.580.5 Mathematica DSolve solution

Solving time : `0.053` (sec)
Leaf size : 22

```
DSolve[{x^2*(4+3*x)*D[y[x],{x,2}]-x*(4-3*x)*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x(c_2 \log(x) + c_1)}{3x + 4}$$

1.581 problem 597

Internal problem ID [8719]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 597
Date solved : `Monday, October 21, 2024 at 05:20:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+3x+1\right)y''+8x^2(3+2x)\,y'+\left(9x^2+3x+1\right)y=0$$

1.581.1 Solved as second order ode using Kovacic algorithm

Time used: 0.191 (sec)

Writing the ode as

$$\left(4x^4+12x^3+4x^2\right)y''+\left(16x^3+24x^2\right)y'+\left(9x^2+3x+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^4+12x^3+4x^2\\B&=16x^3+24x^2\\C&=9x^2+3x+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1107: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0 \\
0 = 0
\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{16x^3+24x^2}{4x^4+12x^3+4x^2}\, dx} \\
&= z_1 e^{-\ln(x^2+3x+1)} \\
&= z_1\left(\frac{1}{x^2+3x+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{x^2+3x+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{16x^3+24x^2}{4x^4+12x^3+4x^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-2\ln(x^2+3x+1)}}{(y_1)^2}\,dx$$
$$= y_1(\ln(x))$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{\sqrt{x}}{x^2+3x+1}\right) + c_2\left(\frac{\sqrt{x}}{x^2+3x+1}(\ln(x))\right)$$

Will add steps showing solving for IC soon.

1.581.2 Maple step by step solution

Let's solve
$4x^2(x^2+3x+1)\left(\frac{d}{dx}y'\right) + 8x^2(3+2x)\,y' + (9x^2+3x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{(9x^2+3x+1)y}{4x^2(x^2+3x+1)} - \frac{2(3+2x)y'}{x^2+3x+1}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + \frac{2(3+2x)y'}{x^2+3x+1} + \frac{(9x^2+3x+1)y}{4x^2(x^2+3x+1)} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{2(3+2x)}{x^2+3x+1}, P_3(x) = \frac{9x^2+3x+1}{4x^2(x^2+3x+1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $4x^2(x^2+3x+1)\left(\frac{d}{dx}y'\right) + 8x^2(3+2x)\,y' + (9x^2+3x+1)\,y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$
  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- Shift index using $k- >k-m$
  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^m \cdot y'$ to series expansion for $m = 2..3$
  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k- >k+1-m$
  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$
  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
- Shift index using $k- >k+2-m$
  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions
  $a_0(-1+2r)^2 x^r + \left(a_1(1+2r)^2 + 3a_0(1+2r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + 3a_{k-1}(2k+2r\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+2r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = \frac{1}{2}$

- Each term must be 0

  $a_1(1+2r)^2 + 3a_0(1+2r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = -3a_0$
- Each term in the series must be 0, giving the recursion relation

  $(2k+2r-1)^2 (a_k + 3a_{k-1} + a_{k-2}) = 0$
- Shift index using $k->k+2$

  $(2k+2r+3)^2 (a_{k+2} + 3a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -3a_{k+1} - a_k$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -3a_{k+1} - a_k$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$

1.581.3 Maple trace

```
Methods for second order ODEs:
```

1.581.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 24

```
dsolve(4*x^2*(x^2+3*x+1)*diff(diff(y(x),x),x)+8*x^2*(3+2*x)*diff(y(x),x)+(9*x^2+3*x+1)
       y(x),singsol=all)
```

$$y = \frac{\sqrt{x}\,(c_2 \ln(x) + c_1)}{x^2 + 3x + 1}$$

1.581.5 Mathematica DSolve solution

Solving time : 0.081 (sec)
Leaf size : 29

```
DSolve[{4*x^2*(1+3*x+x^2)*D[y[x],{x,2}]+8*x^2*(3+2*x)*D[y[x],x]+(1+3*x+9*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}(c_2 \log(x) + c_1)}{x^2 + 3x + 1}$$

1.582 problem 598

Internal problem ID [8720]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 598
Date solved : `Monday, October 21, 2024 at 05:20:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-x)^2 y'' - x\left(-3x^2+2x+1\right) y' + \left(x^2+1\right) y = 0$$

1.582.1 Solved as second order ode using Kovacic algorithm

Time used: 0.201 (sec)

Writing the ode as

$$x^2(-1+x)^2 y'' + \left(3x^3-2x^2-x\right) y' + \left(x^2+1\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(-1+x)^2 \\ B &= 3x^3-2x^2-x \\ C &= x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1109: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx}\\ &= \mathrm{e}^{\int \frac{1}{2x} dx}\\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^3-2x^2-x}{x^2(-1+x)^2} \, dx}\\ &= z_1 e^{-2\ln(-1+x)+\frac{\ln(x)}{2}}\\ &= z_1 \left(\frac{\sqrt{x}}{(-1+x)^2}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{(-1+x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3-2x^2-x}{x^2(-1+x)^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4\ln(-1+x)+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{(-1+x)^2}\right) + c_2 \left(\frac{x}{(-1+x)^2}(\ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.582.2 Maple step by step solution

Let's solve

$x^2(1-x)^2\left(\frac{d}{dx}y'\right) - x(-3x^2+2x+1)\, y' + (x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+1)y}{x^2(-1+x)^2} - \frac{y'(3x+1)}{(-1+x)x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'(3x+1)}{(-1+x)x} + \frac{(x^2+1)y}{x^2(-1+x)^2} = 0$

- □ Check to see if x_0 is a regular singular point
  - ○ Define functions

    $\left[P_2(x) = \frac{3x+1}{x(-1+x)}, P_3(x) = \frac{x^2+1}{x^2(-1+x)^2}\right]$

  - ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$

  - ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$$

- $x = 0$is a regular singular point
  Check to see if x_0 is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $$x^2(-1+x)^2 \left(\frac{d}{dx}y'\right) + x(-1+x)(3x+1)y' + (x^2+1)y = 0$$
- Assume series solution for y
  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$
  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
- Shift index using $k- >k-m$
  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$
  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$
- Shift index using $k- >k+1-m$
  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$
- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$
  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$
- Shift index using $k- >k+2-m$
  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions
  $$a_0(-1+r)^2 x^r + \left(-2a_0 r^2 + a_1 r^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 - 2a_{k-1}(k+r-1)^2 + a_{k-2}(k+r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $$(-1+r)^2 = 0$$
- Values of r that satisfy the indicial equation
  $$r = 1$$

- Each term must be 0

  $-2a_0r^2 + a_1r^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 2a_0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)^2 (a_k - 2a_{k-1} + a_{k-2}) = 0$
- Shift index using $k- >k+2$

  $(k+r+1)^2 (a_{k+2} - 2a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = 2a_{k+1} - a_k$
- Recursion relation for $r = 1$

  $a_{k+2} = 2a_{k+1} - a_k$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = 2a_{k+1} - a_k, a_1 = 2a_0\right]$

1.582.3 Maple trace

```
Methods for second order ODEs:
```

1.582.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 17

```
dsolve(x^2*(1-x)^2*diff(diff(y(x),x),x)-x*(-3*x^2+2*x+1)*diff(y(x),x)+(x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x(c_1 + c_2 \ln(x))}{(-1+x)^2}$$

1.582.5 Mathematica DSolve solution

Solving time : 0.051 (sec)
Leaf size : 20

```
DSolve[{x^2*(1-x)^2*D[y[x],{x,2}]-x*(1+2*x-3*x^2)*D[y[x],x]+(1+x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x(c_2 \log(x) + c_1)}{(x-1)^2}$$

1.583 problem 599

Internal problem ID [8721]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 599
Date solved : `Monday, October 21, 2024 at 05:20:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$9x^2\left(x^2+x+1\right)y''+3x\left(13x^2+7x+1\right)y'+\left(25x^2+4x+1\right)y=0$$

1.583.1 Solved as second order ode using Kovacic algorithm

Time used: 0.311 (sec)

Writing the ode as

$$\left(9x^4+9x^3+9x^2\right)y''+\left(39x^3+21x^2+3x\right)y'+\left(25x^2+4x+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=9x^4+9x^3+9x^2\\B&=39x^3+21x^2+3x\\C&=25x^2+4x+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1111: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0 \\ 0 = 0\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{39x^3+21x^2+3x}{9x^4+9x^3+9x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{6} - \ln(x^2+x+1)} \\ &= z_1 \left(\frac{1}{x^{1/6}(x^2+x+1)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/3}}{x^2 + x + 1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{39x^3+21x^2+3x}{9x^4+9x^3+9x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{3}-2\ln\left(x^2+x+1\right)}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(2x - \frac{19}{24} + (x-1)^2 - \frac{x^5}{3\left(x^2+x+1\right)} - \frac{x^4}{3\left(x^2+x+1\right)} - \frac{x^3}{3\left(x^2+x+1\right)} \right. \\
&\quad \left. + \frac{x^2}{3x^2+3x+3} + \frac{x}{3x^2+3x+3} + \frac{1}{3x^2+3x+3} + \frac{x^3}{3} - x^2 + \ln(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/3}}{x^2+x+1}\right) \\
&\quad + c_2 \left(\frac{x^{1/3}}{x^2+x+1} \left(2x - \frac{19}{24} + (x-1)^2 - \frac{x^5}{3\left(x^2+x+1\right)} - \frac{x^4}{3\left(x^2+x+1\right)} - \frac{x^3}{3\left(x^2+x+1\right)} + \frac{x^2}{3x^2+3x+3} + \right.\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.583.2 Maple step by step solution

Let's solve

$9x^2(x^2+x+1)\left(\frac{d}{dx}y'\right) + 3x(13x^2+7x+1)\, y' + (25x^2+4x+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(25x^2+4x+1)y}{9x^2(x^2+x+1)} - \frac{(13x^2+7x+1)y'}{3x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(13x^2+7x+1)y'}{3x(x^2+x+1)} + \frac{(25x^2+4x+1)y}{9x^2(x^2+x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{13x^2+7x+1}{3x(x^2+x+1)}, P_3(x) = \frac{25x^2+4x+1}{9x^2(x^2+x+1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{3}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{9}$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2(x^2+x+1)\left(\frac{d}{dx}y'\right) + 3x(13x^2+7x+1)\,y' + (25x^2+4x+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ○ Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+3r)^2 x^r + \left(a_1(2+3r)^2 + a_0(2+3r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 + a_{k-1}(3k+3r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+3r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = \frac{1}{3}$
- Each term must be 0
  $a_1(2+3r)^2 + a_0(2+3r)^2 = 0$
- Solve for the dependent coefficient(s)
  $a_1 = -a_0$
- Each term in the series must be 0, giving the recursion relation
  $(3k+3r-1)^2 (a_k + a_{k-1} + a_{k-2}) = 0$
- Shift index using $k- >k+2$
  $(3k+3r+5)^2 (a_{k+2} + a_{k+1} + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -a_{k+1} - a_k$
- Recursion relation for $r = \frac{1}{3}$
  $a_{k+2} = -a_{k+1} - a_k$
- Solution for $r = \frac{1}{3}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -a_{k+1} - a_k, a_1 = -a_0\right]$

1.583.3 Maple trace

```
Methods for second order ODEs:
```

1.583.4 Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 22

```
dsolve(9*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+3*x*(13*x^2+7*x+1)*diff(y(x),x)+(25*x^2+4*
       y(x),singsol=all)
```

$$y = \frac{x^{1/3}(\ln(x) c_2 + c_1)}{x^2 + x + 1}$$

1.583.5 Mathematica DSolve solution

Solving time : 0.074 (sec)
Leaf size : 27

```
DSolve[{9*x^2*(1+x+x^2)*D[y[x],{x,2}]+3*x*(1+7*x+13*x^2)*D[y[x],x]+(1+4*x+25*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[3]{x}(c_2 \log(x) + c_1)}{x^2 + x + 1}$$

1.584 problem 600

Internal problem ID [8722]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 600
Date solved : `Monday, October 21, 2024 at 05:20:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''-x(4-7x)\,y'-(5-3x)\,y=0$$

1.584.1 Solved as second order ode using Kovacic algorithm

Time used: 0.265 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y''+\left(7x^2-4x\right)y'+(3x-5)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+4x^2\\ B&=7x^2-4x\\ C&=3x-5\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 32x + 128}{16(x^2 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 - 32x + 128$$
$$t = 16(x^2 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 32x + 128}{16(x^2 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1113: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{2x} + \frac{5}{2(2+x)} + \frac{2}{x^2} + \frac{45}{16(2+x)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 32x + 128}{16(x^2 + 2x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-3x^2-32x+128}{16\left(x^2+2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{9}{4}$	$-\frac{5}{4}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^+\right)\\
&= \frac{3}{4}-\left(\frac{3}{4}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{5}{4\left(2+x\right)} + \frac{2}{x} + (0) \\
&= -\frac{5}{4\left(2+x\right)} + \frac{2}{x} \\
&= \frac{3x+16}{4x\left(2+x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{5}{4\left(2+x\right)} + \frac{2}{x}\right)(0) + \left(\left(\frac{5}{4\left(2+x\right)^{2}} - \frac{2}{x^{2}}\right) + \left(-\frac{5}{4\left(2+x\right)} + \frac{2}{x}\right)^{2} - \left(\frac{-3x^{2}-32x+128}{16\left(x^{2}+2x\right)^{2}}\right)\right) = 0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{5}{4(2+x)} + \frac{2}{x}\right) dx} \\
&= \frac{x^{2}}{\left(2+x\right)^{5/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^{2}-4x}{2x^{3}+4x^{2}}\, dx} \\
&= z_1 e^{\frac{\ln(x)}{2} - \frac{9\ln(2+x)}{4}} \\
&= z_1 \left(\frac{\sqrt{x}}{\left(2+x\right)^{9/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x^{5/2}}{(2+x)^{7/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2-4x}{2x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{9\ln(2+x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-\frac{11(2+x)^{5/2}}{8} + \frac{10(2+x)^{3/2}}{3} - \frac{5\sqrt{2+x}}{2}}{x^3} - \frac{5\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{16} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{5/2}}{(2+x)^{7/2}} \right) \\
&\quad + c_2 \left(\frac{x^{5/2}}{(2+x)^{7/2}} \left(\frac{-\frac{11(2+x)^{5/2}}{8} + \frac{10(2+x)^{3/2}}{3} - \frac{5\sqrt{2+x}}{2}}{x^3} - \frac{5\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)}{16} \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.584.2 Maple step by step solution

Let's solve

$2x^2(2+x)\left(\frac{d}{dx}y'\right)-x(4-7x)\,y'-(5-3x)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(3x-5)y}{2x^2(2+x)}-\frac{(-4+7x)y'}{2x(2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(-4+7x)y'}{2x(2+x)}+\frac{(3x-5)y}{2x^2(2+x)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{-4+7x}{2x(2+x)}, P_3(x)=\frac{3x-5}{2x^2(2+x)}\right]$

- ◦ $(2+x)\cdot P_2(x)$ is analytic at $x=-2$

$\left((2+x)\cdot P_2(x)\right)\Big|_{x=-2}=\frac{9}{2}$

- ◦ $(2+x)^2\cdot P_3(x)$ is analytic at $x=-2$

$\left((2+x)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

- ◦ $x=-2$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-2$

- Multiply by denominators

$2x^2(2+x)\left(\frac{d}{dx}y'\right)+x(-4+7x)\,y'+(3x-5)\,y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$(2u^3-8u^2+8u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(7u^2-32u+36)\left(\frac{d}{du}y(u)\right)+(3u-11)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ◦ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0 r(7+2r)\, u^{-1+r} + \left(4a_1(1+r)\,(9+2r) - a_0(8r^2+24r+11)\right) u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)\,(2k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $4r(7+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{7}{2}\right\}$
- Each term must be 0
  $4a_1(1+r)\,(9+2r) - a_0(8r^2+24r+11) = 0$
- Each term in the series must be 0, giving the recursion relation
  $2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\, r - 24a_k + a_{k-1} + 44a_{k+1})\, k + 2(-4a_k +$
- Shift index using $k- >k+1$
  $2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\, r - 24a_{k+1} + a_k + 44a_{k+2})\,(k+1) -$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2 a_k - 8r^2 a_{k+1} + 5ka_k - 40ka_{k+1} + 5ra_k - 40ra_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+4kr+2r^2+15k+15r+22)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}$
- Solution for $r = 0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2+15k+22)}, 36a_1 - 11a_0 = 0\right]$
- Revert the change of variables $u = 2+x$

$$\left[y=\sum_{k=0}^{\infty} a_k (2+x)^k, a_{k+2}=-\frac{2k^2 a_k-8k^2 a_{k+1}+5k a_k-40k a_{k+1}+3a_k-43a_{k+1}}{4(2k^2+15k+22)}, 36a_1-11a_0=0\right]$$

- Recursion relation for $r=-\frac{7}{2}$

$$a_{k+2}=-\frac{2k^2 a_k-8k^2 a_{k+1}-9k a_k+16k a_{k+1}+10a_k-a_{k+1}}{4(2k^2+k-6)}$$

- Solution for $r=-\frac{7}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{7}{2}}, a_{k+2}=-\frac{2k^2 a_k-8k^2 a_{k+1}-9k a_k+16k a_{k+1}+10a_k-a_{k+1}}{4(2k^2+k-6)}, -20a_1-25a_0=0\right]$$

- Revert the change of variables $u=2+x$

$$\left[y=\sum_{k=0}^{\infty} a_k (2+x)^{k-\frac{7}{2}}, a_{k+2}=-\frac{2k^2 a_k-8k^2 a_{k+1}-9k a_k+16k a_{k+1}+10a_k-a_{k+1}}{4(2k^2+k-6)}, -20a_1-25a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (2+x)^k\right)+\left(\sum_{k=0}^{\infty} b_k (2+x)^{k-\frac{7}{2}}\right), a_{k+2}=-\frac{2k^2 a_k-8k^2 a_{k+1}+5k a_k-40k a_{k+1}+3a_k-43a_{k+1}}{4(2k^2+15k+22)}, 36a\right.$$

1.584.3 Maple trace

```
Methods for second order ODEs:
```

1.584.4 Maple dsolve solution

Solving time : `0.012` (sec)
Leaf size : 55

```
dsolve(2*x^2*(2+x)*diff(diff(y(x),x),x)-x*(4-7*x)*diff(y(x),x)-(5-3*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{15 \operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right) c_2\, x^3+33\sqrt{2}\left(x^2+\frac{52}{33}x+\frac{32}{33}\right) c_2\sqrt{2+x}+c_1\, x^3}{(2+x)^{7/2}\sqrt{x}}$$

1.584.5 Mathematica DSolve solution

Solving time : `0.16` (sec)
Leaf size : 92

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}]-x*(4-7*x)*D[y[x],x]-(5-3*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{15\sqrt{2}c_2x^3\operatorname{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right) - 48c_1x^3 + 66c_2\sqrt{x+2}x^2 + 104c_2\sqrt{x+2}x + 64c_2\sqrt{x+2}}{48\sqrt{x}(x+2)^{7/2}}$$

1.585 problem 601

Internal problem ID [8723]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 601
Date solved : `Monday, October 21, 2024 at 05:20:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-2x)\,y''+x(8-9x)\,y'+(6-3x)\,y=0$$

1.585.1 Solved as second order ode using Kovacic algorithm

Time used: 0.251 (sec)

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(-9x^2+8x\right)y'+(6-3x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= -9x^2+8x \\ C &= 6-3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 21x^2 - 20x + 24$$
$$t = 4\left(2x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1115: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2} + \frac{19}{x} + \frac{77}{16\left(x - \frac{1}{2}\right)^2} - \frac{19}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{11}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{7}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2
$\frac{1}{2}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{7}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= \frac{7}{4} - \left(\frac{3}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)} + (0) \\
&= -\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)} \\
&= \frac{4+3x}{4x^2-2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)}\right)(1) + \left(\left(\frac{2}{x^2} - \frac{11}{4\left(x-\frac{1}{2}\right)^2}\right) + \left(-\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{21x^2-20x+24}{4\left(2x^2-x\right)^2}\right)\right)
$$

$$
\frac{4-3a_0}{x\left(-1+2x\right)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{4}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{4}{3}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{4}{3}\right) e^{\int \left(-\frac{2}{x} + \frac{11}{4\left(x-\frac{1}{2}\right)}\right) dx} \\
&= \left(x + \frac{4}{3}\right) e^{\frac{11\ln(-1+2x)}{4} - 2\ln(x)} \\
&= \frac{\left(x+\frac{4}{3}\right)\left(-1+2x\right)^{11/4}}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-9x^2+8x}{-2x^3+x^2}\,dx} \\
&= z_1 e^{\frac{7\ln(-1+2x)}{4}-4\ln(x)} \\
&= z_1\left(\frac{(-1+2x)^{7/4}}{x^4}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(-1+2x)^{9/2}\,(4+3x)}{3x^6}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-9x^2+8x}{-2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{7\ln(-1+2x)}{2}-8\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{(231x^3-198x^2+66x-8)\,x^8 \mathrm{e}^{\frac{7\ln(-1+2x)}{2}-8\ln(x)}}{385\,(4+3x)\,(-1+2x)^8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(-1+2x)^{9/2}\,(4+3x)}{3x^6}\right) \\
&\quad + c_2\left(\frac{(-1+2x)^{9/2}\,(4+3x)}{3x^6}\left(-\frac{(231x^3-198x^2+66x-8)\,x^8 \mathrm{e}^{\frac{7\ln(-1+2x)}{2}-8\ln(x)}}{385\,(4+3x)\,(-1+2x)^8}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.585.2 Maple step by step solution

Let's solve

$x^2(1-2x)\left(\frac{d}{dx}y'\right)+x(8-9x)\,y'+(6-3x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{3(-2+x)y}{x^2(-1+2x)}-\frac{(9x-8)y'}{x(-1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(9x-8)y'}{x(-1+2x)}+\frac{3(-2+x)y}{x^2(-1+2x)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{9x-8}{x(-1+2x)},P_3(x)=\frac{3(-2+x)}{x^2(-1+2x)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=8$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=6$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(-1+2x)\left(\frac{d}{dx}y'\right)+x(9x-8)\,y'+(3x-6)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^{m}\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=2..3$

$$x^{m}\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^{m}\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-a_{0}(6+r)\,(1+r)\,x^{r}+\left(\sum_{k=1}^{\infty}\left(-a_{k}(k+r+6)\,(k+r+1)+a_{k-1}(k+2+r)\,(2k-1+2r)\right)x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(6+r)\,(1+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{-6,-1\}$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+2+r)\left(k-\frac{1}{2}+r\right)a_{k-1}-a_{k}(k+r+6)\,(k+r+1)=0$$

- Shift index using $k->k+1$

$$2(k+r+3)\left(k+\frac{1}{2}+r\right)a_{k}-a_{k+1}(k+7+r)\,(k+2+r)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{(k+r+3)(2k+2r+1)a_{k}}{(k+7+r)(k+2+r)}$$

- Recursion relation for $r=-6$; series terminates at $k=3$

$$a_{k+1}=\frac{(k-3)(2k-11)a_{k}}{(k+1)(k-4)}$$

- Apply recursion relation for $k=0$

$$a_{1}=-\frac{33a_{0}}{4}$$

- Apply recursion relation for $k=1$

$$a_{2}=-3a_{1}$$

- Express in terms of a_0

$$a_{2}=\frac{99a_{0}}{4}$$

- Apply recursion relation for $k=2$

$$a_{3}=-\frac{7a_{2}}{6}$$

- Express in terms of a_0

$a_3 = -\frac{231a_0}{8}$

- Terminating series solution of the ODE for $r = -6$. Use reduction of order to find the second

  $y = a_0 \cdot \left(-\frac{231}{8}x^3 + \frac{99}{4}x^2 - \frac{33}{4}x + 1\right)$
- Recursion relation for $r = -1$

  $a_{k+1} = \frac{(k+2)(2k-1)a_k}{(k+6)(k+1)}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{(k+2)(2k-1)a_k}{(k+6)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(-\frac{231}{8}x^3 + \frac{99}{4}x^2 - \frac{33}{4}x + 1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), b_{k+1} = \frac{(k+2)(2k-1)b_k}{(k+6)(k+1)}\right]$

1.585.3 Maple trace

```
Methods for second order ODEs:
```

1.585.4 Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 43

```
dsolve(x^2*(1-2*x)*diff(diff(y(x),x),x)+x*(8-9*x)*diff(y(x),x)+(6-3*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{48c_1\left(x-\frac{1}{2}\right)^4\left(x+\frac{4}{3}\right)\sqrt{-1+2x}+231\left(x^3-\frac{6}{7}x^2+\frac{2}{7}x-\frac{8}{231}\right)c_2}{x^6}$$

1.585.5 Mathematica DSolve solution

Solving time : `0.173` (sec)
Leaf size : 49

```
DSolve[{x^2*(1-2*x)*D[y[x],{x,2}]+x*(8-9*x)*D[y[x],x]+(6-3*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2(231x^3-198x^2+66x-8)+385c_1(3x+4)(1-2x)^{9/2}}{1155x^6}$$

1.586 problem 602

Internal problem ID [8724]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 602
Date solved : `Monday, October 21, 2024 at 05:20:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''+x\left(10x^2+3\right)y'-\left(-14x^2+15\right)y=0$$

1.586.1 Solved as second order ode using Kovacic algorithm

Time used: 0.360 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(10x^3+3x\right)y'+\left(14x^2-15\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 10x^3+3x \\ C &= 14x^2-15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{24x^4 + 66x^2 + 63}{4(x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 24x^4 + 66x^2 + 63$$
$$t = 4(x^3 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{24x^4 + 66x^2 + 63}{4(x^3 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1117: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{63}{4x^2} + \frac{21}{16(x-i)^2} + \frac{21}{16(x+i)^2} + \frac{99i}{16(x-i)} - \frac{99i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{24x^4 + 66x^2 + 63}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = 6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{24x^4 + 66x^2 + 63}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
i	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 3 - (3) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)} + (0) \\
&= \frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)} \\
&= \frac{9}{2x} - \frac{3x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)}\right)(0) + \left(\left(-\frac{9}{2x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2}\right) + \left(\frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)}\right) dx} \\
&= \frac{x^{9/2}}{(x^2+1)^{3/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{10x^3+3x}{x^4+x^2}\,dx} \\ &= z_1 e^{-\frac{7\ln\left(x^2+1\right)}{4}-\frac{3\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\left(x^2+1\right)^{7/4} x^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^3}{\left(x^2+1\right)^{5/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{10x^3+3x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{7\ln\left(x^2+1\right)}{2}-3\ln(x)}}{\left(y_1\right)^2}\,dx \\ &= y_1\left(-\frac{\left(x^2+1\right)^{5/2}}{8x^8}+\frac{\left(x^2+1\right)^{5/2}}{16x^6}-\frac{\left(x^2+1\right)^{5/2}}{64x^4}-\frac{\left(x^2+1\right)^{5/2}}{128x^2}+\frac{\left(x^2+1\right)^{3/2}}{128}\right. \\ &\quad \left.+\frac{3\sqrt{x^2+1}}{128}-\frac{3\,\mathrm{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{128}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^3}{(x^2+1)^{5/2}}\right) \\
&\quad + c_2\left(\frac{x^3}{(x^2+1)^{5/2}}\left(-\frac{(x^2+1)^{5/2}}{8x^8}+\frac{(x^2+1)^{5/2}}{16x^6}-\frac{(x^2+1)^{5/2}}{64x^4}-\frac{(x^2+1)^{5/2}}{128x^2}+\frac{(x^2+1)^{3/2}}{128}+\frac{3\sqrt{x^2+1}}{128}-\frac{3
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.586.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(10x^2+3)\,y'-(-14x^2+15)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(14x^2-15)y}{x^2(x^2+1)}-\frac{(10x^2+3)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(10x^2+3)y'}{x(x^2+1)}+\frac{(14x^2-15)y}{x^2(x^2+1)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{10x^2+3}{x(x^2+1)},P_3(x)=\frac{14x^2-15}{x^2(x^2+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=3$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-15$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(10x^2+3)\,y'+(14x^2-15)\,y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx} y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (5+r)\,(-3+r)\, x^r + a_1 (6+r)\,(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k+r+5)\,(k+r-3) + a_{k-2} (k+r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-5, 3\}$

- Each term must be 0

  $a_1 (6+r)\,(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+5)\,(a_k (k+r-3) + a_{k-2} (k+r)) = 0$

- Shift index using $k- >k + 2$

  $(k+r+7)\,(a_{k+2} (k+r-1) + a_k (k+r+2)) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+2)}{k+r-1}$
- Recursion relation for $r = -5$

  $a_{k+2} = -\frac{a_k(k-3)}{k-6}$
- Series not valid for $r = -5$, division by 0 in the recursion relation at $k = 6$

  $a_{k+2} = -\frac{a_k(k-3)}{k-6}$
- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{a_k(k+5)}{k+2}$
- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k(k+5)}{k+2}, a_1 = 0\right]$

1.586.3 Maple trace

```
Methods for second order ODEs:
```

1.586.4 Maple dsolve solution

Solving time : `0.025` (sec)
Leaf size : 59

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(10*x^2+3)*diff(y(x),x)-(-14*x^2+15)*y(x) =
       y(x),singsol=all)
```

$$y = \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) c_2\, x^8 - c_2(x^2+2)\left(x^4 - \frac{8}{3}x^2 - \frac{8}{3}\right)\sqrt{x^2+1} + c_1\, x^8}{\left(x^2+1\right)^{5/2} x^5}$$

1.586.5 Mathematica DSolve solution

Solving time : `0.159` (sec)
Leaf size : 75

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]+x*(3+10*x^2)*D[y[x],x]-(15-14*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2\left(\sqrt{x^2+1}(3x^6 - 2x^4 - 24x^2 - 16) - 3x^8 \operatorname{arctanh}\left(\sqrt{x^2+1}\right)\right) + 128c_1x^8}{128x^5\left(x^2+1\right)^{5/2}}$$

1.587 problem 603

Internal problem ID [8725]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 603
Date solved : `Monday, October 21, 2024 at 05:20:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(-2x^2+1)\,y''+x(-13x^2+7)\,y'-14x^2y=0$$

1.587.1 Solved as second order ode using Kovacic algorithm

Time used: 0.339 (sec)

Writing the ode as

$$(-2x^4+x^2)\,y''+(-13x^3+7x)\,y'-14x^2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^4+x^2 \\ B &= -13x^3+7x \\ C &= -14x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 5x^4 - 68x^2 + 35 \\
t &= 4\left(2x^3 - x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1119: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4x^2} + \frac{9}{64\left(x - \frac{\sqrt{2}}{2}\right)^2} + \frac{9}{64\left(x + \frac{\sqrt{2}}{2}\right)^2} - \frac{279\sqrt{2}}{64\left(x - \frac{\sqrt{2}}{2}\right)} + \frac{279\sqrt{2}}{64\left(x + \frac{\sqrt{2}}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = \frac{\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8} \end{aligned}$$

For the pole at $x = -\frac{\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$\frac{\sqrt{2}}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$
$-\frac{\sqrt{2}}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} + (-)(0) \\
&= -\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} \\
&= \frac{-x^2+5}{4x^3-2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)}\right)(0) + \left(\left(\frac{5}{2x^2} - \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)^2} - \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)^2}\right) + \left(-\frac{5}{2x} +\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{2x} + \frac{9}{8\left(x - \frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x + \frac{\sqrt{2}}{2}\right)}\right) dx} \\
&= \frac{\left(2x + \sqrt{2}\right)^{9/8} \left(2x - \sqrt{2}\right)^{9/8}}{x^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-13x^3 + 7x}{-2x^4 + x^2} \, dx} \\
&= z_1 e^{-\frac{7 \ln(x)}{2} + \frac{\ln\left(2x^2 - 1\right)}{8}} \\
&= z_1 \left(\frac{\left(2x^2 - 1\right)^{1/8}}{x^{7/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{2\left(2x^2 - 1\right)^{5/4} 2^{1/8}}{x^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-13x^3 + 7x}{-2x^4 + x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-7 \ln(x) + \frac{\ln\left(2x^2 - 1\right)}{4}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\frac{\left(5x^4 - 20x^2 + 8\right) x^7 \mathrm{e}^{-7 \ln(x) + \frac{\ln\left(2x^2 - 1\right)}{4}} 2^{3/4}}{120 \left(2x^2 - 1\right)^{3/2}}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{2(2x^2-1)^{5/4}\, 2^{1/8}}{x^6}\right) \\
&\quad + c_2\left(\frac{2(2x^2-1)^{5/4}\, 2^{1/8}}{x^6}\left(\frac{(5x^4-20x^2+8)\, x^7 \mathrm{e}^{-7\ln(x)+\frac{\ln(2x^2-1)}{4}}\, 2^{3/4}}{120\,(2x^2-1)^{3/2}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.587.2 Maple step by step solution

Let's solve

$x^2(-2x^2+1)\left(\frac{d}{dx}y'\right) + x(-13x^2+7)\, y' - 14x^2 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{14y}{2x^2-1} - \frac{(13x^2-7)y'}{x(2x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(13x^2-7)y'}{x(2x^2-1)} + \frac{14y}{2x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{13x^2-7}{x(2x^2-1)}, P_3(x) = \frac{14}{2x^2-1}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 7$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$x(2x^2-1)\left(\frac{d}{dx}y'\right)+(13x^2-7)\,y'+14yx=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x\cdot y$ to series expansion

$x\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+1}$

- ◦ Shift index using $k->k-1$

$x\cdot y=\sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..2$

$x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=1..3$

$x^m\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

$x^m\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$-a_0r(6+r)\,x^{-1+r}-a_1(1+r)\,(7+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(-a_{k+1}(k+r+1)\,(k+7+r)+a_{k-1}(2k+5+\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(6+r)=0$

- Values of r that satisfy the indicial equation

$r\in\{-6,0\}$

- Each term must be 0

$-a_1(1+r)\,(7+r)=0$

- Each term in the series must be 0, giving the recursion relation

$2\left(\left(k+r+\frac{5}{2}\right)a_{k-1}-\frac{a_{k+1}(k+7+r)}{2}\right)(k+r+1)=0$

- Shift index using $k->k+1$

$$2\left(\left(k+\frac{7}{2}+r\right)a_k-\frac{a_{k+2}(k+8+r)}{2}\right)(k+r+2)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=\frac{(2k+2r+7)a_k}{k+8+r}$$

- Recursion relation for $r=-6$

$$a_{k+2}=\frac{(2k-5)a_k}{k+2}$$

- Solution for $r=-6$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-6}, a_{k+2}=\frac{(2k-5)a_k}{k+2}, 5a_1=0\right]$$

- Recursion relation for $r=0$

$$a_{k+2}=\frac{(2k+7)a_k}{k+8}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+2}=\frac{(2k+7)a_k}{k+8}, -7a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-6}\right)+\left(\sum_{k=0}^{\infty}b_k x^k\right), a_{k+2}=\frac{(2k-5)a_k}{k+2}, 5a_1=0, b_{k+2}=\frac{(2k+7)b_k}{k+8}, -7b_1=0\right]$$

1.587.3 Maple trace

```
Methods for second order ODEs:
```

1.587.4 Maple dsolve solution

Solving time : `0.012` (sec)
Leaf size : 35

```
dsolve(x^2*(-2*x^2+1)*diff(diff(y(x),x),x)+x*(-13*x^2+7)*diff(y(x),x)-14*x^2*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1(2x^2-1)^{5/4}+5c_2x^4-20c_2x^2+8c_2}{x^6}$$

1.587.5 Mathematica DSolve solution

Solving time : 0.147 (sec)
Leaf size : 43

```
DSolve[{x^2*(1-2*x^2)*D[y[x],{x,2}]+x*(7-13*x^2)*D[y[x],x]-14*x^2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{15c_1(1-2x^2)^{5/4} + c_2(-5x^4 + 20x^2 - 8)}{15x^6}$$

1.588 problem 604

Internal problem ID [8726]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 604
Date solved : `Monday, October 21, 2024 at 05:20:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y''+4x(1+2x)\,y'-(1+3x)\,y=0$$

1.588.1 Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(8x^2+4x\right)y'+(-3x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^3+4x^2\\B&=8x^2+4x\\C&=-3x-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3x+4}{4x\left(1+x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3x+4 \\
t &= 4x(1+x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3x+4}{4x\left(1+x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1121: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(1+x)^2} - \frac{1}{1+x} + \frac{1}{x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x+4}{4x(1+x)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x+4}{4x\left(1+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{3}{2} - \left(\frac{3}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} + \frac{1}{2+2x} + (0) \\
&= \frac{1}{x} + \frac{1}{2+2x} \\
&= \frac{1}{x} + \frac{1}{2+2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{x} + \frac{1}{2+2x}\right)(0) + \left(\left(-\frac{1}{x^2} - \frac{1}{2(1+x)^2}\right) + \left(\frac{1}{x} + \frac{1}{2+2x}\right)^2 - \left(\frac{3x+4}{4x(1+x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{x} + \frac{1}{2+2x}\right) dx} \\
&= x\sqrt{1+x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x^2+4x}{4x^3+4x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x(1+x))}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x(1+x)}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x\sqrt{1+x}}{\sqrt{x\,(1+x)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2+4x}{4x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x(1+x))}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{1}{x} - \ln(x) + \ln(1+x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x\sqrt{1+x}}{\sqrt{x\,(1+x)}}\right) + c_2\left(\frac{x\sqrt{1+x}}{\sqrt{x\,(1+x)}}\left(-\frac{1}{x} - \ln(x) + \ln(1+x)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.588.2 Maple step by step solution

Let's solve

$4x^2(1+x)\left(\frac{d}{dx}y'\right) + 4x(1+2x)\,y' - (1+3x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(1+3x)y}{4x^2(1+x)} - \frac{(1+2x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(1+2x)y'}{x(1+x)} - \frac{(1+3x)y}{4x^2(1+x)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{1+2x}{x(1+x)}, P_3(x) = -\frac{1+3x}{4x^2(1+x)}\right]$

- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $$\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1} = 1$$

- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $$\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4x^2(1+x)\left(\frac{d}{dx}y'\right) + 4x(1+2x)\,y' + (-3x-1)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(4u^3-8u^2+4u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (8u^2-12u+4)\left(\frac{d}{du}y(u)\right) + (-3u+2)\,y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $$u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..3$

  $$u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $4a_0r^2u^{-1+r}+\left(4a_1(1+r)^2-2a_0(4r^2+2r-1)\right)u^r+\left(\sum_{k=1}^{\infty}\left(4a_{k+1}(k+1+r)^2-2a_k(4k^2+8kr\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r^2=0$

- Values of r that satisfy the indicial equation

  $r=0$

- Each term must be 0

  $4a_1(1+r)^2-2a_0(4r^2+2r-1)=0$

- Each term in the series must be 0, giving the recursion relation

  $(4k^2-4k-3)\,a_{k-1}+(-8k^2-4k+2)\,a_k+4a_{k+1}(k+1)^2=0$

- Shift index using $k->k+1$

  $\left(4(k+1)^2-4k-7\right)a_k+\left(-8(k+1)^2-4k-2\right)a_{k+1}+4a_{k+2}(k+2)^2=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2}$

- Recursion relation for $r=0$

  $a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2},4a_1+2a_0=0\right]$

- Revert the change of variables $u=1+x$

  $\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k,a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-20ka_{k+1}-3a_k-10a_{k+1}}{4(k+2)^2},4a_1+2a_0=0\right]$

1.588.3 Maple trace

```
Methods for second order ODEs:
```

1.588.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 26

```
dsolve(4*x^2*(1+x)*diff(diff(y(x),x),x)+4*x*(1+2*x)*diff(y(x),x)-(1+3*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 x + \ln(x) c_2 x - \ln(1+x) c_2 x + c_2}{\sqrt{x}}$$

1.588.5 Mathematica DSolve solution

Solving time : `0.069` (sec)
Leaf size : 32

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+4*x*(1+2*x)*D[y[x],x]-(1+3*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1 x + c_2(-x\log(x) + x\log(x+1) - 1)}{\sqrt{x}}$$

1.589 problem 605

Internal problem ID [8727]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 605
Date solved : `Monday, October 21, 2024 at 05:20:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+3x)\,y''+x(4+21x)\,y'-(1-9x)\,y=0$$

1.589.1 Solved as second order ode using Kovacic algorithm

Time used: 0.259 (sec)

Writing the ode as

$$\left(6x^3+4x^2\right)y''+\left(21x^2+4x\right)y'+(9x-1)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=6x^3+4x^2\\B&=21x^2+4x\\C&=9x-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-27x - 48}{16x\,(2 + 3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -27x - 48$$
$$t = 16x(2 + 3x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-27x - 48}{16x\,(2 + 3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1123: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 3 - 1 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x(2+3x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = -\frac{2}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}
$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{3}{4x} + \frac{5}{16\left(x+\frac{2}{3}\right)^2} + \frac{3}{4\left(x+\frac{2}{3}\right)}
$$

For the pole at $x = -\frac{2}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-27x - 48}{16x\,(2+3x)^2}
$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-27x-48}{16x\,(2+3x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
$-\frac{2}{3}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{3}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{3}{4}-\left(\frac{3}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)} + (0) \\
&= \frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)} \\
&= \frac{8+9x}{12x^2+8x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{4\left(x+\frac{2}{3}\right)^2}\right) + \left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right)^2 - \left(\frac{-27x-48}{16x\left(2+3x\right)^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right) dx} \\
&= \frac{x}{\left(2+3x\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{21x^2+4x}{6x^3+4x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} - \frac{5\ln(2+3x)}{4}} \\
&= z_1 \left(\frac{1}{\sqrt{x}\,\left(2+3x\right)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(2+3x)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{21x^2+4x}{6x^3+4x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-\ln(x)-\frac{5\ln(2+3x)}{2}}}{(y_1)^2} dx \\
&= y_1 \left(-\frac{\sqrt{2+3x}}{x} - \frac{3\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{2+3x}\,\sqrt{2}}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{(2+3x)^{3/2}}\right) + c_2 \left(\frac{\sqrt{x}}{(2+3x)^{3/2}} \left(-\frac{\sqrt{2+3x}}{x} - \frac{3\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{2+3x}\,\sqrt{2}}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.589.2 Maple step by step solution

Let's solve

$2x^2(2+3x)\left(\frac{d}{dx}y'\right) + x(4+21x)\,y' - (1-9x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = -\frac{(9x-1)y}{2x^2(2+3x)} - \frac{(4+21x)y'}{2x(2+3x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(4+21x)y'}{2x(2+3x)} + \frac{(9x-1)y}{2x^2(2+3x)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{4+21x}{2x(2+3x)}, P_3(x) = \frac{9x-1}{2x^2(2+3x)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2(2+3x)\left(\frac{d}{dx}y'\right) + x(4+21x)\,y' + (9x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)(2k+2r-1) + 3a_{k-1}(2k+2r+1)(k+r)\right) x^{k+}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(\left(k+r-\frac{1}{2}\right) a_k + \frac{3a_{k-1}(k+r)}{2}\right)\left(k+r+\frac{1}{2}\right) = 0$$

- Shift index using $k- >k+1$

$$4\left(\left(k+r+\frac{1}{2}\right) a_{k+1} + \frac{3a_k(k+r+1)}{2}\right)\left(k+\frac{3}{2}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{3a_k(k+r+1)}{2k+2r+1}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{3a_k\left(k+\frac{3}{2}\right)}{2k+2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{3a_k\left(k+\frac{3}{2}\right)}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}, b_{k+1} = -\frac{3b_k\left(k+\frac{3}{2}\right)}{2k+2}\right]$$

1.589.3 Maple trace

```
Methods for second order ODEs:
```

1.589.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 48

```
dsolve(2*x^2*(2+3*x)*diff(diff(y(x),x),x)+x*(4+21*x)*diff(y(x),x)-(1-9*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{3 \operatorname{arctanh}\left(\frac{\sqrt{2+3x}\,\sqrt{2}}{2}\right) c_2 x + \sqrt{2}\,\sqrt{2+3x}\, c_2 + c_1 x}{\sqrt{x}\, \left(2+3x\right)^{3/2}}$$

1.589.5 Mathematica DSolve solution

Solving time : 0.128 (sec)
Leaf size : 64

```
DSolve[{2*x^2*(2+3*x)*D[y[x],{x,2}]+x*(4+21*x)*D[y[x],x]-(1-9*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{3\sqrt{2}c_2 x \operatorname{arctanh}\left(\sqrt{\frac{3x}{2}+1}\right) - 2c_1 x + 2c_2\sqrt{3x+2}}{2\sqrt{x}(3x+2)^{3/2}}$$

1.590 problem 606

Internal problem ID [8728]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 606
Date solved : `Monday, October 21, 2024 at 05:20:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(2+x)\,y' - (2-3x)\,y = 0$$

1.590.1 Solved as second order ode using Kovacic algorithm

Time used: 0.281 (sec)

Writing the ode as

$$x^2y'' + \left(x^2+2x\right) y' + (3x-2)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+2x \\ C &= 3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 8x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 8x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 8x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1125: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} - \frac{2}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{2}{x} - \frac{2}{x^2} - \frac{8}{x^3} - \frac{36}{x^4} - \frac{176}{x^5} - \frac{912}{x^6} - \frac{4928}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 8x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-8x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-8x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -8. Dividing this by leading coefficient in t which is 4 gives -2. Now b can be found.

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 0\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 0\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 8x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 2$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= 2 - (2) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty\\ &= \frac{2}{x} + (-)\left(\frac{1}{2}\right)\\ &= \frac{2}{x} - \frac{1}{2}\\ &= -\frac{x-4}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{2}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-8x+8}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx}\\ &= \mathrm{e}^{\int \left(\frac{2}{x}-\frac{1}{2}\right)dx}\\ &= x^2 \mathrm{e}^{-\frac{x}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2+2x}{x^2}\, dx}\\ &= z_1 e^{-\frac{x}{2}-\ln(x)}\\ &= z_1\left(\frac{\mathrm{e}^{-\frac{x}{2}}}{x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{x}}{3x^3} - \frac{\mathrm{e}^{x}}{6x^2} - \frac{\mathrm{e}^{x}}{6x} - \frac{\mathrm{Ei}_1(-x)}{6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-x}\right) + c_2 \left(x\,\mathrm{e}^{-x} \left(-\frac{\mathrm{e}^{x}}{3x^3} - \frac{\mathrm{e}^{x}}{6x^2} - \frac{\mathrm{e}^{x}}{6x} - \frac{\mathrm{Ei}_1(-x)}{6}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.590.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + x(2+x)\,y' - (2-3x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(3x-2)y}{x^2} - \frac{(2+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(2+x)y'}{x} + \frac{(3x-2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions
  $\left[P_2(x)=\frac{2+x}{x}, P_3(x)=\frac{3x-2}{x^2}\right]$
- ○ $x \cdot P_2(x)$ is analytic at $x=0$
  $\left(x \cdot P_2(x)\right)\bigg|_{x=0}=2$
- ○ $x^2 \cdot P_3(x)$ is analytic at $x=0$
  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0}=-2$
- ○ $x=0$is a regular singular point
  Check to see if $x_0=0$ is a regular singular point
  $x_0=0$
- • Multiply by denominators
  $x^2\left(\frac{d}{dx}y'\right)+x(2+x)\,y'+(3x-2)\,y=0$
- • Assume series solution for y
  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$
- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m=0..1$
    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ○ Shift index using $k->k-m$
    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ○ Convert $x^m \cdot y'$ to series expansion for $m=1..2$
    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - ○ Shift index using $k->k+1-m$
    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - ○ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
    $x^2 \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$
    Rewrite ODE with series expansions
    $a_0(2+r)\,(-1+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+2)\,(k+r-1)+a_{k-1}(k+r+2)\right)x^{k+r}\right)=0$
- • $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)(-1+r)=0$

- Values of r that satisfy the indicial equation
  $r \in \{-2, 1\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r+2)(a_k(k+r-1)+a_{k-1})=0$
- Shift index using $k->k+1$
  $(k+r+3)(a_{k+1}(k+r)+a_k)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k}{k+r}$
- Recursion relation for $r=-2$
  $a_{k+1}=-\frac{a_k}{k-2}$
- Series not valid for $r=-2$, division by 0 in the recursion relation at $k=2$
  $a_{k+1}=-\frac{a_k}{k-2}$
- Recursion relation for $r=1$
  $a_{k+1}=-\frac{a_k}{k+1}$
- Solution for $r=1$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1}=-\frac{a_k}{k+1}\right]$

1.590.3 Maple trace

```
Methods for second order ODEs:
```

1.590.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 40

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(2+x)*diff(y(x),x)-(2-3*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{\mathrm{Ei}_1(-x)\,\mathrm{e}^{-x} c_2\, x^3+c_1\,\mathrm{e}^{-x} x^3+c_2(x^2+x+2)}{x^2}$$

1.590.5 Mathematica DSolve solution

Solving time : 0.087 (sec)
Leaf size : 46

```
DSolve[{x^2*D[y[x],{x,2}]+x*(2+x)*D[y[x],x]-(2-3*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x}(c_2(x^3 \operatorname{ExpIntegralEi}(x) - e^x(x^2 + x + 2)) + 6c_1x^3)}{6x^2}$$

1.591 problem 607

Internal problem ID [8729]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 607
Date solved : `Monday, October 21, 2024 at 05:20:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y''+4x(3+8x)\,y'-(5-49x)\,y=0$$

1.591.1 Solved as second order ode using Kovacic algorithm

Time used: 0.260 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(32x^2+12x\right)y'+(49x-5)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^3+4x^2\\ B&=32x^2+12x\\ C&=49x-5\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 8x + 8}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 - 8x + 8 \\ t &= 4\left(x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 8x + 8}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1127: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4\left(1+x\right)^2} + \frac{2}{x^2} + \frac{6}{1+x} - \frac{6}{x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 8x + 8}{4\left(x^2+x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2-8x+8}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right)\\
&= \frac{1}{2}-\left(\frac{1}{2}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{2(1+x)} + \frac{2}{x} + (-)(0) \\
&= -\frac{3}{2(1+x)} + \frac{2}{x} \\
&= \frac{x+4}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{3}{2(1+x)} + \frac{2}{x}\right)(0) + \left(\left(\frac{3}{2(1+x)^2} - \frac{2}{x^2}\right) + \left(-\frac{3}{2(1+x)} + \frac{2}{x}\right)^2 - \left(\frac{-x^2-8x+8}{4(x^2+x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{3}{2(1+x)} + \frac{2}{x}\right) dx} \\
&= \frac{x^2}{(1+x)^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{32x^2+12x}{4x^3+4x^2} \, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2} - \frac{5\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{x^{3/2}(1+x)^{5/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{\left(1+x\right)^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{32x^2+12x}{4x^3+4x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-3\ln(x)-5\ln(1+x)}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(-\frac{3}{2x^2} + \ln\left(x\right) - \frac{1}{3x^3} - \frac{3}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{\left(1+x\right)^4}\right) + c_2 \left(\frac{\sqrt{x}}{\left(1+x\right)^4} \left(-\frac{3}{2x^2} + \ln\left(x\right) - \frac{1}{3x^3} - \frac{3}{x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.591.2 Maple step by step solution

Let's solve

$4x^2(1+x)\left(\frac{d}{dx}y'\right) + 4x(3+8x)\,y' - (5-49x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(49x-5)y}{4x^2(1+x)} - \frac{(3+8x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(3+8x)y'}{x(1+x)} + \frac{(49x-5)y}{4x^2(1+x)} = 0$

☐ Check to see if x_0 is a regular singular point

- ∘ Define functions

  $\left[P_2(x) = \frac{3+8x}{x(1+x)}, P_3(x) = \frac{49x-5}{4x^2(1+x)}\right]$

- ∘ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $$((1+x) \cdot P_2(x))\bigg|_{x=-1} = 5$$

- ∘ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $$((1+x)^2 \cdot P_3(x))\bigg|_{x=-1} = 0$$

- ∘ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $$4x^2(1+x)\left(\frac{d}{dx}y'\right) + 4x(3+8x)\,y' + (49x-5)\,y = 0$$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $$(4u^3 - 8u^2 + 4u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (32u^2 - 52u + 20)\left(\frac{d}{du}y(u)\right) + (49u - 54)\,y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- ∘ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- ∘ Shift index using $k- >k-m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- ∘ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- ∘ Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- ∘ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$

  Rewrite ODE with series expansions

  $4 a_{0} r \left(4+r \right) u^{-1+r}+\left(4 a_{1} \left(1+r \right) \left(5+r \right)-2 a_{0} \left(4 r^{2}+22 r +27\right)\right) u^{r}+\left(\sum_{k =1}^{\infty}\left(4 a_{k +1} \left(k +1+r \right) \left(k +\right.\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $4 r \left(4+r \right)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-4, 0\right\}$

- Each term must be 0

  $4 a_{1} \left(1+r \right) \left(5+r \right)-2 a_{0} \left(4 r^{2}+22 r +27\right)=0$

- Each term in the series must be 0, giving the recursion relation

  $4 a_{k +1} \left(k +1+r \right) \left(k +5+r \right)-2 a_{k} \left(4 k^{2}+8 k r +4 r^{2}+22 k +22 r +27\right)+a_{k -1} \left(2 k +5+2 r \right)^{2}=($

- Shift index using $k->k+1$

  $4 a_{k +2} \left(k +2+r \right) \left(k +6+r \right)-2 a_{k +1} \left(4 \left(k +1\right)^{2}+8 \left(k +1\right) r +4 r^{2}+22 k +49+22 r \right)+a_{k} \left(2 k +\right.$

- Recursion relation that defines series solution to ODE

  $a_{k +2}=-\frac{4 k^{2} a_{k}-8 k^{2} a_{k +1}+8 k r a_{k}-16 k r a_{k +1}+4 r^{2} a_{k}-8 r^{2} a_{k +1}+28 k a_{k}-60 k a_{k +1}+28 r a_{k}-60 r a_{k +1}+49 a_{k}-106 a_{k +1}}{4 \left(k +2+r \right) \left(k +6+r \right)}$

- Recursion relation for $r =-4$

  $a_{k +2}=-\frac{4 k^{2} a_{k}-8 k^{2} a_{k +1}-4 k a_{k}+4 k a_{k +1}+a_{k}+6 a_{k +1}}{4 \left(k -2\right) \left(k +2\right)}$

- Series not valid for $r =-4$, division by 0 in the recursion relation at $k =2$

  $a_{k +2}=-\frac{4 k^{2} a_{k}-8 k^{2} a_{k +1}-4 k a_{k}+4 k a_{k +1}+a_{k}+6 a_{k +1}}{4 \left(k -2\right) \left(k +2\right)}$

- Recursion relation for $r =0$

  $a_{k +2}=-\frac{4 k^{2} a_{k}-8 k^{2} a_{k +1}+28 k a_{k}-60 k a_{k +1}+49 a_{k}-106 a_{k +1}}{4 \left(k +2\right) \left(k +6\right)}$

- Solution for $r =0$

  $\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k}, a_{k +2}=-\frac{4 k^{2} a_{k}-8 k^{2} a_{k +1}+28 k a_{k}-60 k a_{k +1}+49 a_{k}-106 a_{k +1}}{4 \left(k +2\right) \left(k +6\right)}, 20 a_{1}-54 a_{0}=0\right]$

- Revert the change of variables $u =1+x$

  $\left[y =\sum_{k =0}^{\infty}a_{k} \left(1+x \right)^{k}, a_{k +2}=-\frac{4 k^{2} a_{k}-8 k^{2} a_{k +1}+28 k a_{k}-60 k a_{k +1}+49 a_{k}-106 a_{k +1}}{4 \left(k +2\right) \left(k +6\right)}, 20 a_{1}-54 a_{0}=0\right]$

1.591.3 Maple trace

```
Methods for second order ODEs:
```

1.591.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 40

```
dsolve(4*x^2*(1+x)*diff(diff(y(x),x),x)+4*x*(3+8*x)*diff(y(x),x)-(5-49*x)*y(x) = 0,
       y(x),singsol=all)
```

$$
y = \frac{c_1 x^3 + 6 \ln(x) c_2 x^3 - 18 c_2 x^2 - 9 c_2 x - 2 c_2}{x^{5/2} (1+x)^4}
$$

1.591.5 Mathematica DSolve solution

Solving time : 0.086 (sec)
Leaf size : 52

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+4*x*(3+8*x)*D[y[x],x]-(5-49*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to \frac{6c_1x^3 + 6c_2x^3 \log(x) - 18c_2x^2 - 9c_2x - 2c_2}{6x^{5/2}(x+1)^4}
$$

1.592 problem 608

Internal problem ID [8730]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 608
Date solved : `Monday, October 21, 2024 at 05:20:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''-x(3+10x)\,y'+30xy=0$$

1.592.1 Solved as second order ode using Kovacic algorithm

Time used: 0.283 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(-10x^2-3x\right)y'+30xy=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\B&=-10x^2-3x\\C&=30x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-48x + 15}{4\left(x^2 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -48x + 15 \\
t &= 4\left(x^2 + x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-48x + 15}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1129: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{63}{4(1+x)^2} - \frac{39}{2x} + \frac{39}{2(1+x)} + \frac{15}{4x^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-48x + 15}{4 \left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{7}{2 (1 + x)} + \frac{5}{2x} + (0) \\ &= -\frac{7}{2 (1 + x)} + \frac{5}{2x} \\ &= -\frac{2x - 5}{2x (1 + x)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{7}{2(1+x)} + \frac{5}{2x}\right)(1) + \left(\left(\frac{7}{2(1+x)^2} - \frac{5}{2x^2}\right) + \left(-\frac{7}{2(1+x)} + \frac{5}{2x}\right)^2 - \left(\frac{-48x+15}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{5+2a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{5}{2}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{5}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{5}{2}\right) e^{\int \left(-\frac{7}{2(1+x)} + \frac{5}{2x}\right) dx} \\
&= \left(x - \frac{5}{2}\right) e^{\frac{5\ln(x)}{2} - \frac{7\ln(1+x)}{2}} \\
&= \frac{\left(x - \frac{5}{2}\right) x^{5/2}}{(1+x)^{7/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-10x^2-3x}{x^2(1+x)} \, dx} \\
&= z_1 e^{\frac{3\ln(x)}{2} + \frac{7\ln(1+x)}{2}} \\
&= z_1 \left(x^{3/2} (1+x)^{7/2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^5 - \frac{5}{2}x^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x^2-3x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)+7\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(x - \frac{823543}{6250\,(2x-5)} - \frac{1}{25x^4} - \frac{52}{125x^3} - \frac{1354}{625x^2} - \frac{27708}{3125x} + 12\ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^5 - \frac{5}{2}x^4\right) \\
&\quad + c_2\left(x^5 - \frac{5}{2}x^4\left(x - \frac{823543}{6250\,(2x-5)} - \frac{1}{25x^4} - \frac{52}{125x^3} - \frac{1354}{625x^2} - \frac{27708}{3125x} + 12\ln(x)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.592.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right) - x(3+10x)\,y' + 30xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{30y}{x(1+x)} + \frac{(3+10x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{(3+10x)y'}{x(1+x)} + \frac{30y}{x(1+x)} = 0$
- ☐ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{3+10x}{x(1+x)}, P_3(x) = \frac{30}{x(1+x)}\right]$
  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$
    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = -7$
  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$
    $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$
  - $x = -1$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = -1$
- Multiply by denominators
  $x(1+x)\left(\frac{d}{dx}y'\right) + (-3-10x)\,y' + 30y = 0$
- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$
  $(u^2 - u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (7-10u)\left(\frac{d}{du}y(u)\right) + 30y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k- >k+1-m$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$
    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k- >k+2-m$
    $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-a_0 r(-8+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k-7+r) + a_k(k+r-5)\,(k+r-6)\right) u^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-8+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 8\}$
- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)\,(k-7+r) + a_k(k+r-5)\,(k+r-6) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-5)(k+r-6)}{(k+1+r)(k-7+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 5$

  $a_{k+1} = \frac{a_k(k-5)(k-6)}{(k+1)(k-7)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{30a_0}{7}$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{5a_1}{3}$
- Express in terms of a_0

  $a_2 = \frac{50a_0}{7}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{4a_2}{5}$
- Express in terms of a_0

  $a_3 = -\frac{40a_0}{7}$
- Apply recursion relation for $k = 3$

  $a_4 = -\frac{3a_3}{8}$
- Express in terms of a_0

  $a_4 = \frac{15a_0}{7}$
- Apply recursion relation for $k = 4$

  $a_5 = -\frac{2a_4}{15}$
- Express in terms of a_0

  $a_5 = -\frac{2a_0}{7}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{30}{7}u + \frac{50}{7}u^2 - \frac{40}{7}u^3 + \frac{15}{7}u^4 - \frac{2}{7}u^5\right)$
- Revert the change of variables $u = 1 + x$

$$\left[y = a_0\left(\frac{5}{7}x^4 - \frac{2}{7}x^5\right)\right]$$

- Recursion relation for $r = 8$

$$a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}$$

- Solution for $r = 8$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+8}, a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+8}, a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0\left(\frac{5}{7}x^4 - \frac{2}{7}x^5\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k+8}\right), b_{k+1} = \frac{b_k(k+3)(k+2)}{(k+9)(k+1)}\right]$$

1.592.3 Maple trace

```
Methods for second order ODEs:
```

1.592.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 65

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)-x*(3+10*x)*diff(y(x),x)+30*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = 3\left(x - \frac{5}{2}\right)x^4 c_2 \ln(x) + \frac{c_2 x^6}{4} + \frac{(16c_1 - 5c_2)x^5}{8} + \frac{(-80c_1 - 299c_2)x^4}{16} + 5c_2 x^3 + \frac{5c_2 x^2}{4} + \frac{c_2 x}{4} + \frac{c_2}{40}$$

1.592.5 Mathematica DSolve solution

Solving time : `0.101` (sec)
Leaf size : 68

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-x*(3+10*x)*D[y[x],x]+30*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1\left(x^5 - \frac{5x^4}{2}\right) + \frac{1}{20}c_2\left(20x^6 - 50x^5 - 1495x^4 + 120(2x-5)x^4\log(x) + 400x^3 + 100x^2 + 20x + 2\right)$$

1.593 problem 609

Internal problem ID [8731]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 609
Date solved : `Monday, October 21, 2024 at 05:20:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(1+x)\,y' - 3(3+x)\,y = 0$$

1.593.1 Solved as second order ode using Kovacic algorithm

Time used: 0.322 (sec)

Writing the ode as

$$x^2y'' + \left(x^2 + x\right)y' + (-3x - 9)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 + x \\ C &= -3x - 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 + 14x + 35}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 14x + 35 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 + 14x + 35}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1131: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4} + \frac{7}{2x} + \frac{35}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{7}{2x} - \frac{7}{2x^2} + \frac{49}{2x^3} - \frac{735}{4x^4} + \frac{5831}{4x^5} - \frac{48363}{4x^6} + \frac{415373}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 14x + 35}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{14x + 35}{4x^2}\right) \\ &= \frac{1}{4} + \frac{14x + 35}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 14. Dividing this by leading coefficient in t which is 4 gives $\frac{7}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{7}{2}\right) - (0) \\ &= \frac{7}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{7}{2}}{\frac{1}{2}} - 0\right) &&= \frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{7}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{7}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 14x + 35}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{7}{2}$	$-\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{7}{2} - \left(\frac{7}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{7}{2x} \\
&= \frac{x+7}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{7}{2x}\right)(0) + \left(\left(-\frac{7}{2x^2}\right) + \left(\frac{1}{2} + \frac{7}{2x}\right)^2 - \left(\frac{x^2 + 14x + 35}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{7}{2x}\right) dx} \\
&= x^{7/2} \mathrm{e}^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-x}}{6x^6} + \frac{\mathrm{e}^{-x}}{30x^5} - \frac{\mathrm{e}^{-x}}{120x^4} + \frac{\mathrm{e}^{-x}}{360x^3} - \frac{\mathrm{e}^{-x}}{720x^2} + \frac{\mathrm{e}^{-x}}{720x} - \frac{\mathrm{Ei}_1(x)}{720} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^3) + c_2 \left(x^3 \left(-\frac{\mathrm{e}^{-x}}{6x^6} + \frac{\mathrm{e}^{-x}}{30x^5} - \frac{\mathrm{e}^{-x}}{120x^4} + \frac{\mathrm{e}^{-x}}{360x^3} - \frac{\mathrm{e}^{-x}}{720x^2} + \frac{\mathrm{e}^{-x}}{720x} - \frac{\mathrm{Ei}_1(x)}{720} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.593.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + x(1+x)\, y' - 3(3+x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{3(3+x)y}{x^2} - \frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{(1+x)y'}{x} - \frac{3(3+x)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{1+x}{x}, P_3(x) = -\frac{3(3+x)}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -9$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x(1+x)\,y' + (-3x-9)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(3+r)(-3+r)x^r+\left(\sum_{k=1}^{\infty}(a_k(k+r+3)(k+r-3)+a_{k-1}(k-4+r))x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(3+r)(-3+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{-3,3\}$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+3)(k+r-3)+a_{k-1}(k-4+r)=0$
- Shift index using $k->k+1$
  $a_{k+1}(k+4+r)(k-2+r)+a_k(k+r-3)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k(k+r-3)}{(k+4+r)(k-2+r)}$
- Recursion relation for $r=-3$; series terminates at $k=6$
  $a_{k+1}=-\frac{a_k(k-6)}{(k+1)(k-5)}$
- Series not valid for $r=-3$, division by 0 in the recursion relation at $k=5$
  $a_{k+1}=-\frac{a_k(k-6)}{(k+1)(k-5)}$
- Recursion relation for $r=3$
  $a_{k+1}=-\frac{a_k k}{(k+7)(k+1)}$
- Solution for $r=3$
  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+3},a_{k+1}=-\frac{a_k k}{(k+7)(k+1)}\right]$

1.593.3 Maple trace

```
Methods for second order ODEs:
```

1.593.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 50

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(1+x)*diff(y(x),x)-3*(3+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{(x^5-x^4+2x^3-6x^2+24x-120)\,c_2\,\mathrm{e}^{-x}+x^6(-c_2\,\mathrm{Ei}_1(x)+c_1)}{x^3}$$

1.593.5 Mathematica DSolve solution

Solving time : `0.074` (sec)
Leaf size : 60

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1+x)*D[y[x],x]-3*(3+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 e^{-x} (e^x x^6 \operatorname{ExpIntegralEi}(-x) + x^5 - x^4 + 2x^3 - 6x^2 + 24x - 120)}{720x^3} + c_1 x^3$$

1.594 problem 610

Internal problem ID [8732]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 610
Date solved : `Monday, October 21, 2024 at 05:20:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+2x)\,y''+x(9+13x)\,y'+(7+5x)\,y=0$$

1.594.1 Solved as second order ode using Kovacic algorithm

Time used: 0.277 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(13x^2+9x\right)y'+(7+5x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+x^2\\B&=13x^2+9x\\C&=7+5x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 77x^2 + 86x + 35$$
$$t = 4\left(2x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1133: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{27}{2x} + \frac{35}{4x^2} + \frac{45}{16\left(x + \frac{1}{2}\right)^2} + \frac{27}{2\left(x + \frac{1}{2}\right)}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2}
\end{aligned}
$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{9}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r=\frac{s}{t}=\frac{77x^2+86x+35}{4\left(2x^2+x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{77}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{11}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{7}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{77x^2+86x+35}{4\left(2x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-\frac{1}{2}$	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=-\frac{7}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= -\frac{7}{4}-\left(-\frac{15}{4}\right)\\
&= 2
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)} + (-)\,(0) \\
&= -\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)} \\
&= \frac{-5-15x}{4x^2+2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right)(2x+a_1) + \left(\left(\frac{5}{2x^2} + \frac{5}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{77x^2+86x}{4\left(2x^2+\right.}\right.\right.
$$

$$
\frac{(11a_1-8)\,x + 26a_0}{2x^2+x}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{20}{143}, a_1 = \frac{8}{11}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + \frac{8}{11}x + \frac{20}{143}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + \frac{8}{11}x + \frac{20}{143}\right) \mathrm{e}^{\int \left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right) dx} \\
&= \left(x^2 + \frac{8}{11}x + \frac{20}{143}\right) \mathrm{e}^{-\frac{5\ln(1+2x)}{4} - \frac{5\ln(x)}{2}} \\
&= \frac{x^2 + \frac{8}{11}x + \frac{20}{143}}{(1+2x)^{5/4}\, x^{5/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{13x^2+9x}{2x^3+x^2}\,dx} \\
&= z_1 e^{\frac{5\ln(1+2x)}{4}-\frac{9\ln(x)}{2}} \\
&= z_1 \left(\frac{(1+2x)^{5/4}}{x^{9/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2+\frac{8}{11}x+\frac{20}{143}}{x^7}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{13x^2+9x}{2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{5\ln(1+2x)}{2}-9\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{143(1+2x)\left(35x^3-45x^2+36x-20\right)x^9 \mathrm{e}^{\frac{5\ln(1+2x)}{2}-9\ln(x)}}{315\left(143x^2+104x+20\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2+\frac{8}{11}x+\frac{20}{143}}{x^7}\right) \\
&\quad + c_2\left(\frac{x^2+\frac{8}{11}x+\frac{20}{143}}{x^7}\left(\frac{143(1+2x)\left(35x^3-45x^2+36x-20\right)x^9 \mathrm{e}^{\frac{5\ln(1+2x)}{2}-9\ln(x)}}{315\left(143x^2+104x+20\right)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.594.2 Maple step by step solution

Let's solve

$x^2(1+2x)\left(\frac{d}{dx}y'\right)+x(9+13x)\,y'+(7+5x)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(7+5x)y}{x^2(1+2x)}-\frac{(9+13x)y'}{x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(9+13x)y'}{x(1+2x)}+\frac{(7+5x)y}{x^2(1+2x)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{9+13x}{x(1+2x)},P_3(x)=\frac{7+5x}{x^2(1+2x)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$(x\cdot P_2(x))\bigg|_{x=0}=9$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2\cdot P_3(x))\bigg|_{x=0}=7$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(1+2x)\left(\frac{d}{dx}y'\right)+x(9+13x)\,y'+(7+5x)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

$x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(7+r)\,(1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+7)\,(k+r+1) + a_{k-1}(k+4+r)\,(2k-1+2r)\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(7+r)\,(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-7, -1\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k-\frac{1}{2}+r\right)(k+4+r)\, a_{k-1} + a_k(k+r+7)\,(k+r+1) = 0$

- Shift index using $k->k+1$

  $2\left(k+\frac{1}{2}+r\right)(k+r+5)\, a_k + a_{k+1}(k+8+r)\,(k+2+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{(2k+2r+1)(k+r+5)a_k}{(k+8+r)(k+2+r)}$

- Recursion relation for $r = -7$; series terminates at $k = 2$

  $a_{k+1} = -\frac{(2k-13)(k-2)a_k}{(k+1)(k-5)}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{26a_0}{5}$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{11a_1}{8}$

- Express in terms of a_0

  $a_2 = \frac{143a_0}{20}$

- Terminating series solution of the ODE for $r = -7$. Use reduction of order to find the second

  $y = a_0 \cdot \left(\frac{143}{20}x^2 + \frac{26}{5}x + 1\right)$

- Recursion relation for $r=-1$

  $a_{k+1}=-\frac{(2k-1)(k+4)a_k}{(k+7)(k+1)}$
- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1}=-\frac{(2k-1)(k+4)a_k}{(k+7)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y=a_0\cdot\left(\frac{143}{20}x^2+\frac{26}{5}x+1\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), b_{k+1}=-\frac{(2k-1)(k+4)b_k}{(k+7)(k+1)}\right]$

1.594.3 Maple trace

```
Methods for second order ODEs:
```

1.594.4 Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 50

```
dsolve(x^2*(1+2*x)*diff(diff(y(x),x),x)+x*(9+13*x)*diff(y(x),x)+(7+5*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{280c_2\left(x^3-\frac{9}{7}x^2+\frac{36}{35}x-\frac{4}{7}\right)\left(x+\frac{1}{2}\right)^3\sqrt{1+2x}+143c_1x^2+104c_1x+20c_1}{x^7}$$

1.594.5 Mathematica DSolve solution

Solving time : `0.132` (sec)
Leaf size : 58

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]+x*(9+13*x)*D[y[x],x]+(7+5*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{c_1(13x(11x+8)+20)}{143x^7}+\frac{c_2(35x^3-45x^2+36x-20)(2x+1)^{7/2}}{315x^7}$$

1.595 problem 611

Internal problem ID [8733]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 611
Date solved : `Monday, October 21, 2024 at 05:20:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+2x)\,y''-2x(4-x)\,y'-(7+5x)\,y=0$$

1.595.1 Solved as second order ode using Kovacic algorithm

Time used: 0.242 (sec)

Writing the ode as

$$\left(8x^3+4x^2\right)y''+\left(2x^2-8x\right)y'+(-5x-7)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=8x^3+4x^2\\ B&=2x^2-8x\\ C&=-5x-7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{33x^2 + 132x + 60}{16\,(2x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 33x^2 + 132x + 60$$
$$t = 16(2x^2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{33x^2 + 132x + 60}{16\,(2x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1135: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{64\left(x + \frac{1}{2}\right)^2} + \frac{27}{4\left(x + \frac{1}{2}\right)} - \frac{27}{4x} + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{9}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{33x^2 + 132x + 60}{16\left(2x^2 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{11}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{33x^2 + 132x + 60}{16\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
$-\frac{1}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= -\frac{3}{8} - \left(-\frac{3}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)} + (-)(0) \\
&= -\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)} \\
&= -\frac{3(x+2)}{4x(1+2x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{9}{8\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{33x^2+132x+60}{16\left(2x^2+x\right)^2}\right.\right.
$$

0

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right) dx} \\
&= \frac{(1+2x)^{9/8}}{x^{3/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2-8x}{8x^3+4x^2}\, dx} \\
&= z_1 e^{\ln(x) - \frac{9\ln(1+2x)}{8}} \\
&= z_1\left(\frac{x}{(1+2x)^{9/8}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2-8x}{8x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2\ln(x)-\frac{9\ln(1+2x)}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2(1+2x)\left(5x^3-10x^2-40x-16\right) e^{2\ln(x)-\frac{9\ln(1+2x)}{4}}}{35x^2} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\sqrt{x}}\right) + c_2 \left(\frac{1}{\sqrt{x}} \left(\frac{2(1+2x)\left(5x^3-10x^2-40x-16\right) e^{2\ln(x)-\frac{9\ln(1+2x)}{4}}}{35x^2} \right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.595.2 Maple step by step solution

Let's solve

$4x^2(1+2x)\left(\frac{d}{dx}y'\right) - 2x(4-x)\, y' - (7+5x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(7+5x)y}{4x^2(1+2x)} - \frac{(x-4)y'}{2x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(x-4)y'}{2x(1+2x)} - \frac{(7+5x)y}{4x^2(1+2x)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{x-4}{2x(1+2x)}, P_3(x) = -\frac{7+5x}{4x^2(1+2x)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{7}{4}$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $4x^2(1+2x)\left(\frac{d}{dx}y'\right) + 2x(x-4)\,y' + (-5x-7)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

    $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)(-7+2r)\, x^r + \left(\sum_{k=1}^{\infty} (a_k(2k+2r+1)(2k+2r-7) + a_{k-1}(2k-1+2r)(4k-9+4r$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-7+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{7}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $8\left(k-\frac{9}{4}+r\right)\left(k-\frac{1}{2}+r\right) a_{k-1} + 4\left(k+r-\frac{7}{2}\right)\left(k+r+\frac{1}{2}\right) a_k = 0$

- Shift index using $k->k+1$

  $8\left(k-\frac{5}{4}+r\right)\left(k+r+\frac{1}{2}\right) a_k + 4\left(k-\frac{5}{2}+r\right)\left(k+\frac{3}{2}+r\right) a_{k+1} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{(4k+4r-5)(2k+2r+1)a_k}{(2k-5+2r)(2k+3+2r)}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{2(4k-7)ka_k}{(2k-6)(2k+2)}$

- Series not valid for $r = -\frac{1}{2}$, division by 0 in the recursion relation at $k = 3$

  $a_{k+1} = -\frac{2(4k-7)ka_k}{(2k-6)(2k+2)}$

- Recursion relation for $r = \frac{7}{2}$

  $a_{k+1} = -\frac{(4k+9)(2k+8)a_k}{(2k+2)(2k+10)}$

- Solution for $r = \frac{7}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{7}{2}}, a_{k+1} = -\frac{(4k+9)(2k+8)a_k}{(2k+2)(2k+10)}\right]$

1.595.3 Maple trace

```
Methods for second order ODEs:
```

1.595.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 34

```
dsolve(4*x^2*(1+2*x)*diff(diff(y(x),x),x)-2*x*(4-x)*diff(y(x),x)-(7+5*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 + \frac{c_2(5x^3-10x^2-40x-16)}{(1+2x)^{5/4}}}{\sqrt{x}}$$

1.595.5 Mathematica DSolve solution

Solving time : 0.099 (sec)
Leaf size : 47

```
DSolve[{4*x^2*(1+2*x)*D[y[x],{x,2}]-2*x*(4-x)*D[y[x],x]-(7+5*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\frac{2c_2(5x^3-10x^2-40x-16)}{(2x+1)^{5/4}} + 35c_1}{35\sqrt{x}}$$

1.596 problem 612

Internal problem ID [8734]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 612
Date solved : `Monday, October 21, 2024 at 05:20:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2(3+x)\,y''-x(15+x)\,y'-20y=0$$

1.596.1 Solved as second order ode using Kovacic algorithm

Time used: 0.260 (sec)

Writing the ode as

$$\left(3x^3+9x^2\right)y''+\left(-x^2-15x\right)y'-20y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^3+9x^2 \\ B &= -x^2-15x \\ C &= -20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 450x + 1215}{36\left(x^2 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 7x^2 + 450x + 1215$$
$$t = 36\left(x^2 + 3x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 450x + 1215}{36\left(x^2 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1137: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{10}{9x} + \frac{10}{9(3+x)} + \frac{15}{4x^2} - \frac{2}{9(3+x)^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(3+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7x^2 + 450x + 1215}{36(x^2+3x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{7}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{6} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{7x^2+450x+1215}{36\left(x^2+3x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-3	2	0	$\frac{2}{3}$	$\frac{1}{3}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=-\frac{1}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= -\frac{1}{6}-\left(-\frac{7}{6}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{9+3x} - \frac{3}{2x} + (-)\,(0) \\
&= \frac{1}{9+3x} - \frac{3}{2x} \\
&= -\frac{7x+27}{6x\,(3+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{9+3x} - \frac{3}{2x}\right)(1) + \left(\left(-\frac{1}{3\,(3+x)^2} + \frac{3}{2x^2}\right) + \left(\frac{1}{9+3x} - \frac{3}{2x}\right)^2 - \left(\frac{7x^2+450x+1215}{36\,(x^2+3x)^2}\right)\right) = \frac{-27+7a_0}{3x\,(3+x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{27}{7}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{27}{7}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x + \frac{27}{7}\right) e^{\int \left(\frac{1}{9+3x} - \frac{3}{2x}\right) dx} \\
&= \left(x + \frac{27}{7}\right) e^{\frac{\ln(3+x)}{3} - \frac{3\ln(x)}{2}} \\
&= \frac{\left(x + \frac{27}{7}\right)(3+x)^{1/3}}{x^{3/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-15x}{3x^3+9x^2}\,dx} \\
&= z_1 e^{-\frac{2\ln(3+x)}{3}+\frac{5\ln(x)}{6}} \\
&= z_1 \left(\frac{x^{5/6}}{(3+x)^{2/3}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{7x+27}{7\left(3+x\right)^{1/3} x^{2/3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-15x}{3x^3+9x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{4\ln(3+x)}{3}+\frac{5\ln(x)}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{21(3+x)^{5/3}\left(x^2-36x-243\right)\mathrm{e}^{-\frac{4\ln(3+x)}{3}+\frac{5\ln(x)}{3}}}{4\left(7x+27\right)x^{5/3}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{7x+27}{7\left(3+x\right)^{1/3} x^{2/3}}\right) \\
&\quad + c_2 \left(\frac{7x+27}{7\left(3+x\right)^{1/3} x^{2/3}} \left(\frac{21(3+x)^{5/3}\left(x^2-36x-243\right)\mathrm{e}^{-\frac{4\ln(3+x)}{3}+\frac{5\ln(x)}{3}}}{4\left(7x+27\right)x^{5/3}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.596.2 Maple step by step solution

Let's solve

$3x^2(3+x)\left(\frac{d}{dx}y'\right) - x(15+x)\,y' - 20y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = \frac{20y}{3x^2(3+x)} + \frac{(15+x)y'}{3x(3+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{(15+x)y'}{3x(3+x)} - \frac{20y}{3x^2(3+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{15+x}{3x(3+x)}, P_3(x) = -\frac{20}{3x^2(3+x)}\right]$

  ◦ $(3+x)\cdot P_2(x)$ is analytic at $x=-3$

  $\left((3+x)\cdot P_2(x)\right)\Big|_{x=-3} = \frac{4}{3}$

  ◦ $(3+x)^2\cdot P_3(x)$ is analytic at $x=-3$

  $\left((3+x)^2\cdot P_3(x)\right)\Big|_{x=-3} = 0$

  ◦ $x=-3$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -3$

- Multiply by denominators

$3x^2(3+x)\left(\frac{d}{dx}y'\right) - x(15+x)\,y' - 20y = 0$

- Change variables using $x = u-3$ so that the regular singular point is at $u=0$

$(3u^3 - 18u^2 + 27u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u^2 - 9u + 36)\left(\frac{d}{du}y(u)\right) - 20y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$9a_0r(1+3r)\, u^{-1+r} + \left(9a_1(1+r)\,(4+3r) - a_0(18r^2-9r+20)\right) u^r + \left(\sum_{k=1}^{\infty} (9a_{k+1}(k+1+r)\,(3k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$9r(1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{3}\right\}$$

- Each term must be 0

$$9a_1(1+r)\,(4+3r) - a_0(18r^2-9r+20) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$3(-6a_k + a_{k-1} + 9a_{k+1})\, k^2 + \left(6(-6a_k + a_{k-1} + 9a_{k+1})\, r + 9a_k - 10a_{k-1} + 63a_{k+1}\right) k + 3(-6a_k +$$

- Shift index using $k- >k+1$

$$3(-6a_{k+1} + a_k + 9a_{k+2})\,(k+1)^2 + \left(6(-6a_{k+1} + a_k + 9a_{k+2})\, r + 9a_{k+1} - 10a_k + 63a_{k+2}\right)(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} + 6kra_k - 36kra_{k+1} + 3r^2a_k - 18r^2a_{k+1} - 4ka_k - 27ka_{k+1} - 4ra_k - 27ra_{k+1} - 29a_{k+1}}{9(3k^2+6kr+3r^2+13k+13r+14)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2+13k+14)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2+13k+14)}, 36a_1 - 20a_0 = 0\right]$$

- Revert the change of variables $u = 3 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(3+x)^k, a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2+13k+14)}, 36a_1 - 20a_0 = 0\right]$$

- Recursion relation for $r = -\frac{1}{3}$

$$a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 6ka_k - 15ka_{k+1} + \frac{5}{3}a_k - 22a_{k+1}}{9(3k^2+11k+10)}$$

- Solution for $r = -\frac{1}{3}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{1}{3}}, a_{k+2}=-\frac{3k^2a_k-18k^2a_{k+1}-6ka_k-15ka_{k+1}+\frac{5}{3}a_k-22a_{k+1}}{9(3k^2+11k+10)}, 18a_1-25a_0=0\right]$$

- Revert the change of variables $u=3+x$

$$\left[y=\sum_{k=0}^{\infty} a_k(3+x)^{k-\frac{1}{3}}, a_{k+2}=-\frac{3k^2a_k-18k^2a_{k+1}-6ka_k-15ka_{k+1}+\frac{5}{3}a_k-22a_{k+1}}{9(3k^2+11k+10)}, 18a_1-25a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k(3+x)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k(3+x)^{k-\frac{1}{3}}\right), a_{k+2}=-\frac{3k^2a_k-18k^2a_{k+1}-4ka_k-27ka_{k+1}-29a_{k+1}}{9(3k^2+13k+14)}, 36a_1 -\right.$$

1.596.3 Maple trace

```
Methods for second order ODEs:
```

1.596.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 31

```
dsolve(3*x^2*(3+x)*diff(diff(y(x),x),x)-x*(15+x)*diff(y(x),x)-20*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1(x^2-36x-243)+\frac{c_2(7x+27)}{(3+x)^{1/3}}}{x^{2/3}}$$

1.596.5 Mathematica DSolve solution

Solving time : `0.127` (sec)
Leaf size : 43

```
DSolve[{3*x^2*(3+x)*D[y[x],{x,2}]-x*(15+x)*D[y[x],x]-20*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{21c_2(x^2-36x-243)+\frac{4c_1(7x+27)}{\sqrt[3]{x+3}}}{28x^{2/3}}$$

1.597 problem 613

Internal problem ID [8735]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 613
Date solved : `Monday, October 21, 2024 at 05:20:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+x(1-10x)\,y'-(9-10x)\,y=0$$

1.597.1 Solved as second order ode using Kovacic algorithm

Time used: 0.285 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(-10x^2+x\right)y'+(10x-9)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x)\\ B &= -10x^2+x\\ C &= 10x-9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{80x^2 - 28x + 35}{4(x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 80x^2 - 28x + 35$$
$$t = 4(x^2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{80x^2 - 28x + 35}{4(x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1139: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{143}{4(1+x)^2} + \frac{49}{2(1+x)} + \frac{35}{4x^2} - \frac{49}{2x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{143}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{11}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{80x^2 - 28x + 35}{4(x^2 + x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=20$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = 5 \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = -4
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{80x^2-28x+35}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{13}{2}$	$-\frac{11}{2}$
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	5	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=5$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= 5-(4) \\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{13}{2(1+x)} - \frac{5}{2x} + (0) \\
&= \frac{13}{2(1+x)} - \frac{5}{2x} \\
&= \frac{8x - 5}{2x(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{13}{2(1+x)} - \frac{5}{2x}\right)(1) + \left(\left(-\frac{13}{2(1+x)^2} + \frac{5}{2x^2}\right) + \left(\frac{13}{2(1+x)} - \frac{5}{2x}\right)^2 - \left(\frac{80x^2 - 28x + 35}{4(x^2+x)^2}\right)\right) = \frac{-5 - 8a_0}{x(1+x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{5}{8}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{5}{8}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{5}{8}\right) e^{\int \left(\frac{13}{2(1+x)} - \frac{5}{2x}\right) dx} \\
&= \left(x - \frac{5}{8}\right) e^{-\frac{5\ln(x)}{2} + \frac{13\ln(1+x)}{2}} \\
&= \frac{\left(x - \frac{5}{8}\right)(1+x)^{13/2}}{x^{5/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10x^2+x}{x^2(1+x)}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}+\frac{11\ln(1+x)}{2}} \\
&= z_1 \left(\frac{(1+x)^{11/2}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(1+x)^{12}\left(x-\frac{5}{8}\right)}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x^2+x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)+11\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{8\,\mathrm{e}^{-\ln(x)+11\ln(1+x)}x(715x^4+572x^3+234x^2+52x+5)}{6435\,(8x-5)\,(1+x)^{23}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(1+x)^{12}\left(x-\frac{5}{8}\right)}{x^3}\right) \\
&\quad + c_2 \left(\frac{(1+x)^{12}\left(x-\frac{5}{8}\right)}{x^3}\left(-\frac{8\,\mathrm{e}^{-\ln(x)+11\ln(1+x)}x(715x^4+572x^3+234x^2+52x+5)}{6435\,(8x-5)\,(1+x)^{23}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.597.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right)+x(1-10x)\,y'-(9-10x)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(10x-9)y}{x^2(1+x)}+\frac{(-1+10x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{(-1+10x)y'}{x(1+x)}+\frac{(10x-9)y}{x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{-1+10x}{x(1+x)},P_3(x)=\frac{10x-9}{x^2(1+x)}\right]$

- ○ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=-11$

- ○ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$x^2(1+x)\left(\frac{d}{dx}y'\right)-x(-1+10x)\,y'+(10x-9)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^3-2u^2+u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(-10u^2+21u-11)\left(\frac{d}{du}y(u)\right)+(10u-19)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ○ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-12+r)\, u^{-1+r} + \left(a_1(1+r)\,(-11+r) - a_0(2r^2-23r+19)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-12+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 12\}$$

- Each term must be 0

$$a_1(1+r)\,(-11+r) - a_0(2r^2-23r+19) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-2a_k + a_{k-1} + a_{k+1})\, k^2 + \left((-4a_k + 2a_{k-1} + 2a_{k+1})\, r + 23a_k - 13a_{k-1} - 10a_{k+1}\right) k + (-2a_k + a$$

- Shift index using $k- >k+1$

$$(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + \left((-4a_{k+1} + 2a_k + 2a_{k+2})\, r + 23a_{k+1} - 13a_k - 10a_{k+2}\right)(k+1) -$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kr a_k - 4kr a_{k+1} + r^2 a_k - 2r^2 a_{k+1} - 11k a_k + 19k a_{k+1} - 11r a_k + 19r a_{k+1} + 10a_k + 2a_{k+1}}{k^2 + 2kr + r^2 - 8k - 8r - 20}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 11k a_k + 19k a_{k+1} + 10a_k + 2a_{k+1}}{k^2 - 8k - 20}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 10$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 11k a_k + 19k a_{k+1} + 10a_k + 2a_{k+1}}{k^2 - 8k - 20}$$

- Recursion relation for $r = 12$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 13k a_k - 29k a_{k+1} + 22a_k - 58a_{k+1}}{k^2 + 16k + 28}$$

- Solution for $r = 12$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+12}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 13k a_k - 29k a_{k+1} + 22a_k - 58a_{k+1}}{k^2 + 16k + 28}, 13a_1 - 31a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+12}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 13k a_k - 29k a_{k+1} + 22a_k - 58a_{k+1}}{k^2 + 16k + 28}, 13a_1 - 31a_0 = 0\right]$$

1.597.3 Maple trace

```
Methods for second order ODEs:
```

1.597.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 82

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)+x*(1-10*x)*diff(y(x),x)-(9-10*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{8c_2 x^{13} + 91c_2 x^{12} + 468c_2 x^{11} + 1430c_2 x^{10} + 2860c_2 x^9 + 3861c_2 x^8 + 3432c_2 x^7 + 1716c_2 x^6 + 715c_1 x^4}{x^3}$$

1.597.5 Mathematica DSolve solution

Solving time : `0.126` (sec)
Leaf size : 51

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+x*(1-10*x)*D[y[x],x]-(9-10*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{6435c_1(x+1)^{12}(8x-5) - 8c_2(715x^4 + 572x^3 + 234x^2 + 52x + 5)}{51480x^3}$$

1.598 problem 614

Internal problem ID [8736]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 614
Date solved : `Monday, October 21, 2024 at 05:20:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+3x^2y'-(6-x)\,y=0$$

1.598.1 Solved as second order ode using Kovacic algorithm

Time used: 0.244 (sec)

Writing the ode as

$$x^2(1+x)\,y''+3x^2y'+(x-6)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= 3x^2 \\ C &= x-6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 20x + 24}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 20x + 24 \\ t &= 4\left(x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 20x + 24}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1141: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(1+x)^2} + \frac{6}{x^2} + \frac{7}{1+x} - \frac{7}{x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 20x + 24}{4(x^2+x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^2+20x+24}{4\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{3}{2\,(1+x)} - \frac{2}{x} + (-)\,(0) \\ &= \frac{3}{2\,(1+x)} - \frac{2}{x} \\ &= -\frac{x+4}{2x\,(1+x)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{2\,(1+x)} - \frac{2}{x}\right)(1) + \left(\left(-\frac{3}{2\,(1+x)^2} + \frac{2}{x^2}\right) + \left(\frac{3}{2\,(1+x)} - \frac{2}{x}\right)^2 - \left(\frac{-x^2+20x+24}{4\,(x^2+x)^2}\right)\right) = 0$$

$$\frac{-4+a_0}{x\,(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 4$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\,dx} \\ &= (x+4)\,\mathrm{e}^{\int \left(\frac{3}{2(1+x)} - \frac{2}{x}\right)dx} \\ &= (x+4)\,\mathrm{e}^{-2\ln(x) + \frac{3\ln(1+x)}{2}} \\ &= \frac{(x+4)\,(1+x)^{3/2}}{x^2}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^2}{x^2(1+x)}\,dx} \\
&= z_1 e^{-\frac{3\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{(1+x)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x+4}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{256}{27(x+4)} + \ln(1+x) - \frac{1}{18(1+x)^2} + \frac{14}{27(1+x)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x+4}{x^2}\right) + c_2\left(\frac{x+4}{x^2}\left(\frac{256}{27(x+4)} + \ln(1+x) - \frac{1}{18(1+x)^2} + \frac{14}{27(1+x)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.598.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right)+3x^2y'-(6-x)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(x-6)y}{x^2(1+x)}-\frac{3y'}{1+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{3y'}{1+x}+\frac{(x-6)y}{x^2(1+x)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{3}{1+x}, P_3(x)=\frac{x-6}{x^2(1+x)}\right]$

- ○ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=3$

- ○ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

- ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\left(\frac{d}{dx}y'\right)+3x^2y'+(x-6)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(3u^2-6u+3)\left(\frac{d}{du}y(u)\right)+(u-7)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- ○ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r)\, u^{-1+r} + \left(a_1(1+r)\,(3+r) - a_0(2r^2+4r+7)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+3+r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-2, 0\}$$

- Each term must be 0

$$a_1(1+r)\,(3+r) - a_0(2r^2+4r+7) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k-1}(k+r)^2 + a_{k+1}(k+r+1)\,(k+3+r) - 2\left(k^2+(2r+2)\,k+r^2+2r+\frac{7}{2}\right) a_k = 0$$

- Shift index using $k->k+1$

$$a_k(k+r+1)^2 + a_{k+2}(k+r+2)\,(k+4+r) - 2\left((k+1)^2+(2r+2)\,(k+1)+r^2+2r+\frac{7}{2}\right) a_{k+1}$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2kra_k - 4kra_{k+1} + r^2a_k - 2r^2a_{k+1} + 2ka_k - 8ka_{k+1} + 2ra_k - 8ra_{k+1} + a_k - 13a_{k+1}}{(k+r+2)(k+4+r)}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 2ka_k + a_k - 5a_{k+1}}{k(k+2)}$$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 2ka_k + a_k - 5a_{k+1}}{k(k+2)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2k a_k - 8k a_{k+1} + a_k - 13 a_{k+1}}{(k+2)(k+4)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2k a_k - 8k a_{k+1} + a_k - 13 a_{k+1}}{(k+2)(k+4)}, 3a_1 - 7a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2k a_k - 8k a_{k+1} + a_k - 13 a_{k+1}}{(k+2)(k+4)}, 3a_1 - 7a_0 = 0\right]$$

1.598.3 Maple trace

```
Methods for second order ODEs:
```

1.598.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 45

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)+3*x^2*diff(y(x),x)-(6-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1(x+4) + \frac{c_2\left(6(x+4)(1+x)^2 \ln(1+x) + 60x^2 + 129x + 68\right)}{(1+x)^2}}{x^2}$$

1.598.5 Mathematica DSolve solution

Solving time : 0.113 (sec)
Leaf size : 49

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+3*x^2*D[y[x],x]-(6-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\frac{c_2(60x^2+129x+68)}{(x+1)^2} + 6c_1(x+4) + 6c_2(x+4)\log(x+1)}{6x^2}$$

1.599 problem 615

Internal problem ID [8737]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 615
Date solved : `Monday, October 21, 2024 at 05:20:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+2x)\,y''-2x(3+14x)\,y'+(6+100x)\,y=0$$

1.599.1 Solved as second order ode using Kovacic algorithm

Time used: 0.237 (sec)

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(-28x^2-6x\right)y'+(6+100x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+x^2 \\ B &= -28x^2-6x \\ C &= 6+100x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{24x^2 - 16x + 6}{(2x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 24x^2 - 16x + 6$$
$$t = (2x^2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{24x^2 - 16x + 6}{(2x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1143: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{20}{\left(x + \frac{1}{2}\right)^2} + \frac{40}{x + \frac{1}{2}} - \frac{40}{x} + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 5 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -4 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{24x^2 - 16x + 6}{(2x^2 + x)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{24x^2 - 16x + 6}{(2x^2 + x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2
$-\frac{1}{2}$	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 3 - (3) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\ &= -\frac{2}{x} + \frac{5}{x+\frac{1}{2}} + (0) \\ &= -\frac{2}{x} + \frac{5}{x+\frac{1}{2}} \\ &= \frac{-2+6x}{2x^2+x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{2}{x} + \frac{5}{x+\frac{1}{2}}\right)(0) + \left(\left(\frac{2}{x^2} - \frac{5}{\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{2}{x} + \frac{5}{x+\frac{1}{2}}\right)^2 - \left(\frac{24x^2-16x+6}{\left(2x^2+x\right)^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{2}{x} + \frac{5}{x+\frac{1}{2}}\right) dx} \\ &= \frac{(1+2x)^5}{x^2}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-28x^2-6x}{2x^3+x^2}\, dx} \\ &= z_1 e^{4\ln(1+2x)+3\ln(x)} \\ &= z_1\left((1+2x)^4 x^3\right)\end{aligned}$$

Which simplifies to

$$y_1 = (1+2x)^9 x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-28x^2-6x}{2x^3+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{8\ln(1+2x)+6\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(2016x^4+672x^3+144x^2+18x+1)\, e^{8\ln(1+2x)+6\ln(x)}}{20160\,(1+2x)^{17}\, x^6} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left((1+2x)^9 x\right) \\ &\quad + c_2 \left((1+2x)^9 x \left(-\frac{(2016x^4+672x^3+144x^2+18x+1)\, e^{8\ln(1+2x)+6\ln(x)}}{20160\,(1+2x)^{17}\, x^6} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.599.2 Maple step by step solution

Let's solve

$x^2(1+2x)\left(\frac{d}{dx}y'\right) - 2x(3+14x)\, y' + (6+100x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2(3+50x)y}{x^2(1+2x)} + \frac{2(3+14x)y'}{x(1+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$\frac{d}{dx}y' - \frac{2(3+14x)y'}{x(1+2x)} + \frac{2(3+50x)y}{x^2(1+2x)} = 0$$

□ Check to see if x_0 is a regular singular point

- Define functions

  $$\left[P_2(x) = -\frac{2(3+14x)}{x(1+2x)}, P_3(x) = \frac{2(3+50x)}{x^2(1+2x)}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -6$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 6$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2(1+2x)\left(\frac{d}{dx}y'\right) - 2x(3+14x)\,y' + (6+100x)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+r)(-6+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)(k+r-6) + 2a_{k-1}(k+r-6)(k-11+r)\right) x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)(-6+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 6\}$
- Each term in the series must be 0, giving the recursion relation

  $((2k+2r-22)\, a_{k-1} + a_k(k+r-1))(k+r-6) = 0$
- Shift index using $k->k+1$

  $((2k+2r-20)\, a_k + a_{k+1}(k+r))(k+r-5) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2(k+r-10)a_k}{k+r}$
- Recursion relation for $r=1$; series terminates at $k=9$

  $a_{k+1} = -\frac{2(k-9)a_k}{k+1}$
- Recursion relation that defines the terminating series solution of the ODE for $r=1$

  $\left[y = \sum_{k=0}^{8} a_k x^{k+1}, a_{k+1} = -\frac{2(k-9)a_k}{k+1}\right]$
- Recursion relation for $r=6$; series terminates at $k=4$

  $a_{k+1} = -\frac{2(k-4)a_k}{k+6}$
- Apply recursion relation for $k=0$

  $a_1 = \frac{4a_0}{3}$
- Apply recursion relation for $k=1$

  $a_2 = \frac{6a_1}{7}$
- Express in terms of a_0

  $a_2 = \frac{8a_0}{7}$
- Apply recursion relation for $k=2$

  $a_3 = \frac{a_2}{2}$
- Express in terms of a_0

  $a_3 = \frac{4a_0}{7}$
- Apply recursion relation for $k=3$

  $a_4 = \frac{2a_3}{9}$

- Express in terms of a_0

  $a_4 = \frac{8a_0}{63}$
- Terminating series solution of the ODE for $r = 6$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 + \frac{4}{3}x + \frac{8}{7}x^2 + \frac{4}{7}x^3 + \frac{8}{63}x^4\right)$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{8} a_k x^{k+1}\right) + b_0 \cdot \left(1 + \frac{4}{3}x + \frac{8}{7}x^2 + \frac{4}{7}x^3 + \frac{8}{63}x^4\right), a_{k+1} = -\frac{2(k-9)a_k}{k+1}\right]$

1.599.3 Maple trace

```
Methods for second order ODEs:
```

1.599.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 62

```
dsolve(x^2*(1+2*x)*diff(diff(y(x),x),x)-2*x*(3+14*x)*diff(y(x),x)+(6+100*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = 8c_2 x^{10} + 36c_2 x^9 + 72c_2 x^8 + 84c_2 x^7 + 63c_2 x^6 + 2016c_1 x^5 + 672c_1 x^4 + 144c_1 x^3 + 18c_1 x^2 + c_1 x$$

1.599.5 Mathematica DSolve solution

Solving time : `0.097` (sec)
Leaf size : 44

```
DSolve[{x^2*(1+2*x)*D[y[x],{x,2}]-2*x*(3+14*x)*D[y[x],x]+(6+100*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x(2x+1)^9 - \frac{c_2 x(2016x^4 + 672x^3 + 144x^2 + 18x + 1)}{20160}$$

1.600 problem 616

Internal problem ID [8738]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 616
Date solved : `Monday, October 21, 2024 at 05:20:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y'' - x(6+11x)\,y' + (6+32x)\,y = 0$$

1.600.1 Solved as second order ode using Kovacic algorithm

Time used: 0.268 (sec)

Writing the ode as

$$x^2(1+x)\,y'' + \left(-11x^2-6x\right)y' + (6+32x)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= x^2(1+x)\\ B &= -11x^2-6x\\ C &= 6+32x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 + 4x + 24}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15x^2 + 4x + 24$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 + 4x + 24}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1145: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{35}{4(1+x)^2} + \frac{11}{1+x} - \frac{11}{x} + \frac{6}{x^2}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{15x^2 + 4x + 24}{4(x^2 + x)^2}
$$

Since the gcd$(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{15x^2+4x+24}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}\right) \\
&= \frac{5}{2}-\left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{7}{2(1+x)} - \frac{2}{x} + (0) \\
&= \frac{7}{2(1+x)} - \frac{2}{x} \\
&= \frac{3x-4}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{2(1+x)} - \frac{2}{x}\right)(1) + \left(\left(-\frac{7}{2(1+x)^2} + \frac{2}{x^2}\right) + \left(\frac{7}{2(1+x)} - \frac{2}{x}\right)^2 - \left(\frac{15x^2+4x+24}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{-4-3a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{4}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{4}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{7}{2(1+x)} - \frac{2}{x}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{-2\ln(x) + \frac{7\ln(1+x)}{2}} \\
&= \frac{\left(x - \frac{4}{3}\right)(1+x)^{7/2}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-11x^2-6x}{x^2(1+x)}\,dx} \\
&= z_1 e^{3\ln(x)+\frac{5\ln(1+x)}{2}} \\
&= z_1\left(x^3(1+x)^{5/2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x(1+x)^6\left(x-\frac{4}{3}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-11x^2-6x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{6\ln(x)+5\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{3\,\mathrm{e}^{6\ln(x)+5\ln(1+x)}(35x^3+42x^2+21x+4)}{140\,(3x-4)\,x^6\,(1+x)^{11}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x(1+x)^6\left(x-\frac{4}{3}\right)\right) \\
&\quad + c_2\left(x(1+x)^6\left(x-\frac{4}{3}\right)\left(-\frac{3\,\mathrm{e}^{6\ln(x)+5\ln(1+x)}(35x^3+42x^2+21x+4)}{140\,(3x-4)\,x^6\,(1+x)^{11}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.600.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right) - x(6+11x)\,y' + (6+32x)\,y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = -\frac{2(3+16x)y}{x^2(1+x)} + \frac{(6+11x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{(6+11x)y'}{x(1+x)} + \frac{2(3+16x)y}{x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x) = -\frac{6+11x}{x(1+x)}, P_3(x) = \frac{2(3+16x)}{x^2(1+x)}\right]$

- ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

$\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = -5$

- ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -1$

- Multiply by denominators

$x^2(1+x)\left(\frac{d}{dx}y'\right) - x(6+11x)\,y' + (6+32x)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

$(u^3-2u^2+u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-11u^2+16u-5)\left(\frac{d}{du}y(u)\right) + (-26+32u)\,y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- ◦ Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-6+r)\, u^{-1+r} + (a_1(1+r)\,(-5+r) - 2a_0(r^2-9r+13))\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-5\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-6+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 6\}$$

- Each term must be 0

$$a_1(1+r)\,(-5+r) - 2a_0(r^2-9r+13) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-2a_k + a_{k-1} + a_{k+1})\, k^2 + 2((-2a_k + a_{k-1} + a_{k+1})\, r + 9a_k - 7a_{k-1} - 2a_{k+1})\, k + (-2a_k + a_{k-1} -$$

- Shift index using $k->k+1$

$$(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + 2((-2a_{k+1} + a_k + a_{k+2})\, r + 9a_{k+1} - 7a_k - 2a_{k+2})\,(k+1) + (-2$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kr a_k - 4kr a_{k+1} + r^2 a_k - 2r^2 a_{k+1} - 12k a_k + 14k a_{k+1} - 12r a_k + 14r a_{k+1} + 32a_k - 10a_{k+1}}{k^2 + 2kr + r^2 - 2k - 2r - 8}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 12k a_k + 14k a_{k+1} + 32a_k - 10a_{k+1}}{k^2 - 2k - 8}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 4$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 12k a_k + 14k a_{k+1} + 32a_k - 10a_{k+1}}{k^2 - 2k - 8}$$

- Recursion relation for $r = 6$

$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10k a_{k+1} - 4a_k + 2a_{k+1}}{k^2+10k+16}$

- Solution for $r = 6$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10k a_{k+1} - 4a_k + 2a_{k+1}}{k^2+10k+16}, 7a_1 + 10a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+6}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10k a_{k+1} - 4a_k + 2a_{k+1}}{k^2+10k+16}, 7a_1 + 10a_0 = 0\right]$$

1.600.3 Maple trace

```
Methods for second order ODEs:
```

1.600.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 45

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)-x*(6+11*x)*diff(y(x),x)+(6+32*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = 3c_1 x^8 + 14c_1 x^7 + 21c_1 x^6 + 35c_2 x^4 + 42c_2 x^3 + 21c_2 x^2 + 4c_2 x$$

1.600.5 Mathematica DSolve solution

Solving time : 0.109 (sec)
Leaf size : 45

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-x*(6+11*x)*D[y[x],x]+(6+32*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3}c_1 x(x+1)^6(3x-4) - \frac{1}{140}c_2 x\left(35x^3 + 42x^2 + 21x + 4\right)$$

1.601 problem 617

Internal problem ID [8739]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 617
Date solved : `Monday, October 21, 2024 at 05:20:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2(1+x)\,y''+4x(1+4x)\,y'-(49+27x)\,y=0$$

1.601.1 Solved as second order ode using Kovacic algorithm

Time used: 0.264 (sec)

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(16x^2+4x\right)y'+(-27x-49)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^3+4x^2\\ B&=16x^2+4x\\ C&=-27x-49\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35x^2 + 80x + 48}{4(x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35x^2 + 80x + 48$$
$$t = 4(x^2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35x^2 + 80x + 48}{4(x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1147: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{12}{x^2} + \frac{4}{1+x} - \frac{4}{x} + \frac{3}{4(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35x^2 + 80x + 48}{4(x^2 + x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{5}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{35x^2+80x+48}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{7}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= \frac{7}{2}-\left(\frac{7}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(1+x)} + \frac{4}{x} + (0) \\
&= -\frac{1}{2(1+x)} + \frac{4}{x} \\
&= \frac{7x+8}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(1+x)} + \frac{4}{x}\right)(0) + \left(\left(\frac{1}{2(1+x)^2} - \frac{4}{x^2}\right) + \left(-\frac{1}{2(1+x)} + \frac{4}{x}\right)^2 - \left(\frac{35x^2 + 80x + 48}{4(x^2+x)^2}\right)\right) = 0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(1+x)} + \frac{4}{x}\right) dx} \\
&= \frac{x^4}{\sqrt{1+x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{16x^2+4x}{4x^3+4x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} - \frac{3\ln(1+x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}\,(1+x)^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{7/2}}{(1+x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{16x^2+4x}{4x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)-3\ln(1+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(7x+6)(1+x)^3 \mathrm{e}^{-\ln(x)-3\ln(1+x)}}{42x^6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^{7/2}}{(1+x)^2}\right) + c_2 \left(\frac{x^{7/2}}{(1+x)^2}\left(-\frac{(7x+6)(1+x)^3 \mathrm{e}^{-\ln(x)-3\ln(1+x)}}{42x^6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.601.2 Maple step by step solution

Let's solve

$4x^2(1+x)\left(\frac{d}{dx}y'\right) + 4x(1+4x)\, y' - (49+27x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(49+27x)y}{4x^2(1+x)} - \frac{(1+4x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(1+4x)y'}{x(1+x)} - \frac{(49+27x)y}{4x^2(1+x)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{1+4x}{x(1+x)}, P_3(x) = -\frac{49+27x}{4x^2(1+x)}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = 3$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4x^2(1+x)\left(\frac{d}{dx}y'\right) + 4x(1+4x)\,y' + (-27x-49)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(4u^3 - 8u^2 + 4u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (16u^2 - 28u + 12)\left(\frac{d}{du}y(u)\right) + (-27u - 22)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- >k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k- > k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $4a_0r(2+r)\,u^{-1+r} + (4a_1(1+r)(3+r) - 2a_0(4r^2+10r+11))\,u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)(k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- Each term must be 0

  $4a_1(1+r)(3+r) - 2a_0(4r^2+10r+11) = 0$

- Each term in the series must be 0, giving the recursion relation

  $4(-2a_k + a_{k-1} + a_{k+1})\,k^2 + 4(2(-2a_k + a_{k-1} + a_{k+1})\,r - 5a_k + a_{k-1} + 4a_{k+1})\,k + 4(-2a_k + a_{k-}$

- Shift index using $k- > k+1$

  $4(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + 4(2(-2a_{k+1} + a_k + a_{k+2})\,r - 5a_{k+1} + a_k + 4a_{k+2})\,(k+1) + 4($

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 8kra_k - 16kra_{k+1} + 4r^2a_k - 8r^2a_{k+1} + 12ka_k - 36ka_{k+1} + 12ra_k - 36ra_{k+1} - 27a_k - 50a_{k+1}}{4(k^2+2kr+r^2+6k+6r+8)}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 4ka_{k+1} - 35a_k - 10a_{k+1}}{4(k^2+2k)}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 4ka_{k+1} - 35a_k - 10a_{k+1}}{4(k^2+2k)}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 12ka_k - 36ka_{k+1} - 27a_k - 50a_{k+1}}{4(k^2+6k+8)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 12ka_k - 36ka_{k+1} - 27a_k - 50a_{k+1}}{4(k^2+6k+8)}, 12a_1 - 22a_0 = 0\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 12ka_k - 36ka_{k+1} - 27a_k - 50a_{k+1}}{4(k^2+6k+8)}, 12a_1 - 22a_0 = 0\right]$

1.601.3 Maple trace

```
Methods for second order ODEs:
```

1.601.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 26

```
dsolve(4*x^2*(1+x)*diff(diff(y(x),x),x)+4*x*(1+4*x)*diff(y(x),x)-(49+27*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 x^7 + 7c_2 x + 6c_2}{(1+x)^2 x^{7/2}}$$

1.601.5 Mathematica DSolve solution

Solving time : 0.081 (sec)
Leaf size : 36

```
DSolve[{4*x^2*(1+x)*D[y[x],{x,2}]+4*x*(1+4*x)*D[y[x],x]-(49+27*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{42c_1 x^7 - 7c_2 x - 6c_2}{42x^{7/2}(x+1)^2}$$

1.602 problem 618

Internal problem ID [8740]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 618
Date solved : `Monday, October 21, 2024 at 05:20:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2 \left(x^2+1\right) y''-x \left(-2 x^2+7\right) y'+12 y=0$$

1.602.1 Solved as second order ode using Kovacic algorithm

Time used: 0.340 (sec)

Writing the ode as

$$\left(x^4+x^2\right) y''+\left(2 x^3-7 x\right) y'+12 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 2 x^3-7 x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2 A} d x}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-30x^2 + 15}{4(x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -30x^2 + 15$$
$$t = 4(x^3 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-30x^2 + 15}{4(x^3 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1149: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2} + \frac{45}{16 (x - i)^2} + \frac{45}{16 (x + i)^2} + \frac{75i}{16 (x - i)} - \frac{75i}{16 (x + i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{9}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-30x^2 + 15}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{9}{4}$	$-\frac{5}{4}$
$-i$	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)} + (0) \\ &= \frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)} \\ &= \frac{5}{2x(x^2+1)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)}\right)(0) + \left(\left(-\frac{5}{2x^2} + \frac{5}{4(x-i)^2} + \frac{5}{4(x+i)^2}\right) + \left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)}\right) dx} \\ &= \frac{x^{5/2}}{(x^2+1)^{5/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3-7x}{x^4+x^2}\,dx} \\
&= z_1 e^{\frac{7\ln(x)}{2}-\frac{9\ln\left(x^2+1\right)}{4}} \\
&= z_1\left(\frac{x^{7/2}}{\left(x^2+1\right)^{9/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^6}{\left(x^2+1\right)^{7/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-7x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{7\ln(x)-\frac{9\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{\left(x^2+1\right)^{7/2}}{4x^4}-\frac{3\left(x^2+1\right)^{7/2}}{8x^2}+\frac{3\left(x^2+1\right)^{5/2}}{8}+\frac{5\left(x^2+1\right)^{3/2}}{8}+\frac{15\sqrt{x^2+1}}{8}\right. \\
&\qquad \left. -\frac{15\,\mathrm{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^6}{(x^2+1)^{7/2}}\right) \\
&\quad + c_2 \left(\frac{x^6}{(x^2+1)^{7/2}} \left(-\frac{(x^2+1)^{7/2}}{4x^4} - \frac{3(x^2+1)^{7/2}}{8x^2} + \frac{3(x^2+1)^{5/2}}{8} + \frac{5(x^2+1)^{3/2}}{8} + \frac{15\sqrt{x^2+1}}{8} - \frac{15 \arctan
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.602.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right) - x(-2x^2+7)\,y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{12y}{x^2(x^2+1)} - \frac{(2x^2-7)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(2x^2-7)y'}{x(x^2+1)} + \frac{12y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x^2-7}{x(x^2+1)}, P_3(x) = \frac{12}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -7$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 12$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d}{dx}y'\right) + x(2x^2-7)\,y' + 12y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$
  - ○ Shift index using $k- >k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$
  - ○ Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

    $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$
  - ○ Shift index using $k- >k+2-m$

    $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-2+r)\,(-6+r)\, x^r + a_1(-1+r)\,(-5+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-6) + a_{k-2}(k+$$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-6+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{2, 6\}$
- Each term must be 0

  $a_1(-1+r)\,(-5+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_k(k+r-6) + a_{k-2}(k+r-1)) = 0$
- Shift index using $k- >k+2$

  $(k+r)\,(a_{k+2}(k-4+r) + a_k(k+r+1)) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+1)}{k-4+r}$
- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k(k+3)}{k-2}$
- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 2$

$a_{k+2} = -\frac{a_k(k+3)}{k-2}$

- Recursion relation for $r = 6$

  $a_{k+2} = -\frac{a_k(k+7)}{k+2}$

- Solution for $r = 6$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+2} = -\frac{a_k(k+7)}{k+2}, a_1 = 0\right]$

1.602.3 Maple trace

```
Methods for second order ODEs:
```

1.602.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 56

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-2*x^2+7)*diff(y(x),x)+12*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^2\left(-15 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) c_2\, x^4 + c_2(8x^4 - 9x^2 - 2)\sqrt{x^2+1} + c_1\, x^4\right)}{(x^2+1)^{7/2}}$$

1.602.5 Mathematica DSolve solution

Solving time : `0.149` (sec)
Leaf size : 88

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(7-2*x^2)*D[y[x],x]+12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-15c_2x^6\operatorname{arctanh}\left(\sqrt{x^2+1}\right) - 2c_2\sqrt{x^2+1}x^2 + 8x^6\left(c_2\sqrt{x^2+1} + c_1\right) - 9c_2\sqrt{x^2+1}x^4}{8\left(x^2+1\right)^{7/2}}$$

1.603 problem 619

Internal problem ID [8741]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 619
Date solved : `Monday, October 21, 2024 at 05:20:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x\left(-x^2+7\right)y' + 12y = 0$$

1.603.1 Solved as second order ode using Kovacic algorithm

Time used: 0.309 (sec)

Writing the ode as

$$x^2y'' + \left(x^3-7x\right)y' + 12y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3-7x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 12x^2 + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 12x^2 + 15$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 12x^2 + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1151: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 4\\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 3 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{x} - \frac{21}{4x^3} - \frac{63}{2x^5} - \frac{3465}{16x^7} - \frac{13041}{8x^9} - \frac{417501}{32x^{11}} - \frac{1744659}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 12x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{x^2}{4} - 3\right) + \left(\frac{15}{4x^2}\right) \\ &= \frac{x^2}{4} - 3 + \frac{15}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -3. Now b can be found.

$$\begin{aligned} b &= (-3) - (0) \\ &= -3 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{\frac{1}{2}} - 1\right) &&= -\frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{\frac{1}{2}} - 1\right) &&= \frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 12x^2 + 15}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	$-\frac{7}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{5}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{5}{2x} + (-)\left(\frac{x}{2}\right) \\
&= \frac{5}{2x} - \frac{x}{2} \\
&= \frac{5}{2x} - \frac{x}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{5}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{5}{2x^2} - \frac{1}{2}\right) + \left(\frac{5}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 12x^2 + 15}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{5}{2x} - \frac{x}{2}\right) dx} \\
&= x^{5/2} e^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^3 - 7x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{4} + \frac{7 \ln(x)}{2}} \\
&= z_1 \left(x^{7/2} e^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^6 \mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-7x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}+7\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{4x^4} - \frac{\mathrm{e}^{\frac{x^2}{2}}}{8x^2} - \frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{16}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^6 \mathrm{e}^{-\frac{x^2}{2}}\right) + c_2 \left(x^6 \mathrm{e}^{-\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{4x^4} - \frac{\mathrm{e}^{\frac{x^2}{2}}}{8x^2} - \frac{\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)}{16}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.603.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x(-x^2+7)\, y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{12y}{x^2} - \frac{(x^2-7)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(x^2-7)y'}{x} + \frac{12y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{x^2-7}{x}, P_3(x) = \frac{12}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -7$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 12$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x(x^2-7)\,y' + 12y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-6+r)\,x^r + a_1(-1+r)\,(-5+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-6) + a_{k-2}(k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-6+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 6\}$

- Each term must be 0
  $a_1(-1+r)(-5+r)=0$
- Solve for the dependent coefficient(s)
  $a_1=0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-2)\left(a_k(k+r-6)+a_{k-2}\right)=0$
- Shift index using $k->k+2$
  $(k+r)\left(a_{k+2}(k-4+r)+a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=-\frac{a_k}{k-4+r}$
- Recursion relation for $r=2$
  $a_{k+2}=-\frac{a_k}{k-2}$
- Series not valid for $r=2$, division by 0 in the recursion relation at $k=2$
  $a_{k+2}=-\frac{a_k}{k-2}$
- Recursion relation for $r=6$
  $a_{k+2}=-\frac{a_k}{k+2}$
- Solution for $r=6$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+2}=-\frac{a_k}{k+2}, a_1=0\right]$

1.603.3 Maple trace

```
Methods for second order ODEs:
```

1.603.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 47

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(-x^2+7)*diff(y(x),x)+12*y(x) = 0,
       y(x),singsol=all)
```

$$y=x^2\left(\mathrm{Ei}_1\left(-\frac{x^2}{2}\right)\mathrm{e}^{-\frac{x^2}{2}}c_2\,x^4+c_1\,x^4\mathrm{e}^{-\frac{x^2}{2}}+2c_2\,x^2+4c_2\right)$$

1.603.5 Mathematica DSolve solution

Solving time : `0.109` (sec)
Leaf size : 61

```
DSolve[{x^2*D[y[x],{x,2}]-x*(7-x^2)*D[y[x],x]+12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{16} c_2 e^{-\frac{x^2}{2}} x^6 \, \text{ExpIntegralEi}\left(\frac{x^2}{2}\right) - \frac{1}{8} c_2 \left(x^2+2\right) x^2 + c_1 e^{-\frac{x^2}{2}} x^6$$

1.604 problem 620

Internal problem ID [8742]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 620
Date solved : `Monday, October 21, 2024 at 05:21:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(2x^2+1)\,y' - (-10x^2+1)\,y = 0$$

1.604.1 Solved as second order ode using Kovacic algorithm

Time used: 0.372 (sec)

Writing the ode as

$$x^2y'' + (2x^3+x)\,y' + (10x^2-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^3+x \\ C &= 10x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 32x^2 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^4 - 32x^2 + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 32x^2 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1153: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = x^2 - 8 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{4}{x} - \frac{61}{8x^3} - \frac{61}{2x^5} - \frac{19337}{128x^7} - \frac{26779}{32x^9} - \frac{5083557}{1024x^{11}} - \frac{7896633}{256x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 32x^2 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(x^2 - 8\right) + \left(\frac{3}{4x^2}\right) \\ &= x^2 - 8 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -8. Now b can be found.

$$\begin{aligned} b &= (-8) - (0) \\ &= -8 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= x \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-8}{1} - 1\right) &&= -\frac{9}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-8}{1} - 1\right) &&= \frac{7}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4 - 32x^2 + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	$-\frac{9}{2}$	$\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{7}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{7}{2} - \left(\frac{3}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{3}{2x} + (-)(x) \\
&= \frac{3}{2x} - x \\
&= \frac{3}{2x} - x
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{3}{2x} - x\right)(2x + a_1) + \left(\left(-\frac{3}{2x^2} - 1\right) + \left(\frac{3}{2x} - x\right)^2 - \left(\frac{4x^4 - 32x^2 + 3}{4x^2}\right)\right) = 0
$$

$$
\frac{2x^2 a_1 + (4a_0 + 8)\,x + 3a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -2, a_1 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 2\right) e^{\int \left(\frac{3}{2x} - x\right) dx} \\
&= \left(x^2 - 2\right) e^{-\frac{x^2}{2} + \frac{3 \ln(x)}{2}} \\
&= \left(x^2 - 2\right) x^{3/2} e^{-\frac{x^2}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x^2}{2}-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\,\mathrm{e}^{-x^2}\left(x^2-2\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x^2-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-x^2-\ln(x)}\mathrm{e}^{2x^2}}{x^2\left(x^2-2\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x\,\mathrm{e}^{-x^2}\left(x^2-2\right)\right) + c_2\left(x\,\mathrm{e}^{-x^2}\left(x^2-2\right)\left(\int \frac{\mathrm{e}^{-x^2-\ln(x)}\mathrm{e}^{2x^2}}{x^2\left(x^2-2\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.604.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+x(2x^2+1)\,y'-(-10x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(10x^2-1)y}{x^2}-\frac{(2x^2+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(2x^2+1)y'}{x}+\frac{(10x^2-1)y}{x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{2x^2+1}{x},P_3(x)=\frac{10x^2-1}{x^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$(x\cdot P_2(x))\Big|_{x=0}=1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2\cdot P_3(x))\Big|_{x=0}=-1$

- ○ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right)+x(2x^2+1)\,y'+(10x^2-1)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ○ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+r)\, x^r + a_1(2+r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)\,(k+r-1) + 2a_{k-2}(k+3+r)\right) x^{k}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+r)\,(-1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 1\}$
- Each term must be 0
  $a_1(2+r)\, r = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+1)\,(k+r-1) + 2a_{k-2}(k+3+r) = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(k+3+r)\,(k+r+1) + 2a_k(k+r+5) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{2a_k(k+r+5)}{(k+3+r)(k+r+1)}$
- Recursion relation for $r = -1$
  $a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}$
- Solution for $r = -1$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}, a_1 = 0\right]$
- Recursion relation for $r = 1$
  $a_{k+2} = -\frac{2a_k(k+6)}{(k+4)(k+2)}$
- Solution for $r = 1$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_k(k+6)}{(k+4)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}, a_1 = 0, b_{k+2} = -\frac{2b_k(k+6)}{(k+4)(k+2)}, b_1 = 0\right]$$

1.604.3 Maple trace

```
Methods for second order ODEs:
```

1.604.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(2*x^2+1)*diff(y(x),x)-(-10*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{x\,\mathrm{e}^{-x^2}(x^2-2)(c_1-2c_2)}{2}$$

1.604.5 Mathematica DSolve solution

Solving time : `0.141` (sec)
Leaf size : 68

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1+2*x^2)*D[y[x],x]-(1-10*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x^2}\left(c_2(x^2-2)\,x^2\,\text{ExpIntegralEi}\left(x^2\right) + 4c_1x^4 - x^2\left(c_2e^{x^2} + 8c_1\right) + c_2e^{x^2}\right)}{4x}$$

1.605 problem 621

Internal problem ID [8743]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 621
Date solved : `Monday, October 21, 2024 at 05:21:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x\left(-2x^2+1\right)y' - 4\left(2x^2+1\right)y = 0$$

1.605.1 Solved as second order ode using Kovacic algorithm

Time used: 0.300 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^3+x\right)y' + \left(-8x^2-4\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^3+x \\ C &= -8x^2-4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 + 24x^2 + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^4 + 24x^2 + 15$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 + 24x^2 + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1155: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = x^2 + 6 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{x} - \frac{21}{8x^3} + \frac{63}{8x^5} - \frac{3465}{128x^7} + \frac{13041}{128x^9} - \frac{417501}{1024x^{11}} + \frac{1744659}{1024x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 + 24x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (x^2 + 6) + \left(\frac{15}{4x^2}\right) \\ &= x^2 + 6 + \frac{15}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 6. Now b can be found.

$$\begin{aligned} b &= (6) - (0) \\ &= 6 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= x \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{6}{1} - 1\right) &= \frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{6}{1} - 1\right) &= -\frac{7}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4 + 24x^2 + 15}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	$\frac{5}{2}$	$-\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{5}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{5}{2x} + (x) \\
&= \frac{5}{2x} + x \\
&= \frac{5}{2x} + x
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{5}{2x} + x\right)(0) + \left(\left(-\frac{5}{2x^2} + 1\right) + \left(\frac{5}{2x} + x\right)^2 - \left(\frac{4x^4 + 24x^2 + 15}{4x^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{5}{2x} + x\right) dx} \\
&= x^{5/2} \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x^3 + x}{x^2} \, dx} \\
&= z_1 e^{\frac{x^2}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x^2}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^2 \mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x^2-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-x^2}}{4x^4} + \frac{\mathrm{e}^{-x^2}}{4x^2} - \frac{\mathrm{Ei}_1(x^2)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^2 \mathrm{e}^{x^2}\right) + c_2 \left(x^2 \mathrm{e}^{x^2} \left(-\frac{\mathrm{e}^{-x^2}}{4x^4} + \frac{\mathrm{e}^{-x^2}}{4x^2} - \frac{\mathrm{Ei}_1(x^2)}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.605.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + x(-2x^2+1)\, y' - 4(2x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{4(2x^2+1)y}{x^2} + \frac{(2x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' - \frac{(2x^2-1)y'}{x} - \frac{4(2x^2+1)y}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2x^2-1}{x}, P_3(x) = -\frac{4(2x^2+1)}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -4$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - x(2x^2-1)\,y' + (-8x^2-4)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

    Rewrite ODE with series expansions

$$a_0(2+r)(-2+r)x^r + a_1(3+r)(-1+r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)(k+r-2) - 2a_{k-2}(k+r\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(2+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 2\}$
- Each term must be 0
  $a_1(3+r)(-1+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k+r+2)(a_k(k+r-2) - 2a_{k-2}) = 0$
- Shift index using $k- >k+2$
  $(k+r+4)(a_{k+2}(k+r) - 2a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{2a_k}{k+r}$
- Recursion relation for $r = -2$
  $a_{k+2} = \frac{2a_k}{k-2}$
- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$
  $a_{k+2} = \frac{2a_k}{k-2}$
- Recursion relation for $r = 2$
  $a_{k+2} = \frac{2a_k}{k+2}$
- Solution for $r = 2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2a_k}{k+2}, a_1 = 0\right]$

1.605.3 Maple trace

```
Methods for second order ODEs:
```

1.605.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 41

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(-2*x^2+1)*diff(y(x),x)-4*(2*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-\operatorname{Ei}_1\left(x^2\right) \mathrm{e}^{x^2} c_2 \, x^4 + c_1 \, x^4 \mathrm{e}^{x^2} + c_2 \, x^2 - c_2}{x^2}$$

1.605.5 Mathematica DSolve solution

Solving time : 0.086 (sec)
Leaf size : 46

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1-2*x^2)*D[y[x],x]-4*(1+2*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 \left(e^{x^2} x^4 \operatorname{ExpIntegralEi}\left(-x^2\right) + x^2 - 1\right)}{4x^2} + c_1 e^{x^2} x^2$$

1.606 problem 622

Internal problem ID [8744]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 622
Date solved : `Monday, October 21, 2024 at 05:21:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x\left(-3x^2+1\right)y' - 4\left(-3x^2+1\right)y = 0$$

1.606.1 Solved as second order ode using Kovacic algorithm

Time used: 0.627 (sec)

Writing the ode as

$$x^2y'' + \left(-3x^3+x\right)y' + \left(12x^2-4\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x^3+x \\ C &= 12x^2-4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^4 - 60x^2 + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9x^4 - 60x^2 + 15$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^4 - 60x^2 + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1157: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9x^2}{4} - 15 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3x}{2} - \frac{5}{x} - \frac{85}{12x^3} - \frac{425}{18x^5} - \frac{41225}{432x^7} - \frac{278375}{648x^9} - \frac{1787125}{864x^{11}} - \frac{40534375}{3888x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{3x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{9x^4 - 60x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{9x^2}{4} - 15\right) + \left(\frac{15}{4x^2}\right) \\ &= \frac{9x^2}{4} - 15 + \frac{15}{4x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is -15. Now b can be found.

$$\begin{aligned}b &= (-15) - (0) \\ &= -15\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-15}{\frac{3}{2}} - 1\right) &= -\frac{11}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-15}{\frac{3}{2}} - 1\right) = \frac{9}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{9x^4 - 60x^2 + 15}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{3x}{2}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{9}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{9}{2} - \left(\frac{5}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{5}{2x} + (-)\left(\frac{3x}{2}\right) \\
&= \frac{5}{2x} - \frac{3x}{2} \\
&= \frac{5}{2x} - \frac{3x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{5}{2x} - \frac{3x}{2}\right)(2x + a_1) + \left(\left(-\frac{5}{2x^2} - \frac{3}{2}\right) + \left(\frac{5}{2x} - \frac{3x}{2}\right)^2 - \left(\frac{9x^4 - 60x^2 + 15}{4x^2}\right)\right) = 0
$$

$$
\frac{3x^2 a_1 + 6(2 + a_0)\, x + 5a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -2, a_1 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 2\right) e^{\int \left(\frac{5}{2x} - \frac{3x}{2}\right) dx} \\
&= \left(x^2 - 2\right) e^{-\frac{3x^2}{4} + \frac{5\ln(x)}{2}} \\
&= \left(x^2 - 2\right) x^{5/2} e^{-\frac{3x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x^3+x}{x^2}\,dx} \\
&= z_1 e^{\frac{3x^2}{4}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{3x^2}{4}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2-2\right)x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+x}{x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{3x^2}{2}-\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}-\ln(x)}}{\left(x^2-2\right)^2 x^4}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2-2\right)x^2\right) + c_2\left(\left(x^2-2\right)x^2\left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}-\ln(x)}}{\left(x^2-2\right)^2 x^4}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.606.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+x(-3x^2+1)\,y'-4(-3x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{4(3x^2-1)y}{x^2}+\frac{(3x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{(3x^2-1)y'}{x}+\frac{4(3x^2-1)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{3x^2-1}{x}, P_3(x)=\frac{4(3x^2-1)}{x^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=-4$

- ○ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right)-(3x^2-1)\,xy'+(12x^2-4)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ○ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

◦ Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)\,(-2+r)\, x^r + a_1(3+r)\,(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(k+r-2) - 3a_{k-2}(k-6\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(2+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 2\}$
- Each term must be 0
  $a_1(3+r)\,(-1+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+2)\,(k+r-2) - 3a_{k-2}(k-6+r) = 0$
- Shift index using $k->k+2$
  $a_{k+2}(k+4+r)\,(k+r) - 3a_k(k+r-4) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{3a_k(k+r-4)}{(k+4+r)(k+r)}$
- Recursion relation for $r = -2$; series terminates at $k = 6$
  $a_{k+2} = \frac{3a_k(k-6)}{(k+2)(k-2)}$
- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$
  $a_{k+2} = \frac{3a_k(k-6)}{(k+2)(k-2)}$
- Recursion relation for $r = 2$; series terminates at $k = 2$
  $a_{k+2} = \frac{3a_k(k-2)}{(k+6)(k+2)}$
- Solution for $r = 2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{3a_k(k-2)}{(k+6)(k+2)}, a_1 = 0\right]$

1.606.3 Maple trace

```
Methods for second order ODEs:
```

1.606.4 Maple dsolve solution

Solving time : 0.012 (sec)
Leaf size : 19

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(-3*x^2+1)*diff(y(x),x)-4*(-3*x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{x^2(x^2-2)(c_1-c_2)}{2}$$

1.606.5 Mathematica DSolve solution

Solving time : 0.14 (sec)
Leaf size : 89

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1-3*x^2)*D[y[x],x]-4*(1-3*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{64}\left(27c_2(x^2-2)\,x^2\,\text{ExpIntegralEi}\left(\frac{3x^2}{2}\right)+64c_1x^4-2x^2\left(9c_2e^{\frac{3x^2}{2}}+64c_1\right)+24c_2e^{\frac{3x^2}{2}}+\frac{8c_2e^{\frac{3x^2}{2}}}{x^2}\right)$$

1.607 problem 623

Internal problem ID [8745]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 623
Date solved : `Monday, October 21, 2024 at 05:21:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2+1\right)y''+x\left(11x^2+5\right)y'+24x^2y=0$$

1.607.1 Solved as second order ode using Kovacic algorithm

Time used: 0.418 (sec)

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(11x^3+5x\right)y'+24x^2y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^4+x^2\\B&=11x^3+5x\\C&=24x^2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^4 + 6x^2 + 15$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1159: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{9i}{4(x-i)} - \frac{9i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{3}{2} - \left(\frac{3}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (0) \\
&= -\frac{3}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(x+i)} \\
&= -\frac{3}{2x} + \frac{3x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{3}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(x+i)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{3}{2\,(x-i)^2} - \frac{3}{2\,(x+i)^2}\right) + \left(-\frac{3}{2x} + \frac{3}{2\,(x-i)} + \right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{\left(x^2+1\right)^{3/2}}{x^{3/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^3+5x}{x^4+x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+1\right)}{2}-\frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\left(x^2+1\right)^{3/2} x^{5/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^4}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+5x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln\left(x^2+1\right)-5\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{\left(x^2+1\right)\left(2x^2+1\right)x^5 \mathrm{e}^{-3\ln\left(x^2+1\right)-5\ln(x)}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^4}\right) + c_2 \left(\frac{1}{x^4}\left(-\frac{\left(x^2+1\right)\left(2x^2+1\right)x^5 \mathrm{e}^{-3\ln\left(x^2+1\right)-5\ln(x)}}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.607.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(11x^2+5)\,y'+24x^2y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{24y}{x^2+1}-\frac{(11x^2+5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(11x^2+5)y'}{x(x^2+1)}+\frac{24y}{x^2+1}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{11x^2+5}{x(x^2+1)},P_3(x)=\frac{24}{x^2+1}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$\left.(x\cdot P_2(x))\right|_{x=0}=5$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

- ○ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x(x^2+1)\left(\frac{d}{dx}y'\right)+(11x^2+5)\,y'+24yx=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x\cdot y$ to series expansion

$x\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+1}$

- ○ Shift index using $k->k-1$

$x\cdot y=\sum_{k=1}^{\infty}a_{k-1}x^{k+r}$

- ○ Convert $x^m\cdot y'$ to series expansion for $m=0..2$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(4+r)\, x^{-1+r} + a_1(1+r)\,(5+r)\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)\,(k+5+r) + a_{k-1}(k+5+r)\,(k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-4, 0\}$

- Each term must be 0

  $a_1(1+r)\,(5+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+5+r)\,(a_{k+1}(k+r+1) + a_{k-1}(k+3+r)) = 0$

- Shift index using $k- >k+1$

  $(k+r+6)\,(a_{k+2}(k+2+r) + a_k(k+r+4)) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+4)}{k+2+r}$

- Recursion relation for $r = -4$

  $a_{k+2} = -\frac{a_k k}{k-2}$

- Series not valid for $r = -4$, division by 0 in the recursion relation at $k = 2$

  $a_{k+2} = -\frac{a_k k}{k-2}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k(k+4)}{k+2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{k+2}, 5a_1 = 0\right]$

1.607.3 Maple trace

```
Methods for second order ODEs:
```

1.607.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 28

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(11*x^2+5)*diff(y(x),x)+24*x^2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 x^4 + 2c_2 x^2 + c_2}{(x^2+1)^2 x^4}$$

1.607.5 Mathematica DSolve solution

Solving time : 0.075 (sec)
Leaf size : 36

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]+x*(5+11*x^2)*D[y[x],x]+24*x^2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{-4c_1 x^4 + 2c_2 x^2 + c_2}{4x^4 (x^2+1)^2}$$

1.608 problem 624

Internal problem ID [8746]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 624
Date solved : `Monday, October 21, 2024 at 05:21:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+1\right)y''+8xy'-\left(-x^2+35\right)y=0$$

1.608.1 Solved as second order ode using Kovacic algorithm

Time used: 0.394 (sec)

Writing the ode as

$$\left(4x^4+4x^2\right)y''+8xy'+\left(x^2-35\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 8x \\ C &= x^2-35 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 + 22x^2 + 35}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^4 + 22x^2 + 35$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 + 22x^2 + 35}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1161: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{21i}{4(x-i)} - \frac{21i}{4(x+i)} + \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-x^4 + 22x^2 + 35}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^4 + 22x^2 + 35}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + \frac{3}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{5}{2x} + \frac{3}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\
&= -\frac{5}{2x} + \frac{3x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{5}{2x} + \frac{3}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)}\right)(0) + \left(\left(\frac{5}{2x^2} - \frac{3}{2\left(x-i\right)^2} - \frac{3}{2\left(x+i\right)^2}\right) + \left(-\frac{5}{2x} + \frac{3}{2\left(x-i\right)} + \right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{5}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{\left(x^2+1\right)^{3/2}}{x^{5/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x}{4x^4+4x^2}\,dx} \\
&= z_1 e^{-\ln(x)+\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{\sqrt{x^2+1}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+1\right)^2}{x^{7/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x}{4x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)+\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{1}{x^2+1} + \frac{\ln\left(x^2+1\right)}{2} - \frac{1}{4\left(x^2+1\right)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x^2+1\right)^2}{x^{7/2}}\right) + c_2 \left(\frac{\left(x^2+1\right)^2}{x^{7/2}} \left(\frac{1}{x^2+1} + \frac{\ln\left(x^2+1\right)}{2} - \frac{1}{4\left(x^2+1\right)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.608.2 Maple step by step solution

Let's solve

$4x^2(x^2+1)\left(\frac{d}{dx}y'\right)+8xy'-(-x^2+35)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(x^2-35)y}{4x^2(x^2+1)}-\frac{2y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{2y'}{x(x^2+1)}+\frac{(x^2-35)y}{4x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{2}{x(x^2+1)},P_3(x)=\frac{x^2-35}{4x^2(x^2+1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$(x\cdot P_2(x))\bigg|_{x=0}=2$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2\cdot P_3(x))\bigg|_{x=0}=-\frac{35}{4}$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$4x^2(x^2+1)\left(\frac{d}{dx}y'\right)+8xy'+(x^2-35)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ◦ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ◦ Convert $x\cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(7+2r)\,(-5+2r)\, x^r + a_1(9+2r)\,(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+7)\,(2k+2r-5) + a_k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(7+2r)\,(-5+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{7}{2}, \frac{5}{2}\right\}$$

- Each term must be 0

  $$a_1(9+2r)\,(-3+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$4\left(\left(k+r-\frac{5}{2}\right) a_{k-2} + a_k\left(k+r+\frac{7}{2}\right)\right)\left(k+r-\frac{5}{2}\right) = 0$$

- Shift index using $k- >k+2$

  $$4\left(\left(k-\frac{1}{2}+r\right) a_k + a_{k+2}\left(k+\frac{11}{2}+r\right)\right)\left(k-\frac{1}{2}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{(2k+2r-1)a_k}{2k+11+2r}$$

- Recursion relation for $r = -\frac{7}{2}$; series terminates at $k = 4$

  $$a_{k+2} = -\frac{(2k-8)a_k}{2k+4}$$

- Solution for $r = -\frac{7}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{7}{2}}, a_{k+2} = -\frac{(2k-8)a_k}{2k+4}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{5}{2}$

  $$a_{k+2} = -\frac{(2k+4)a_k}{2k+16}$$

- Solution for $r = \frac{5}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{(2k+4)a_k}{2k+16}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{7}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = -\frac{(2k-8)a_k}{2k+4}, a_1 = 0, b_{k+2} = -\frac{(2k+4)b_k}{2k+16}, b_1 = 0\right]$$

1.608.3 Maple trace

```
Methods for second order ODEs:
```

1.608.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 42

```
dsolve(4*x^2*(x^2+1)*diff(diff(y(x),x),x)+8*x*diff(y(x),x)-(-x^2+35)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(x^2+1)^2 c_2 \ln(x^2+1) + \left(2x^2 + \frac{3}{2}\right) c_2 + c_1 (x^2+1)^2}{x^{7/2}}$$

1.608.5 Mathematica DSolve solution

Solving time : `0.112` (sec)
Leaf size : 53

```
DSolve[{4*x^2*(1+x^2)*D[y[x],{x,2}]+8*x*D[y[x],x]-(35-x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1(x^2+1)^2 + c_2(4x^2+3) + 2c_2(x^2+1)^2 \log(x^2+1)}{4x^{7/2}}$$

1.609 problem 625

Internal problem ID [8747]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 625
Date solved : `Monday, October 21, 2024 at 05:21:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2 \left(x^2+1\right) y''-x \left(-x^2+5\right) y'-\left(25 x^2+7\right) y=0$$

1.609.1 Solved as second order ode using Kovacic algorithm

Time used: 0.389 (sec)

Writing the ode as

$$\left(x^4+x^2\right) y''+\left(x^3-5 x\right) y'+\left(-25 x^2-7\right) y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= x^3-5 x \\ C &= -25 x^2-7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 99x^4 + 150x^2 + 63 \\
t &= 4\left(x^3 + x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1163: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 6 - 4\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{63}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} - \frac{15i}{4(x-i)} + \frac{15i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2}\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{99}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{11}{2}$	$-\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{9}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= -\frac{9}{2} - \left(-\frac{9}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{7}{2x} - \frac{1}{2\,(x - i)} - \frac{1}{2\,(x + i)} + (-)\,(0) \\ &= -\frac{7}{2x} - \frac{1}{2\,(x - i)} - \frac{1}{2\,(x + i)} \\ &= -\frac{7}{2x} - \frac{x}{x^2 + 1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{7}{2x} - \frac{1}{2\,(x - i)} - \frac{1}{2\,(x + i)}\right)(0) + \left(\left(\frac{7}{2x^2} + \frac{1}{2\,(x - i)^2} + \frac{1}{2\,(x + i)^2}\right) + \left(-\frac{7}{2x} - \frac{1}{2\,(x - i)} - \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right) dx} \\ &= \frac{1}{\sqrt{x^2 + 1}\, x^{7/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3-5x}{x^4+x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+1\right)}{2}+\frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{x^{5/2}}{\left(x^2+1\right)^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{\left(x^2+1\right)^2 x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-5x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln\left(x^2+1\right)+5\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{x^3\left(4x^2+5\right)\left(x^2+1\right)^3 \mathrm{e}^{-3\ln\left(x^2+1\right)+5\ln(x)}}{40}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\left(x^2+1\right)^2 x}\right) + c_2 \left(\frac{1}{\left(x^2+1\right)^2 x}\left(\frac{x^3\left(4x^2+5\right)\left(x^2+1\right)^3 \mathrm{e}^{-3\ln\left(x^2+1\right)+5\ln(x)}}{40}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.609.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)-x(-x^2+5)\,y'-(25x^2+7)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{(25x^2+7)y}{x^2(x^2+1)}-\frac{(x^2-5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(x^2-5)y'}{x(x^2+1)}-\frac{(25x^2+7)y}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- Define functions

$\left[P_2(x)=\frac{x^2-5}{x(x^2+1)},P_3(x)=-\frac{25x^2+7}{x^2(x^2+1)}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

$(x\cdot P_2(x))\bigg|_{x=0}=-5$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2\cdot P_3(x))\bigg|_{x=0}=-7$

- $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(x^2-5)\,y'+(-25x^2-7)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)(-7+r)\, x^r + a_1(2+r)(-6+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)(k+r-7) + a_{k-2}(k+3-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)(-7+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 7\}$$

- Each term must be 0

$$a_1(2+r)(-6+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-7)\left(a_k(k+r+1) + a_{k-2}(k+3+r)\right) = 0$$

- Shift index using $k->k+2$

$$(k+r-5)\left(a_{k+2}(k+3+r) + a_k(k+r+5)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+5)}{k+3+r}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{a_k(k+4)}{k+2}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0\right]$$

- Recursion relation for $r = 7$

$$a_{k+2} = -\frac{a_k(k+12)}{k+10}$$

- Solution for $r = 7$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+7}, a_{k+2} = -\frac{a_k(k+12)}{k+10}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+7}\right), a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0, b_{k+2} = -\frac{b_k(k+12)}{k+10}, b_1 = 0\right]$$

1.609.3 Maple trace

```
Methods for second order ODEs:
```

1.609.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 29

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)-x*(-x^2+5)*diff(y(x),x)-(25*x^2+7)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{4c_2 x^{10} + 5c_2 x^8 + c_1}{(x^2 + 1)^2 x}$$

1.609.5 Mathematica DSolve solution

Solving time : `0.087` (sec)
Leaf size : 37

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]-x*(5-x^2)*D[y[x],x]-(7+25*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2(4x^2 + 5) x^8 + 40c_1}{40x (x^2 + 1)^2}$$

1.610 problem 626

Internal problem ID [8748]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 626
Date solved : `Monday, October 21, 2024 at 05:21:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2+1)\,y''+x(2x^2+5)\,y'-21y=0$$

1.610.1 Solved as second order ode using Kovacic algorithm

Time used: 0.372 (sec)

Writing the ode as

$$(x^4+x^2)\,y''+(2x^3+5x)\,y'-21y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 2x^3+5x \\ C &= -21 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{78x^2 + 99}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 78x^2 + 99$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{78x^2 + 99}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1165: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16(x-i)^2} + \frac{21}{16(x+i)^2} + \frac{219i}{16(x-i)} - \frac{219i}{16(x+i)} + \frac{99}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{99}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{78x^2 + 99}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{11}{2}$	$-\frac{9}{2}$
i	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= 1 - (-1) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{9}{2x} + \frac{7}{4\,(x-i)} + \frac{7}{4\,(x+i)} + (-)\,(0) \\
&= -\frac{9}{2x} + \frac{7}{4\,(x-i)} + \frac{7}{4\,(x+i)} \\
&= -\frac{9}{2x} + \frac{7x}{2x^2+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{9}{2x} + \frac{7}{4\,(x-i)} + \frac{7}{4\,(x+i)}\right)(2x + a_1) + \left(\left(\frac{9}{2x^2} - \frac{7}{4\,(x-i)^2} - \frac{7}{4\,(x+i)^2}\right) + \left(-\frac{9}{2x} + \frac{7}{4\,(x-\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + 8\right) e^{\int \left(-\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)}\right) dx} \\
&= \left(x^2 + 8\right) e^{-\frac{9\ln(x)}{2} + \frac{7\ln\left(x^2+1\right)}{4}} \\
&= \frac{(x^2+8)\,(x^2+1)^{7/4}}{x^{9/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+5x}{x^4+x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{2}+\frac{3\ln\left(x^2+1\right)}{4}} \\
&= z_1\left(\frac{\left(x^2+1\right)^{3/4}}{x^{5/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+1\right)^{5/2}\left(x^2+8\right)}{x^7}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+5x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-5\ln(x)+\frac{3\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{\left(35x^6+140x^4+168x^2+64\right)x^5 \mathrm{e}^{-5\ln(x)+\frac{3\ln\left(x^2+1\right)}{2}}}{35\left(x^2+1\right)^4\left(x^2+8\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x^2+1\right)^{5/2}\left(x^2+8\right)}{x^7}\right) \\
&\quad + c_2\left(\frac{\left(x^2+1\right)^{5/2}\left(x^2+8\right)}{x^7}\left(-\frac{\left(35x^6+140x^4+168x^2+64\right)x^5 \mathrm{e}^{-5\ln(x)+\frac{3\ln\left(x^2+1\right)}{2}}}{35\left(x^2+1\right)^4\left(x^2+8\right)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.610.2 Maple step by step solution

Let's solve

$x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(2x^2+5)\,y'-21y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{21y}{x^2(x^2+1)}-\frac{(2x^2+5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(2x^2+5)y'}{x(x^2+1)}-\frac{21y}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2x^2+5}{x(x^2+1)},P_3(x)=-\frac{21}{x^2(x^2+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=5$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-21$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x^2+1)\left(\frac{d}{dx}y'\right)+x(2x^2+5)\,y'-21y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(7+r)(-3+r)\, x^r + a_1(8+r)(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+7)(k+r-3) + a_{k-2}(k-2-$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(7+r)(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-7, 3\}$
- Each term must be 0

  $a_1(8+r)(-2+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+7)(k+r-3) + a_{k-2}(k-2+r)(k+r-1) = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+9+r)(k+r-1) + a_k(k+r)(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r)(k+r+1)}{(k+9+r)(k+r-1)}$
- Recursion relation for $r = -7$; series terminates at $k = 6$

  $a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}$
- Solution for $r = -7$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-7}, a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}, a_1 = 0\right]$
- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{a_k(k+3)(k+4)}{(k+12)(k+2)}$
- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k(k+3)(k+4)}{(k+12)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-7}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2}=-\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}, a_1=0, b_{k+2}=-\frac{b_k(k+3)(k+4)}{(k+12)(k+2)}, b_1=0\right]$$

1.610.3 Maple trace

```
Methods for second order ODEs:
```

1.610.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 41

```
dsolve(x^2*(x^2+1)*diff(diff(y(x),x),x)+x*(2*x^2+5)*diff(y(x),x)-21*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1(x^2+8)\left(x^2+1\right)^{5/2}+35\left(x^6+4x^4+\frac{24}{5}x^2+\frac{64}{35}\right)c_2}{x^7}$$

1.610.5 Mathematica DSolve solution

Solving time : `0.162` (sec)
Leaf size : 52

```
DSolve[{x^2*(1+x^2)*D[y[x],{x,2}]+x*(5+2*x^2)*D[y[x],x]-21*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{35c_1(x^2+1)^{5/2}(x^2+8)-c_2(35x^6+140x^4+168x^2+64)}{35x^7}$$

1.611 problem 627

Internal problem ID [8749]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 627
Date solved : `Monday, October 21, 2024 at 05:21:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2\left(x^2+1\right)y''+4x\left(x^2+2\right)y'-\left(x^2+15\right)y=0$$

1.611.1 Solved as second order ode using Kovacic algorithm

Time used: 0.299 (sec)

Writing the ode as

$$\left(4x^4+4x^2\right)y''+\left(4x^3+8x\right)y'+\left(-x^2-15\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 4x^3+8x \\ C &= -x^2-15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{10x^2 + 15}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 10x^2 + 15$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{10x^2 + 15}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1167: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16(x-i)^2} + \frac{5}{16(x+i)^2} + \frac{35i}{16(x-i)} - \frac{35i}{16(x+i)} + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{10x^2 + 15}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + \frac{5}{4\left(x-i\right)} + \frac{5}{4\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{3}{2x} + \frac{5}{4\left(x-i\right)} + \frac{5}{4\left(x+i\right)} \\
&= -\frac{3}{2x} + \frac{5x}{2x^2+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x} + \frac{5}{4\left(x-i\right)} + \frac{5}{4\left(x+i\right)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{5}{4\left(x-i\right)^2} - \frac{5}{4\left(x+i\right)^2}\right) + \left(-\frac{3}{2x} + \frac{5}{4\left(x-i\right)} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)}\right) dx} \\
&= \frac{\left(x^2+1\right)^{5/4}}{x^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^3+8x}{4x^4+4x^2}\,dx} \\
&= z_1 e^{-\ln(x)+\frac{\ln(x^2+1)}{4}} \\
&= z_1\left(\frac{(x^2+1)^{1/4}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x^2+1)^{3/2}}{x^{5/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^3+8x}{4x^4+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)+\frac{\ln(x^2+1)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{(3x^2+2)\,x^2 \mathrm{e}^{-2\ln(x)+\frac{\ln(x^2+1)}{2}}}{3\,(x^2+1)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x^2+1)^{3/2}}{x^{5/2}}\right) + c_2\left(\frac{(x^2+1)^{3/2}}{x^{5/2}}\left(-\frac{(3x^2+2)\,x^2 \mathrm{e}^{-2\ln(x)+\frac{\ln(x^2+1)}{2}}}{3\,(x^2+1)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.611.2 Maple step by step solution

Let's solve

$4x^2(x^2+1)\left(\frac{d}{dx}y'\right)+4x(x^2+2)\,y'-(x^2+15)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{(x^2+15)y}{4x^2(x^2+1)}-\frac{(x^2+2)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(x^2+2)y'}{x(x^2+1)}-\frac{(x^2+15)y}{4x^2(x^2+1)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{x^2+2}{x(x^2+1)},P_3(x)=-\frac{x^2+15}{4x^2(x^2+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=2$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=-\frac{15}{4}$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+1)\left(\frac{d}{dx}y'\right)+4x(x^2+2)\,y'+(-x^2-15)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ○ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(5+2r)\,(-3+2r)\, x^r + a_1(7+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+5)\,(2k+2r-3) + a_{k\text{-}}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(5+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{5}{2}, \frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(7+2r)\,(-1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(\left(k+r-\frac{5}{2}\right) a_{k-2} + a_k\left(k+r+\frac{5}{2}\right)\right)\left(k+r-\frac{3}{2}\right) = 0$$

- Shift index using $k->k+2$

$$4\left(\left(k-\frac{1}{2}+r\right) a_k + a_{k+2}\left(k+\frac{9}{2}+r\right)\right)\left(k+\frac{1}{2}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{(2k+2r-1)a_k}{2k+9+2r}$$

- Recursion relation for $r = -\frac{5}{2}$

$$a_{k+2} = -\frac{(2k-6)a_k}{2k+4}$$

- Solution for $r = -\frac{5}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{(2k-6)a_k}{2k+4}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{3}{2}$

$a_{k+2} = -\frac{(2k+2)a_k}{2k+12}$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{(2k+2)a_k}{2k+12}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -\frac{(2k-6)a_k}{2k+4}, a_1 = 0, b_{k+2} = -\frac{(2k+2)b_k}{2k+12}, b_1 = 0\right]$$

1.611.3 Maple trace

```
Methods for second order ODEs:
```

1.611.4 Maple dsolve solution

Solving time : `0.014` (sec)
Leaf size : 27

```
dsolve(4*x^2*(x^2+1)*diff(diff(y(x),x),x)+4*x*(x^2+2)*diff(y(x),x)-(x^2+15)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 (x^2+1)^{3/2} + 3c_1 x^2 + 2c_1}{x^{5/2}}$$

1.611.5 Mathematica DSolve solution

Solving time : `0.115` (sec)
Leaf size : 39

```
DSolve[{4*x^2*(1+x^2)*D[y[x],{x,2}]+4*x*(2+x^2)*D[y[x],x]-(15+x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3c_1 (x^2+1)^{3/2} - c_2(3x^2+2)}{3x^{5/2}}$$

1.612 problem 628

Internal problem ID [8750]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 628
Date solved : `Monday, October 21, 2024 at 05:21:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - \frac{2(t+1)\, y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0$$

1.612.1 Solved as second order ode using Kovacic algorithm

Time used: 0.312 (sec)

Writing the ode as

$$y'' + \frac{(-2t-2)\, y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{-2t-2}{t^2+2t-1} \\ C &= \frac{2}{t^2+2t-1} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{(t^2 + 2t - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 6$$
$$t = (t^2 + 2t - 1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{6}{(t^2 + 2t - 1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1169: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2 + 2t - 1)^2$. There is a pole at $t = \sqrt{2} - 1$ of order 2. There is a pole at $t = -1 - \sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(t - \sqrt{2} + 1\right)^2} + \frac{3}{4\left(t + 1 + \sqrt{2}\right)^2} - \frac{3\sqrt{2}}{8\left(t - \sqrt{2} + 1\right)} + \frac{3\sqrt{2}}{8\left(t + 1 + \sqrt{2}\right)}$$

For the pole at $t = \sqrt{2} - 1$ let b be the coefficient of $\frac{1}{\left(t-\sqrt{2}+1\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $t = -1 - \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(t+1+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_{\infty} &= 0 \\
\alpha_{\infty}^{+} &= 0 \\
\alpha_{\infty}^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{6}{(t^2 + 2t - 1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\sqrt{2} - 1$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-1 - \sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_{\infty}$	α_{∞}^{+}	α_{∞}^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_{\infty}$ and its associated $\alpha_{\infty}^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{t-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)} + (-)\,(0) \\
&= -\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)} \\
&= \frac{t+1-2\sqrt{2}}{t^2+2t-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)}\right)(0) + \left(\left(\frac{1}{2\left(t-\sqrt{2}+1\right)^2} - \frac{3}{2\left(t+1+\sqrt{2}\right)^2}\right) + \left(-\frac{}{2\,(t-}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)}\right) dt} \\
&= \frac{\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{-2t-2}{t^2+2t-1}}{1} \, dt} \\
&= z_1 e^{\frac{\ln\left(t^2+2t-1\right)}{2}} \\
&= z_1 \left(\sqrt{t^2+2t-1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{t^2+2t-1}\,\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{-2t-2}{t^2+2t-1}}{1}\, dt}}{\left(y_1\right)^2}\, dt \\
&= y_1 \int \frac{e^{\ln\left(t^2+2t-1\right)}}{\left(y_1\right)^2}\, dt \\
&= y_1 \left(-\frac{1}{t+1+\sqrt{2}} + \frac{\sqrt{2}}{\left(t+1+\sqrt{2}\right)^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{t^2+2t-1}\,\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}}\right) \\
&\quad + c_2 \left(\frac{\sqrt{t^2+2t-1}\,\left(t+1+\sqrt{2}\right)^{3/2}}{\sqrt{t-\sqrt{2}+1}} \left(-\frac{1}{t+1+\sqrt{2}} + \frac{\sqrt{2}}{\left(t+1+\sqrt{2}\right)^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.612.2 Maple step by step solution

Let's solve

$\frac{d}{dt}y' - \frac{2(t+1)y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- □ Check to see if t_0 is a regular singular point
  - Define functions

    $\left[P_2(t) = -\frac{2(t+1)}{t^2+2t-1}, P_3(t) = \frac{2}{t^2+2t-1}\right]$
  - $(t+1+\sqrt{2}) \cdot P_2(t)$ is analytic at $t = -1-\sqrt{2}$

    $\left((t+1+\sqrt{2}) \cdot P_2(t)\right)\Big|_{t=-1-\sqrt{2}} = 0$
  - $(t+1+\sqrt{2})^2 \cdot P_3(t)$ is analytic at $t = -1-\sqrt{2}$

    $\left((t+1+\sqrt{2})^2 \cdot P_3(t)\right)\Big|_{t=-1-\sqrt{2}} = 0$
  - $t = -1-\sqrt{2}$is a regular singular point

    Check to see if t_0 is a regular singular point

    $t_0 = -1-\sqrt{2}$

- Multiply by denominators

  $(t^2+2t-1)\left(\frac{d}{dt}y'\right) + (-2t-2)\,y' + 2y = 0$

- Change variables using $t = u-1-\sqrt{2}$ so that the regular singular point is at $u = 0$

  $(u^2-2u\sqrt{2})\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-2u+2\sqrt{2})\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$u^{m} \cdot \left(\frac{d}{du}\frac{d}{du}y \left(u \right)\right) = \sum_{k =0}^{\infty} a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$

- Shift index using $k->k +2-m$

  $u^{m} \cdot \left(\frac{d}{du}\frac{d}{du}y \left(u \right)\right) = \sum_{k =-2+m}^{\infty} a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$

  Rewrite ODE with series expansions

  $-2 \sqrt{2}\, r \left(r -2\right) a_{0} u^{r -1}+\left(\sum_{k =0}^{\infty} \left(-2 \sqrt{2}\, \left(k +1+r \right) \left(k +r -1\right) a_{k +1}+a_{k} \left(k +r -1\right) \left(k +r -2\right)\right) u$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-2 \sqrt{2}\, r \left(r -2\right) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, 2\right\}$

- Each term in the series must be 0, giving the recursion relation

  $\left(k +r -1\right) \left(-2 a_{k +1} \left(k +1+r \right) \sqrt{2}+a_{k} \left(k +r -2\right)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k +1} = \frac{a_{k} \left(k +r -2\right) \sqrt{2}}{4 \left(k +1+r \right)}$

- Recursion relation for $r =0$; series terminates at $k =2$

  $a_{k +1} = \frac{a_{k} \left(k -2\right) \sqrt{2}}{4 \left(k +1\right)}$

- Apply recursion relation for $k =0$

  $a_{1} = -\frac{a_{0} \sqrt{2}}{2}$

- Apply recursion relation for $k =1$

  $a_{2} = -\frac{a_{1} \sqrt{2}}{8}$

- Express in terms of a_{0}

  $a_{2} = \frac{a_{0}}{8}$

- Terminating series solution of the ODE for $r =0$. Use reduction of order to find the second li

  $y \left(u \right) = a_{0} \cdot \left(1-\frac{u \sqrt{2}}{2}+\frac{u^{2}}{8}\right)$

- Revert the change of variables $u = t +1+\sqrt{2}$

  $\left[y = a_{0} \left(\frac{\left(-2 t -2\right) \sqrt{2}}{8}+\frac{t^{2}}{8}+\frac{t}{4}+\frac{3}{8}\right)\right]$

- Recursion relation for $r =2$

  $a_{k +1} = \frac{a_{k} k \sqrt{2}}{4 \left(k +3\right)}$

- Solution for $r =2$

  $\left[y \left(u \right) = \sum_{k =0}^{\infty} a_{k} u^{k +2}, a_{k +1} = \frac{a_{k} k \sqrt{2}}{4 \left(k +3\right)}\right]$

- Revert the change of variables $u = t+1+\sqrt{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(t+1+\sqrt{2}\right)^{k+2}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0\left(\frac{(-2t-2)\sqrt{2}}{8} + \frac{t^2}{8} + \frac{t}{4} + \frac{3}{8}\right) + \left(\sum_{k=0}^{\infty} b_k \left(t+1+\sqrt{2}\right)^{k+2}\right), b_{k+1} = \frac{b_k k\sqrt{2}}{4(k+3)}\right]$$

1.612.3 Maple trace

```
Methods for second order ODEs:
```

1.612.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 15

```
dsolve(diff(diff(y(t),t),t)-2*(t+1)/(t^2+2*t-1)*diff(y(t),t)+2/(t^2+2*t-1)*y(t) = 0,
       y(t),singsol=all)
```

$$y = c_2 t^2 + c_1 t + c_1 + c_2$$

1.612.5 Mathematica DSolve solution

Solving time : `0.299` (sec)
Leaf size : 64

```
DSolve[{D[y[t],{t,2}]-2*(t+1)/(t^2+2*t-1)*D[y[t],t]+2/(t^2+2*t-1)*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{\sqrt{t^2+2t-1}\left(c_1\left(t^2-2\left(\sqrt{2}-1\right)t-2\sqrt{2}+3\right)+c_2(t+1)\right)}{\sqrt{-t^2-2t+1}}$$

1.613 problem 629

Internal problem ID [8751]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 629
Date solved : `Monday, October 21, 2024 at 05:21:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4ty' + \left(4t^2 - 2\right) y = 0$$

1.613.1 Solved as second order ode using Kovacic algorithm

Time used: 0.086 (sec)

Writing the ode as

$$y'' - 4ty' + \left(4t^2 - 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4t \\ C &= 4t^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1171: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4t}{1}\, dt} \\
&= z_1 e^{t^2} \\
&= z_1 \left(\mathrm{e}^{t^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{t^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4t}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{2t^2}}{(y_1)^2}\, dt \\
&= y_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{t^2}\right) + c_2 \left(\mathrm{e}^{t^2} (t)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.613.2 Maple step by step solution

Let's solve

$\frac{d}{dt}y' - 4ty' + (4t^2 - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^k$

□ Rewrite ODE with series expansions

  - Convert $t^m \cdot y$ to series expansion for $m = 0..2$

    $t^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k t^{k+m}$

  - Shift index using $k- >k - m$

    $t^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} t^k$

  - Convert $t \cdot y'$ to series expansion

    $t \cdot y' = \sum_{k=0}^{\infty} a_k k\, t^k$

  - Convert $\frac{d}{dt}y'$ to series expansion

    $\frac{d}{dt}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, t^{k-2}$

  - Shift index using $k- >k + 2$

    $\frac{d}{dt}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, t^k$

Rewrite ODE with series expansions

$2a_2 - 2a_0 + (6a_3 - 6a_1)\, t + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2})\, t^k\right) = 0$

- The coefficients of each power of t must be 0

  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = a_0, a_3 = a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $\left((k + 2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k + 2) - 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+4} = \frac{2(2ka_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$

1.613.3 Maple trace

```
Methods for second order ODEs:
```

1.613.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(t),t),t)-4*t*diff(y(t),t)+(4*t^2-2)*y(t) = 0,
       y(t),singsol=all)
```

$$y = e^{t^2}(tc_2 + c_1)$$

1.613.5 Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 18

```
DSolve[{D[y[t],{t,2}]-4*t*D[y[t],t]+(4*t^2-2)*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^{t^2}(c_2 t + c_1)$$

1.614 problem 630

Internal problem ID [8752]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 630
Date solved : `Monday, October 21, 2024 at 05:21:11 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-t^2+1\right) y''-2ty'+2y=0$$

1.614.1 Solved as second order ode using Kovacic algorithm

Time used: 0.283 (sec)

Writing the ode as

$$\left(-t^2+1\right) y''-2ty'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -t^2+1 \\ B &= -2t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2t^2 - 3}{(t^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2t^2 - 3$$
$$t = (t^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{2t^2 - 3}{(t^2 - 1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1173: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2 - 1)^2$. There is a pole at $t = 1$ of order 2. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(t+1)^2} - \frac{5}{4(t+1)} - \frac{1}{4(t-1)^2} + \frac{5}{4(t-1)}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(t+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2t^2 - 3}{(t^2 - 1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{2t^2-3}{\left(t^2-1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= 2-(1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{t-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2t-2} + \frac{1}{2t+2} + (0) \\
&= \frac{1}{2t-2} + \frac{1}{2t+2} \\
&= \frac{t}{t^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)(1) + \left(\left(-\frac{1}{2(t-1)^2} - \frac{1}{2(t+1)^2}\right) + \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)^2 - \left(\frac{2t^2-3}{(t^2-1)^2}\right)\right) = -\frac{2a_0}{t^2-1} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= (t)\, e^{\int \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right) dt} \\
&= (t)\sqrt{(t-1)(t+1)} \\
&= t\sqrt{t^2-1}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2t}{-t^2+1}\,dt} \\
&= z_1 e^{-\frac{\ln(t-1)}{2}-\frac{\ln(t+1)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{t-1}\,\sqrt{t+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{-t^2+1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\ln(t-1)-\ln(t+1)}}{(y_1)^2}\,dt \\
&= y_1\left(\frac{\ln(t-1)}{2}+\frac{1}{t}-\frac{\ln(t+1)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}\right) + c_2\left(\frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}\left(\frac{\ln(t-1)}{2}+\frac{1}{t}-\frac{\ln(t+1)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.614.2 Maple step by step solution

Let's solve

$(-t^2+1)\left(\frac{d}{dt}y'\right)-2ty'+2y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dt}y'$

- Isolate 2nd derivative

$\frac{d}{dt}y'=\frac{2y}{t^2-1}-\frac{2ty'}{t^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y'+\frac{2ty'}{t^2-1}-\frac{2y}{t^2-1}=0$

□ Check to see if t_0 is a regular singular point

- ◦ Define functions

$\left[P_2(t)=\frac{2t}{t^2-1}, P_3(t)=-\frac{2}{t^2-1}\right]$

- ◦ $(t+1)\cdot P_2(t)$ is analytic at $t=-1$

$\left((t+1)\cdot P_2(t)\right)\bigg|_{t=-1}=1$

- ◦ $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$

$\left((t+1)^2\cdot P_3(t)\right)\bigg|_{t=-1}=0$

- ◦ $t=-1$is a regular singular point

Check to see if t_0 is a regular singular point

$t_0=-1$

- Multiply by denominators

$(t^2-1)\left(\frac{d}{dt}y'\right)+2ty'-2y=0$

- Change variables using $t=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+2)\,(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-2a_{k+1}(k+1)^2 + a_k(k+2)\,(k-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$$a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$$

- Apply recursion relation for $k = 0$

$$a_1 = -a_0$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot (-u+1)$$

- Revert the change of variables $u = t+1$

$$[y = -a_0t]$$

1.614.3 Maple trace

```
Methods for second order ODEs:
```

1.614.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 25

```
dsolve((-t^2+1)*diff(diff(y(t),t),t)-2*t*diff(y(t),t)+2*y(t) = 0,
       y(t),singsol=all)
```

$$y = \frac{\ln (t-1) c_2 t}{2} - \frac{\ln (t+1) c_2 t}{2} + c_1 t + c_2$$

1.614.5 Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 33

```
DSolve[{(1-t^2)*D[y[t],{t,2}]-2*t*D[y[t],t]+2*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_1 t - \frac{1}{2} c_2 (t \log(1-t) - t \log(t+1) + 2)$$

1.615 problem 631

Internal problem ID [8753]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 631
Date solved : `Monday, October 21, 2024 at 05:21:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(t^2+1\right) y''-2ty'+2y=0$$

1.615.1 Solved as second order ode using Kovacic algorithm

Time used: 0.303 (sec)

Writing the ode as

$$\left(t^2+1\right) y''-2ty'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=t^2+1\\ B&=-2t\\ C&=2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(t^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (t^2+1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{3}{(t^2+1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1175: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2+1)^2$. There is a pole at $t = i$ of order 2. There is a pole at $t = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4(t-i)^2} + \frac{3}{4(t+i)^2} + \frac{3i}{4(t-i)} - \frac{3i}{4(t+i)}
$$

For the pole at $t = i$ let b be the coefficient of $\frac{1}{(t-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $t = -i$ let b be the coefficient of $\frac{1}{(t+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(t^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(t-i\right)} + \frac{3}{2\left(t+i\right)} + (-)\,(0) \\
&= -\frac{1}{2\left(t-i\right)} + \frac{3}{2\left(t+i\right)} \\
&= \frac{t-2i}{t^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\,(t-i)} + \frac{3}{2\,(t+i)}\right)(0) + \left(\left(\frac{1}{2\,(t-i)^2} - \frac{3}{2\,(t+i)^2}\right) + \left(-\frac{1}{2\,(t-i)} + \frac{3}{2\,(t+i)}\right)^2 - \left(-\frac{3}{(t^2 \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega\, dt} \\
&= e^{\int \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right) dt} \\
&= \frac{(t^2+1)^{3/2}}{(it+1)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2t}{t^2+1}\, dt} \\
&= z_1 e^{\frac{\ln(t^2+1)}{2}} \\
&= z_1\left(\sqrt{t^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(t^2+1)^2}{(it+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{t^2+1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{\ln(t^2+1)}}{(y_1)^2}\,dt \\
&= y_1 \left(-\frac{t}{(t+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(t^2+1)^2}{(it+1)^2}\right) + c_2 \left(\frac{(t^2+1)^2}{(it+1)^2} \left(-\frac{t}{(t+i)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.615.2 Maple step by step solution

1.615.3 Maple trace

```
Methods for second order ODEs:
```

1.615.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
dsolve((t^2+1)*diff(diff(y(t),t),t)-2*t*diff(y(t),t)+2*y(t) = 0,
       y(t),singsol=all)
```

$$
y = c_2 t^2 + c_1 t - c_2
$$

1.615.5 Mathematica DSolve solution

Solving time : 0.071 (sec)
Leaf size : 21

```
DSolve[{(1+t^2)*D[y[t],{t,2}]-2*t*D[y[t],t]+2*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_2 t - c_1 (t - i)^2$$

1.616 problem 632

Internal problem ID [8754]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 632
Date solved : `Monday, October 21, 2024 at 05:21:13 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-t^2+1\right) y''-2ty'+6y=0$$

1.616.1 Solved as second order ode using Kovacic algorithm

Time used: 0.309 (sec)

Writing the ode as

$$\left(-t^2+1\right) y''-2ty'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -t^2+1 \\ B &= -2t \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{6t^2 - 7}{(t^2 - 1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 6t^2 - 7 \\
t &= (t^2 - 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{6t^2 - 7}{(t^2 - 1)^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1176: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2 - 1)^2$. There is a pole at $t = 1$ of order 2. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{13}{4(t+1)} + \frac{13}{4(t-1)} - \frac{1}{4(t-1)^2} - \frac{1}{4(t+1)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(t+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6t^2 - 7}{(t^2 - 1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=3 \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{6t^2-7}{(t^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=3$ then

$$\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= 3-(1) \\
&= 2
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{t-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2t-2} + \frac{1}{2t+2} + (0) \\
&= \frac{1}{2t-2} + \frac{1}{2t+2} \\
&= \frac{t}{t^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(t) = t^2 + a_1 t + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)(2t + a_1) + \left(\left(-\frac{1}{2(t-1)^2} - \frac{1}{2(t+1)^2}\right) + \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)^2 - \left(\frac{6t^2-7}{(t^2-1)^2}\right.\right.
$$

$$
\frac{-4a_1 t - 6a_0 - 2}{t^2-1}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{1}{3}, a_1 = 0\right\}
$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$
p(t) = t^2 - \frac{1}{3}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^2 - \frac{1}{3}\right) e^{\int \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right) dt} \\
&= \left(t^2 - \frac{1}{3}\right) \sqrt{(t-1)(t+1)} \\
&= \frac{(3t^2-1)\sqrt{t^2-1}}{3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2t}{-t^2+1}\,dt} \\
&= z_1 e^{-\frac{\ln(t-1)}{2}-\frac{\ln(t+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t-1}\,\sqrt{t+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(3t^2-1)\sqrt{t^2-1}}{3\sqrt{t-1}\,\sqrt{t+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{-t^2+1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\ln(t-1)-\ln(t+1)}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{9\ln(t-1)}{8} - \frac{9\ln(t+1)}{8} + \frac{9t}{4\left(t^2-\frac{1}{3}\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(3t^2-1)\sqrt{t^2-1}}{3\sqrt{t-1}\,\sqrt{t+1}}\right) \\
&\quad + c_2 \left(\frac{(3t^2-1)\sqrt{t^2-1}}{3\sqrt{t-1}\,\sqrt{t+1}} \left(\frac{9\ln(t-1)}{8} - \frac{9\ln(t+1)}{8} + \frac{9t}{4\left(t^2-\frac{1}{3}\right)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.616.2 Maple step by step solution

Let's solve

$(-t^2+1)\left(\frac{d}{dt}y'\right) - 2ty' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = \frac{6y}{t^2-1} - \frac{2ty'}{t^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' + \frac{2ty'}{t^2-1} - \frac{6y}{t^2-1} = 0$

☐ Check to see if t_0 is a regular singular point

- ○ Define functions

  $\left[P_2(t) = \frac{2t}{t^2-1}, P_3(t) = -\frac{6}{t^2-1}\right]$

- ○ $(t+1)\cdot P_2(t)$ is analytic at $t=-1$

  $\left((t+1)\cdot P_2(t)\right)\Big|_{t=-1} = 1$

- ○ $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$

  $\left((t+1)^2\cdot P_3(t)\right)\Big|_{t=-1} = 0$

- ○ $t=-1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = -1$

- Multiply by denominators

  $(t^2-1)\left(\frac{d}{dt}y'\right) + 2ty' - 6y = 0$

- Change variables using $t = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 6y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r^2 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+3)\,(k+r-2)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2a_{k+1}(k+1)^2 + a_k(k+3)\,(k-2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+3)(k-2)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 2$
  $a_{k+1} = \frac{a_k(k+3)(k-2)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -3a_0$
- Apply recursion relation for $k = 1$
  $a_2 = -\frac{a_1}{2}$
- Express in terms of a_0
  $a_2 = \frac{3a_0}{2}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y(u) = a_0 \cdot \left(1 - 3u + \frac{3}{2}u^2\right)$
- Revert the change of variables $u = t + 1$
  $\left[y = a_0\left(\frac{3t^2}{2} - \frac{1}{2}\right)\right]$

1.616.3 Maple trace

```
Methods for second order ODEs:
```

1.616.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 44

```
dsolve((-t^2+1)*diff(diff(y(t),t),t)-2*t*diff(y(t),t)+6*y(t) = 0,
       y(t),singsol=all)
```

$$y = \frac{c_2(3t^2-1)\ln(t-1)}{2} + \frac{(-3t^2+1)\,c_2\ln(t+1)}{2} - 3c_1\,t^2 + 3c_2t + c_1$$

1.616.5 Mathematica DSolve solution

Solving time : 0.035 (sec)
Leaf size : 55

```
DSolve[{(1-t^2)*D[y[t],{t,2}]-2*t*D[y[t],t]+6*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{1}{2}c_1(3t^2-1) - \frac{1}{4}c_2\left((3t^2-1)\log(1-t) + (1-3t^2)\log(t+1) + 6t\right)$$

1.617 problem 633

Internal problem ID [8755]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 633
Date solved : `Monday, October 21, 2024 at 05:21:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2t+1)\,y''-4(t+1)\,y'+4y=0$$

1.617.1 Solved as second order ode using Kovacic algorithm

Time used: 0.267 (sec)

Writing the ode as

$$(2t+1)\,y''+(-4t-4)\,y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t+1 \\ B &= -4t-4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4t^2 + 2}{(2t + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4t^2 + 2$$
$$t = (2t + 1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{4t^2 + 2}{(2t + 1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1178: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2t+1)^2$. There is a pole at $t = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(t+\frac{1}{2}\right)^2} - \frac{1}{t+\frac{1}{2}}$$

For the pole at $t = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(t+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2t} + \frac{1}{2t^2} - \frac{1}{4t^3} + \frac{3}{32t^4} - \frac{3}{64t^5} + \frac{1}{32t^6} - \frac{1}{64t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4t^2 + 2}{4t^2 + 4t + 1} \\ &= Q + \frac{R}{4t^2 + 4t + 1} \\ &= (1) + \left(\frac{-4t + 1}{4t^2 + 4t + 1}\right) \\ &= 1 + \frac{-4t + 1}{4t^2 + 4t + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4t^2 + 2}{(2t + 1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - (\alpha_{c_1}^{-}) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(t+\frac{1}{2}\right)} + (1) \\
&= -\frac{1}{2\left(t+\frac{1}{2}\right)} + 1 \\
&= \frac{2t}{2t+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{2\left(t+\frac{1}{2}\right)} + 1\right)(0) + \left(\left(\frac{1}{2\left(t+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(t+\frac{1}{2}\right)} + 1\right)^2 - \left(\frac{4t^2+2}{(2t+1)^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(-\frac{1}{2\left(t+\frac{1}{2}\right)} + 1\right) dt} \\
&= \frac{e^t}{\sqrt{2t+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4t-4}{2t+1} \, dt} \\
&= z_1 e^{t + \frac{\ln(2t+1)}{2}} \\
&= z_1 \left(\sqrt{2t+1}\, e^t\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = e^{2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4t-4}{2t+1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{2t+\ln(2t+1)}}{(y_1)^2}\, dt \\ &= y_1 \left(-\frac{(t+1)\, e^{2t+\ln(2t+1)} e^{-4t}}{2t+1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{2t}\right) + c_2 \left(e^{2t} \left(-\frac{(t+1)\, e^{2t+\ln(2t+1)} e^{-4t}}{2t+1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.617.2 Maple step by step solution

Let's solve

$(2t+1)\left(\frac{d}{dt}y'\right) - 4(t+1)\,y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{4y}{2t+1} + \frac{4(t+1)y'}{2t+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' - \frac{4(t+1)y'}{2t+1} + \frac{4y}{2t+1} = 0$

☐ Check to see if $t_0 = -\frac{1}{2}$ is a regular singular point

- Define functions

  $\left[P_2(t) = -\frac{4(t+1)}{2t+1}, P_3(t) = \frac{4}{2t+1}\right]$

- $\left(t+\frac{1}{2}\right) \cdot P_2(t)$ is analytic at $t = -\frac{1}{2}$

  $\left(\left(t+\frac{1}{2}\right) \cdot P_2(t)\right)\bigg|_{t=-\frac{1}{2}} = -1$

- $\left(t+\frac{1}{2}\right)^2 \cdot P_3(t)$ is analytic at $t = -\frac{1}{2}$

  $\left(\left(t+\frac{1}{2}\right)^2 \cdot P_3(t)\right)\bigg|_{t=-\frac{1}{2}} = 0$

- $t = -\frac{1}{2}$is a regular singular point

  Check to see if $t_0 = -\frac{1}{2}$ is a regular singular point

  $t_0 = -\frac{1}{2}$

- Multiply by denominators

  $(2t+1)\left(\frac{d}{dt}y'\right) + (-4t-4)\,y' + 4y = 0$

- Change variables using $t = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $2u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-4u-2)\left(\frac{d}{du}y(u)\right) + 4y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- Shift index using $k- >k+1$

  $u \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$2a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(2a_{k+1}(k+1+r)\,(k+r-1)-4a_k(k+r-1)\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $2r(-2+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{0,2\}$
- Each term in the series must be 0, giving the recursion relation
  $2(k+r-1)\,(a_{k+1}(k+1+r)-2a_k)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{2a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{2a_k}{k+1}$
- Solution for $r=0$
  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{2a_k}{k+1}\right]$
- Revert the change of variables $u=t+\frac{1}{2}$
  $\left[y=\sum_{k=0}^{\infty}a_k\left(t+\frac{1}{2}\right)^k, a_{k+1}=\frac{2a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{2a_k}{k+3}$
- Solution for $r=2$
  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+2}, a_{k+1}=\frac{2a_k}{k+3}\right]$
- Revert the change of variables $u=t+\frac{1}{2}$
  $\left[y=\sum_{k=0}^{\infty}a_k\left(t+\frac{1}{2}\right)^{k+2}, a_{k+1}=\frac{2a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty}a_k\left(t+\frac{1}{2}\right)^k\right)+\left(\sum_{k=0}^{\infty}b_k\left(t+\frac{1}{2}\right)^{k+2}\right), a_{k+1}=\frac{2a_k}{k+1}, b_{k+1}=\frac{2b_k}{k+3}\right]$

1.617.3 Maple trace

```
Methods for second order ODEs:
```

1.617.4 Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 15

```
dsolve((2*t+1)*diff(diff(y(t),t),t)-4*(t+1)*diff(y(t),t)+4*y(t) = 0,
       y(t),singsol=all)
```

$$y = c_2 \mathrm{e}^{2t} + c_1 t + c_1$$

1.617.5 Mathematica DSolve solution

Solving time : 0.075 (sec)
Leaf size : 23

```
DSolve[{(2*t+1)*D[y[t],{t,2}]-4*(t+1)*D[y[t],t]+4*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_1 e^{2t+1} - c_2(t+1)$$

1.618 problem 634

Internal problem ID [8756]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 634
Date solved : `Monday, October 21, 2024 at 05:21:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right) y = 0$$

1.618.1 Solved as second order ode using Kovacic algorithm

Time used: 0.175 (sec)

Writing the ode as

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t \\ C &= t^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1180: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{t}{t^2}\,dt} \\ &= z_1 e^{-\frac{\ln(t)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{t}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(t)}{\sqrt{t}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{t^2}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-\ln(t)}}{(y_1)^2}\,dt \\ &= y_1(\tan(t)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos (t)}{\sqrt{t}}\right) + c_2 \left(\frac{\cos (t)}{\sqrt{t}} (\tan (t))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.618.2 Maple step by step solution

Let's solve

$t^2 \left(\frac{d}{dt} y'\right) + t y' + \left(t^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt} y'$

- Isolate 2nd derivative

  $\frac{d}{dt} y' = -\frac{(4t^2-1)y}{4t^2} - \frac{y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt} y' + \frac{y'}{t} + \frac{(4t^2-1)y}{4t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = \frac{1}{t}, P_3(t) = \frac{4t^2-1}{4t^2}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left.(t \cdot P_2(t))\right|_{t=0} = 1$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left.(t^2 \cdot P_3(t))\right|_{t=0} = -\frac{1}{4}$

  ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $4t^2 \left(\frac{d}{dt} y'\right) + 4ty' + (4t^2 - 1) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k t^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $t^m \cdot y$ to series expansion for $m = 0..2$

$$t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$$

- Shift index using $k- >k - m$

$$t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$$

- Convert $t \cdot y'$ to series expansion

$$t \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r}$$

- Convert $t^2 \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

$$t^2 \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\, t^r + a_1(3+2r)\,(1+2r)\, t^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-2}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0
  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$
  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$
  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k t^{k-\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k t^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

1.618.3 Maple trace

```
Methods for second order ODEs:
```

1.618.4 Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 17

```
dsolve(t^2*diff(diff(y(t),t),t)+t*diff(y(t),t)+(t^2-1/4)*y(t) = 0,
       y(t),singsol=all)
```

$$y=\frac{c_1 \sin(t)+c_2 \cos(t)}{\sqrt{t}}$$

1.618.5 Mathematica DSolve solution

Solving time : `0.051` (sec)
Leaf size : 39

```
DSolve[{t^2*D[y[t],{t,2}]+t*D[y[t],t]+(t^2-1/4)*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t)\to\frac{e^{-it}(2c_1-ic_2e^{2it})}{2\sqrt{t}}$$

1.619 problem 635

Internal problem ID [8757]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 635
Date solved : `Monday, October 21, 2024 at 05:21:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - \frac{2ty'}{t^2+1} + \frac{2y}{t^2+1} = 0$$

1.619.1 Solved as second order ode using Kovacic algorithm

Time used: 0.304 (sec)

Writing the ode as

$$y'' - \frac{2ty'}{t^2+1} + \frac{2y}{t^2+1} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{2t}{t^2+1} \\ C &= \frac{2}{t^2+1} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(t^2 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (t^2 + 1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{3}{(t^2 + 1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1182: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2+1)^2$. There is a pole at $t = i$ of order 2. There is a pole at $t = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(t-i)^2} + \frac{3}{4(t+i)^2} + \frac{3i}{4(t-i)} - \frac{3i}{4(t+i)}$$

For the pole at $t = i$ let b be the coefficient of $\frac{1}{(t-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $t = -i$ let b be the coefficient of $\frac{1}{(t+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(t^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(t-i)} + \frac{3}{2(t+i)} + (-)(0) \\ &= -\frac{1}{2(t-i)} + \frac{3}{2(t+i)} \\ &= \frac{t-2i}{t^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)(0) + \left(\left(\frac{1}{2(t-i)^2} - \frac{3}{2(t+i)^2}\right) + \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)^2 - \left(-\frac{3}{(t^2 -}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right) dt} \\
&= \frac{\left(t^2+1\right)^{3/2}}{\left(it+1\right)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-\frac{2t}{t^2+1}}{1} \, dt} \\
&= z_1 e^{\frac{\ln\left(t^2+1\right)}{2}} \\
&= z_1 \left(\sqrt{t^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(t^2+1\right)^2}{\left(it+1\right)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{2t}{t^2+1}}{1} dt}}{(y_1)^2} \, dt \\
&= y_1 \int \frac{e^{\ln(t^2+1)}}{(y_1)^2} \, dt \\
&= y_1 \left(-\frac{t}{(t+i)^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(t^2+1)^2}{(it+1)^2}\right) + c_2 \left(\frac{(t^2+1)^2}{(it+1)^2} \left(-\frac{t}{(t+i)^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.619.2 Maple step by step solution

1.619.3 Maple trace

```
Methods for second order ODEs:
```

1.619.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 16

```
dsolve(diff(diff(y(t),t),t)-2*t/(t^2+1)*diff(y(t),t)+2/(t^2+1)*y(t) = 0,
       y(t),singsol=all)
```

$$y = c_2 t^2 + c_1 t - c_2$$

1.619.5 Mathematica DSolve solution

Solving time : 0.059 (sec)
Leaf size : 21

```
DSolve[{D[y[t],{t,2}]-2*t/(1+t^2)*D[y[t],t]+2/(1+t^2)*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_2 t - c_1 (t - i)^2$$

1.620 problem 636

Internal problem ID [8758]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 636
Date solved : `Monday, October 21, 2024 at 05:21:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + \left(t^2 + 2t + 1\right) y' - (4 + 4t)\, y = 0$$

1.620.1 Solved as second order ode using Kovacic algorithm

Time used: 0.298 (sec)

Writing the ode as

$$y'' + (1+t)^2 y' + (-4 - 4t)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= (1+t)^2 \\ C &= -4 - 4t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 + 4t^3 + 6t^2 + 24t + 21}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^4 + 4t^3 + 6t^2 + 24t + 21$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1183: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 4 \\
&= -4
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{2} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t^2}{2} + t + \frac{1}{2} + \frac{5}{t} - \frac{5}{t^2} + \frac{5}{t^3} - \frac{30}{t^4} + \frac{105}{t^5} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i t^i \\
&= \frac{1}{2} t^2 + t + \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^1 = t$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2 + t + \frac{1}{4}$$

This shows that the coefficient of t in the above is 1. Now we need to find the coefficient of t in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of t in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^4 + 4t^3 + 6t^2 + 24t + 21}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2\right) + (0) \\
&= \frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{t}$ in the quotient is 6. Now b can be found.

$$\begin{aligned}
b &= (6) - (1) \\
&= 5
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}t^2 + t + \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{5}{\frac{1}{2}} - 2\right) = 4 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{5}{\frac{1}{2}} - 2\right) = -6
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21}{4} + 6t + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{1}{2}t^2+t+\frac{1}{2}$	4	-6

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right) \\ &= \frac{1}{2}t^2 + t + \frac{1}{2} \\ &= \frac{(1+t)^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 4$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(t) = t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12t^2 + 6ta_3 + 2a_2\right) + 2\left(\frac{1}{2}t^2 + t + \frac{1}{2}\right)\left(4t^3 + 3t^2a_3 + 2ta_2 + a_1\right) + \left((1+t) + \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right)^2 - \left(\frac{21}{4} + \right.\right.$$

$$(-a_3+4)\,t^4 + 2(2-a_2+a_3)\,t^3 + 3(4-a_1+a_3)\,t^2 + 2(-2a_0-a_1+a_2+3a_3)$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 5, a_1 = 8, a_2 = 6, a_3 = 4\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^4 + 4t^3 + 6t^2 + 8t + 5$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \left(t^4 + 4t^3 + 6t^2 + 8t + 5\right) \mathrm{e}^{\int \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right) dt} \\ &= \left(t^4 + 4t^3 + 6t^2 + 8t + 5\right) \mathrm{e}^{\frac{(1+t)^3}{6}} \\ &= (1+t)\left(t^3 + 3t^2 + 3t + 5\right) \mathrm{e}^{\frac{(1+t)^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{(1+t)^2}{1} \, dt} \\ &= z_1 e^{-\frac{(1+t)^3}{6}} \\ &= z_1 \left(\mathrm{e}^{-\frac{(1+t)^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (1+t)\left(t^3 + 3t^2 + 3t + 5\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{(1+t)^2}{1} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{-\frac{(1+t)^3}{3}}}{(y_1)^2} \, dt \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{(1+t)^3}{3}}}{(1+t)^2 \left(t^3 + 3t^2 + 3t + 5\right)^2} dt\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((1+t)\left(t^3+3t^2+3t+5\right)\right) \\
&\quad + c_2\left((1+t)\left(t^3+3t^2+3t+5\right)\left(\int \frac{\mathrm{e}^{-\frac{(1+t)^3}{3}}}{(1+t)^2\left(t^3+3t^2+3t+5\right)^2}dt\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.620.2 Maple step by step solution

Let's solve

$\frac{d}{dt}y' + (t^2+2t+1)\,y' - (4+4t)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -(t^2+2t+1)\,y' + (4+4t)\,y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' + (t^2+2t+1)\,y' + (-4-4t)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^k$

☐ Rewrite ODE with series expansions

- ○ Convert $t^m \cdot y$ to series expansion for $m = 0..1$

  $t^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k t^{k+m}$

- ○ Shift index using $k- > k-m$

  $t^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} t^k$

- ○ Convert $t^m \cdot y'$ to series expansion for $m = 0..2$

  $t^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, t^{k-1+m}$

- ○ Shift index using $k- > k+1-m$

  $t^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, t^k$

- Convert $\frac{d}{dt}y'$ to series expansion

  $\frac{d}{dt}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, t^{k-2}$

- Shift index using $k- >k+2$

  $\frac{d}{dt}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, t^{k}$

  Rewrite ODE with series expansions

  $$2a_2 + a_1 - 4a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k+1}(k+1) + 2a_k(k-2) + a_{k-1}(k-5)\right) t^k\right) = 0$$

- Each term must be 0

  $2a_2 + a_1 - 4a_0 = 0$

- Each term in the series must be 0, giving the recursion relation

  $k^2 a_{k+2} + (2a_k + a_{k-1} + a_{k+1} + 3a_{k+2})\, k - 4a_k - 5a_{k-1} + a_{k+1} + 2a_{k+2} = 0$

- Shift index using $k- >k+1$

  $(k+1)^2 a_{k+3} + (2a_{k+1} + a_k + a_{k+2} + 3a_{k+3})\,(k+1) - 4a_{k+1} - 5a_k + a_{k+2} + 2a_{k+3} = 0$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+3} = -\frac{a_k k + 2a_{k+1} k + k a_{k+2} - 4a_k - 2a_{k+1} + 2a_{k+2}}{k^2+5k+6}, 2a_2 + a_1 - 4a_0 = 0\right]$$

1.620.3 Maple trace

```
Methods for second order ODEs:
```

1.620.4 Maple dsolve solution

Solving time : `0.038` (sec)
Leaf size : 60

```
dsolve(diff(diff(y(t),t),t)+(t^2+2*t+1)*diff(y(t),t)-(4+4*t)*y(t) = 0,
       y(t),singsol=all)
```

$$y = (1+t)\left(t^3 + 3t^2 + 3t + 5\right)\left(\left(\int \frac{e^{-\frac{t\left(t^2+3t+3\right)}{3}}}{(1+t)^2 \left(t^3 + 3t^2 + 3t + 5\right)^2} dt\right) c_2 + c_1\right)$$

1.620.5 Mathematica DSolve solution

Solving time : 0.428 (sec)
Leaf size : 132

```
DSolve[{D[y[t],{t,2}]+(t^2+2*t+1)*D[y[t],t]-(4+4*t)*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{1}{36}e^{-\frac{1}{3}t(t^2+3t+3)}\left(-3c_2(t^3+3t^2+3t+4) + 3^{2/3}c_2e^{\frac{1}{3}(t+1)^3}\sqrt[3]{(t+1)^3}(t^3+3t^2+3t+5)\,\Gamma\left(\frac{2}{3},\frac{1}{3}(t+1)^3\right) + 36c_1e^{\frac{t^3}{3}+t^2+t}(t^4+4t^3+6t^2+8t+5)\right)$$

1.621 problem 638

Internal problem ID [8759]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 638
Date solved : `Monday, October 21, 2024 at 05:21:18 PM`
CAS classification : `[_Laguerre]`

Solve

$$2ty'' + (1-2t)\,y' - y = 0$$

1.621.1 Solved as second order ode using Kovacic algorithm

Time used: 0.260 (sec)

Writing the ode as

$$2ty'' + (1-2t)\,y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t \\ B &= 1-2t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4t^2 + 4t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4t^2 + 4t - 3$$
$$t = 16t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{4t^2 + 4t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1185: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{4t} - \frac{3}{16t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4t} - \frac{1}{4t^2} + \frac{1}{8t^3} - \frac{1}{8t^4} + \frac{1}{8t^5} - \frac{9}{64t^6} + \frac{21}{128t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{4t^2 + 4t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4t - 3}{16t^2}\right) \\ &= \frac{1}{4} + \frac{4t - 3}{16t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4t^2 + 4t - 3}{16t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - (\alpha_{c_1}^{-}) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{4t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{4t} \\
&= \frac{1}{2} + \frac{1}{4t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{1}{4t}\right)(0) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{2} + \frac{1}{4t}\right)^2 - \left(\frac{4t^2+4t-3}{16t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(\frac{1}{2} + \frac{1}{4t}\right) dt} \\
&= t^{1/4} e^{\frac{t}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-2t}{2t} \, dt} \\
&= z_1 e^{\frac{t}{2} - \frac{\ln(t)}{4}} \\
&= z_1 \left(\frac{e^{\frac{t}{2}}}{t^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1-2t}{2t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t-\frac{\ln(t)}{2}}}{(y_1)^2}\, dt \\ &= y_1\left(\sqrt{\pi}\ \mathrm{erf}\left(\sqrt{t}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^t\right) + c_2\left(e^t\left(\sqrt{\pi}\ \mathrm{erf}\left(\sqrt{t}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.621.2 Maple step by step solution

Let's solve

$2t\left(\frac{d}{dt}y'\right) + (1-2t)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = \frac{y}{2t} + \frac{(2t-1)y'}{2t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' - \frac{(2t-1)y'}{2t} - \frac{y}{2t} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

  - Define functions

$\left[P_2(t) = -\frac{2t-1}{2t}, P_3(t) = -\frac{1}{2t}\right]$

- $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\Big|_{t=0} = \frac{1}{2}$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 0$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $2t\left(\frac{d}{dt}y'\right) + (1-2t)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $t^m \cdot y'$ to series expansion for $m = 0..1$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,t^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,t^{k+r}$

- Convert $t \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

  $t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,t^{k+r-1}$

- Shift index using $k- >k+1$

  $t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+2r)\,t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+2r+1) - a_k(2k+2r+1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation
  $2(a_{k+1}(k+1+r)-a_k)\left(k+r+\frac{1}{2}\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y=\sum_{k=0}^{\infty} a_k t^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1}=\frac{a_k}{k+\frac{3}{2}}$
- Solution for $r=\frac{1}{2}$
  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1}=\frac{a_k}{k+\frac{3}{2}}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k t^k\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+\frac{3}{2}}\right]$

1.621.3 Maple trace

```
Methods for second order ODEs:
```

1.621.4 Maple dsolve solution

Solving time : `0.071` (sec)
Leaf size : 15

```
dsolve(2*t*diff(diff(y(t),t),t)+(1-2*t)*diff(y(t),t)-y(t) = 0,
       y(t),singsol=all)
```

$$y=e^{t}\left(\operatorname{erf}\left(\sqrt{t}\right) c_1+c_2\right)$$

1.621.5 Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 21

```
DSolve[{2*t*D[y[t],{t,2}]+(1-2*t)*D[y[t],t]-y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^t \left(c_1 - c_2 \Gamma\left(\frac{1}{2}, t\right)\right)$$

1.622 problem 639

Internal problem ID [8760]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 639
Date solved : `Monday, October 21, 2024 at 05:21:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2ty'' + (1+t)\,y' - 2y = 0$$

1.622.1 Solved as second order ode using Kovacic algorithm

Time used: 0.500 (sec)

Writing the ode as

$$2ty'' + (1+t)\,y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t \\ B &= 1+t \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 18t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 + 18t - 3$$
$$t = 16t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 18t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1187: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{9}{8t} - \frac{3}{16t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{9}{4t} - \frac{21}{2t^2} + \frac{189}{2t^3} - \frac{1071}{t^4} + \frac{13608}{t^5} - \frac{370629}{2t^6} + \frac{5288409}{2t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 + 18t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{18t - 3}{16t^2}\right) \\ &= \frac{1}{16} + \frac{18t - 3}{16t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 18. Dividing this by leading coefficient in t which is 16 gives $\frac{9}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{9}{8}\right) - (0) \\ &= \frac{9}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{9}{8}}{\frac{1}{4}} - 0\right) &&= \frac{9}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{9}{8}}{\frac{1}{4}} - 0\right) &&= -\frac{9}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 18t - 3}{16t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$\frac{9}{4}$	$-\frac{9}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{9}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= \frac{9}{4} - \left(\frac{1}{4}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{4t} + \left(\frac{1}{4}\right) \\
&= \frac{1}{4t} + \frac{1}{4} \\
&= \frac{1+t}{4t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{4t} + \frac{1}{4}\right)(2t + a_1) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{4t} + \frac{1}{4}\right)^2 - \left(\frac{t^2 + 18t - 3}{16t^2}\right) \right) = 0$$

$$\frac{(-a_1 + 6)\,t - 2a_0 + a_1}{2t} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = 6\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 + 6t + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^2 + 6t + 3\right) e^{\int \left(\frac{1}{4t} + \frac{1}{4}\right) dt} \\
&= \left(t^2 + 6t + 3\right) e^{\frac{t}{4} + \frac{\ln(t)}{4}} \\
&= \left(t^2 + 6t + 3\right) t^{1/4} e^{\frac{t}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+t}{2t}\,dt} \\
&= z_1 e^{-\frac{t}{4}-\frac{\ln(t)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{t}{4}}}{t^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^2 + 6t + 3
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+t}{2t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\frac{t}{2}-\frac{\ln(t)}{2}}}{(y_1)^2}\,dt \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{t}{2}-\frac{\ln(t)}{2}}}{(t^2+6t+3)^2}dt\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (t^2+6t+3) + c_2 \left(t^2+6t+3 \left(\int \frac{\mathrm{e}^{-\frac{t}{2}-\frac{\ln(t)}{2}}}{(t^2+6t+3)^2}dt\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.622.2 Maple step by step solution

Let's solve

$2t\left(\frac{d}{dt}y'\right)+(1+t)\,y'-2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y'=\frac{y}{t}-\frac{(1+t)y'}{2t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y'+\frac{(1+t)y'}{2t}-\frac{y}{t}=0$

- □ Check to see if $t_0=0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(t)=\frac{1+t}{2t}, P_3(t)=-\frac{1}{t}\right]$

  - ◦ $t\cdot P_2(t)$ is analytic at $t=0$

    $\left(t\cdot P_2(t)\right)\bigg|_{t=0}=\frac{1}{2}$

  - ◦ $t^2\cdot P_3(t)$ is analytic at $t=0$

    $\left(t^2\cdot P_3(t)\right)\bigg|_{t=0}=0$

  - ◦ $t=0$is a regular singular point

    Check to see if $t_0=0$ is a regular singular point

    $t_0=0$

- Multiply by denominators

  $2t\left(\frac{d}{dt}y'\right)+(1+t)\,y'-2y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k t^{k+r}$

- □ Rewrite ODE with series expansions
  - ◦ Convert $t^m\cdot y'$ to series expansion for $m=0..1$

    $t^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,t^{k+r-1+m}$

  - ◦ Shift index using $k->k+1-m$

    $t^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,t^{k+r}$

  - ◦ Convert $t\cdot\left(\frac{d}{dt}y'\right)$ to series expansion

$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, t^{k+r-1}$

- Shift index using $k->k+1$

  $t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+2r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(2k+1+2r) + a_k(k+r-2)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + a_k(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-2)}{(2k+1+2r)(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = -\frac{a_k(k-2)}{(2k+1)(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = 2a_0$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{a_1}{6}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{3}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 + 2t + \frac{1}{3}t^2\right)$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 + 2t + \frac{1}{3}t^2\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), b_{k+1} = -\frac{b_k\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

1.622.3 Maple trace

```
Methods for second order ODEs:
```

1.622.4 Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 56

```
dsolve(2*t*diff(diff(y(t),t),t)+(1+t)*diff(y(t),t)-2*y(t) = 0,
       y(t),singsol=all)
```

$$y = c_1\sqrt{\pi}\,(t^2+6t+3)\,\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{t}}{2}\right)+5\left(\sqrt{t}+\frac{t^{3/2}}{5}\right)c_1\sqrt{2}\,\mathrm{e}^{-\frac{t}{2}}+c_2(t^2+6t+3)$$

1.622.5 Mathematica DSolve solution

Solving time : 0.244 (sec)
Leaf size : 71

```
DSolve[{2*t*D[y[t],{t,2}]+(1+t)*D[y[t],t]-2*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t)\to \frac{1}{24}\left(\sqrt{2\pi}c_2(t^2+6t+3)\,\mathrm{erf}\left(\frac{\sqrt{t}}{\sqrt{2}}\right)+24c_1(t^2+6t+3)+2c_2e^{-t/2}\sqrt{t}(t+5)\right)$$

1.623 problem 640

Internal problem ID [8761]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 640
Date solved : `Monday, October 21, 2024 at 05:21:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2t^2y'' - ty' + (1+t)\,y = 0$$

1.623.1 Solved as second order ode using Kovacic algorithm

Time used: 0.254 (sec)

Writing the ode as

$$2t^2y'' - ty' + (1+t)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2 \\ B &= -t \\ C &= 1+t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3 - 8t}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 - 8t \\ t &= 16t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{-3 - 8t}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1189: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 1 \\
&= 1
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [2]
$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{3}{16t^2} - \frac{1}{2t}
$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\
&= \{1, 2, 3\}
\end{aligned}
$$

Since the order of r at ∞ is $1 < 2$ then

$$
E_\infty = \{1\}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$
e_1 = 1, \; e_\infty = 1
$$

Gives a non negative integer d (the degree of the polynomial $p(t)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{t-c} \\ &= \frac{1}{2}\left(\frac{1}{(t-(0))}\right) \\ &= \frac{1}{2t} \end{aligned}$$

Now we search for a monic polynomial $p(t)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(t)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2t} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2t} + \frac{1+8t}{16t^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+2\sqrt{2}\,\sqrt{-t}}{4t}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{2}\,\sqrt{-t}}{4t} dt} \\ &= t^{1/4} \mathrm{e}^{\sqrt{2}\,\sqrt{-t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t}{2t^2} \, dt} \\ &= z_1 e^{\frac{\ln(t)}{4}} \\ &= z_1 \left(t^{1/4}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t}{2t^2} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{\frac{\ln(t)}{2}}}{(y_1)^2} \, dt \\ &= y_1 \left(-\frac{\sqrt{2}\,\sqrt{-t}\left(1 - \mathrm{e}^{-2\sqrt{2}\,\sqrt{-t}}\right)}{2\sqrt{t}} \right) \end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}}\right) + c_2\left(\sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}}\left(-\frac{\sqrt{2}\,\sqrt{-t}\left(1-\mathrm{e}^{-2\sqrt{2}\,\sqrt{-t}}\right)}{2\sqrt{t}}\right)\right)$$

Will add steps showing solving for IC soon.

1.623.2 Maple step by step solution

Let's solve
$2t^2\left(\frac{d}{dt}y'\right) - ty' + (1+t)\,y = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d}{dt}y'$
- Isolate 2nd derivative
$\frac{d}{dt}y' = -\frac{(1+t)y}{2t^2} + \frac{y'}{2t}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
$\frac{d}{dt}y' - \frac{y'}{2t} + \frac{(1+t)y}{2t^2} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point
  ◦ Define functions
  $\left[P_2(t) = -\frac{1}{2t}, P_3(t) = \frac{1+t}{2t^2}\right]$
  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$
  $\left.(t \cdot P_2(t))\right|_{t=0} = -\frac{1}{2}$
  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$
  $\left.(t^2 \cdot P_3(t))\right|_{t=0} = \frac{1}{2}$
  ◦ $t = 0$is a regular singular point
  Check to see if $t_0 = 0$ is a regular singular point
  $t_0 = 0$

- Multiply by denominators
$2t^2\left(\frac{d}{dt}y'\right) - ty' + (1+t)\,y = 0$
- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k t^{k+r}$$

- □ Rewrite ODE with series expansions
  - ○ Convert $t^m \cdot y$ to series expansion for $m = 0..1$

    $$t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$$
  - ○ Shift index using $k- >k - m$

    $$t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$$
  - ○ Convert $t \cdot y'$ to series expansion

    $$t \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, t^{k+r}$$
  - ○ Convert $t^2 \cdot \left(\frac{d}{dt} y'\right)$ to series expansion

    $$t^2 \cdot \left(\frac{d}{dt} y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, t^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1 + 2r)\,(-1 + r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k (2k + 2r - 1)\,(k + r - 1) + a_{k-1}\right) t^{k+r}\right) = 0$$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + 2r)\,(-1 + r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k + r - 1)\left(k + r - \frac{1}{2}\right) a_k + a_{k-1} = 0$
- Shift index using $k- >k + 1$

  $2(k + r)\left(k + \frac{1}{2} + r\right) a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+r)(2k+1+2r)}$
- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{(k+1)(2k+3)}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{(k+1)(2k+3)}\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{\left(k+\frac{1}{2}\right)(2k+2)}$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{(k+1)(2k+3)}, b_{k+1} = -\frac{b_k}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

1.623.3 Maple trace

```
Methods for second order ODEs:
```

1.623.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 29

```
dsolve(2*t^2*diff(diff(y(t),t),t)-t*diff(y(t),t)+(1+t)*y(t) = 0,
       y(t),singsol=all)
```

$$y = \sqrt{t}\left(c_1 \sin\left(\sqrt{t}\sqrt{2}\right) + c_2 \cos\left(\sqrt{t}\sqrt{2}\right)\right)$$

1.623.5 Mathematica DSolve solution

Solving time : 0.102 (sec)
Leaf size : 62

```
DSolve[{2*t^2*D[y[t],{t,2}]-t*D[y[t],t]+(1+t)*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{1}{2} e^{-i\sqrt{2}\sqrt{t}}\sqrt{t}\left(2c_1 e^{2i\sqrt{2}\sqrt{t}} + i\sqrt{2}c_2\right)$$

1.624 problem 641

Internal problem ID [8762]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 641
Date solved : `Monday, October 21, 2024 at 05:21:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2t^2y'' + (t^2 - t)\, y' + y = 0$$

1.624.1 Solved as second order ode using Kovacic algorithm

Time used: 0.274 (sec)

Writing the ode as

$$2t^2y'' + (t^2 - t)\, y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2 \\ B &= t^2 - t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 2t - 3$$
$$t = 16t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1191: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} - \frac{3}{16t^2} - \frac{1}{8t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{1}{4t} - \frac{1}{2t^2} - \frac{1}{2t^3} - \frac{1}{t^4} - \frac{2}{t^5} - \frac{9}{2t^6} - \frac{21}{2t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 - 2t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-2t-3}{16t^2}\right) \\ &= \frac{1}{16} + \frac{-2t-3}{16t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 16 gives $-\frac{1}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{8}\right) - (0) \\ &= -\frac{1}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{8}}{\frac{1}{4}} - 0\right) &= -\frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{8}}{\frac{1}{4}} - 0\right) &= \frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t - 3}{16t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{4t} + (-)\left(\frac{1}{4}\right) \\
&= \frac{1}{4t} - \frac{1}{4} \\
&= -\frac{t-1}{4t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{4t} - \frac{1}{4}\right)(0) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{4t} - \frac{1}{4}\right)^2 - \left(\frac{t^2-2t-3}{16t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(\frac{1}{4t} - \frac{1}{4}\right) dt} \\
&= t^{1/4} e^{-\frac{t}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2-t}{2t^2} \, dt} \\
&= z_1 e^{-\frac{t}{4} + \frac{\ln(t)}{4}} \\
&= z_1 \left(t^{1/4} e^{-\frac{t}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-t}{2t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\frac{t}{2}+\frac{\ln(t)}{2}}}{(y_1)^2}\,dt \\
&= y_1 \left(-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\right) + c_2\left(\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\left(-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.624.2 Maple step by step solution

Let's solve

$2t^2\left(\frac{d}{dt}y'\right) + (t^2 - t)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{y}{2t^2} - \frac{(t-1)y'}{2t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' + \frac{(t-1)y'}{2t} + \frac{y}{2t^2} = 0$

- ☐ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(t) = \frac{t-1}{2t}, P_3(t) = \frac{1}{2t^2}\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$
    $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -\frac{1}{2}$
  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$
    $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = \frac{1}{2}$
  - $t = 0$is a regular singular point
    Check to see if $t_0 = 0$ is a regular singular point
    $t_0 = 0$
- Multiply by denominators
  $2t^2\left(\frac{d}{dt}y'\right) + t(t-1)\,y' + y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $t^m \cdot y'$ to series expansion for $m = 1..2$
    $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,t^{k+r-1+m}$
  - Shift index using $k- >k+1-m$
    $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,t^{k+r}$
  - Convert $t^2 \cdot \left(\frac{d}{dt}y'\right)$ to series expansion
    $t^2 \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,t^{k+r}$
    Rewrite ODE with series expansions
    $a_0(-1+2r)\,(-1+r)\,t^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) + a_{k-1}(k+r-1)\right)t^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+2r)\,(-1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{1, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

$2(k+r-1)\left(\left(k+r-\frac{1}{2}\right)a_k+\frac{a_{k-1}}{2}\right)=0$

- Shift index using $k->k+1$

  $2(k+r)\left(\left(k+\frac{1}{2}+r\right)a_{k+1}+\frac{a_k}{2}\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k}{2k+1+2r}$

- Recursion relation for $r=1$

  $a_{k+1}=-\frac{a_k}{2k+3}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty}a_k t^{k+1}, a_{k+1}=-\frac{a_k}{2k+3}\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=-\frac{a_k}{2k+2}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k t^{k+\frac{1}{2}}, a_{k+1}=-\frac{a_k}{2k+2}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k t^{k+1}\right)+\left(\sum_{k=0}^{\infty}b_k t^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{a_k}{2k+3}, b_{k+1}=-\frac{b_k}{2k+2}\right]$

1.624.3 Maple trace

```
Methods for second order ODEs:
```

1.624.4 Maple dsolve solution

Solving time : `0.009` (sec)
Leaf size : 47

```
dsolve(2*t^2*diff(diff(y(t),t),t)+(t^2-t)*diff(y(t),t)+y(t) = 0,
       y(t),singsol=all)
```

$$y=\frac{e^{-\frac{t}{2}}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{-t}}{2}\right)2^{3/4}\sqrt{\pi}\,c_1 t+4\sqrt{-t}\,\sqrt{t}\,c_2\right)}{4\sqrt{-t}}$$

1.624.5 Mathematica DSolve solution

Solving time : 0.022 (sec)
Leaf size : 46

```
DSolve[{2*t^2*D[y[t],{t,2}]+(t^2-t)*D[y[t],t]+y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^{-t/2}\left(c_2\sqrt{t} + \sqrt{2}c_1\sqrt{-t}\Gamma\left(\frac{1}{2}, -\frac{t}{2}\right)\right)$$

1.625 problem 642

Internal problem ID [8763]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 642
Date solved : `Monday, October 21, 2024 at 05:21:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2 y'' + \left(-t^2 + t\right) y' - y = 0$$

1.625.1 Solved as second order ode using Kovacic algorithm

Time used: 0.256 (sec)

Writing the ode as

$$t^2 y'' + \left(-t^2 + t\right) y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 + t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 2t + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1193: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2t} \\
&= \frac{t-1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2 - 2t + 3}{4t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\
&= \frac{e^{\frac{t}{2}}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t^2+t}{t^2}\,dt} \\
&= z_1 e^{\frac{t}{2}-\frac{\ln(t)}{2}} \\
&= z_1\left(\frac{e^{\frac{t}{2}}}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^t}{t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2+t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{t-\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(-(1+t)\,t\,e^{t-\ln(t)}e^{-2t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^t}{t}\right) + c_2\left(\frac{e^t}{t}\left(-(1+t)\,t\,e^{t-\ln(t)}e^{-2t}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.625.2 Maple step by step solution

Let's solve

$t^2\left(\frac{d}{dt}y'\right)+\left(-t^2+t\right)y'-y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dt}y'$

- Isolate 2nd derivative

$\frac{d}{dt}y'=\frac{y}{t^2}+\frac{(t-1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y'-\frac{(t-1)y'}{t}-\frac{y}{t^2}=0$

□ Check to see if $t_0=0$ is a regular singular point

- ◦ Define functions

$\left[P_2(t)=-\frac{t-1}{t}, P_3(t)=-\frac{1}{t^2}\right]$

- ◦ $t\cdot P_2(t)$ is analytic at $t=0$

$\left(t\cdot P_2(t)\right)\bigg|_{t=0}=1$

- ◦ $t^2\cdot P_3(t)$ is analytic at $t=0$

$\left(t^2\cdot P_3(t)\right)\bigg|_{t=0}=-1$

- ◦ $t=0$is a regular singular point

Check to see if $t_0=0$ is a regular singular point

$t_0=0$

- Multiply by denominators

$t^2\left(\frac{d}{dt}y'\right)-t(t-1)y'-y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k t^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $t^m\cdot y'$ to series expansion for $m=1..2$

$t^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,t^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$t^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,t^{k+r}$

- ◦ Convert $t^2\cdot\left(\frac{d}{dt}y'\right)$ to series expansion

$$t^2\cdot\left(\frac{d}{dt}y'\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,t^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)(-1+r)\,t^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+1)(k+r-1)-a_{k-1}(k+r-1)\right)t^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+r)(-1+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{-1,1\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)(a_k(k+r+1)-a_{k-1})=0$
- Shift index using $k->k+1$
  $(k+r)(a_{k+1}(k+2+r)-a_k)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+2+r}$
- Recursion relation for $r=-1$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=-1$
  $\left[y=\sum_{k=0}^{\infty}a_k t^{k-1},a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=1$
  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=1$
  $\left[y=\sum_{k=0}^{\infty}a_k t^{k+1},a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty}a_k t^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k t^{k+1}\right),a_{k+1}=\frac{a_k}{k+1},b_{k+1}=\frac{b_k}{k+3}\right]$

1.625.3 Maple trace

```
Methods for second order ODEs:
```

1.625.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(t^2*diff(diff(y(t),t),t)+(-t^2+t)*diff(y(t),t)-y(t) = 0,
       y(t),singsol=all)
```

$$y = \frac{c_2 \,\mathrm{e}^{t} + c_1 t + c_1}{t}$$

1.625.5 Mathematica DSolve solution

Solving time : 0.018 (sec)
Leaf size : 23

```
DSolve[{t^2*D[y[t],{t,2}]+(t-t^2)*D[y[t],t]-y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{c_2 e^t - c_1 (t+1)}{t}$$

1.626 problem 643

Internal problem ID [8764]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 643
Date solved : `Monday, October 21, 2024 at 05:21:23 PM`
CAS classification : `[_Lienard]`

Solve

$$ty'' - \left(t^2+2\right) y' + ty = 0$$

1.626.1 Solved as second order ode using Kovacic algorithm

Time used: 0.299 (sec)

Writing the ode as

$$ty'' + \left(-t^2-2\right) y' + ty = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t^2-2 \\ C &= t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 - 2t^2 + 8}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^4 - 2t^2 + 8$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^4 - 2t^2 + 8}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1195: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - \frac{1}{2} + \frac{2}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{1} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - \frac{1}{2t} + \frac{7}{4t^3} + \frac{7}{4t^5} - \frac{21}{16t^7} - \frac{119}{16t^9} - \frac{189}{32t^{11}} + \frac{791}{32t^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= \frac{t}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^4 - 2t^2 + 8}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{t^2}{4} - \frac{1}{2}\right) + \left(\frac{2}{t^2}\right) \\ &= \frac{t^2}{4} - \frac{1}{2} + \frac{2}{t^2}\end{aligned}$$

We see that the coefficient of the term t in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{t}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^4 - 2t^2 + 8}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{t}{2}$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{t} + \left(\frac{t}{2}\right) \\
&= -\frac{1}{t} + \frac{t}{2} \\
&= -\frac{1}{t} + \frac{t}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{t} + \frac{t}{2}\right)(0) + \left(\left(\frac{1}{t^2} + \frac{1}{2}\right) + \left(-\frac{1}{t} + \frac{t}{2}\right)^2 - \left(\frac{t^4 - 2t^2 + 8}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{t} + \frac{t}{2}\right) dt} \\
&= \frac{\mathrm{e}^{\frac{t^2}{4}}}{t}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-t^2-2}{t} \, dt} \\
&= z_1 e^{\frac{t^2}{4} + \ln(t)} \\
&= z_1 \left(t \, \mathrm{e}^{\frac{t^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{t^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-2}{t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{\frac{t^2}{2}+2\ln(t)}}{(y_1)^2}\, dt \\
&= y_1 \left(-t\,\mathrm{e}^{-\frac{t^2}{2}} + \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{t^2}{2}}\right) + c_2\left(\mathrm{e}^{\frac{t^2}{2}}\left(-t\,\mathrm{e}^{-\frac{t^2}{2}} + \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.626.2 Maple step by step solution

Let's solve

$t\left(\frac{d}{dt}y'\right) - (t^2+2)\,y' + ty = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -y + \frac{(t^2+2)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y' - \frac{(t^2+2)y'}{t} + y = 0$

- ☐ Check to see if $t_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(t) = -\frac{t^2+2}{t}, P_3(t) = 1\right]$
  - $t \cdot P_2(t)$ is analytic at $t = 0$

    $(t \cdot P_2(t))\bigg|_{t=0} = -2$
  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $(t^2 \cdot P_3(t))\bigg|_{t=0} = 0$
  - $t = 0$is a regular singular point

    Check to see if $t_0 = 0$ is a regular singular point

    $t_0 = 0$
- Multiply by denominators

  $t\left(\frac{d}{dt}y'\right) + (-t^2 - 2)\,y' + ty = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $t \cdot y$ to series expansion

    $t \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+1}$
  - Shift index using $k- >k - 1$

    $t \cdot y = \sum_{k=1}^{\infty} a_{k-1} t^{k+r}$
  - Convert $t^m \cdot y'$ to series expansion for $m = 0..2$

    $t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, t^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$

    $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, t^{k+r}$
  - Convert $t \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

    $t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, t^{k+r-1}$
  - Shift index using $k- >k + 1$

$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\,t^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-3+r)\,t^{-1+r} + a_1(1+r)\,(-2+r)\,t^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k-2+r) - a_{k-1}(k-2+r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 3\}$
- Each term must be 0
  $a_1(1+r)\,(-2+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(k-2+r)\,(a_{k+1}(k+r+1) - a_{k-1}) = 0$
- Shift index using $k- >k+1$
  $(k+r-1)\,(a_{k+2}(k+2+r) - a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_k}{k+2+r}$
- Recursion relation for $r = 0$
  $a_{k+2} = \frac{a_k}{k+2}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = \frac{a_k}{k+2}, -2a_1 = 0\right]$
- Recursion relation for $r = 3$
  $a_{k+2} = \frac{a_k}{k+5}$
- Solution for $r = 3$
  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+3}, a_{k+2} = \frac{a_k}{k+5}, 4a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+3}\right), a_{k+2} = \frac{a_k}{k+2}, -2a_1 = 0, b_{k+2} = \frac{b_k}{k+5}, 4b_1 = 0\right]$

1.626.3 Maple trace

```
Methods for second order ODEs:
```

1.626.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 34

```
dsolve(t*diff(diff(y(t),t),t)-(t^2+2)*diff(y(t),t)+t*y(t) = 0,
       y(t),singsol=all)
```

$$y = \left(c_2\pi \operatorname{erf}\left(\frac{\sqrt{2}\, t}{2}\right) + c_1\right) e^{\frac{t^2}{2}} - \sqrt{\pi}\, \sqrt{2}\, c_2 t$$

1.626.5 Mathematica DSolve solution

Solving time : 0.118 (sec)
Leaf size : 52

```
DSolve[{t*D[y[t],{t,2}]-(t^2+2)*D[y[t],t]+t*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \sqrt{\frac{\pi}{2}} c_2 e^{\frac{t^2}{2}} \operatorname{erf}\left(\frac{t}{\sqrt{2}}\right) + c_1 e^{\frac{t^2}{2}} - c_2 t$$

1.627 problem 644

Internal problem ID [8765]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 644
Date solved : `Monday, October 21, 2024 at 05:21:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' + t(t+1)\,y' - y = 0$$

1.627.1 Solved as second order ode using Kovacic algorithm

Time used: 0.248 (sec)

Writing the ode as

$$t^2y'' + \left(t^2+t\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t^2 + t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 + 2t + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1197: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4t^2} + \frac{1}{2t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2t} + \frac{1}{2t^2} - \frac{1}{2t^3} + \frac{1}{4t^4} + \frac{1}{4t^5} - \frac{3}{4t^6} + \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 + 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - (\alpha_{c_1}^{-}) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2t} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{2t} - \frac{1}{2} \\
&= -\frac{t+1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2t} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(-\frac{1}{2t} - \frac{1}{2}\right)^2 - \left(\frac{t^2+2t+3}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(-\frac{1}{2t} - \frac{1}{2}\right) dt} \\
&= \frac{e^{-\frac{t}{2}}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2+t}{t^2}\,dt} \\
&= z_1 e^{-\frac{t}{2}-\frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{e^{-\frac{t}{2}}}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{-t}}{t}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2+t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-t-\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left((-1+t)\,t\,e^{-t-\ln(t)}e^{2t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{-t}}{t}\right) + c_2\left(\frac{e^{-t}}{t}\left((-1+t)\,t\,e^{-t-\ln(t)}e^{2t}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.627.2 Maple step by step solution

Let's solve

$t^2\left(\frac{d}{dt}y'\right)+t(t+1)\,y'-y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dt}y'$

- Isolate 2nd derivative

$\frac{d}{dt}y'=\frac{y}{t^2}-\frac{(t+1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y'+\frac{(t+1)y'}{t}-\frac{y}{t^2}=0$

☐ Check to see if $t_0=0$ is a regular singular point

- ◦ Define functions

$\left[P_2(t)=\frac{t+1}{t},P_3(t)=-\frac{1}{t^2}\right]$

- ◦ $t\cdot P_2(t)$ is analytic at $t=0$

$$\left.(t\cdot P_2(t))\right|_{t=0}=1$$

- ◦ $t^2\cdot P_3(t)$ is analytic at $t=0$

$$\left.(t^2\cdot P_3(t))\right|_{t=0}=-1$$

- ◦ $t=0$is a regular singular point

Check to see if $t_0=0$ is a regular singular point

$t_0=0$

- Multiply by denominators

$t^2\left(\frac{d}{dt}y'\right)+t(t+1)\,y'-y=0$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty}a_k t^{k+r}$$

☐ Rewrite ODE with series expansions

- ◦ Convert $t^m\cdot y'$ to series expansion for $m=1..2$

$$t^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,t^{k+r-1+m}$$

- ◦ Shift index using $k->k+1-m$

$$t^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,t^{k+r}$$

- ◦ Convert $t^2\cdot\left(\frac{d}{dt}y'\right)$ to series expansion

$t^2 \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,t^{k+r}$

Rewrite ODE with series expansions

$a_0(1+r)\,(-1+r)\,t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) + a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r+1) + a_{k-1}) = 0$
- Shift index using $k- > k+1$

  $(k+r)\,(a_{k+1}(k+2+r) + a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k+2+r}$
- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k}{k+1}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$
- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{k+3}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = -\frac{a_k}{k+1}, b_{k+1} = -\frac{b_k}{k+3}\right]$

1.627.3 Maple trace

```
Methods for second order ODEs:
```

1.627.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 20

```
dsolve(t^2*diff(diff(y(t),t),t)+t*(t+1)*diff(y(t),t)-y(t) = 0,
       y(t),singsol=all)
```

$$y = \frac{c_2 \, \mathrm{e}^{-t} + c_1(-1+t)}{t}$$

1.627.5 Mathematica DSolve solution

Solving time : 0.018 (sec)
Leaf size : 26

```
DSolve[{t^2*D[y[t],{t,2}]+t*(t+1)*D[y[t],t]-y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{e^{-t}(c_1 e^t (t-1) + c_2)}{t}$$

1.628 problem 645

Internal problem ID [8766]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 645
Date solved : `Monday, October 21, 2024 at 05:21:25 PM`
CAS classification : `[_Laguerre]`

Solve

$$ty'' - (4+t)\,y' + 2y = 0$$

1.628.1 Solved as second order ode using Kovacic algorithm

Time used: 0.290 (sec)

Writing the ode as

$$ty'' + (-4-t)\,y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -4-t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 24}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 + 24 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 24}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1199: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{6}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{6}{t^2} - \frac{36}{t^4} + \frac{432}{t^6} - \frac{6480}{t^8} + \frac{108864}{t^{10}} - \frac{1959552}{t^{12}} + \frac{36951552}{t^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 + 24}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{6}{t^2}\right) \\ &= \frac{1}{4} + \frac{6}{t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &&= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 24}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{t} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{2}{t} - \frac{1}{2} \\
&= -\frac{4+t}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = t^2 + a_1 t + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(2) + 2\left(-\frac{2}{t} - \frac{1}{2}\right)(2t + a_1) + \left(\left(\frac{2}{t^2}\right) + \left(-\frac{2}{t} - \frac{1}{2}\right)^2 - \left(\frac{t^2+24}{4t^2}\right)\right) &= 0 \\
\frac{(a_1 - 6)\, t + 2a_0 - 4a_1}{t} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 12, a_1 = 6\}
$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$
p(t) = t^2 + 6t + 12
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^2 + 6t + 12\right) e^{\int \left(-\frac{2}{t} - \frac{1}{2}\right) dt} \\
&= \left(t^2 + 6t + 12\right) e^{-\frac{t}{2} - 2\ln(t)} \\
&= \frac{\left(t^2 + 6t + 12\right) e^{-\frac{t}{2}}}{t^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4-t}{t}\,dt} \\
&= z_1 e^{\frac{t}{2}+2\ln(t)} \\
&= z_1 \left(t^2 \mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^2 + 6t + 12
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4-t}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{t+4\ln(t)}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{(t^2-6t+12)\,\mathrm{e}^{t+4\ln(t)}}{(t^2+6t+12)\,t^4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(t^2+6t+12\right) + c_2\left(t^2+6t+12\left(\frac{(t^2-6t+12)\,\mathrm{e}^{t+4\ln(t)}}{(t^2+6t+12)\,t^4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.628.2 Maple step by step solution

Let's solve

$t\left(\frac{d}{dt}y'\right)-(4+t)\,y'+2y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dt}y'$

- Isolate 2nd derivative

$\frac{d}{dt}y'=-\frac{2y}{t}+\frac{(4+t)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y'-\frac{(4+t)y'}{t}+\frac{2y}{t}=0$

☐ Check to see if $t_0=0$ is a regular singular point

- ◦ Define functions

$\left[P_2(t)=-\frac{4+t}{t},P_3(t)=\frac{2}{t}\right]$

- ◦ $t\cdot P_2(t)$ is analytic at $t=0$

$$\left(t\cdot P_2(t)\right)\bigg|_{t=0}=-4$$

- ◦ $t^2\cdot P_3(t)$ is analytic at $t=0$

$$\left(t^2\cdot P_3(t)\right)\bigg|_{t=0}=0$$

- ◦ $t=0$is a regular singular point

Check to see if $t_0=0$ is a regular singular point

$t_0=0$

- Multiply by denominators

$t\left(\frac{d}{dt}y'\right)+(-4-t)\,y'+2y=0$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty}a_k t^{k+r}$$

☐ Rewrite ODE with series expansions

- ◦ Convert $t^m\cdot y'$ to series expansion for $m=0..1$

$$t^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,t^{k+r-1+m}$$

- ◦ Shift index using $k->k+1-m$

$$t^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,t^{k+r}$$

- ◦ Convert $t\cdot\left(\frac{d}{dt}y'\right)$ to series expansion

$$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, t^{k+r-1}$$

- Shift index using $k- > k+1$

$$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-5+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k-4+r) - a_k(k+r-2)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-5+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 5\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k-4+r) - a_k(k+r-2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+r-2)}{(k+1+r)(k-4+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 2$
  $a_{k+1} = \frac{a_k(k-2)}{(k+1)(k-4)}$
- Apply recursion relation for $k = 0$
  $a_1 = \frac{a_0}{2}$
- Apply recursion relation for $k = 1$
  $a_2 = \frac{a_1}{6}$
- Express in terms of a_0
  $a_2 = \frac{a_0}{12}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y = a_0 \cdot \left(1 + \frac{1}{2}t + \frac{1}{12}t^2\right)$
- Recursion relation for $r = 5$
  $a_{k+1} = \frac{a_k(k+3)}{(k+6)(k+1)}$
- Solution for $r = 5$
  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+5}, a_{k+1} = \frac{a_k(k+3)}{(k+6)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y = a_0 \cdot \left(1 + \frac{1}{2}t + \frac{1}{12}t^2\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+5}\right), b_{k+1} = \frac{b_k(k+3)}{(k+6)(k+1)}\right]$

1.628.3 Maple trace

```
Methods for second order ODEs:
```

1.628.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 27

```
dsolve(t*diff(diff(y(t),t),t)-(4+t)*diff(y(t),t)+2*y(t) = 0,
       y(t),singsol=all)
```

$$y = c_1\left(t^2+6t+12\right)+c_2\left(t^2-6t+12\right)\mathrm{e}^t$$

1.628.5 Mathematica DSolve solution

Solving time : 0.084 (sec)
Leaf size : 85

```
DSolve[{t*D[y[t],{t,2}]-(4+t)*D[y[t],t]+2*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{2e^{t/2}\sqrt{t}\left((c_2t^2-6ic_1t+12c_2)\cosh\left(\frac{t}{2}\right)+i(c_1(t^2+12)+6ic_2t)\sinh\left(\frac{t}{2}\right)\right)}{\sqrt{\pi}\sqrt{-it}}$$

1.629 problem 646

Internal problem ID [8767]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 646
Date solved : `Monday, October 21, 2024 at 05:21:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' + \left(t^2 - 3t\right) y' + 3y = 0$$

1.629.1 Solved as second order ode using Kovacic algorithm

Time used: 0.277 (sec)

Writing the ode as

$$t^2y'' + \left(t^2 - 3t\right) y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t^2 - 3t \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 6t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 6t + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 6t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1201: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2t} - \frac{3}{2t^2} - \frac{9}{2t^3} - \frac{63}{4t^4} - \frac{243}{4t^5} - \frac{999}{4t^6} - \frac{4293}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 6t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-6t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 - 6t + 3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{3}{2} - \left(\frac{3}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{t - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{2t} + (-) \left(\frac{1}{2} \right) \\
&= \frac{3}{2t} - \frac{1}{2} \\
&= -\frac{t - 3}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r \right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{3}{2t} - \frac{1}{2} \right)(0) + \left(\left(-\frac{3}{2t^2} \right) + \left(\frac{3}{2t} - \frac{1}{2} \right)^2 - \left(\frac{t^2 - 6t + 3}{4t^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{3}{2t} - \frac{1}{2} \right) dt} \\
&= t^{3/2} \mathrm{e}^{-\frac{t}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t^2 - 3t}{t^2} \, dt} \\
&= z_1 e^{-\frac{t}{2} + \frac{3 \ln(t)}{2}} \\
&= z_1 \left(t^{3/2} \mathrm{e}^{-\frac{t}{2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = t^3 \mathrm{e}^{-t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-3t}{t^2}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-t+3\ln(t)}}{(y_1)^2}\, dt \\
&= y_1 \left(-\frac{\mathrm{e}^t}{2t^2} - \frac{\mathrm{e}^t}{2t} - \frac{\mathrm{Ei}_1(-t)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^3 \mathrm{e}^{-t}\right) + c_2 \left(t^3 \mathrm{e}^{-t} \left(-\frac{\mathrm{e}^t}{2t^2} - \frac{\mathrm{e}^t}{2t} - \frac{\mathrm{Ei}_1(-t)}{2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.629.2 Maple step by step solution

Let's solve

$t^2 \left(\frac{d}{dt} y'\right) + (t^2 - 3t)\, y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt} y'$

- Isolate 2nd derivative

  $\frac{d}{dt} y' = -\frac{3y}{t^2} - \frac{(t-3)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt} y' + \frac{(t-3)y'}{t} + \frac{3y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

- Define functions
  $\left[P_2(t)=\frac{t-3}{t}, P_3(t)=\frac{3}{t^2}\right]$
- $t\cdot P_2(t)$ is analytic at $t=0$
  $$\left(t\cdot P_2(t)\right)\Big|_{t=0}=-3$$
- $t^2\cdot P_3(t)$ is analytic at $t=0$
  $$\left(t^2\cdot P_3(t)\right)\Big|_{t=0}=3$$
- $t=0$is a regular singular point
  Check to see if $t_0=0$ is a regular singular point
  $t_0=0$

- Multiply by denominators
  $t^2\left(\frac{d}{dt}y'\right)+t(t-3)\,y'+3y=0$
- Assume series solution for y
  $$y=\sum_{k=0}^{\infty}a_k t^{k+r}$$

☐ Rewrite ODE with series expansions
  - Convert $t^m\cdot y'$ to series expansion for $m=1..2$
    $$t^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,t^{k+r-1+m}$$
  - Shift index using $k->k+1-m$
    $$t^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,t^{k+r}$$
  - Convert $t^2\cdot\left(\frac{d}{dt}y'\right)$ to series expansion
    $$t^2\cdot\left(\frac{d}{dt}y'\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,t^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0(-1+r)\,(-3+r)\,t^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r-1)\,(k+r-3)+a_{k-1}(k+r-1)\right)t^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)\,(-3+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{1,3\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_k(k+r-3)+a_{k-1}\right)=0$
- Shift index using $k->k+1$

$$(k+r)\left(a_{k+1}(k-2+r)+a_k\right)=0$$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k}{k-2+r}$
- Recursion relation for $r=1$

  $a_{k+1}=-\frac{a_k}{k-1}$
- Series not valid for $r=1$, division by 0 in the recursion relation at $k=1$

  $a_{k+1}=-\frac{a_k}{k-1}$
- Recursion relation for $r=3$

  $a_{k+1}=-\frac{a_k}{k+1}$
- Solution for $r=3$

  $\left[y=\sum_{k=0}^{\infty}a_k t^{k+3}, a_{k+1}=-\frac{a_k}{k+1}\right]$

1.629.3 Maple trace

```
Methods for second order ODEs:
```

1.629.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 34

```
dsolve(t^2*diff(diff(y(t),t),t)+(t^2-3*t)*diff(y(t),t)+3*y(t) = 0,
       y(t),singsol=all)
```

$$y=t\left(\mathrm{Ei}_1\left(-t\right)\mathrm{e}^{-t}c_2\,t^2+c_1\,t^2\mathrm{e}^{-t}+c_2t+c_2\right)$$

1.629.5 Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 41

```
DSolve[{t^2*D[y[t],{t,2}]+(t^2-3*t)*D[y[t],t]+3*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t)\to\frac{1}{2}e^{-t}\left(c_1t^3\,\mathrm{ExpIntegralEi}(t)+2c_2t^3-c_1e^t(t+1)t\right)$$

1.630 problem 647

Internal problem ID [8768]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 647
Date solved : `Monday, October 21, 2024 at 05:21:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$ty'' + ty' + 2y = 0$$

1.630.1 Solved as second order ode using Kovacic algorithm

Time used: 0.265 (sec)

Writing the ode as

$$ty'' + ty' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t - 8}{4t} \tag{6}$$

Comparing the above to (5) shows that

$$s = t - 8$$
$$t = 4t$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t - 8}{4t}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1203: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t$. There is a pole at $t = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Looking at poles of order 1. For the pole at $t = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{2}{t} - \frac{4}{t^2} - \frac{16}{t^3} - \frac{80}{t^4} - \frac{448}{t^5} - \frac{2688}{t^6} - \frac{16896}{t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t-8}{4t} \\
&= Q + \frac{R}{4t} \\
&= \left(\frac{1}{4}\right) + \left(-\frac{2}{t}\right) \\
&= \frac{1}{4} - \frac{2}{t}
\end{aligned}
$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is -8. Dividing this by leading coefficient in t which is 4 gives -2. Now b can be found.

$$
\begin{aligned}
b &= (-2) - (0) \\
&= -2
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 0\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 0\right) &&= 2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t-8}{4t}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{t} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{t} - \frac{1}{2} \\ &= \frac{1}{t} - \frac{1}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{t} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{t^2}\right) + \left(\frac{1}{t} - \frac{1}{2}\right)^2 - \left(\frac{t-8}{4t}\right)\right) &= 0 \\ \frac{2+a_0}{t} &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = -2 + t$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega \, dt} \\ &= (-2+t)\, e^{\int \left(\frac{1}{t} - \frac{1}{2}\right) dt} \\ &= (-2+t)\, e^{-\frac{t}{2} + \ln(t)} \\ &= (-2+t)\, t\, e^{-\frac{t}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{t}{t}\,dt} \\ &= z_1 e^{-\frac{t}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-t}(-2+t)\,t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{t}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-t}}{(y_1)^2}\,dt \\ &= y_1\left(-\frac{\mathrm{e}^{t}(-t+1)}{2\,(2-t)\,t} - \frac{\mathrm{Ei}_1(-t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-t}(-2+t)\,t\right) + c_2\left(\mathrm{e}^{-t}(-2+t)\,t\left(-\frac{\mathrm{e}^{t}(-t+1)}{2\,(2-t)\,t} - \frac{\mathrm{Ei}_1(-t)}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.630.2 Maple step by step solution

Let's solve

$t\left(\frac{d}{dt}y'\right)+ty'+2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y'=-\frac{2y}{t}-y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y'+y'+\frac{2y}{t}=0$

- ☐ Check to see if $t_0=0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(t)=1, P_3(t)=\frac{2}{t}\right]$

  - ◦ $t\cdot P_2(t)$ is analytic at $t=0$

    $\left(t\cdot P_2(t)\right)\bigg|_{t=0}=0$

  - ◦ $t^2\cdot P_3(t)$ is analytic at $t=0$

    $\left(t^2\cdot P_3(t)\right)\bigg|_{t=0}=0$

  - ◦ $t=0$is a regular singular point

  Check to see if $t_0=0$ is a regular singular point

  $t_0=0$

- Multiply by denominators

  $t\left(\frac{d}{dt}y'\right)+ty'+2y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k t^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ◦ Convert $t\cdot y'$ to series expansion

    $t\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r}$

  - ◦ Convert $t\cdot\left(\frac{d}{dt}y'\right)$ to series expansion

    $t\cdot\left(\frac{d}{dt}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$

  - ◦ Shift index using $k->k+1$

$$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,t^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+r)\,t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r) + a_k(k+r+2)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 1\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k+r) + a_k(k+r+2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k(k+r+2)}{(k+1+r)(k+r)}$
- Recursion relation for $r = 0$
  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}\right]$
- Recursion relation for $r = 1$
  $a_{k+1} = -\frac{a_k(k+3)}{(k+2)(k+1)}$
- Solution for $r = 1$
  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k(k+3)}{(k+2)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}, b_{k+1} = -\frac{b_k(k+3)}{(k+2)(k+1)}\right]$

1.630.3 Maple trace

```
Methods for second order ODEs:
```

1.630.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 35

```
dsolve(t*diff(diff(y(t),t),t)+t*diff(y(t),t)+2*y(t) = 0,
       y(t),singsol=all)
```

$$y = tc_2 \,\mathrm{e}^{-t}(-2+t)\,\mathrm{Ei}_1(-t) + c_1\,\mathrm{e}^{-t}(-2+t)\,t + c_2(t-1)$$

1.630.5 Mathematica DSolve solution

Solving time : 0.097 (sec)
Leaf size : 51

```
DSolve[{t*D[y[t],{t,2}]+t*D[y[t],t]+2*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{1}{2}e^{-t}\left(c_2(t-2)t\,\mathrm{ExpIntegralEi}(t) + 2c_1t^2 - t\left(c_2e^t + 4c_1\right) + c_2e^t\right)$$

1.631 problem 648

Internal problem ID [8769]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 648
Date solved : `Monday, October 21, 2024 at 05:21:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$ty'' + \left(-t^2 + 1\right) y' + 4ty = 0$$

1.631.1 Solved as second order ode using Kovacic algorithm

Time used: 0.702 (sec)

Writing the ode as

$$ty'' + \left(-t^2 + 1\right) y' + 4ty = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t^2 + 1 \\ C &= 4t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{t^4 - 20t^2 - 1}{4t^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= t^4 - 20t^2 - 1 \\
t &= 4t^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{t^4 - 20t^2 - 1}{4t^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1205: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - 5 - \frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{1} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - \frac{5}{t} - \frac{101}{4t^3} - \frac{505}{2t^5} - \frac{50601}{16t^7} - \frac{355015}{8t^9} - \frac{21351501}{32t^{11}} - \frac{168167525}{16t^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= \frac{t}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^4 - 20t^2 - 1}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{t^2}{4} - 5\right) + \left(-\frac{1}{4t^2}\right) \\ &= \frac{t^2}{4} - 5 - \frac{1}{4t^2} \end{aligned}$$

We see that the coefficient of the term t in the quotient is -5. Now b can be found.

$$\begin{aligned} b &= (-5) - (0) \\ &= -5 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{t}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-5}{\frac{1}{2}} - 1\right) &= -\frac{11}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-5}{\frac{1}{2}} - 1\right) &= \frac{9}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^4 - 20t^2 - 1}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{t}{2}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{9}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{9}{2} - \left(\frac{1}{2}\right) \\
&= 4
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{t-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2t} + (-)\left(\frac{t}{2}\right) \\
&= \frac{1}{2t} - \frac{t}{2} \\
&= \frac{1}{2t} - \frac{t}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 4$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(12t^2 + 6ta_3 + 2a_2) + 2\left(\frac{1}{2t} - \frac{t}{2}\right)(4t^3 + 3a_3t^2 + 2a_2t + a_1) + \left(\left(-\frac{1}{2t^2} - \frac{1}{2}\right) + \left(\frac{1}{2t} - \frac{t}{2}\right)^2 - \left(\frac{t^4 - 20t}{4t^2}\right.\right.
$$

$$
\frac{t^4 a_3 + 2(8 + a_2)\, t^3 + 3(a_1 + 3a_3)\, t^2 + 4(a_0 + a_2)\, t}{t}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 8, a_1 = 0, a_2 = -8, a_3 = 0\}
$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$
p(t) = t^4 - 8t^2 + 8
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^4 - 8t^2 + 8\right) e^{\int \left(\frac{1}{2t} - \frac{t}{2}\right) dt} \\
&= \left(t^4 - 8t^2 + 8\right) e^{-\frac{t^2}{4} + \frac{\ln(t)}{2}} \\
&= \left(t^4 - 8t^2 + 8\right) \sqrt{t}\, e^{-\frac{t^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t^2+1}{t}\,dt} \\
&= z_1 e^{\frac{t^2}{4}-\frac{\ln(t)}{2}} \\
&= z_1\left(\frac{\mathrm{e}^{\frac{t^2}{4}}}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^4 - 8t^2 + 8
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2+1}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{\frac{t^2}{2}-\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(\int \frac{\mathrm{e}^{\frac{t^2}{2}-\ln(t)}}{(t^4-8t^2+8)^2}dt\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(t^4-8t^2+8\right) + c_2\left(t^4-8t^2+8\left(\int \frac{\mathrm{e}^{\frac{t^2}{2}-\ln(t)}}{(t^4-8t^2+8)^2}dt\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.631.2 Maple step by step solution

Let's solve

$t\left(\frac{d}{dt}y'\right)+\left(-t^{2}+1\right)y'+4ty=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y'=-4y+\frac{(t^{2}-1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y'-\frac{(t^{2}-1)y'}{t}+4y=0$

□ Check to see if $t_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t)=-\frac{t^{2}-1}{t}, P_3(t)=4\right]$

  ◦ $t\cdot P_2(t)$ is analytic at $t=0$

    $\left(t\cdot P_2(t)\right)\bigg|_{t=0}=1$

  ◦ $t^{2}\cdot P_3(t)$ is analytic at $t=0$

    $\left(t^{2}\cdot P_3(t)\right)\bigg|_{t=0}=0$

  ◦ $t=0$is a regular singular point

    Check to see if $t_0=0$ is a regular singular point

    $t_0=0$

- Multiply by denominators

  $t\left(\frac{d}{dt}y'\right)+\left(-t^{2}+1\right)y'+4ty=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t\cdot y$ to series expansion

    $t\cdot y=\sum_{k=0}^{\infty}a_k t^{k+r+1}$

  ◦ Shift index using $k->k-1$

    $t\cdot y=\sum_{k=1}^{\infty}a_{k-1}t^{k+r}$

  ◦ Convert $t^{m}\cdot y'$ to series expansion for $m=0..2$

$t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$

- Shift index using $k->k+1-m$

$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r-1}$

- Shift index using $k->k+1$

$t \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, t^{k+r}$

Rewrite ODE with series expansions

$a_0 r^2 t^{-1+r} + a_1 (1+r)^2 t^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 - a_{k-1}(k-5+r)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term must be 0

  $a_1 (1+r)^2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 - a_{k-1}(k-5) = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2)^2 - a_k (k-4) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k (k-4)}{(k+2)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 4$

  $a_{k+2} = \frac{a_k (k-4)}{(k+2)^2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = \frac{a_k (k-4)}{(k+2)^2}, a_1 = 0\right]$

1.631.3 Maple trace

```
Methods for second order ODEs:
```

1.631.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 21

```
dsolve(t*diff(diff(y(t),t),t)+(-t^2+1)*diff(y(t),t)+4*t*y(t) = 0,
       y(t),singsol=all)
```

$$y = \frac{(t^4 - 8t^2 + 8)(c_1 + 2c_2)}{8}$$

1.631.5 Mathematica DSolve solution

Solving time : 0.174 (sec)
Leaf size : 61

```
DSolve[{t*D[y[t],{t,2}]+(1-t^2)*D[y[t],t]+4*t*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{1}{128}c_2\left((t^4 - 8t^2 + 8)\,\text{ExpIntegralEi}\left(\frac{t^2}{2}\right) - 2e^{\frac{t^2}{2}}(t^2 - 6)\right) + c_1(t^4 - 8t^2 + 8)$$

1.632 problem 649

Internal problem ID [8770]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 649
Date solved : `Monday, October 21, 2024 at 05:21:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$t^2y'' - t(1+t)\,y' + y = 0$$

1.632.1 Solved as second order ode using Kovacic algorithm

Time used: 0.263 (sec)

Writing the ode as

$$t^2y'' + \left(-t^2 - t\right)y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 - t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 2t - 1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 + 2t - 1$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 2t - 1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1207: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4t^2} + \frac{1}{2t}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2t} - \frac{1}{2t^2} + \frac{1}{2t^3} - \frac{3}{4t^4} + \frac{5}{4t^5} - \frac{9}{4t^6} + \frac{17}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 + 2t - 1}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2t-1}{4t^2}\right) \\ &= \frac{1}{4} + \frac{2t-1}{4t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 + 2t - 1}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{2t} \\
&= \frac{1+t}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2} + \frac{1}{2t}\right)^2 - \left(\frac{t^2 + 2t - 1}{4t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{1}{2t}\right) dt} \\
&= \sqrt{t}\, \mathrm{e}^{\frac{t}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-t^2 - t}{t^2} \, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(t)}{2}} \\
&= z_1 \left(\sqrt{t}\, \mathrm{e}^{\frac{t}{2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = t\,\mathrm{e}^{t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-t}{t^2}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\,dt \\ &= y_1(-\operatorname{Ei}_1(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(t\,\mathrm{e}^{t}\right) + c_2\left(t\,\mathrm{e}^{t}(-\operatorname{Ei}_1(t))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.632.2 Maple step by step solution

Let's solve

$t^2\left(\frac{d}{dt}y'\right) - t(1+t)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y' = -\frac{y}{t^2} + \frac{(1+t)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y' - \frac{(1+t)y'}{t} + \frac{y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = -\frac{1+t}{t}, P_3(t) = \frac{1}{t^2}\right]$

- $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -1$
- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 1$
- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

• Multiply by denominators

$t^2\left(\frac{d}{dt}y'\right) - t(1+t)\,y' + y = 0$

• Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,t^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,t^{k+r}$
- Convert $t^2 \cdot \left(\frac{d}{dt}y'\right)$ to series expansion

  $t^2 \cdot \left(\frac{d}{dt}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2\,t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)^2 - a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$

• $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)^2 = 0$

• Values of r that satisfy the indicial equation

$r = 1$

• Each term in the series must be 0, giving the recursion relation

$(k+r-1)\left(a_k(k+r-1) - a_{k-1}\right) = 0$

• Shift index using $k->k+1$

$(k+r)\left(a_{k+1}(k+r) - a_k\right) = 0$

• Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k}{k+r}$

- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$

1.632.3 Maple trace

```
Methods for second order ODEs:
```

1.632.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 15

```
dsolve(t^2*diff(diff(y(t),t),t)-t*(1+t)*diff(y(t),t)+y(t) = 0,
       y(t),singsol=all)
```

$$y = t\,\mathrm{e}^{t}(c_1 + \mathrm{Ei}_1(t)\, c_2)$$

1.632.5 Mathematica DSolve solution

Solving time : `0.018` (sec)
Leaf size : 20

```
DSolve[{t^2*D[y[t],{t,2}]-t*(1+t)*D[y[t],t]+y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^t t(c_1\,\text{ExpIntegralEi}(-t) + c_2)$$

1.633 problem 650

Internal problem ID [8771]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 650
Date solved : `Monday, October 21, 2024 at 05:21:30 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + \left(4x^2 + 6\right) y = 0$$

1.633.1 Solved as second order ode using Kovacic algorithm

Time used: 0.187 (sec)

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 6\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1209: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\ &= z_1 e^{-x^2} \\ &= z_1 \left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2} \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x^2} \cos(2x)\right) + c_2 \left(\mathrm{e}^{-x^2} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.633.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 4xy' + (4x^2 + 6)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k + 2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + 6a_0 + (6a_3 + 10a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) + 2a_k(2k+3) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 6a_0 = 0, 6a_3 + 10a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = -3a_0, a_3 = -\frac{5a_1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 6a_k + 4a_{k-2} = 0$
- Shift index using $k- >k+2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 6a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2k a_{k+2} + 2a_k + 7a_{k+2})}{k^2+7k+12}, a_2 = -3a_0, a_3 = -\frac{5a_1}{3}\right]$

1.633.3 Maple trace

```
Methods for second order ODEs:
```

1.633.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 24

```
dsolve(diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(4*x^2+6)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-x^2}(c_1 \cos(2x) + c_2 \sin(2x))$$

1.633.5 Mathematica DSolve solution

Solving time : `0.056` (sec)
Leaf size : 37

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2+6)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4} e^{-x(x+2i)}\left(4c_1 - ic_2 e^{4ix}\right)$$

1.634 problem 651

Internal problem ID [8772]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 651
Date solved : `Monday, October 21, 2024 at 05:21:31 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-z^{2}+1\right) y''-3 z y'+y=0$$

1.634.1 Solved as second order ode using Kovacic algorithm

Time used: 0.399 (sec)

Writing the ode as

$$\left(-z^{2}+1\right) y''-3 z y'+y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -z^{2}+1 \\ B &= -3 z \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z)=y e^{\int \frac{B}{2 A} d z}$$

Then (2) becomes

$$z''(z)=r z(z) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7z^2 - 10}{4(z^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 7z^2 - 10$$
$$t = 4(z^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{7z^2 - 10}{4(z^2 - 1)^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1211: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(z^2-1)^2$. There is a pole at $z = 1$ of order 2. There is a pole at $z = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{17}{16(z+1)} - \frac{3}{16(z+1)^2} + \frac{17}{16(z-1)} - \frac{3}{16(z-1)^2}$$

For the pole at $z = 1$ let b be the coefficient of $\frac{1}{(z-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\ &= \{1, 2, 3\}\end{aligned}$$

For the pole at $z = -1$ let b be the coefficient of $\frac{1}{(z+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\ &= \{1, 2, 3\}\end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{z^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7z^2 - 10}{4(z^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{7}{4}$. Hence

$$\begin{aligned}E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\ &= \{2\}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1,2,3\}$
-1	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(z)$), which is generated using

$$\begin{aligned}d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\ &= \frac{1}{2}(2-(1+(1)))\\ &= 0\end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{z-c}\\ &= \frac{1}{2}\left(\frac{1}{(z-(1))}+\frac{1}{(z-(-1))}\right)\\ &= \frac{1}{2z-2}+\frac{1}{2z+2}\end{aligned}$$

Now we search for a monic polynomial $p(z)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(z)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2z-2} + \frac{1}{2z+2}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2z-2} + \frac{1}{2z+2}\right) w + \frac{-7z^2+8}{4\left(z^2-1\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{z + 2\sqrt{2z^2-2}}{2\left(z-1\right)\left(z+1\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(z) &= e^{\int \omega \, dz} \\ &= \mathrm{e}^{\int \frac{z+2\sqrt{2z^2-2}}{2(z-1)(z+1)} dz} \\ &= \left(z^2-1\right)^{1/4} 2^{\frac{\sqrt{2}}{2}} \left(\sqrt{z^2-1} + z\right)^{\sqrt{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3z}{-z^2+1} \, dz} \\ &= z_1 e^{-\frac{3\ln(z-1)}{4} - \frac{3\ln(z+1)}{4}} \\ &= z_1 \left(\frac{1}{\left(z-1\right)^{3/4}\left(z+1\right)^{3/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(z^2-1)^{1/4}\, 2^{\frac{\sqrt{2}}{2}} \left(\sqrt{z^2-1}+z\right)^{\sqrt{2}}}{(z-1)^{3/4}\,(z+1)^{3/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{y_1^2}\, dz$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3z}{-z^2+1}\, dz}}{(y_1)^2}\, dz \\
&= y_1 \int \frac{e^{-\frac{3\ln(z-1)}{2}-\frac{3\ln(z+1)}{2}}}{(y_1)^2}\, dz \\
&= y_1 \left(-\frac{2^{-\sqrt{2}}\sqrt{2}\left(\sqrt{z^2-1}+z\right)^{-2\sqrt{2}}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(z^2-1)^{1/4}\, 2^{\frac{\sqrt{2}}{2}} \left(\sqrt{z^2-1}+z\right)^{\sqrt{2}}}{(z-1)^{3/4}\,(z+1)^{3/4}}\right) + c_2 \left(\frac{(z^2-1)^{1/4}\, 2^{\frac{\sqrt{2}}{2}} \left(\sqrt{z^2-1}+z\right)^{\sqrt{2}}}{(z-1)^{3/4}\,(z+1)^{3/4}} \left(-\frac{2^{-\sqrt{2}}\sqrt{2}\left(\sqrt{z^2-1}+z\right.}{4}\right.\right.
\end{aligned}$$

Will add steps showing solving for IC soon.

1.634.2 Maple step by step solution

Let's solve

$(-z^2+1)\left(\frac{d}{dz}y'\right) - 3zy' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dz}y'$

- Isolate 2nd derivative

  $\frac{d}{dz}y' = \frac{y}{z^2-1} - \frac{3zy'}{z^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dz}y' + \frac{3zy'}{z^2-1} - \frac{y}{z^2-1} = 0$

☐ Check to see if z_0 is a regular singular point
  - Define functions
    $\left[P_2(z) = \frac{3z}{z^2-1}, P_3(z) = -\frac{1}{z^2-1}\right]$
  - $(z+1) \cdot P_2(z)$ is analytic at $z = -1$
    $$\left((z+1) \cdot P_2(z)\right)\bigg|_{z=-1} = \frac{3}{2}$$
  - $(z+1)^2 \cdot P_3(z)$ is analytic at $z = -1$
    $$\left((z+1)^2 \cdot P_3(z)\right)\bigg|_{z=-1} = 0$$
  - $z = -1$is a regular singular point
    Check to see if z_0 is a regular singular point
    $z_0 = -1$

- Multiply by denominators
  $(z^2-1)\left(\frac{d}{dz}y'\right) + 3zy' - y = 0$
- Change variables using $z = u - 1$ so that the regular singular point is at $u = 0$
  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (3u-3)\left(\frac{d}{du}y(u)\right) - y(u) = 0$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$
  - Shift index using $k- >k+1-m$
    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$
  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$
    $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$
  - Shift index using $k- >k+2-m$
    $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

$$-a_0 r(1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+3+2r) + a_k(k^2+2kr+r^2+2k+2r-1)\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-r(1+2r)=0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $-2\left(k+\frac{3}{2}+r\right)(k+1+r)\, a_{k+1} + a_k\left(k^2+(2r+2)\, k+r^2+2r-1\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k\left(k^2+2kr+r^2+2k+2r-1\right)}{(2k+3+2r)(k+1+r)}$
- Recursion relation for $r=0$
  $a_{k+1} = \frac{a_k\left(k^2+2k-1\right)}{(2k+3)(k+1)}$
- Solution for $r=0$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k\left(k^2+2k-1\right)}{(2k+3)(k+1)}\right]$
- Revert the change of variables $u=z+1$
  $\left[y = \sum_{k=0}^{\infty} a_k(z+1)^k, a_{k+1} = \frac{a_k\left(k^2+2k-1\right)}{(2k+3)(k+1)}\right]$
- Recursion relation for $r=-\frac{1}{2}$
  $a_{k+1} = \frac{a_k\left(k^2+k-\frac{7}{4}\right)}{(2k+2)\left(k+\frac{1}{2}\right)}$
- Solution for $r=-\frac{1}{2}$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k^2+k-\frac{7}{4}\right)}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$
- Revert the change of variables $u=z+1$
  $\left[y = \sum_{k=0}^{\infty} a_k(z+1)^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k^2+k-\frac{7}{4}\right)}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k(z+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(z+1)^{k-\frac{1}{2}}\right), a_{k+1} = \frac{a_k\left(k^2+2k-1\right)}{(2k+3)(k+1)}, b_{k+1} = \frac{b_k\left(k^2+k-\frac{7}{4}\right)}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$

1.634.3 Maple trace

```
Methods for second order ODEs:
```

1.634.4 Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 45

```
dsolve((-z^2+1)*diff(diff(y(z),z),z)-3*z*diff(y(z),z)+y(z) = 0,
       y(z),singsol=all)
```

$$y = \frac{c_2 \left(\sqrt{z^2-1}+z\right)^{-\sqrt{2}} + c_1 \left(\sqrt{z^2-1}+z\right)^{\sqrt{2}}}{\sqrt{z^2-1}}$$

1.634.5 Mathematica DSolve solution

Solving time : 0.093 (sec)
Leaf size : 90

```
DSolve[{(1-z^2)*D[y[z],{z,2}]-3*z*D[y[z],z]+y[z]==0,{}},
      y[z],z,IncludeSingularSolutions->True]
```

$$y(z) \to \frac{\sqrt{2} c_1 \cos\left(2\sqrt{2}\arcsin\left(\frac{\sqrt{1-z}}{\sqrt{2}}\right)\right) + \sqrt{\pi} c_2 \sqrt[4]{1-z^2} Q^{\frac{1}{2}}_{-\frac{1}{2}+\sqrt{2}}(z)}{\sqrt{\pi}\sqrt[4]{-\left(z^2-1\right)^2}}$$

1.635 problem 652

Internal problem ID [8773]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 652
Date solved : `Monday, October 21, 2024 at 05:21:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4zy'' + 2(1-z)\,y' - y = 0$$

1.635.1 Solved as second order ode using Kovacic algorithm

Time used: 0.271 (sec)

Writing the ode as

$$4zy'' + (-2z+2)\,y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4z \\ B &= -2z+2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = y e^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z) = r z(z) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{z^2 + 2z - 3}{16z^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= z^2 + 2z - 3 \\
t &= 16z^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(z) = \left(\frac{z^2 + 2z - 3}{16z^2}\right) z(z) \tag{7}
$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$
y = z(z)\, e^{-\int \frac{B}{2A}\, dz}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1213: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{1}{8z} - \frac{3}{16z^2}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{0} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{1}{4z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{z^4} + \frac{2}{z^5} - \frac{9}{2z^6} + \frac{21}{2z^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{z^2 + 2z - 3}{16z^2} \\ &= Q + \frac{R}{16z^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{2z - 3}{16z^2}\right) \\ &= \frac{1}{16} + \frac{2z - 3}{16z^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is 2. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{8}\right) - (0) \\ &= \frac{1}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{8}}{\frac{1}{4}} - 0\right) &= \frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{8}}{\frac{1}{4}} - 0\right) &= -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{z^2 + 2z - 3}{16z^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{z - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{4z} + \left(\frac{1}{4} \right) \\
&= \frac{1}{4} + \frac{1}{4z} \\
&= \frac{z+1}{4z}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 0$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(z) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{4} + \frac{1}{4z}\right)(0) + \left(\left(-\frac{1}{4z^2}\right) + \left(\frac{1}{4} + \frac{1}{4z}\right)^2 - \left(\frac{z^2 + 2z - 3}{16z^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(z) &= p e^{\int \omega \, dz} \\
&= e^{\int \left(\frac{1}{4} + \frac{1}{4z}\right) dz} \\
&= z^{1/4} e^{\frac{z}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dz} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2z+2}{4z} \, dz} \\
&= z_1 e^{\frac{z}{4} - \frac{\ln(z)}{4}} \\
&= z_1 \left(\frac{e^{\frac{z}{4}}}{z^{1/4}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{z}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{y_1^2}\, dz$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2z+2}{4z}\, dz}}{(y_1)^2}\, dz \\ &= y_1 \int \frac{e^{\frac{z}{2}-\frac{\ln(z)}{2}}}{(y_1)^2}\, dz \\ &= y_1 \left(\sqrt{\pi}\,\sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{\frac{z}{2}}\right) + c_2 \left(\mathrm{e}^{\frac{z}{2}} \left(\sqrt{\pi}\,\sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.635.2 Maple step by step solution

Let's solve

$4z\left(\frac{d}{dz}y'\right) + 2(1-z)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dz}y'$

- Isolate 2nd derivative

  $\frac{d}{dz}y' = \frac{y}{4z} + \frac{(z-1)y'}{2z}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dz}y' - \frac{(z-1)y'}{2z} - \frac{y}{4z} = 0$

- ☐ Check to see if $z_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(z) = -\frac{z-1}{2z}, P_3(z) = -\frac{1}{4z}\right]$
  - $z \cdot P_2(z)$ is analytic at $z = 0$

    $\left(z \cdot P_2(z)\right)\Big|_{z=0} = \frac{1}{2}$
  - $z^2 \cdot P_3(z)$ is analytic at $z = 0$

    $\left(z^2 \cdot P_3(z)\right)\Big|_{z=0} = 0$
  - $z = 0$is a regular singular point

    Check to see if $z_0 = 0$ is a regular singular point

    $z_0 = 0$
- Multiply by denominators

  $4z\left(\frac{d}{dz}y'\right) + (-2z+2)\, y' - y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k z^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $z^m \cdot y'$ to series expansion for $m = 0..1$

    $z^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, z^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $z^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, z^{k+r}$
  - Convert $z \cdot \left(\frac{d}{dz}y'\right)$ to series expansion

    $z \cdot \left(\frac{d}{dz}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r-1}$
  - Shift index using $k->k+1$

    $z \cdot \left(\frac{d}{dz}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, z^{k+r}$

    Rewrite ODE with series expansions

    $2a_0 r(-1+2r)\, z^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)\,(2k+2r+1) - a_k(2k+2r+1)\right) z^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-1+2r) = 0$
- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r+\frac{1}{2}\right)\left(a_{k+1}(k+1+r)-\frac{a_k}{2}\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{2(k+1+r)}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{2(k+1)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k z^k, a_{k+1}=\frac{a_k}{2(k+1)}\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=\frac{a_k}{2\left(k+\frac{3}{2}\right)}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k z^{k+\frac{1}{2}}, a_{k+1}=\frac{a_k}{2\left(k+\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k z^k\right)+\left(\sum_{k=0}^{\infty} b_k z^{k+\frac{1}{2}}\right), a_{k+1}=\frac{a_k}{2(k+1)}, b_{k+1}=\frac{b_k}{2\left(k+\frac{3}{2}\right)}\right]$

1.635.3 Maple trace

```
Methods for second order ODEs:
```

1.635.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 22

```
dsolve(4*z*diff(diff(y(z),z),z)+2*(1-z)*diff(y(z),z)-y(z) = 0,
      y(z),singsol=all)
```

$$y = e^{\frac{z}{2}}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right)c_1+c_2\right)$$

1.635.5 Mathematica DSolve solution

Solving time : 0.058 (sec)
Leaf size : 34

```
DSolve[{4*z*D[y[z],{z,2}]+2*(1-z)*D[y[z],z]-y[z]==0,{}},
       y[z],z,IncludeSingularSolutions->True]
```

$$y(z) \to e^{z/2}\left(c_1 - \sqrt{2}c_2\Gamma\left(\frac{1}{2}, \frac{z}{2}\right)\right)$$

1.636 problem 653

Internal problem ID [8774]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 653
Date solved : `Monday, October 21, 2024 at 05:21:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$f'' + 2(z-1)\, f' + 4f = 0$$

1.636.1 Solved as second order ode using Kovacic algorithm

Time used: 0.269 (sec)

Writing the ode as

$$f'' + (2z-2)\, f' + 4f = 0 \tag{1}$$
$$Af'' + Bf' + Cf = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2z-2 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = f e^{\int \frac{B}{2A}\, dz}$$

Then (2) becomes

$$z''(z) = r z(z) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{z^2 - 2z - 2}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= z^2 - 2z - 2 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(z) = \left(z^2 - 2z - 2\right) z(z) \tag{7}
$$

Equation (7) is now solved. After finding $z(z)$ then f is found using the inverse transformation

$$
f = z(z)\, e^{-\int \frac{B}{2A}\, dz}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1215: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\
&= \sum_{i=0}^{1} a_i z^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $z^v = z^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx z - 1 - \frac{3}{2z} - \frac{3}{2z^2} - \frac{21}{8z^3} - \frac{39}{8z^4} - \frac{159}{16z^5} - \frac{339}{16z^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i z^i \\
&= z - 1 \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = z^2 - 2z + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{z^2 - 2z - 2}{1} \\
&= Q + \frac{R}{1} \\
&= \left(z^2 - 2z - 2\right) + (0) \\
&= z^2 - 2z - 2
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{z}$ in the quotient is -2. Now b can be found.

$$\begin{aligned}
b &= (-2) - (1) \\
&= -3
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= z - 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-3}{1} - 1\right) &= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-3}{1} - 1\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = z^2 - 2z - 2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$z - 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\,(z-1) \\ &= 1 - z \\ &= 1 - z \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(z) = z + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2(1-z)\,(1) + \left((-1) + (1-z)^2 - \left(z^2 - 2z - 2\right)\right) &= 0 \\ 2 + 2a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$p(z) = z - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(z) &= p e^{\int \omega \, dz} \\ &= (z-1)\,\mathrm{e}^{\int (1-z)dz} \\ &= (z-1)\,\mathrm{e}^{z-\frac{1}{2}z^2} \\ &= (z-1)\,\mathrm{e}^{-\frac{z(-2+z)}{2}} \end{aligned}$$

The first solution to the original ode in f is found from

$$\begin{aligned} f_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2z-2}{1}\, dz} \\ &= z_1 e^{z-\frac{1}{2}z^2} \\ &= z_1 \left(\mathrm{e}^{-\frac{z(-2+z)}{2}}\right) \end{aligned}$$

Which simplifies to

$$f_1 = \mathrm{e}^{-z(-2+z)}(z-1)$$

The second solution f_2 to the original ode is found using reduction of order

$$f_2 = f_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{f_1^2}\, dz$$

Substituting gives

$$\begin{aligned} f_2 &= f_1 \int \frac{e^{\int -\frac{2z-2}{1}\, dz}}{(f_1)^2}\, dz \\ &= f_1 \int \frac{e^{-z^2+2z}}{(f_1)^2}\, dz \\ &= f_1 \left(-\frac{\mathrm{e}^{(z-1)^2-1}}{z-1} - i\sqrt{\pi}\,\mathrm{e}^{-1}\operatorname{erf}\left(i(z-1)\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} f &= c_1 f_1 + c_2 f_2 \\ &= c_1\left(\mathrm{e}^{-z(-2+z)}(z-1)\right) + c_2\left(\mathrm{e}^{-z(-2+z)}(z-1)\left(-\frac{\mathrm{e}^{(z-1)^2-1}}{z-1} - i\sqrt{\pi}\,\mathrm{e}^{-1}\operatorname{erf}\left(i(z-1)\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.636.2 Maple step by step solution

Let's solve

$\frac{d}{dz}f' + 2(z-1)\, f' + 4f = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dz}f'$

- Isolate 2nd derivative

$\frac{d}{dz}f' = -2(z-1)\, f' - 4f$

- Group terms with f on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dz}f' + (2z-2)\, f' + 4f = 0$

- Assume series solution for f

$f = \sum_{k=0}^{\infty} a_k z^k$

☐ Rewrite DE with series expansions

- ○ Convert $z^m \cdot f'$ to series expansion for $m = 0..1$

$z^m \cdot f' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, z^{k-1+m}$

- ○ Shift index using $k- >k+1-m$

$z^m \cdot f' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, z^k$

- ○ Convert $\frac{d}{dz}f'$ to series expansion

$\frac{d}{dz}f' = \sum_{k=2}^{\infty} a_k k(k-1)\, z^{k-2}$

- ○ Shift index using $k- >k+2$

$\frac{d}{dz}f' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, z^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_{k+1}(k+1) + 2a_k(k+2)\right) z^k = 0$

- Each term in the series must be 0, giving the recursion relation

$k^2 a_{k+2} + (2a_k - 2a_{k+1} + 3a_{k+2})\, k + 4a_k - 2a_{k+1} + 2a_{k+2} = 0$

- Recursion relation that defines the series solution to the ODE

$\left[f = \sum_{k=0}^{\infty} a_k z^k, a_{k+2} = -\frac{2(k a_k - k a_{k+1} + 2a_k - a_{k+1})}{k^2+3k+2}\right]$

1.636.3 Maple trace

```
Methods for second order ODEs:
```

1.636.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 42

```
dsolve(diff(diff(f(z),z),z)+2*(z-1)*diff(f(z),z)+4*f(z) = 0,
       f(z),singsol=all)
```

$$f = \sqrt{\pi}\, \operatorname{erf}\left(i(z-1)\right) c_2 (z-1)\, \mathrm{e}^{-(z-1)^2} + c_1\, \mathrm{e}^{-z(-2+z)}(z-1) - ic_2$$

1.636.5 Mathematica DSolve solution

Solving time : 0.206 (sec)
Leaf size : 72

```
DSolve[{D[ f[z],{z,2}]+2*(z-a)*D[ f[z],z]+4*f[z]==0,{}},
       f[z],z,IncludeSingularSolutions->True]
```

$$f(z) \to e^{z(2a-z)}\left(-\sqrt{\pi}c_2\sqrt{(a-z)^2}\text{erfi}\left(\sqrt{(a-z)^2}\right) + c_2 e^{(a-z)^2} - 2ac_1 + 2c_1 z\right)$$

1.637 problem 654

Internal problem ID [8775]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 654
Date solved : `Monday, October 21, 2024 at 05:21:34 PM`
CAS classification : `[_Lienard]`

Solve

$$zy'' - 2y' + zy = 0$$

1.637.1 Solved as second order ode using Kovacic algorithm

Time used: 0.311 (sec)

Writing the ode as

$$zy'' - 2y' + zy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= z \\ B &= -2 \\ C &= z \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\, dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$r = \frac{s}{t}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-z^2 + 2}{z^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -z^2 + 2$$
$$t = z^2$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{-z^2 + 2}{z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1217: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{z^2}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{0} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{z^2} - \frac{i}{2z^4} - \frac{i}{2z^6} - \frac{5i}{8z^8} - \frac{7i}{8z^{10}} - \frac{21i}{16z^{12}} - \frac{33i}{16z^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-z^2 + 2}{z^2} \\ &= Q + \frac{R}{z^2} \\ &= (-1) + \left(\frac{2}{z^2}\right) \\ &= -1 + \frac{2}{z^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= i\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-z^2 + 2}{z^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right)\\
&= 0 - (-1)\\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{z-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{z} + (-)\left(i\right) \\
&= -\frac{1}{z} - i \\
&= -\frac{1}{z} - i
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(z) = z + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{z} - i\right)(1) + \left(\left(\frac{1}{z^2}\right) + \left(-\frac{1}{z} - i\right)^2 - \left(\frac{-z^2+2}{z^2}\right)\right) &= 0 \\
\frac{2ia_0 - 2}{z} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -i\}
$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$
p(z) = z - i
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(z) &= p e^{\int \omega \, dz} \\
&= (z - i)\, e^{\int \left(-\frac{1}{z} - i\right) dz} \\
&= (z - i)\, e^{-\ln(z) - iz} \\
&= \frac{(z - i)\, e^{-iz}}{z}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dz} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{z}\,dz} \\
&= z_1 e^{\ln(z)} \\
&= z_1(z)
\end{aligned}$$

Which simplifies to

$$y_1 = (z - i)\,\mathrm{e}^{-iz}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dz}}{y_1^2}\,dz$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{z}\,dz}}{(y_1)^2}\,dz \\
&= y_1 \int \frac{e^{2\ln(z)}}{(y_1)^2}\,dz \\
&= y_1 \left(\frac{(iz-1)\,\mathrm{e}^{2iz}}{-2z+2i}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((z-i)\,\mathrm{e}^{-iz}\right) + c_2\left((z-i)\,\mathrm{e}^{-iz}\left(\frac{(iz-1)\,\mathrm{e}^{2iz}}{-2z+2i}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.637.2 Maple step by step solution

Let's solve

$z\left(\frac{d}{dz}y'\right) - 2y' + zy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dz}y'$

- Isolate 2nd derivative

  $\frac{d}{dz}y' = -y + \frac{2y'}{z}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dz}y' - \frac{2y'}{z} + y = 0$

□ Check to see if $z_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(z) = -\frac{2}{z}, P_3(z) = 1\right]$

  ◦ $z \cdot P_2(z)$ is analytic at $z = 0$

  $(z \cdot P_2(z))\Big|_{z=0} = -2$

  ◦ $z^2 \cdot P_3(z)$ is analytic at $z = 0$

  $(z^2 \cdot P_3(z))\Big|_{z=0} = 0$

  ◦ $z = 0$is a regular singular point

  Check to see if $z_0 = 0$ is a regular singular point

  $z_0 = 0$

- Multiply by denominators

  $z\left(\frac{d}{dz}y'\right) - 2y' + zy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k z^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $z \cdot y$ to series expansion

  $z \cdot y = \sum_{k=0}^{\infty} a_k z^{k+r+1}$

  ◦ Shift index using $k- >k-1$

  $z \cdot y = \sum_{k=1}^{\infty} a_{k-1} z^{k+r}$

  ◦ Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k(k+r)\, z^{k+r-1}$$

- Shift index using $k- >k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, z^{k+r}$$

- Convert $z \cdot \left(\frac{d}{dz}y'\right)$ to series expansion

$$z \cdot \left(\frac{d}{dz}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r-1}$$

- Shift index using $k- >k+1$

$$z \cdot \left(\frac{d}{dz}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, z^{k+r}$$

Rewrite ODE with series expansions

$$a_0r(-3+r)\, z^{-1+r} + a_1(1+r)\,(-2+r)\, z^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k-2+r) + a_{k-1}\right) z^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 3\}$
- Each term must be 0
  $a_1(1+r)\,(-2+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+r+1)\,(k-2+r) + a_{k-1} = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2+r)\,(k+r-1) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+r-1)}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{a_k}{(k+2)(k-1)}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k z^k, a_{k+2} = -\frac{a_k}{(k+2)(k-1)}, -2a_1 = 0\right]$
- Recursion relation for $r = 3$
  $a_{k+2} = -\frac{a_k}{(k+5)(k+2)}$
- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k z^{k+3}, a_{k+2} = -\frac{a_k}{(k+5)(k+2)}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k z^k\right) + \left(\sum_{k=0}^{\infty} b_k z^{k+3}\right), a_{k+2} = -\frac{a_k}{(k+2)(k-1)}, -2a_1 = 0, b_{k+2} = -\frac{b_k}{(k+5)(k+2)}, 4b_1 = 0\right]$$

1.637.3 Maple trace

```
Methods for second order ODEs:
```

1.637.4 Maple dsolve solution

Solving time : `0.011` (sec)
Leaf size : 23

```
dsolve(z*diff(diff(y(z),z),z)-2*diff(y(z),z)+z*y(z) = 0,
       y(z),singsol=all)
```

$$y = (c_1 z + c_2) \cos(z) + \sin(z) (c_2 z - c_1)$$

1.637.5 Mathematica DSolve solution

Solving time : `0.078` (sec)
Leaf size : 39

```
DSolve[{z*D[y[z],{z,2}]-2*D[y[z],z]+z*y[z]==0,{}},
       y[z],z,IncludeSingularSolutions->True]
```

$$y(z) \to -\sqrt{\frac{2}{\pi}}((c_1 z + c_2) \cos(z) + (c_2 z - c_1) \sin(z))$$

1.638 problem 655

Internal problem ID [8776]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 655
Date solved : `Monday, October 21, 2024 at 05:21:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$zy'' + (2z-3)\,y' + \frac{4y}{z} = 0$$

1.638.1 Solved as second order ode using Kovacic algorithm

Time used: 0.303 (sec)

Writing the ode as

$$zy'' + (2z-3)\,y' + \frac{4y}{z} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= z \\ B &= 2z-3 \\ C &= \frac{4}{z} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4z^2 - 12z - 1}{4z^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4z^2 - 12z - 1 \\
t &= 4z^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(z) = \left(\frac{4z^2 - 12z - 1}{4z^2}\right) z(z) \tag{7}
$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$
y = z(z)\, e^{-\int \frac{B}{2A}\, dz}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1219: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{3}{z} - \frac{1}{4z^2}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{0} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{3}{2z} - \frac{5}{4z^2} - \frac{15}{8z^3} - \frac{115}{32z^4} - \frac{495}{64z^5} - \frac{2285}{128z^6} - \frac{11055}{256z^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4z^2 - 12z - 1}{4z^2} \\ &= Q + \frac{R}{4z^2} \\ &= (1) + \left(\frac{-12z - 1}{4z^2}\right) \\ &= 1 + \frac{-12z - 1}{4z^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is -12. Dividing this by leading coefficient in t which is 4 gives -3. Now b can be

found.

$$\begin{aligned} b &= (-3) - (0) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{1} - 0\right) &&= -\frac{3}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{1} - 0\right) &&= \frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4z^2 - 12z - 1}{4z^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{z - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2z} + (-)\,(1) \\
&= \frac{1}{2z} - 1 \\
&= \frac{1}{2z} - 1
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(z) = z + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2z} - 1\right)(1) + \left(\left(-\frac{1}{2z^2}\right) + \left(\frac{1}{2z} - 1\right)^2 - \left(\frac{4z^2 - 12z - 1}{4z^2}\right)\right) = 0
$$

$$
\frac{1 + 2a_0}{z} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{1}{2}\right\}
$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$
p(z) = z - \frac{1}{2}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(z) &= p e^{\int \omega \, dz} \\
&= \left(z - \frac{1}{2}\right) e^{\int \left(\frac{1}{2z} - 1\right) dz} \\
&= \left(z - \frac{1}{2}\right) e^{-z + \frac{\ln(z)}{2}} \\
&= \frac{(-1 + 2z)\sqrt{z}\, e^{-z}}{2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dz} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2z-3}{z}\,dz} \\
&= z_1 e^{-z+\frac{3\ln(z)}{2}} \\
&= z_1\left(z^{3/2}\mathrm{e}^{-z}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{z^2\mathrm{e}^{-2z}(-1+2z)}{2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dz}}{y_1^2}\,dz
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2z-3}{z}\,dz}}{(y_1)^2}\,dz \\
&= y_1 \int \frac{e^{-2z+3\ln(z)}}{(y_1)^2}\,dz \\
&= y_1\left(-\frac{4\,\mathrm{e}^{2z}}{-1+2z} - 4\,\mathrm{Ei}_1\left(-2z\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{z^2\mathrm{e}^{-2z}(-1+2z)}{2}\right) + c_2\left(\frac{z^2\mathrm{e}^{-2z}(-1+2z)}{2}\left(-\frac{4\,\mathrm{e}^{2z}}{-1+2z} - 4\,\mathrm{Ei}_1\left(-2z\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.638.2 Maple step by step solution

Let's solve

$z\left(\frac{d}{dz}y'\right)+(2z-3)\,y'+\frac{4y}{z}=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dz}y'$

- Isolate 2nd derivative

  $\frac{d}{dz}y'=-\frac{4y}{z^2}-\frac{(2z-3)y'}{z}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dz}y'+\frac{(2z-3)y'}{z}+\frac{4y}{z^2}=0$

□ Check to see if $z_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(z)=\frac{2z-3}{z}, P_3(z)=\frac{4}{z^2}\right]$

  ◦ $z\cdot P_2(z)$ is analytic at $z=0$

  $(z\cdot P_2(z))\bigg|_{z=0}=-3$

  ◦ $z^2\cdot P_3(z)$ is analytic at $z=0$

  $(z^2\cdot P_3(z))\bigg|_{z=0}=4$

  ◦ $z=0$is a regular singular point

  Check to see if $z_0=0$ is a regular singular point

  $z_0=0$

- Multiply by denominators

  $z^2\left(\frac{d}{dz}y'\right)+z(2z-3)\,y'+4y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k z^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $z^m\cdot y'$ to series expansion for $m=1..2$

  $z^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,z^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $z^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,z^{k+r}$

  ◦ Convert $z^2\cdot\left(\frac{d}{dz}y'\right)$ to series expansion

$$z^2 \cdot \left(\frac{d}{dz}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1) z^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)^2 z^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 + 2a_{k-1}(k+r-1)\right) z^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 + 2a_{k-1}(k+r-1) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+r-1)^2 + 2a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k(k+r)}{(k+r-1)^2}$
- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{2a_k(k+2)}{(k+1)^2}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k z^{k+2}, a_{k+1} = -\frac{2a_k(k+2)}{(k+1)^2}\right]$

1.638.3 Maple trace

```
Methods for second order ODEs:
```

1.638.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 36

```
dsolve(z*diff(diff(y(z),z),z)+(2*z-3)*diff(y(z),z)+4/z*y(z) = 0,
       y(z),singsol=all)
```

$$y = 2z^2\left(c_2\,e^{-2z}\left(z - \frac{1}{2}\right)\mathrm{Ei}_1(-2z) + \left(z - \frac{1}{2}\right)c_1\,e^{-2z} + \frac{c_2}{2}\right)$$

1.638.5 Mathematica DSolve solution

Solving time : 0.125 (sec)
Leaf size : 47

```
DSolve[{z*D[y[z],{z,2}]+(2*z-3)*D[y[z],z]+4/z*y[z]==0,{}},
       y[z],z,IncludeSingularSolutions->True]
```

$$y(z) \to -\frac{1}{2}e^{-2z}z^2\left(4c_2(1-2z)\,\text{ExpIntegralEi}(2z) - 2c_1z + 4c_2e^{2z} + c_1\right)$$

1.639 problem 656

Internal problem ID [8777]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 656
Date solved : `Monday, October 21, 2024 at 05:21:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0$$

1.639.1 Solved as second order ode using Kovacic algorithm

Time used: 0.184 (sec)

Writing the ode as

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1-2x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1221: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1-2x}{x} \, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{1-2x}{x}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx$$
$$= y_1(\ln(x))$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\ln(x)))$$

Will add steps showing solving for IC soon.

1.639.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + (1-2x)\,y' + (x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x-1)y}{x} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x\left(\frac{d}{dx}y'\right) + (1-2x)\, y' + (x-1)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- ○ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- ○ Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- ○ Shift index using $k- >k+1$

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(a_1 (1+r)^2 - a_0 (1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1} (k+1+r)^2 - a_k (2k+2r+1) + a_{k-1}\right) x^{k}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

$a_1 (1+r)^2 - a_0 (1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 + (-2k-1)\,a_k + a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$a_{k+2}(k+2)^2 + (-2k-3)\,a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$$

1.639.3 Maple trace

```
Methods for second order ODEs:
```

1.639.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 13

```
dsolve(x*diff(diff(y(x),x),x)+(1-2*x)*diff(y(x),x)+(x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^x (c_1 + \ln(x)\, c_2)$$

1.639.5 Mathematica DSolve solution

Solving time : `0.038` (sec)
Leaf size : 17

```
DSolve[{x*D[y[x],{x,2}]+(1-2*x)*D[y[x],x]+(x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^x (c_2 \log(x) + c_1)$$

1.640 problem 657

Internal problem ID [8778]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 657
Date solved : `Monday, October 21, 2024 at 05:21:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + \left(x^2+2\right) y = 0$$

1.640.1 Solved as second order ode using Kovacic algorithm

Time used: 0.157 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1223: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x}{x^2} \, dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}$$

Which simplifies to

$$y_1 = x \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2} \, dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.640.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)(-2+r) x^r + a_1 r(-1+r) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2) + a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}(k+1+r)(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 2$

$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

1.640.3 Maple trace

```
Methods for second order ODEs:
```

1.640.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+(x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x(c_1 \sin(x) + c_2 \cos(x))$$

1.640.5 Mathematica DSolve solution

Solving time : `0.045` (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

1.641 problem 658

Internal problem ID [8779]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 658
Date solved : `Monday, October 21, 2024 at 05:21:38 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-2xy'+2y=0$$

1.641.1 Solved as second order ode using Kovacic algorithm

Time used: 0.277 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-2xy'+2y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 3}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2 - 3 \\ t &= (x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 3}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1225: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4(x+1)} - \frac{1}{4(x+1)^2} + \frac{5}{4(x-1)} - \frac{1}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^2-3}{(x^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=2$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2-(1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(x+1)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)^2 - \left(\frac{2x^2-3}{(x^2-1)^2}\right)\right.
$$

$$
-\frac{2a_0}{x^2-1} =
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= (x) \sqrt{(x-1)(x+1)} \\
&= x\sqrt{x^2-1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \frac{1}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}\right) + c_2 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}} \left(\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \frac{1}{x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.641.2 Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d}{dx}y'\right)-2xy'+2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{2y}{x^2-1}-\frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{2xy'}{x^2-1}-\frac{2y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{2}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=1$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d}{dx}y'\right)+2xy'-2y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+2)\,(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(k+2)\,(k-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u+1)$
- Revert the change of variables $u = x+1$

  $[y = -a_0x]$

1.641.3 Maple trace

```
Methods for second order ODEs:
```

1.641.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 25

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$
y = \frac{\ln \left(x-1\right) c_{2} x}{2}-\frac{\ln \left(x+1\right) c_{2} x}{2}+c_{1} x +c_{2}
$$

1.641.5 Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 33

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x)\to c_1x-\frac{1}{2}c_2(x\log(1-x)-x\log(x+1)+2)
$$

1.642 problem 659

Internal problem ID [8780]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 659
Date solved : Monday, October 21, 2024 at 05:21:39 PM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$4x^2y'' + 4xy' + \left(4x^2 - 1\right) y = 0$$

1.642.1 Solved as second order ode using Kovacic algorithm

Time used: 0.164 (sec)

Writing the ode as

$$4x^2y'' + 4xy' + \left(4x^2 - 1\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 4x \\ C &= 4x^2 - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1227: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{4x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.642.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) + 4xy' + (4x^2-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 4xy' + (4x^2-1)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$$

1.642.3 Maple trace

```
Methods for second order ODEs:
```

1.642.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 17

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(4*x^2-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{\sqrt{x}}$$

1.642.5 Mathematica DSolve solution

Solving time : `0.049` (sec)
Leaf size : 39

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

1.643 problem 660

Internal problem ID [8781]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 660
Date solved : `Monday, October 21, 2024 at 05:21:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (2x + 1)\, y' + 2y = 0$$

1.643.1 Solved as second order ode using Kovacic algorithm

Time used: 0.234 (sec)

Writing the ode as

$$xy'' + (-2x - 1)\, y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x - 1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 - 4x + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1229: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} + \frac{1}{4x^2} + \frac{1}{8x^3} + \frac{1}{32x^4} - \frac{1}{64x^5} - \frac{3}{128x^6} - \frac{3}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-4x + 3}{4x^2}\right) \\ &= 1 + \frac{-4x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$
\begin{aligned}
b &= (-1) - (0)\\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &= -\frac{1}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 - 4x + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right)\\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (1) \\
&= 1 - \frac{1}{2x} \\
&= 1 - \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(1 - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(1 - \frac{1}{2x}\right)^2 - \left(\frac{4x^2 - 4x + 3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(1 - \frac{1}{2x}\right) dx} \\
&= \frac{\mathrm{e}^{x}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x-1}{x} \, dx} \\
&= z_1 e^{x + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-1}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2x+\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(2x+1)\,\mathrm{e}^{2x+\ln(x)}\mathrm{e}^{-4x}}{4x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{2x}\right) + c_2 \left(\mathrm{e}^{2x}\left(-\frac{(2x+1)\,\mathrm{e}^{2x+\ln(x)}\mathrm{e}^{-4x}}{4x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.643.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) - (2x+1)\,y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2y}{x} + \frac{(2x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(2x+1)y'}{x} + \frac{2y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{2x+1}{x}, P_3(x) = \frac{2}{x}\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\bigg|_{x=0} = -1$$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + (-2x - 1)\,y' + 2y = 0$
- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$
- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$
- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$
- Shift index using $k- >k+1$

  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) - 2a_k(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_{k+1}(k+1+r)-2a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{2a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{2a_k}{k+1}$
- Solution for $r=0$
  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{2a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{2a_k}{k+3}$
- Solution for $r=2$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=\frac{2a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=\frac{2a_k}{k+1}, b_{k+1}=\frac{2b_k}{k+3}\right]$

1.643.3 Maple trace

```
Methods for second order ODEs:
```

1.643.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 16

```
dsolve(x*diff(diff(y(x),x),x)-(2*x+1)*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2\,\mathrm{e}^{2x}+2c_1x+c_1$$

1.643.5 Mathematica DSolve solution

Solving time : `0.053` (sec)
Leaf size : 25

```
DSolve[{x*D[y[x],{x,2}]-(2*x+1)*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{2x} - \frac{1}{4} c_2 (2x + 1)$$

1.644 problem 661

Internal problem ID [8782]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 661
Date solved : `Monday, October 21, 2024 at 05:21:41 PM`
CAS classification : `[_erf]`

Solve

$$y'' + 2xy' + 4y = 0$$

1.644.1 Solved as second order ode using Kovacic algorithm

Time used: 0.220 (sec)

Writing the ode as

$$y'' + 2xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 3$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 - 3)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1231: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{3}{2x} - \frac{9}{8x^3} - \frac{27}{16x^5} - \frac{405}{128x^7} - \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} - \frac{72171}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 3}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2 - 3) + (0) \\ &= x^2 - 3 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -3. Now b can be found.

$$\begin{aligned} b &= (-3) - (0) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{1} - 1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{1} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 - 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = 1$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_{\infty} \\
&= 0 + (-)\,(x) \\
&= -x \\
&= -x
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2(-x)\,(1) + \left((-1) + (-x)^2 - \left(x^2 - 3\right)\right) &= 0 \\
2a_0 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int -x dx} \\
&= (x)\, \mathrm{e}^{-\frac{x^2}{2}} \\
&= x\, \mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{1}\, dx} \\
&= z_1 e^{-\frac{x^2}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x^2} x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x^2}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{x^2}}{x} + \sqrt{\pi}\, \mathrm{erfi}(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x^2} x\right) + c_2 \left(\mathrm{e}^{-x^2} x \left(-\frac{\mathrm{e}^{x^2}}{x} + \sqrt{\pi}\, \mathrm{erfi}(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.644.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 2xy' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(ka_{k+2} + 2a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k}{k+1}\right]$

1.644.3 Maple trace

```
Methods for second order ODEs:
```

1.644.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve(diff(diff(y(x),x),x)+2*x*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = x\left(c_2\sqrt{\pi}\,\operatorname{erfi}(x) + c_1\right) \mathrm{e}^{-x^2} - c_2$$

1.644.5 Mathematica DSolve solution

Solving time : 0.057 (sec)
Leaf size : 51

```
DSolve[{D[y[x],{x,2}]+2*x*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x^2}\left(-\sqrt{\pi}c_2\sqrt{x^2}\operatorname{erfi}\left(\sqrt{x^2}\right) + c_2e^{x^2} + 2c_1x\right)$$

1.645 problem 662

Internal problem ID [8783]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 662
Date solved : `Monday, October 21, 2024 at 05:21:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 3y = 0$$

1.645.1 Solved as second order ode using Kovacic algorithm

Time used: 0.260 (sec)

Writing the ode as

$$y'' + xy' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 10$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1233: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \qquad (9)
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) = 0$$

$$a_1 x + 2a_0 + 2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{-\frac{x^2}{4}} \\
&= \left(x^2 - 1\right) \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \,dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \,dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}} \left(x^2 - 1\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \,dx}}{y_1^2} \,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \,dx}}{\left(y_1\right)^2} \,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{\left(y_1\right)^2} \,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^2 - 1\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} \left(x^2-1\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} \left(x^2-1\right) \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^2-1\right)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.645.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+3)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $\left(k^2+3k+2\right) a_{k+2} + a_k(k+3) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+3)}{k^2+3k+2}\right]$

1.645.3 Maple trace

```
Methods for second order ODEs:
```

1.645.4 Maple dsolve solution

Solving time : 0.073 (sec)
Leaf size : 42

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)+3*y(x) = 0,
       y(x),singsol=all)
```

$$y = -(x-1)(x+1)\left(c_1\sqrt{\pi}\,\sqrt{2}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)-c_2\right)\mathrm{e}^{-\frac{x^2}{2}}+2c_1x$$

1.645.5 Mathematica DSolve solution

Solving time : 0.171 (sec)
Leaf size : 65

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{1}{4}e^{-\frac{x^2}{2}}\left(\sqrt{2\pi}c_2(x^2-1)\,\mathrm{erfi}\left(\frac{x}{\sqrt{2}}\right)+4c_1(x^2-1)-2c_2e^{\frac{x^2}{2}}x\right)$$

1.646 problem 663

Internal problem ID [8784]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 663
Date solved : `Monday, October 21, 2024 at 05:21:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2 y' - 3xy = 0$$

1.646.1 Solved as second order ode using Kovacic algorithm

Time used: 0.267 (sec)

Writing the ode as

$$y'' - x^2 y' - 3xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= -3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3 + 8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x(x^3 + 8) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3 + 8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1235: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} + \frac{2}{x} - \frac{4}{x^4} + \frac{16}{x^7} - \frac{80}{x^{10}} + \frac{448}{x^{13}} - \frac{2688}{x^{16}} + \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x(x^3+8)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^4 + 2x\right) + (0) \\
&= \frac{1}{4}x^4 + 2x
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 2. Now b can be found.

$$\begin{aligned}
b &= (2) - (0) \\
&= 2
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^2}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 2\right) &&= 1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 2\right) &&= -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3+8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x^2}{2}\right) \\ &= \frac{x^2}{2} \\ &= \frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{x^2}{2}\right)(1) + \left((x) + \left(\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3+8)}{4}\right)\right) = 0$$

$$-xa_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, \mathrm{e}^{\int \frac{x^2}{2} dx} \\ &= (x) \, \mathrm{e}^{\frac{x^3}{6}} \\ &= x \, \mathrm{e}^{\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x^2}{1} \, dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x^3}{3}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x^3}{3}} x\right) + c_2 \left(\mathrm{e}^{\frac{x^3}{3}} x \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.646.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - x^2y' - 3xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - ◦ Shift index using $k- > k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - ◦ Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^{k+1}$

  - ◦ Shift index using $k- > k-1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1) \, x^k$

  - ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$

  - ◦ Shift index using $k- > k+2$

$\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_{k-1}(k+2))x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+2)(ka_{k+2} - a_{k-1} + a_{k+2}) = 0$
- Shift index using $k- >k+1$

  $(k+3)((k+1)a_{k+3} - a_k + a_{k+3}) = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k+2}, 2a_2 = 0\right]$

1.646.3 Maple trace

```
Methods for second order ODEs:
```

1.646.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 58

```
dsolve(diff(diff(y(x),x),x)-x^2*diff(y(x),x)-3*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{9\,\mathrm{WhittakerM}\left(\frac{1}{3}, \frac{5}{6}, \frac{x^3}{3}\right) \mathrm{e}^{\frac{x^3}{6}} c_2\, x^3 + 9c_1\, x^2 \mathrm{e}^{\frac{x^3}{3}} + 5\, 3^{2/3} c_2 (x^3)^{1/3} (x^3+2)}{9x}$$

1.646.5 Mathematica DSolve solution

Solving time : 0.094 (sec)
Leaf size : 51

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]-3*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{9} e^{\frac{x^3}{3}} \left(9 c_1 x - 3^{2/3} c_2 \sqrt[3]{x^3} \Gamma\left(-\frac{1}{3}, \frac{x^3}{3}\right)\right)$$

1.647 problem 664

Internal problem ID [8785]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 664
Date solved : `Monday, October 21, 2024 at 05:21:43 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-4x^2+1\right)y''-20xy'-16y=0$$

1.647.1 Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$\left(-4x^2+1\right)y''-20xy'-16y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -4x^2+1 \\ B &= -20x \\ C &= -16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4x^2 + 6}{(4x^2 - 1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4x^2 + 6 \\
t &= (4x^2 - 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-4x^2 + 6}{(4x^2 - 1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1237: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (4x^2 - 1)^2$. There is a pole at $x = \frac{1}{2}$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\left(x - \frac{1}{2}\right)^2} - \frac{7}{8\left(x - \frac{1}{2}\right)} + \frac{5}{16\left(x + \frac{1}{2}\right)^2} + \frac{7}{8\left(x + \frac{1}{2}\right)}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4x^2+6}{(4x^2-1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4x^2+6}{(4x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)} + (-)\,(0) \\
&= -\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)} \\
&= -\frac{2x}{4x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right)(1) + \left(\left(\frac{1}{4\left(x-\frac{1}{2}\right)^2} + \frac{1}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right)^2 -
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right) dx} \\
&= (x) \frac{1}{\left((2x-1)\,(2x+1)\right)^{1/4}} \\
&= \frac{x}{\left(4x^2-1\right)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-20x}{-4x^2+1}\, dx} \\
&= z_1 e^{-\frac{5\ln\left(4x^2-1\right)}{4}} \\
&= z_1\left(\frac{1}{\left(4x^2-1\right)^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{\left(4x^2-1\right)^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-20x}{-4x^2+1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(4x^2-1\right)}{2}}}{\left(y_1\right)^2}\, dx \\
&= y_1\left(\frac{\left(4x^2-1\right)^{3/2}}{x} - 4x\sqrt{4x^2-1} + \ln\left(x\sqrt{4}+\sqrt{4x^2-1}\right)\sqrt{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{\left(4x^2-1\right)^{3/2}}\right) \\
&\quad + c_2\left(\frac{x}{\left(4x^2-1\right)^{3/2}}\left(\frac{\left(4x^2-1\right)^{3/2}}{x} - 4x\sqrt{4x^2-1} + \ln\left(x\sqrt{4}+\sqrt{4x^2-1}\right)\sqrt{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.647.2 Maple step by step solution

Let's solve

$\left(-4x^2+1\right)\left(\frac{d}{dx}y'\right)-20xy'-16y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{16y}{4x^2-1}-\frac{20xy'}{4x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{20xy'}{4x^2-1}+\frac{16y}{4x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{20x}{4x^2-1},P_3(x)=\frac{16}{4x^2-1}\right]$

- ◦ $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}}=\frac{5}{2}$

- ◦ $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}}=0$

- ◦ $x=-\frac{1}{2}$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-\frac{1}{2}$

- Multiply by denominators

$\left(4x^2-1\right)\left(\frac{d}{dx}y'\right)+20xy'+16y=0$

- Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

$\left(4u^2-4u\right)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\left(20u-10\right)\left(\frac{d}{du}y(u)\right)+16y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- ○ Convert $u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

$$u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- ○ Shift index using $k->k+2-m$

$$u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_{0}r(3+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(2k+5+2r)+4a_{k}(k+r+2)^{2}\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(3+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4a_{k}(k+r+2)^{2}-4\left(k+\frac{5}{2}+r\right)a_{k+1}(k+1+r)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{2a_{k}(k+r+2)^{2}}{(2k+5+2r)(k+1+r)}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{2a_{k}(k+2)^{2}}{(2k+5)(k+1)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k},a_{k+1}=\frac{2a_{k}(k+2)^{2}}{(2k+5)(k+1)}\right]$$

- Revert the change of variables $u=x+\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_{k}\left(x+\frac{1}{2}\right)^{k},a_{k+1}=\frac{2a_{k}(k+2)^{2}}{(2k+5)(k+1)}\right]$$

- Recursion relation for $r=-\frac{3}{2}$

$$a_{k+1}=\frac{2a_{k}\left(k+\frac{1}{2}\right)^{2}}{(2k+2)\left(k-\frac{1}{2}\right)}$$

- Solution for $r=-\frac{3}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k-\frac{3}{2}},a_{k+1}=\frac{2a_{k}\left(k+\frac{1}{2}\right)^{2}}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$$

- Revert the change of variables $u=x+\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k-\frac{3}{2}}, a_{k+1}=\frac{2 a_k \left(k+\frac{1}{2}\right)^2}{\left(2k+2\right)\left(k-\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k \left(x+\frac{1}{2}\right)^{k-\frac{3}{2}}\right), a_{k+1}=\frac{2 a_k \left(k+2\right)^2}{\left(2k+5\right)\left(k+1\right)}, b_{k+1}=\frac{2 b_k \left(k+\frac{1}{2}\right)^2}{\left(2k+2\right)\left(k-\frac{1}{2}\right)}\right]$$

1.647.3 Maple trace

```
Methods for second order ODEs:
```

1.647.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 48

```
dsolve((-4*x^2+1)*diff(diff(y(x),x),x)-20*x*diff(y(x),x)-16*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{2 c_2 \ln\left(2x+\sqrt{4x^2-1}\right) x-\sqrt{4x^2-1}\, c_2+c_1 x}{\left(4x^2-1\right)^{3/2}}$$

1.647.5 Mathematica DSolve solution

Solving time : `0.167` (sec)
Leaf size : 68

```
DSolve[{(1-4*x^2)*D[y[x],{x,2}]-20*x*D[y[x],x]-16*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{-2 c_2 x \arctan\left(\frac{2x}{\sqrt{1-4x^2}}\right)-c_2\sqrt{1-4x^2}+c_1 x}{\sqrt[4]{1-4x^2}\left(4x^2-1\right)^{5/4}}$$

1.648 problem 665

Internal problem ID [8786]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 665
Date solved : `Monday, October 21, 2024 at 05:21:44 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(x^{2}-1\right) y''-6 x y'+12 y = 0$$

1.648.1 Solved as second order ode using Kovacic algorithm

Time used: 0.222 (sec)

Writing the ode as

$$\left(x^{2}-1\right) y''-6 x y'+12 y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}-1 \\ B &= -6 x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2 A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1239: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{15}{4(x-1)^2} + \frac{15}{4(x+1)} - \frac{15}{4(x-1)} + \frac{15}{4(x+1)^2}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}
\end{aligned}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2(x-1)} + \frac{5}{2(x+1)} + (-)(0) \\ &= -\frac{3}{2(x-1)} + \frac{5}{2(x+1)} \\ &= \frac{x-4}{x^2-1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2(x-1)} + \frac{5}{2(x+1)}\right)(0) + \left(\left(\frac{3}{2(x-1)^2} - \frac{5}{2(x+1)^2}\right) + \left(-\frac{3}{2(x-1)} + \frac{5}{2(x+1)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{3}{2(x-1)} + \frac{5}{2(x+1)}\right) dx} \\ &= \frac{(x+1)^{5/2}}{(x-1)^{3/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-6x}{x^2-1} \, dx} \\ &= z_1 e^{\frac{3\ln(x-1)}{2} + \frac{3\ln(x+1)}{2}} \\ &= z_1 \left((x-1)^{3/2} (x+1)^{3/2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (x+1)^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-6x}{x^2-1}\, dx}}{(y_1)^2}\, dx$$
$$= y_1 \int \frac{e^{3\ln(x-1)+3\ln(x+1)}}{(y_1)^2}\, dx$$
$$= y_1 \left(-\frac{x(x^2+1)\,\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7\,(x-1)^3}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left((x+1)^4\right) + c_2\left((x+1)^4\left(-\frac{x(x^2+1)\,\mathrm{e}^{3\ln(x-1)+3\ln(x+1)}}{(x+1)^7\,(x-1)^3}\right)\right)$$

Will add steps showing solving for IC soon.

1.648.2 Maple step by step solution

Let's solve

$(x^2-1)\left(\frac{d}{dx}y'\right) - 6xy' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{12y}{x^2-1} + \frac{6xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{6xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{6x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = -3$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d}{dx}y'\right) - 6xy' + 12y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-6u+6)\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

- □ Rewrite ODE with series expansions

  - ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

  - ◦ Shift index using $k- > k+1-m$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

  - ◦ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

    $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

  - ◦ Shift index using $k- > k+2-m$

    $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0 r(-4+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r-3) + a_k(k+r-3)(k+r-4)\right) u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 4\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-3)\left((-2k-2r-2)\, a_{k+1} + a_k(k+r-4)\right) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r-4)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 4$

  $a_{k+1} = \frac{a_k(k-4)}{2(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -2a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{3a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{3a_0}{2}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2}{3}$
- Express in terms of a_0

  $a_3 = -\frac{a_0}{2}$
- Apply recursion relation for $k = 3$

  $a_4 = -\frac{a_3}{8}$
- Express in terms of a_0

  $a_4 = \frac{a_0}{16}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 2u + \frac{3}{2}u^2 - \frac{1}{2}u^3 + \frac{1}{16}u^4\right)$
- Revert the change of variables $u = x + 1$

  $\left[y = \frac{a_0(x-1)^4}{16}\right]$
- Recursion relation for $r = 4$

  $a_{k+1} = \frac{a_k k}{2(k+5)}$
- Solution for $r = 4$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+4}, a_{k+1} = \frac{a_k k}{2(k+5)}\right]$
- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+4}, a_{k+1} = \frac{a_k k}{2(k+5)}\right]$
- Combine solutions and rename parameters

  $\left[y = \frac{a_0(x-1)^4}{16} + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+4}\right), b_{k+1} = \frac{b_k k}{2(k+5)}\right]$

1.648.3 Maple trace

```
Methods for second order ODEs:
```

1.648.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve((x^2-1)*diff(diff(y(x),x),x)-6*x*diff(y(x),x)+12*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2 x^4 + c_1 x^3 + 6c_2 x^2 + c_1 x + c_2$$

1.648.5 Mathematica DSolve solution

Solving time : 0.184 (sec)
Leaf size : 45

```
DSolve[{(x^2-1)*D[y[x],{x,2}]-6*x*D[y[x],x]+12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{x^2-1}(c_2 x(x^2+1) + c_1(x-1)^4)}{\sqrt{1-x^2}}$$

1.649 problem 666

Internal problem ID [8787]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 666
Date solved : `Monday, October 21, 2024 at 05:21:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + (2 + x)\, y = 0$$

1.649.1 Solved as second order ode using Kovacic algorithm

Time used: 0.293 (sec)

Writing the ode as

$$y'' + xy' + (2 + x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 + x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x - 6$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1241: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - 1 - \frac{5}{2x} - \frac{5}{x^2} - \frac{65}{4x^3} - \frac{115}{2x^4} - \frac{885}{4x^5} - \frac{1785}{2x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} - 1 \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 - x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right) + (0) \\ &= \frac{1}{4}x^2 - x - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (1) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} - 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 - x - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}-1$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} - 1\right) \\ &= 1 - \frac{x}{2} \\ &= 1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(1 - \frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right)\right) &= 0 \\ (2 + x)\, a_1 + 4x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = -4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 4x + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= \left(x^2 - 4x + 3\right) \mathrm{e}^{\int \left(1 - \frac{x}{2}\right) dx} \\
&= \left(x^2 - 4x + 3\right) \mathrm{e}^{x - \frac{1}{4}x^2} \\
&= \left(x^2 - 4x + 3\right) \mathrm{e}^{-\frac{x(-4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1}\, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \left(x^2 - 4x + 3\right) \mathrm{e}^{-\frac{x(-2+x)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}} \mathrm{e}^{x(-2+x)}}{\left(x^2 - 4x + 3\right)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\left(x^2 - 4x + 3\right) e^{-\frac{x(-2+x)}{2}} \right) + c_2 \left(\left(x^2 - 4x + 3\right) e^{-\frac{x(-2+x)}{2}} \left(\int \frac{e^{-\frac{x^2}{2}} e^{x(-2+x)}}{\left(x^2 - 4x + 3\right)^2} dx \right) \right)$$

Will add steps showing solving for IC soon.

1.649.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' + (2 + x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k + 2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2+2a_0+\left(\sum_{k=1}^{\infty}\left(a_{k+2}(k+2)\left(k+1\right)+a_k(k+2)+a_{k-1}\right)x^k\right)=0$$

- Each term must be 0
  $2a_2+2a_0=0$
- Each term in the series must be 0, giving the recursion relation
  $(k^2+3k+2)\,a_{k+2}+a_k k+2a_k+a_{k-1}=0$
- Shift index using $k->k+1$
  $\left((k+1)^2+3k+5\right)a_{k+3}+a_{k+1}(k+1)+2a_{k+1}+a_k=0$
- Recursion relation that defines the series solution to the ODE
  $\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+3}=-\frac{k a_{k+1}+a_k+3a_{k+1}}{k^2+5k+6}, 2a_2+2a_0=0\right]$

1.649.3 Maple trace

```
Methods for second order ODEs:
```

1.649.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 78

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)+(2+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\left(\left(x-3\right)c_2(-1+x)\,\mathrm{e}^{-\frac{(-2+x)^2}{2}}\left(\mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{-\left(-2+x\right)^2}}{2}\right)-1\right)\sqrt{\pi}-\sqrt{2}\,\sqrt{-\left(-2+x\right)^2}\,c_2-c_1\,\mathrm{e}^{-\frac{(-2+x)^2}{2}}(-1+x)\left(x-3\right)\right)\mathrm{e}^{-x}$$

1.649.5 Mathematica DSolve solution

Solving time : 0.338 (sec)
Leaf size : 94

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+(2+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}e^{-\frac{x^2}{2}+x-\frac{9}{2}}\left(e^{5/2}\sqrt{2\pi}c_2\left(x^2-4x+3\right)\operatorname{erfi}\left(\frac{x-2}{\sqrt{2}}\right) + 4e^{9/2}c_1\left(x^2-4x+3\right) - 2c_2e^{\frac{1}{2}(x-3)^2+x}(x-2)\right)$$

1.650 problem 667

Internal problem ID [8788]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 667
Date solved : `Monday, October 21, 2024 at 05:21:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(2x^2+1\right)y''+7xy'+2y=0$$

1.650.1 Solved as second order ode using Kovacic algorithm

Time used: 0.351 (sec)

Writing the ode as

$$\left(2x^2+1\right)y''+7xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^2+1\\ B&=7x\\ C&=2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 6}{4\left(2x^2 + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 + 6$$
$$t = 4\left(2x^2 + 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 6}{4\left(2x^2 + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1243: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{7}{64\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{17i\sqrt{2}}{64\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{17i\sqrt{2}}{64\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2+6}{4(2x^2+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2+6}{4(2x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - (\alpha_{c_1}^- + \alpha_{c_2}^-)\\ &= \frac{5}{4} - \left(\frac{1}{4}\right)\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} + (0) \\
&= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} \\
&= \frac{x}{4x^2 + 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}\right)(1) + \left(\left(-\frac{1}{8\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{8\left(x + \frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4\cdots}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x)\, e^{\int \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}\right) dx} \\
&= (x) \left(\left(i\sqrt{2} - 2x\right) \left(2x + i\sqrt{2}\right) \right)^{1/8} \\
&= x\left(-4x^2 - 2\right)^{1/8}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x}{2x^2+1}\,dx} \\
&= z_1 e^{-\frac{7\ln(2x^2+1)}{8}} \\
&= z_1 \left(\frac{1}{(2x^2+1)^{7/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{7/8} x(-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x}{2x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(2x^2+1)}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\,(2x^2+1)^{7/4}\,x^2\,(-4x^2-2)^{1/4}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2^{7/8} x(-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}}\right) + c_2 \left(\frac{2^{7/8} x(-4x^2-2)^{1/8}}{(4x^2+2)^{7/8}} \left(\int \frac{2^{1/4}(4x^2+2)^{7/4}}{4\,(2x^2+1)^{7/4}\,x^2\,(-4x^2-2)^{1/4}} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.650.2 Maple step by step solution

1.650.3 Maple trace

```
Methods for second order ODEs:
```

1.650.4 Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 37

```
dsolve((2*x^2+1)*diff(diff(y(x),x),x)+7*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \operatorname{LegendreP}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\,x\right) + c_2 \operatorname{LegendreQ}\left(\frac{1}{4}, \frac{3}{4}, i\sqrt{2}\,x\right)}{\left(2x^2+1\right)^{3/8}}$$

1.650.5 Mathematica DSolve solution

Solving time : 0.105 (sec)
Leaf size : 66

```
DSolve[{(1+2*x^2)*D[y[x],{x,2}]+7*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 Q_{\frac{1}{4}}^{\frac{3}{4}}\left(i\sqrt{2}x\right)}{\left(2x^2+1\right)^{3/8}} + \frac{2i\sqrt[4]{2}c_1 x}{\left(2x^2+1\right)^{3/4} \operatorname{Gamma}\left(\frac{1}{4}\right)}$$

1.651 problem 668

Internal problem ID [8789]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 668
Date solved : `Monday, October 21, 2024 at 05:21:47 PM`
CAS classification : `[_Lienard]`

Solve

$$4y'' + xy' + 4y = 0$$

1.651.1 Solved as second order ode using Kovacic algorithm

Time used: 0.267 (sec)

Writing the ode as

$$4y'' + xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 56}{64} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 56$$
$$t = 64$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{64} - \frac{7}{8}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1244: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 2\\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{8} - \frac{7}{2x} - \frac{49}{x^3} - \frac{1372}{x^5} - \frac{48020}{x^7} - \frac{1882384}{x^9} - \frac{79060128}{x^{11}} - \frac{3478645632}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i\\ &= \frac{x}{8}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{64}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 56}{64} \\ &= Q + \frac{R}{64} \\ &= \left(\frac{x^2}{64} - \frac{7}{8}\right) + (0) \\ &= \frac{x^2}{64} - \frac{7}{8} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{7}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{8}\right) - (0) \\ &= -\frac{7}{8} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{8} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{7}{8}}{\frac{1}{8}} - 1\right) &= -4 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{7}{8}}{\frac{1}{8}} - 1\right) &= 3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{64} - \frac{7}{8}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{8}$	-4	3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 3$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{8}\right) \\ &= -\frac{x}{8} \\ &= -\frac{x}{8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (6x + 2a_2) + 2\left(-\frac{x}{8}\right)(3x^2 + 2xa_2 + a_1) + \left(\left(-\frac{1}{8}\right) + \left(-\frac{x}{8}\right)^2 - \left(\frac{x^2}{64} - \frac{7}{8}\right)\right) = 0 \\ 6x + 2a_2 + \frac{1}{4}a_2x^2 + \frac{1}{2}a_1x + \frac{3}{4}a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = -12, a_2 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 - 12x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^3 - 12x\right) \mathrm{e}^{\int -\frac{x}{8} dx} \\ &= \left(x^3 - 12x\right) \mathrm{e}^{-\frac{x^2}{16}} \\ &= x\left(x^2 - 12\right) \mathrm{e}^{-\frac{x^2}{16}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{4} \, dx} \\ &= z_1 e^{-\frac{x^2}{16}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{16}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{8}} x\left(x^2 - 12\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{4} \, dx}}{\left(y_1\right)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{8}}}{\left(y_1\right)^2} \, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{8}}}{x^2 \left(x^2 - 12\right)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^2}{8}} x\left(x^2-12\right)\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{8}} x\left(x^2-12\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{8}}}{x^2\left(x^2-12\right)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.651.2 Maple step by step solution

Let's solve

$4\frac{d}{dx}y' + xy' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{xy'}{4} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{xy'}{4} + y = 0$

- Multiply by denominators

  $4\frac{d}{dx}y' + xy' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(4a_{k+2}(k+2)(k+1) + a_k(k+4)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4(k^2 + 3k + 2)\, a_{k+2} + a_k(k+4) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{4(k^2+3k+2)}\right]$$

1.651.3 Maple trace

```
Methods for second order ODEs:
```

1.651.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 34

```
dsolve(4*diff(diff(y(x),x),x)+x*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{\mathrm{e}^{-\frac{x^2}{8}} \left(-12\,\mathrm{hypergeom}\left(\left[-\frac{3}{2}\right], \left[\frac{1}{2}\right], \frac{x^2}{8}\right) c_2 + x(x^2-12)\, c_1\right)}{12}$$

1.651.5 Mathematica DSolve solution

Solving time : `0.152` (sec)
Leaf size : 122

```
DSolve[{4*D[y[x],{x,2}]+x*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{x^2}{8}} \left(\sqrt{2\pi} c_2 (x^2-12)\, x^2 \operatorname{erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{2}}\right) + 4\sqrt{x^2}\left(2\sqrt{2} c_1 x^3 - c_2 e^{\frac{x^2}{8}} x^2 + 8 c_2 e^{\frac{x^2}{8}} - 24\sqrt{2} c_1 x\right)\right)}{32\sqrt{x^2}}$$

1.652 problem 669

Internal problem ID [8790]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 669
Date solved : `Monday, October 21, 2024 at 05:21:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' - 4y = 0$$

1.652.1 Solved as second order ode using Kovacic algorithm

Time used: 0.239 (sec)

Writing the ode as

$$y'' + xy' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 18}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 18 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{9}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1246: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{9}{2x} - \frac{81}{4x^3} + \frac{729}{4x^5} - \frac{32805}{16x^7} + \frac{413343}{16x^9} - \frac{11160261}{32x^{11}} + \frac{157837977}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 18}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} + \frac{9}{2}\right) + (0) \\
&= \frac{x^2}{4} + \frac{9}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{9}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{9}{2}\right) - (0) \\
&= \frac{9}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{9}{2}}{\frac{1}{2}} - 1\right) &&= 4 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{9}{2}}{\frac{1}{2}} - 1\right) &&= -5
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{9}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	4	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 4$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^{+} \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{x}{2}\right)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{9}{2}\right)\right) = 0$$

$$-a_3x^3 + \left(-2a_2 + 12\right)x^2 + \left(-3a_1 + 6a_3\right)x - 4a_0 + 2a_2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = 0, a_2 = 6, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 + 6x^2 + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 + 6x^2 + 3\right) \mathrm{e}^{\int \frac{x}{2} dx} \\
&= \left(x^4 + 6x^2 + 3\right) \mathrm{e}^{\frac{x^2}{4}} \\
&= \left(x^4 + 6x^2 + 3\right) \mathrm{e}^{\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 + 6x^2 + 3
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^4 + 6x^2 + 3\right)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 + 6x^2 + 3\right) + c_2 \left(x^4 + 6x^2 + 3 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^4 + 6x^2 + 3\right)^2} dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.652.2 Maple trace

```
Methods for second order ODEs:
```

1.652.3 Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 47

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y = xc_1(x^2+5)\sqrt{2}\,\mathrm{e}^{-\frac{x^2}{2}} + (x^4+6x^2+3)\left(\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{\pi}\,c_1 + c_2\right)$$

1.652.4 Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 43

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{2}}\,\mathrm{HermiteH}\left(-5,\frac{x}{\sqrt{2}}\right) + \frac{1}{3}c_2(x^4+6x^2+3)$$

1.653 problem 670

Internal problem ID [8791]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 670
Date solved : `Monday, October 21, 2024 at 05:21:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4xy'' - xy' + 2y = 0$$

1.653.1 Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$4xy'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x - 32}{64x} \tag{6}$$

Comparing the above to (5) shows that

$$s = x - 32$$
$$t = 64x$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x - 32}{64x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1247: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{8} - \frac{2}{x} - \frac{16}{x^2} - \frac{256}{x^3} - \frac{5120}{x^4} - \frac{114688}{x^5} - \frac{2752512}{x^6} - \frac{69206016}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{8} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x-32}{64x} \\ &= Q + \frac{R}{64x} \\ &= \left(\frac{1}{64}\right) + \left(-\frac{1}{2x}\right) \\ &= \frac{1}{64} - \frac{1}{2x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is -32. Dividing this by leading coefficient in t which is 64 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{8} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{8}} - 0\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{8}} - 0\right) &&= 2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x - 32}{64x}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{8}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty}\\ &= \frac{1}{x} + (-)\left(\frac{1}{8}\right)\\ &= \frac{1}{x} - \frac{1}{8}\\ &= \frac{1}{x} - \frac{1}{8}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{8}\right)(1) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} - \frac{1}{8}\right)^2 - \left(\frac{x-32}{64x}\right)\right) = 0$$

$$\frac{8 + a_0}{4x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -8\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -8 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx}\\ &= (-8 + x)\, \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{8}\right) dx}\\ &= (-8 + x)\, \mathrm{e}^{-\frac{x}{8} + \ln(x)}\\ &= (-8 + x)\, x\, \mathrm{e}^{-\frac{x}{8}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{4x}\,dx} \\
&= z_1 e^{\frac{x}{8}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (-8+x)\,x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{4x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{\frac{x}{4}}}{256\left(-2+\frac{x}{4}\right)} - \frac{\mathrm{e}^{\frac{x}{4}}}{64x} - \frac{\mathrm{Ei}_1\left(-\frac{x}{4}\right)}{128}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1((-8+x)\,x) + c_2\left((-8+x)\,x\left(-\frac{\mathrm{e}^{\frac{x}{4}}}{256\left(-2+\frac{x}{4}\right)} - \frac{\mathrm{e}^{\frac{x}{4}}}{64x} - \frac{\mathrm{Ei}_1\left(-\frac{x}{4}\right)}{128}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.653.2 Maple step by step solution

1.653.3 Maple trace

```
Methods for second order ODEs:
```

1.653.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 33

```
dsolve(4*x*diff(diff(y(x),x),x)-x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 x(-8+x)\,\mathrm{Ei}_1\left(-\frac{x}{4}\right)}{16}+\frac{c_2(x-4)\,\mathrm{e}^{\frac{x}{4}}}{4}+c_1(-8+x)\,x$$

1.653.5 Mathematica DSolve solution

Solving time : 0.082 (sec)
Leaf size : 43

```
DSolve[{4*x*D[y[x],{x,2}]-x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{1}{128}c_2\left((x-8)x\,\text{ExpIntegralEi}\left(\frac{x}{4}\right)-4e^{x/4}(x-4)\right)+c_1(x-8)x$$

1.654 problem 671

Internal problem ID [8792]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 671
Date solved : `Monday, October 21, 2024 at 05:21:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$6x^2y'' + x(1+18x)\,y' + (1+12x)\,y = 0$$

1.654.1 Solved as second order ode using Kovacic algorithm

Time used: 0.296 (sec)

Writing the ode as

$$6x^2y'' + \left(18x^2 + x\right)y' + (1+12x)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= 18x^2 + x \\ C &= 1 + 12x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{324x^2 - 252x - 35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 324x^2 - 252x - 35$$
$$t = 144x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{324x^2 - 252x - 35}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1248: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} - \frac{7}{4x} - \frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{7}{12x} - \frac{7}{36x^2} - \frac{49}{648x^3} - \frac{245}{5832x^4} - \frac{343}{13122x^5} - \frac{66199}{3779136x^6} - \frac{837949}{68024448x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{324x^2 - 252x - 35}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{9}{4}\right) + \left(\frac{-252x - 35}{144x^2}\right) \\ &= \frac{9}{4} + \frac{-252x - 35}{144x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -252. Dividing this by leading coefficient in t which is 144 gives $-\frac{7}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{4}\right) - (0) \\ &= -\frac{7}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{3}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{4}}{\frac{3}{2}} - 0\right) &&= -\frac{7}{12} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{4}}{\frac{3}{2}} - 0\right) &&= \frac{7}{12} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{324x^2 - 252x - 35}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{2}$	$-\frac{7}{12}$	$\frac{7}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{7}{12}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{7}{12} - \left(\frac{7}{12}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{7}{12x} + (-)\left(\frac{3}{2}\right) \\
&= \frac{7}{12x} - \frac{3}{2} \\
&= \frac{7}{12x} - \frac{3}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{12x} - \frac{3}{2}\right)(0) + \left(\left(-\frac{7}{12x^2}\right) + \left(\frac{7}{12x} - \frac{3}{2}\right)^2 - \left(\frac{324x^2 - 252x - 35}{144x^2}\right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{7}{12x} - \frac{3}{2}\right) dx} \\
&= x^{7/12} e^{-\frac{3x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{18x^2 + x}{6x^2} \, dx} \\
&= z_1 e^{-\frac{3x}{2} - \frac{\ln(x)}{12}} \\
&= z_1 \left(\frac{e^{-\frac{3x}{2}}}{x^{1/12}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\, \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{18x^2+x}{6x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3x-\frac{\ln(x)}{6}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-3x-\frac{\ln(x)}{6}}\, \mathrm{e}^{6x}}{x} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\sqrt{x}\, \mathrm{e}^{-3x} \right) + c_2 \left(\sqrt{x}\, \mathrm{e}^{-3x} \left(\int \frac{\mathrm{e}^{-3x-\frac{\ln(x)}{6}}\, \mathrm{e}^{6x}}{x} dx \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.654.2 Maple step by step solution

Let's solve

$6x^2 \left(\frac{d}{dx} y'\right) + x(1+18x)\, y' + (1+12x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(1+12x)y}{6x^2} - \frac{(1+18x)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{(1+18x)y'}{6x} + \frac{(1+12x)y}{6x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{1+18x}{6x}, P_3(x) = \frac{1+12x}{6x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{6}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{6}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2\left(\frac{d}{dx}y'\right) + x(1+18x)\, y' + (1+12x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-1)\,(2k+2r-1) + 6a_{k-1}(3k+3r-1)\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $6\left(\left(k+r-\frac{1}{2}\right)a_k + 3a_{k-1}\right)\left(k+r-\frac{1}{3}\right) = 0$

- Shift index using $k->k+1$

  $6\left(\left(k+\frac{1}{2}+r\right)a_{k+1} + 3a_k\right)\left(k+\frac{2}{3}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{6a_k}{2k+1+2r}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{6a_k}{2k+2}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{6a_k}{2k+2}\right]$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+1} = -\frac{6a_k}{2k+\frac{5}{3}}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{6a_k}{2k+\frac{5}{3}}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+1} = -\frac{6a_k}{2k+2}, b_{k+1} = -\frac{6b_k}{2k+\frac{5}{3}}\right]$

1.654.3 Maple trace

```
Methods for second order ODEs:
```

1.654.4 Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 40

```
dsolve(6*x^2*diff(diff(y(x),x),x)+x*(1+18*x)*diff(y(x),x)+(1+12*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-\frac{c_2(-x)^{5/6}3^{5/6}}{3} + x\,\mathrm{e}^{-3x}\left(c_2\Gamma\left(\frac{5}{6}\right) - c_2\Gamma\left(\frac{5}{6}, -3x\right) + c_1\right)}{\sqrt{x}}$$

1.654.5 Mathematica DSolve solution

Solving time : 0.094 (sec)
Leaf size : 47

```
DSolve[{6*x^2*D[y[x],{x,2}]+x*(1+18*x)*D[y[x],x]+(1+12*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-3x}\left(\frac{\sqrt[6]{3}c_2x^{4/3}\Gamma\left(-\frac{1}{6}, -3x\right)}{(-x)^{5/6}} + c_1\sqrt{x}\right)$$

1.655 problem 672

Internal problem ID [8793]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 672
Date solved : `Monday, October 21, 2024 at 05:21:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3x^2y'' - x(x+8)\,y' + 6y = 0$$

1.655.1 Solved as second order ode using Kovacic algorithm

Time used: 3.604 (sec)

Writing the ode as

$$3x^2y'' + \left(-x^2 - 8x\right) y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= -x^2 - 8x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 16x + 40}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 16x + 40$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 16x + 40}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1250: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{10}{9x^2} + \frac{4}{9x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{10}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{2}{3} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{4}{3x} - \frac{2}{x^2} + \frac{16}{x^3} - \frac{140}{x^4} + \frac{1312}{x^5} - \frac{12944}{x^6} + \frac{132736}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{6}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 16x + 40}{36x^2} \\
&= Q + \frac{R}{36x^2} \\
&= \left(\frac{1}{36}\right) + \left(\frac{16x + 40}{36x^2}\right) \\
&= \frac{1}{36} + \frac{16x + 40}{36x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 16. Dividing this by leading coefficient in t which is 36 gives $\frac{4}{9}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{4}{9}\right) - (0) \\ &= \frac{4}{9} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{4}{9}}{\frac{1}{6}} - 0\right) &= \frac{4}{3} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{4}{9}}{\frac{1}{6}} - 0\right) &= -\frac{4}{3} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 16x + 40}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{3}$	$-\frac{2}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{4}{3}$	$-\frac{4}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{4}{3}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= \frac{4}{3} - \left(-\frac{2}{3}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{2}{3x} + \left(\frac{1}{6} \right) \\
&= -\frac{2}{3x} + \frac{1}{6} \\
&= \frac{-4 + x}{6x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{3x} + \frac{1}{6}\right)(2x + a_1) + \left(\left(\frac{2}{3x^2} \right) + \left(-\frac{2}{3x} + \frac{1}{6} \right)^2 - \left(\frac{x^2 + 16x + 40}{36x^2} \right) \right) = 0$$

$$\frac{(-a_1 - 2)\, x - 2a_0 - 4a_1}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4, a_1 = -2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2x + 4$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 2x + 4\right) \mathrm{e}^{\int \left(-\frac{2}{3x} + \frac{1}{6}\right) dx} \\
&= \left(x^2 - 2x + 4\right) \mathrm{e}^{\frac{x}{6} - \frac{2\ln(x)}{3}} \\
&= \frac{\left(x^2 - 2x + 4\right) \mathrm{e}^{\frac{x}{6}}}{x^{2/3}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-8x}{3x^2}\,dx} \\
&= z_1 e^{\frac{x}{6}+\frac{4\ln(x)}{3}} \\
&= z_1 \left(x^{4/3}\mathrm{e}^{\frac{x}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{2/3}\mathrm{e}^{\frac{x}{3}}\left(x^2-2x+4\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-8x}{3x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x}{3}+\frac{8\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x}{3}+\frac{8\ln(x)}{3}}\mathrm{e}^{-\frac{2x}{3}}}{x^{4/3}\left(x^2-2x+4\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{2/3}\mathrm{e}^{\frac{x}{3}}\left(x^2-2x+4\right)\right) + c_2\left(x^{2/3}\mathrm{e}^{\frac{x}{3}}\left(x^2-2x+4\right)\left(\int \frac{\mathrm{e}^{\frac{x}{3}+\frac{8\ln(x)}{3}}\mathrm{e}^{-\frac{2x}{3}}}{x^{4/3}\left(x^2-2x+4\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.655.2 Maple step by step solution

Let's solve

$3x^2\left(\frac{d}{dx}y'\right) - x(x+8)\,y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = -\frac{2y}{x^2} + \frac{(x+8)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{(x+8)y'}{3x} + \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = -\frac{x+8}{3x}, P_3(x) = \frac{2}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{8}{3}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 2$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$3x^2\left(\frac{d}{dx}y'\right) - x(x+8)\,y' + 6y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+3r)(-3+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-2)(k+r-3) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-2+3r)(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{3, \frac{2}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3(k+r-3)\left(k+r-\frac{2}{3}\right) a_k - a_{k-1}(k+r-1) = 0$$

- Shift index using $k->k+1$

$$3(k-2+r)\left(k+\frac{1}{3}+r\right) a_{k+1} - a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r)}{(k-2+r)(3k+1+3r)}$$

- Recursion relation for $r = 3$

$$a_{k+1} = \frac{a_k(k+3)}{(k+1)(3k+10)}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{a_k(k+3)}{(k+1)(3k+10)}\right]$$

- Recursion relation for $r = \frac{2}{3}$

$$a_{k+1} = \frac{a_k\left(k+\frac{2}{3}\right)}{\left(k-\frac{4}{3}\right)(3k+3)}$$

- Solution for $r = \frac{2}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{2}{3}}, a_{k+1} = \frac{a_k\left(k+\frac{2}{3}\right)}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{2}{3}}\right), a_{k+1} = \frac{a_k(k+3)}{(k+1)(3k+10)}, b_{k+1} = \frac{b_k\left(k+\frac{2}{3}\right)}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$$

1.655.3 Maple trace

```
Methods for second order ODEs:
```

1.655.4 Maple dsolve solution

Solving time : 0.027 (sec)
Leaf size : 38

```
dsolve(3*x^2*diff(diff(y(x),x),x)-x*(x+8)*diff(y(x),x)+6*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2\left(x^{2/3} - \frac{x^{5/3}}{2} + \frac{x^{8/3}}{4}\right) e^{\frac{x}{3}} + c_1 \,\mathrm{hypergeom}\left([3], \left[\frac{10}{3}\right], \frac{x}{3}\right) x^3$$

1.655.5 Mathematica DSolve solution

Solving time : 0.255 (sec)
Leaf size : 79

```
DSolve[{3*x^2*D[y[x],{x,2}]-x*(x+8)*D[y[x],x]+6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{x/3} x^{2/3} (x^2 - 2x + 4) - \frac{c_2 e^{x/3} x^{2/3} (x^2 - 2x + 4)\, \Gamma\left(\frac{1}{3}, \frac{x}{3}\right)}{6\ 3^{2/3}} + \frac{1}{6} c_2 (x-4) x$$

1.656 problem 673

Internal problem ID [8794]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 673
Date solved : `Monday, October 21, 2024 at 05:21:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' - x(1+2x)\,y' + 2(4x-1)\,y = 0$$

1.656.1 Solved as second order ode using Kovacic algorithm

Time used: 0.689 (sec)

Writing the ode as

$$2x^2y'' + \left(-2x^2 - x\right)y' + (8x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -2x^2 - x \\ C &= 8x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 60x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 - 60x + 21$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 60x + 21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1252: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{15}{4x} + \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{15}{4x} - \frac{51}{4x^2} - \frac{765}{8x^3} - \frac{3519}{4x^4} - \frac{144585}{16x^5} - \frac{6358527}{64x^6} - \frac{146409525}{128x^7} + \dots \quad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \quad (10)$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{4x^2 - 60x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-60x + 21}{16x^2}\right) \\ &= \frac{1}{4} + \frac{-60x + 21}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -60. Dividing this by leading coefficient in t which is 16 gives $-\frac{15}{4}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{15}{4}\right) - (0) \\
&= -\frac{15}{4}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{15}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{15}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{15}{4}}{\frac{1}{2}} - 0\right) &&= \frac{15}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 - 60x + 21}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{15}{4}$	$\frac{15}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{15}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{15}{4} - \left(\frac{7}{4}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{7}{4x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{7}{4x} - \frac{1}{2} \\ &= \frac{7}{4x} - \frac{1}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{7}{4x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{7}{4x^2}\right) + \left(\frac{7}{4x} - \frac{1}{2}\right)^2 - \left(\frac{4x^2 - 60x + 21}{16x^2}\right)\right) = 0$$

$$\frac{2(9 + a_1)\, x + 4a_0 + 7a_1}{2x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{63}{4}, a_1 = -9\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 9x + \frac{63}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 9x + \frac{63}{4}\right) \mathrm{e}^{\int \left(\frac{7}{4x} - \frac{1}{2}\right) dx} \\
&= \left(x^2 - 9x + \frac{63}{4}\right) \mathrm{e}^{-\frac{x}{2} + \frac{7\ln(x)}{4}} \\
&= \frac{(4x^2 - 36x + 63)\, x^{7/4} \mathrm{e}^{-\frac{x}{2}}}{4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x^2 - x}{2x^2} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{4}} \\
&= z_1 \left(x^{1/4} \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 - 9x^3 + \frac{63}{4} x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2 - x}{2x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x + \frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{x + \frac{\ln(x)}{2}}}{\left(x^4 - 9x^3 + \frac{63}{4} x^2\right)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^4 - 9x^3 + \frac{63}{4}x^2\right) + c_2\left(x^4 - 9x^3 + \frac{63}{4}x^2\left(\int \frac{\mathrm{e}^{x+\frac{\ln(x)}{2}}}{\left(x^4 - 9x^3 + \frac{63}{4}x^2\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.656.2 Maple step by step solution

Let's solve

$2x^2\left(\frac{d}{dx}y'\right) - x(1+2x)\,y' + 2(4x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x-1)y}{x^2} + \frac{(1+2x)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(1+2x)y'}{2x} + \frac{(4x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1+2x}{2x}, P_3(x) = \frac{4x-1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -\frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -1$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d}{dx}y'\right) - x(1+2x)\,y' + (8x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(k+r-2) - 2a_{k-1}(k-5+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{2, -\frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{1}{2}\right)(k+r-2)\, a_k - 2a_{k-1}(k-5+r) = 0$
- Shift index using $k- >k + 1$

  $2\left(k+\frac{3}{2}+r\right)(k+r-1)\, a_{k+1} - 2a_k(k+r-4) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r-4)}{(2k+3+2r)(k+r-1)}$
- Recursion relation for $r = 2$; series terminates at $k = 2$

  $a_{k+1} = \frac{2a_k(k-2)}{(2k+7)(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{4a_0}{7}$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{9}$
- Express in terms of a_0

  $a_2 = \frac{4a_0}{63}$

- Terminating series solution of the ODE for $r = 2$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1-\frac{4}{7}x+\frac{4}{63}x^2\right)$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = \frac{2a_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{2a_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1-\frac{4}{7}x+\frac{4}{63}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), b_{k+1} = \frac{2b_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}\right]$

1.656.3 Maple trace

```
Methods for second order ODEs:
```

1.656.4 Maple dsolve solution

Solving time : 0.012 (sec)
Leaf size : 32

```
dsolve(2*x^2*diff(diff(y(x),x),x)-x*(1+2*x)*diff(y(x),x)+2*(4*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1(4x^2-36x+63)\,x^2}{63} + \frac{c_2 \operatorname{hypergeom}\left(\left[-\frac{9}{2}\right], \left[-\frac{3}{2}\right], x\right)}{\sqrt{x}}$$

1.656.5 Mathematica DSolve solution

Solving time : 0.169 (sec)
Leaf size : 89

```
DSolve[{2*x^2*D[y[x],{x,2}]-x*(1+2*x)*D[y[x],x]+2*(4*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1\left(x^4 - 9x^3 + \frac{63x^2}{4}\right) - \frac{4c_2\left(\sqrt{\pi}(-4x^2 + 36x - 63)\, x^{5/2}\text{erfi}\left(\sqrt{x}\right) + 2e^x(2x^4 - 17x^3 + 24x^2 + 6x + 3)\right)}{945\sqrt{x}}$$

1.657 problem 674

Internal problem ID [8795]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 674
Date solved : `Monday, October 21, 2024 at 05:21:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' - 4x^2y' + (1+2x)\,y = 0$$

1.657.1 Solved as second order ode using Kovacic algorithm

Time used: 0.240 (sec)

Writing the ode as

$$4x^2y'' - 4x^2y' + (1+2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 \\ C &= 1+2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 2x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1254: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} - \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{3}{4x^4} - \frac{5}{4x^5} - \frac{9}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{x-1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-2x-1}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x^2}{4x^2} \, dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2}{4x^2} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^x}{(y_1)^2} dx \\ &= y_1(-\operatorname{Ei}_1(-x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}(-\operatorname{Ei}_1(-x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.657.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) - 4x^2y' + (1+2x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(1+2x)y}{4x^2} + y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - y' + \frac{(1+2x)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -1, P_3(x) = \frac{1+2x}{4x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{4}$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$4x^2\left(\frac{d}{dx}y'\right) - 4x^2y' + (1+2x)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^2 \cdot y'$ to series expansion

    $$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$$

  - Shift index using $k- >k-1$

    $$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$$

  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+2r)^2\,x^r + \left(\sum_{k=1}^{\infty}\left(a_k(2k+2r-1)^2 - 2a_{k-1}(2k-3+2r)\right)x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$r = \frac{1}{2}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r-1)^2 + (-4k+6-4r)\, a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+1}(2k+1+2r)^2 + a_k(-4k-4r+2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(2k+2r-1)}{(2k+1+2r)^2}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{4a_k k}{(2k+2)^2}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{4a_k k}{(2k+2)^2}\right]$

1.657.3 Maple trace

```
Methods for second order ODEs:
```

1.657.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 17

```
dsolve(4*x^2*diff(diff(y(x),x),x)-4*x^2*diff(y(x),x)+(1+2*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = (c_2\,\mathrm{Ei}_1(-x) + c_1)\sqrt{x}$$

1.657.5 Mathematica DSolve solution

Solving time : `0.043` (sec)
Leaf size : 19

```
DSolve[{4*x^2*D[y[x],{x,2}]-4*x^2*D[y[x],x]+(1+2*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{x}(c_2\,\mathrm{ExpIntegralEi}(x) + c_1)$$

1.658 problem 675

Internal problem ID [8796]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 675
Date solved : `Monday, October 21, 2024 at 05:21:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(3-2x)\,y' + (1-2x)\,y = 0$$

1.658.1 Solved as second order ode using Kovacic algorithm

Time used: 0.278 (sec)

Writing the ode as

$$x^2y'' + \left(-2x^2+3x\right)y' + (1-2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2+3x \\ C &= 1-2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4x^2 - 4x - 1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4x^2 - 4x - 1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{4x^2 - 4x - 1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1256: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = 1 - \frac{1}{x} - \frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} - \frac{1}{4x^2} - \frac{1}{8x^3} - \frac{3}{32x^4} - \frac{5}{64x^5} - \frac{9}{128x^6} - \frac{17}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-4x - 1}{4x^2}\right) \\ &= 1 + \frac{-4x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$\begin{aligned}b &= (-1) - (0)\\ &= -1\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 1\\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &= -\frac{1}{2}\\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &= \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 - 4x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right)\\ &= \frac{1}{2} - \left(\frac{1}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\,(1) \\
&= \frac{1}{2x} - 1 \\
&= \frac{1}{2x} - 1
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x} - 1\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - 1\right)^2 - \left(\frac{4x^2 - 4x - 1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - 1\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^2+3x}{x^2} \, dx} \\
&= z_1 e^{x - \frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{x}}{x^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2+3x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2x-3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(-\operatorname{Ei}_1(-2x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}(-\operatorname{Ei}_1(-2x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.658.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + x(3-2x)\,y' + (1-2x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(2x-1)y}{x^2} + \frac{(2x-3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(2x-3)y'}{x} - \frac{(2x-1)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

$\left[P_2(x) = -\frac{2x-3}{x}, P_3(x) = -\frac{2x-1}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2\left(\frac{d}{dx}y'\right) - x(2x-3)\,y' + (1-2x)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)^2 - 2a_{k-1}(k+r)\right)x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)^2 = 0$$

- Values of r that satisfy the indicial equation
  $r = -1$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+1)^2 - 2a_{k-1}(k+r) = 0$
- Shift index using $k- > k+1$
  $a_{k+1}(k+2+r)^2 - 2a_k(k+r+1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{2a_k(k+r+1)}{(k+2+r)^2}$
- Recursion relation for $r = -1$
  $a_{k+1} = \frac{2a_k k}{(k+1)^2}$
- Solution for $r = -1$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{2a_k k}{(k+1)^2}\right]$

1.658.3 Maple trace

```
Methods for second order ODEs:
```

1.658.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(3-2*x)*diff(y(x),x)+(1-2*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \operatorname{Ei}_1(-2x) + c_1}{x}$$

1.658.5 Mathematica DSolve solution

Solving time : `0.051` (sec)
Leaf size : 19

```
DSolve[{x^2*D[y[x],{x,2}]+x*(3-2*x)*D[y[x],x]+(1-2*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 \operatorname{ExpIntegralEi}(2x) + c_1}{x}$$

1.659 problem 676

Internal problem ID [8797]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 676
Date solved : `Monday, October 21, 2024 at 05:21:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(3+x)\,y' + (4-x)\,y = 0$$

1.659.1 Solved as second order ode using Kovacic algorithm

Time used: 0.329 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2-3x\right)y' + (4-x)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2-3x \\ C &= 4-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 10x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 10x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1258: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4x^2} + \frac{5}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{5}{2x} - \frac{13}{2x^2} + \frac{65}{2x^3} - \frac{819}{4x^4} + \frac{5785}{4x^5} - \frac{43797}{4x^6} + \frac{347425}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 10x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{10x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{10x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 10. Dividing this by leading coefficient in t which is 4 gives $\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{2}\right) - (0) \\ &= \frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= \frac{5}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 10x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{5}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - (\alpha_{c_1}^{+}) \\ &= \frac{5}{2} - \left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2x} + \frac{1}{2} \\ &= \frac{1+x}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 10x - 1}{4x^2}\right)\right) = 0$$

$$\frac{(-a_1 + 4)\, x - 2a_0 + a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2, a_1 = 4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 4x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= (x^2 + 4x + 2)\, e^{\int \left(\frac{1}{2x} + \frac{1}{2}\right) dx} \\ &= (x^2 + 4x + 2)\, e^{\frac{x}{2} + \frac{\ln(x)}{2}} \\ &= (x^2 + 4x + 2)\, \sqrt{x}\, e^{\frac{x}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{3\ln(x)}{2}} \\
&= z_1\left(x^{3/2}\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2\mathrm{e}^x\left(x^2+4x+2\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{-x}(-x-3)}{4\left(x^2+4x+2\right)}-\frac{\mathrm{Ei}_1(x)}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\mathrm{e}^x\left(x^2+4x+2\right)\right) + c_2\left(x^2\mathrm{e}^x\left(x^2+4x+2\right)\left(-\frac{\mathrm{e}^{-x}(-x-3)}{4\left(x^2+4x+2\right)}-\frac{\mathrm{Ei}_1(x)}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.659.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x(3+x)\,y' + (4-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(-4+x)y}{x^2} + \frac{(3+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(3+x)y'}{x} - \frac{(-4+x)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{3+x}{x}, P_3(x) = -\frac{-4+x}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - x(3+x)\,y' + (4-x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 2$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-2)^2 - a_{k-1}(k+r) = 0$$

- Shift index using $k->k+1$

$$a_{k+1}(k+r-1)^2 - a_k(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r+1)}{(k+r-1)^2}$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k(k+3)}{(k+1)^2}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+3)}{(k+1)^2}\right]$$

1.659.3 Maple trace

```
Methods for second order ODEs:
```

1.659.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 42

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(3+x)*diff(y(x),x)+(4-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \left(c_2\,\mathrm{e}^{x}\left(x^2+4x+2\right)\mathrm{Ei}_1\left(x\right)+c_1\,\mathrm{e}^{x}\left(x^2+4x+2\right)-c_2(3+x)\right)x^2$$

1.659.5 Mathematica DSolve solution

Solving time : 0.168 (sec)
Leaf size : 52

```
DSolve[{x^2*D[y[x],{x,2}]-x*(3+x)*D[y[x],x]+(4-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{1}{4}x^2\left(c_2e^x\left(x^2+4x+2\right)\text{ExpIntegralEi}(-x)+4c_1e^x\left(x^2+4x+2\right)+c_2(x+3)\right)$$

1.660 problem 677

Internal problem ID [8798]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 677
Date solved : `Monday, October 21, 2024 at 05:22:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(3-x)\,y' + y = 0$$

1.660.1 Solved as second order ode using Kovacic algorithm

Time used: 0.303 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2+3x\right)y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2+3x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1260: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2x} - \frac{5}{2x^2} - \frac{15}{2x^3} - \frac{115}{4x^4} - \frac{495}{4x^5} - \frac{2285}{4x^6} - \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-6x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{-1 + x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 6x - 1}{4x^2}\right)\right) = 0$$

$$\frac{1 + a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (-1 + x)\, e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= (-1 + x)\, e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= (-1 + x)\, \sqrt{x}\, e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2+3x}{x^2}\, dx} \\
&= z_1 e^{\frac{x}{2} - \frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{2}}}{x^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{-1+x}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+3x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x-3\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{x}}{-1+x} - \mathrm{Ei}_1(-x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{-1+x}{x}\right) + c_2 \left(\frac{-1+x}{x}\left(-\frac{\mathrm{e}^{x}}{-1+x} - \mathrm{Ei}_1(-x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.660.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+x(3-x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{y}{x^2}+\frac{(x-3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{(x-3)y'}{x}+\frac{y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{x-3}{x},P_3(x)=\frac{1}{x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=3$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=1$

- ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right)-x(x-3)\,y'+y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^2\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = -1$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)^2 - a_{k-1}(k+r-1) = 0$
- Shift index using $k- > k+1$

  $a_{k+1}(k+2+r)^2 - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+2+r)^2}$
- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k-1)}{(k+1)^2}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot (1-x)$

1.660.3 Maple trace

```
Methods for second order ODEs:
```

1.660.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 28

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(3-x)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(-1+x) c_2 \operatorname{Ei}_1(-x) + c_2 e^x + c_1(-1+x)}{x}$$

1.660.5 Mathematica DSolve solution

Solving time : 0.08 (sec)
Leaf size : 31

```
DSolve[{x^2*D[y[x],{x,2}]+x*(3-x)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2(x-1)\,\text{ExpIntegralEi}(x) + c_1(x-1) - c_2e^x}{x}$$

1.661 problem 678

Internal problem ID [8799]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 678
Date solved : `Monday, October 21, 2024 at 05:22:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - \left(2\sqrt{5}-1\right)xy' + \left(\frac{19}{4}-3x^2\right)y = 0$$

1.661.1 Solved as second order ode using Kovacic algorithm

Time used: 0.174 (sec)

Writing the ode as

$$x^2y'' + \left(-2x\sqrt{5}+x\right)y' + \left(\frac{19}{4}-3x^2\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x\sqrt{5}+x \\ C &= \frac{19}{4}-3x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 3z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1262: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 3$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\sqrt{3}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x\sqrt{5}+x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)\sqrt{5}-\frac{\ln(x)}{2}} \\ &= z_1 \left(x^{\sqrt{5}-\frac{1}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^{\sqrt{5}-\frac{1}{2}} \mathrm{e}^{-\sqrt{3}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x\sqrt{5}+x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x)\left(2\sqrt{5}-1\right)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\mathrm{e}^{\ln(x)\left(2\sqrt{5}-1\right)} x^{1-2\sqrt{5}} \mathrm{e}^{2\sqrt{3}\, x}\sqrt{3}}{6}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{\sqrt{5}-\frac{1}{2}}\mathrm{e}^{-\sqrt{3}\,x}\right) + c_2\left(x^{\sqrt{5}-\frac{1}{2}}\mathrm{e}^{-\sqrt{3}\,x}\left(\frac{\mathrm{e}^{\ln(x)\left(2\sqrt{5}-1\right)}x^{1-2\sqrt{5}}\mathrm{e}^{2\sqrt{3}\,x}\sqrt{3}}{6}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.661.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - \left(2\sqrt{5}-1\right)xy' + \left(\frac{19}{4} - 3x^2\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(12x^2-19)y}{4x^2} + \frac{\left(2\sqrt{5}-1\right)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{\left(2\sqrt{5}-1\right)y'}{x} - \frac{(12x^2-19)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2\sqrt{5}-1}{x}, P_3(x) = -\frac{12x^2-19}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1 - 2\sqrt{5}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{19}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4\left(2\sqrt{5}-1\right)xy' + (-12x^2+19)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$\left(-1+2\sqrt{5}-2r\right)\left(1+2\sqrt{5}-2r\right) a_0 x^r + \left(-3+2\sqrt{5}-2r\right)\left(-1+2\sqrt{5}-2r\right) a_1 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(\left(-\right.\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\left(-1+2\sqrt{5}-2r\right)\left(1+2\sqrt{5}-2r\right) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{2}+\sqrt{5}, \sqrt{5}-\frac{1}{2}\right\}$$

- Each term must be 0

$$\left(-3+2\sqrt{5}-2r\right)\left(-1+2\sqrt{5}-2r\right) a_1 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-8a_k(k+r)\sqrt{5} + \left(4k^2+8kr+4r^2+19\right) a_k - 12a_{k-2} = 0$$

- Shift index using $k->k+2$

$$-8a_{k+2}(k+2+r)\sqrt{5} + \left(4(k+2)^2+8(k+2)\,r+4r^2+19\right) a_{k+2} - 12a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{12a_k}{8k\sqrt{5}+8\sqrt{5}\,r-4k^2-8kr-4r^2+16\sqrt{5}-16k-16r-35}$$

- Recursion relation for $r = \frac{1}{2}+\sqrt{5}$

$$a_{k+2} = -\frac{12a_k}{8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-16k-43}$$

- Solution for $r=\frac{1}{2}+\sqrt{5}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}+\sqrt{5}}, a_{k+2}=-\frac{12a_k}{8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-16k-43}, a_1=0\right]$$

- Recursion relation for $r=\sqrt{5}-\frac{1}{2}$

$$a_{k+2}=-\frac{12a_k}{8k\sqrt{5}+8\sqrt{5}\left(\sqrt{5}-\frac{1}{2}\right)-4k^2-8k\left(\sqrt{5}-\frac{1}{2}\right)-4\left(\sqrt{5}-\frac{1}{2}\right)^2-16k-27}$$

- Solution for $r=\sqrt{5}-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\sqrt{5}-\frac{1}{2}}, a_{k+2}=-\frac{12a_k}{8k\sqrt{5}+8\sqrt{5}\left(\sqrt{5}-\frac{1}{2}\right)-4k^2-8k\left(\sqrt{5}-\frac{1}{2}\right)-4\left(\sqrt{5}-\frac{1}{2}\right)^2-16k-27}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}+\sqrt{5}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\sqrt{5}-\frac{1}{2}}\right), a_{k+2}=-\frac{12a_k}{8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-1}\right.$$

1.661.3 Maple trace

```
Methods for second order ODEs:
```

1.661.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 29

```
dsolve(x^2*diff(diff(y(x),x),x)-(2*5^(1/2)-1)*x*diff(y(x),x)+(19/4-3*x^2)*y(x) = 0,
       y(x),singsol=all)
```

$$y=x^{\sqrt{5}-\frac{1}{2}}\left(c_1\sinh\left(\sqrt{3}\,x\right)+c_2\cosh\left(\sqrt{3}\,x\right)\right)$$

1.661.5 Mathematica DSolve solution

Solving time : `0.117` (sec)
Leaf size : 53

```
DSolve[{x^2*D[y[x],{x,2}]-(2*Sqrt[5]-1)*x*D[y[x],x]+(19/4-3*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{1}{6}e^{-\sqrt{3}x}x^{\sqrt{5}-\frac{1}{2}}\left(\sqrt{3}c_2e^{2\sqrt{3}x}+6c_1\right)$$

1.662 problem 679

Internal problem ID [8800]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 679
Date solved : `Monday, October 21, 2024 at 05:22:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(x-3)\,y' + (4-x)\,y = 0$$

1.662.1 Solved as second order ode using Kovacic algorithm

Time used: 0.252 (sec)

Writing the ode as

$$x^2y'' + \left(x^2-3x\right)y' + (4-x)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2-3x \\ C &= 4-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 2x - 1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 2x - 1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 - 2x - 1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1264: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4x^2} - \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} - \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{3}{4x^4} - \frac{5}{4x^5} - \frac{9}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{2x} - \frac{1}{2} \\ &= -\frac{x-1}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-2x-1}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\ &= \sqrt{x}\, \mathrm{e}^{-\frac{x}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2-3x}{x^2}\, dx} \\ &= z_1 e^{-\frac{x}{2} + \frac{3\ln(x)}{2}} \\ &= z_1 \left(x^{3/2} \mathrm{e}^{-\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = x^2 \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-3x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x+3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(-\operatorname{Ei}_1(-x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^2 \mathrm{e}^{-x}\right) + c_2\left(x^2 \mathrm{e}^{-x}(-\operatorname{Ei}_1(-x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.662.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + x(x-3)\,y' + (4-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(x-4)y}{x^2} - \frac{(x-3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(x-3)y'}{x} - \frac{(x-4)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x-3}{x}, P_3(x) = -\frac{x-4}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\Big|_{x=0} = -3$$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 4$$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- • Multiply by denominators

  $$x^2\left(\frac{d}{dx}y'\right) + x(x-3)\,y' + (4-x)\,y = 0$$

- • Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ◦ Shift index using $k->k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - ◦ Shift index using $k->k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

    Rewrite ODE with series expansions

    $$a_0(-2+r)^2\,x^r + \left(\sum_{k=1}^{\infty}\left(a_k(k+r-2)^2 + a_{k-1}(k+r-2)\right)x^{k+r}\right) = 0$$

- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-2+r)^2 = 0$$

- • Values of r that satisfy the indicial equation

$r = 2$

- Each term in the series must be 0, giving the recursion relation
  $(k+r-2)\left(a_k(k+r-2)+a_{k-1}\right) = 0$
- Shift index using $k->k+1$
  $(k+r-1)\left(a_{k+1}(k+r-1)+a_k\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k}{k+r-1}$
- Recursion relation for $r = 2$
  $a_{k+1} = -\frac{a_k}{k+1}$
- Solution for $r = 2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k}{k+1}\right]$

1.662.3 Maple trace

```
Methods for second order ODEs:
```

1.662.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x-3)*diff(y(x),x)+(4-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^2 \mathrm{e}^{-x}\left(c_1 + \mathrm{Ei}_1\left(-x\right) c_2\right)$$

1.662.5 Mathematica DSolve solution

Solving time : 0.06 (sec)
Leaf size : 22

```
DSolve[{x^2*D[y[x],{x,2}]+x*(x-3)*D[y[x],x]+(4-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x} x^2 (c_2 \,\text{ExpIntegralEi}(x) + c_1)$$

1.663 problem 680

Internal problem ID [8801]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 680
Date solved : `Monday, October 21, 2024 at 05:22:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x^2y' - (2 + x)\, y = 0$$

1.663.1 Solved as second order ode using Kovacic algorithm

Time used: 0.227 (sec)

Writing the ode as

$$x^2y'' + x^2y' + (-x - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 \\ C &= -x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1266: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{3}{x^4} - \frac{2}{x^5} - \frac{6}{x^6} + \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{\frac{1}{2}} - 0\right) &&= 1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{\frac{1}{2}} - 0\right) &&= -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -1$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{x} - \frac{1}{2} \\
&= -\frac{2+x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2+4x+8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x} - \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2} \, dx \\ &= y_1 \left(\left(x^2 - 2x + 2\right) \mathrm{e}^{x} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\mathrm{e}^{-x}}{x} \right) + c_2 \left(\frac{\mathrm{e}^{-x}}{x} \left(\left(x^2 - 2x + 2\right) \mathrm{e}^{x} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.663.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y' \right) + x^2 y' - (2 + x) \, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{(2+x)y}{x^2} - y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + y' - \frac{(2+x)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x)=1, P_3(x)=-\frac{2+x}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x=0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -2$

- $x=0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x^2y' + (-x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

- Shift index using $k->k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation
  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-2)\,(a_k(k+r+1)+a_{k-1})=0$
- Shift index using $k->k+1$
  $(k-1+r)\,(a_{k+1}(k+2+r)+a_k)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=-\frac{a_k}{k+2+r}$
- Recursion relation for $r=-1$
  $a_{k+1}=-\frac{a_k}{k+1}$
- Solution for $r=-1$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1}=-\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=-\frac{a_k}{k+4}$
- Solution for $r=2$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=-\frac{a_k}{k+4}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=-\frac{a_k}{k+1}, b_{k+1}=-\frac{b_k}{k+4}\right]$

1.663.3 Maple trace

```
Methods for second order ODEs:
```

1.663.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 25

```
dsolve(x^2*diff(diff(y(x),x),x)+x^2*diff(y(x),x)-(2+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1\,\mathrm{e}^{-x}+c_2(x^2-2x+2)}{x}$$

1.663.5 Mathematica DSolve solution

Solving time : 0.058 (sec)
Leaf size : 31

```
DSolve[{x^2*D[y[x],{x,2}]+x^2*D[y[x],x]-(2+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x}(c_2 e^x (x^2 - 2x + 2) + c_1)}{x}$$

1.664 problem 681

Internal problem ID [8802]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 681
Date solved : `Monday, October 21, 2024 at 05:22:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right) y = 0$$

1.664.1 Solved as second order ode using Kovacic algorithm

Time used: 0.235 (sec)

Writing the ode as

$$x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 \\ C &= x - \frac{3}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 - 4x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1268: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{1}{x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} + \frac{1}{4x^2} + \frac{1}{8x^3} + \frac{1}{32x^4} - \frac{1}{64x^5} - \frac{3}{128x^6} - \frac{3}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-4x + 3}{4x^2}\right) \\ &= 1 + \frac{-4x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 - 4x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (1) \\
&= 1 - \frac{1}{2x} \\
&= 1 - \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(1 - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(1 - \frac{1}{2x}\right)^2 - \left(\frac{4x^2 - 4x + 3}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(1 - \frac{1}{2x}\right) dx} \\
&= \frac{\mathrm{e}^{x}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^2}{x^2} \, dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(1+2x)\,\mathrm{e}^{-2x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\sqrt{x}}\right) + c_2 \left(\frac{1}{\sqrt{x}}\left(-\frac{(1+2x)\,\mathrm{e}^{-2x}}{4}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.664.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + 2x^2 y' + \left(x - \frac{3}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x-3)y}{4x^2} - 2y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + 2y' + \frac{(4x-3)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x)=2, P_3(x)=\frac{4x-3}{4x^2}\right]$

- $x\cdot P_2(x)$ is analytic at $x=0$

  $$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=0$$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

  $$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-\frac{3}{4}$$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right)+8x^2y'+(4x-3)\,y=0$

- Assume series solution for y

  $$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $$x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k->k-m$

    $$x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^2\cdot y'$ to series expansion

    $$x^2\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$$

  - Shift index using $k->k-1$

    $$x^2\cdot y'=\sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$$

  - Convert $x^2\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

    $$x^2\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-3+2r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(2k+2r+1)\,(2k+2r-3)+4a_{k-1}(2k-1+2r)\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+2r)(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-\frac{1}{2}, \frac{3}{2}\}$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r+\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right)a_k + 8\left(k-\frac{1}{2}+r\right)a_{k-1} = 0$
- Shift index using $k- >k+1$

  $4\left(k+\frac{3}{2}+r\right)\left(k-\frac{1}{2}+r\right)a_{k+1} + 8\left(k+r+\frac{1}{2}\right)a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4(2k+2r+1)a_k}{(2k+3+2r)(2k-1+2r)}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{8ka_k}{(2k+2)(2k-2)}$
- Series not valid for $r = -\frac{1}{2}$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{8ka_k}{(2k+2)(2k-2)}$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = -\frac{4(2k+4)a_k}{(2k+6)(2k+2)}$
- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{4(2k+4)a_k}{(2k+6)(2k+2)}\right]$

1.664.3 Maple trace

```
Methods for second order ODEs:
```

1.664.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 24

```
dsolve(x^2*diff(diff(y(x),x),x)+2*x^2*diff(y(x),x)+(x-3/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{2c_2 \,\mathrm{e}^{-2x} x + c_2 \,\mathrm{e}^{-2x} + c_1}{\sqrt{x}}$$

1.664.5 Mathematica DSolve solution

Solving time : 0.054 (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]+2*x^2*D[y[x],x]+(x-3/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1 - c_2e^{-2x}(2x+1)}{4\sqrt{x}}$$

1.665 problem 682

Internal problem ID [8803]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 682
Date solved : `Monday, October 21, 2024 at 05:22:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''+x^2y'-2y=0$$

1.665.1 Solved as second order ode using Kovacic algorithm

Time used: 0.216 (sec)

Writing the ode as

$$x^2(1+x)\,y''+x^2y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= x^2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-x^2 + 8x + 8}{4\left(x^2 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -x^2 + 8x + 8 \\
t &= 4\left(x^2 + x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-x^2 + 8x + 8}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1270: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} - \frac{1}{4\,(1+x)^2} - \frac{2}{x} + \frac{2}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 8x + 8}{4\,(x^2 + x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^2+8x+8}{4\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2+2x} - \frac{1}{x} + (-)(0) \\
&= \frac{1}{2+2x} - \frac{1}{x} \\
&= -\frac{x+2}{2x(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2+2x} - \frac{1}{x}\right)(1) + \left(\left(-\frac{1}{2(1+x)^2} + \frac{1}{x^2}\right) + \left(\frac{1}{2+2x} - \frac{1}{x}\right)^2 - \left(\frac{-x^2+8x+8}{4(x^2+x)^2}\right)\right) = 0
$$

$$
\frac{-2+a_0}{x(1+x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, e^{\int \left(\frac{1}{2+2x} - \frac{1}{x}\right) dx} \\
&= (x+2)\, e^{\frac{\ln(1+x)}{2} - \ln(x)} \\
&= \frac{(x+2)\sqrt{1+x}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2}{x^2(1+x)}\,dx} \\
&= z_1 e^{-\frac{\ln(1+x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{1+x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x+2}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1\left(\ln(1+x) + \frac{4}{x+2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x+2}{x}\right) + c_2\left(\frac{x+2}{x}\left(\ln(1+x) + \frac{4}{x+2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.665.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right)+x^2y'-2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{2y}{x^2(1+x)}-\frac{y'}{1+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{y'}{1+x}-\frac{2y}{x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{1}{1+x},P_3(x)=-\frac{2}{x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=1$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\left(\frac{d}{dx}y'\right)+x^2y'-2y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(u^2-2u+1)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty}a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- ◦ Convert $u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)$ to series expansion for $m =1..3$

$$u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- ◦ Shift index using $k->k +2-m$

$$u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$a_{0} r^{2} u^{-1+r}+\left(a_{1} \left(1+r \right)^{2}-2 a_{0} \left(r^{2}+1\right)\right) u^{r}+\left(\sum_{k =1}^{\infty}\left(a_{k +1} \left(k +1+r \right)^{2}-2 a_{k} \left(k^{2}+2 k r +r^{2}+1\right)+\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$r^{2}=0$$

- Values of r that satisfy the indicial equation

$$r =0$$

- Each term must be 0

$$a_{1} \left(1+r \right)^{2}-2 a_{0} \left(r^{2}+1\right)=0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k +1} \left(k +1\right)^{2}-2 a_{k} \left(k^{2}+1\right)+a_{k -1} \left(k -1\right)^{2}=0$$

- Shift index using $k->k +1$

$$a_{k +2} \left(k +2\right)^{2}-2 a_{k +1} \left(\left(k +1\right)^{2}+1\right)+k^{2} a_{k}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k +2}=-\frac{k^{2} a_{k}-2 k^{2} a_{k +1}-4 k a_{k +1}-4 a_{k +1}}{\left(k +2\right)^{2}}$$

- Recursion relation for $r =0$

$$a_{k +2}=-\frac{k^{2} a_{k}-2 k^{2} a_{k +1}-4 k a_{k +1}-4 a_{k +1}}{\left(k +2\right)^{2}}$$

- Solution for $r =0$

$$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k}, a_{k +2}=-\frac{k^{2} a_{k}-2 k^{2} a_{k +1}-4 k a_{k +1}-4 a_{k +1}}{\left(k +2\right)^{2}}, a_{1}-2 a_{0}=0\right]$$

- Revert the change of variables $u =1+x$

$$\left[y =\sum_{k =0}^{\infty}a_{k} \left(1+x \right)^{k}, a_{k +2}=-\frac{k^{2} a_{k}-2 k^{2} a_{k +1}-4 k a_{k +1}-4 a_{k +1}}{\left(k +2\right)^{2}}, a_{1}-2 a_{0}=0\right]$$

1.665.3 Maple trace

```
Methods for second order ODEs:
```

1.665.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 27

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)+x^2*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2(x+2)\ln(1+x) + c_1x + 2c_1 + 4c_2}{x}$$

1.665.5 Mathematica DSolve solution

Solving time : 0.089 (sec)
Leaf size : 30

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]+x^2*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1(x+2) + c_2(x+2)\log(x+1) + 4c_2}{x}$$

1.666 problem 683

Internal problem ID [8804]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 683
Date solved : `Monday, October 21, 2024 at 05:22:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x\left(x^2+6\right)y' + 6y = 0$$

1.666.1 Solved as second order ode using Kovacic algorithm

Time used: 0.414 (sec)

Writing the ode as

$$x^2y'' + \left(x^3+6x\right)y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3+6x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 14}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 14$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{7}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1272: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{7}{2x} - \frac{49}{4x^3} + \frac{343}{4x^5} - \frac{12005}{16x^7} + \frac{117649}{16x^9} - \frac{2470629}{32x^{11}} + \frac{27176919}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 14}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} + \frac{7}{2}\right) + (0) \\ &= \frac{x^2}{4} + \frac{7}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{7}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{7}{2}\right) - (0) \\ &= \frac{7}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{7}{2}}{\frac{1}{2}} - 1\right) &&= 3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{7}{2}}{\frac{1}{2}} - 1\right) &&= -4 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{7}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	3	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (6x + 2a_2) + 2\left(\frac{x}{2}\right)(3x^2 + 2xa_2 + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{7}{2}\right)\right) = 0 \\ -a_2 x^2 + (-2a_1 + 6)\, x - 3a_0 + 2a_2 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = 3, a_2 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 + 3x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^3 + 3x)\, e^{\int \frac{x}{2} dx} \\
&= (x^3 + 3x)\, e^{\frac{x^2}{4}} \\
&= x(x^2 + 3)\, e^{\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3+6x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{4} - 3\ln(x)} \\
&= z_1 \left(\frac{e^{-\frac{x^2}{4}}}{x^3}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 + 3}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3+6x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2} - 6\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{-\frac{x^2}{2} - 6\ln(x)} x^4}{(x^2 + 3)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2+3}{x^2}\right) + c_2 \left(\frac{x^2+3}{x^2}\left(\int \frac{e^{-\frac{x^2}{2}-6\ln(x)}x^4}{(x^2+3)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.666.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + x(x^2+6)\,y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{6y}{x^2} - \frac{(x^2+6)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(x^2+6)y'}{x} + \frac{6y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x^2+6}{x}, P_3(x) = \frac{6}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 6$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 6$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + x(x^2+6)\,y' + 6y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(3+r)\,(2+r)\, x^r + a_1(4+r)\,(3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+3)\,(k+r+2) + a_{k-2}(k-2+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(3+r)\,(2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-3, -2\}$$

- Each term must be 0

$$a_1(4+r)\,(3+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+3)\,(k+r+2) + a_{k-2}(k-2+r) = 0$$

- Shift index using $k->k+2$

$$a_{k+2}(k+5+r)\,(k+4+r) + a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r)}{(k+5+r)(k+4+r)}$$

- Recursion relation for $r = -3$

$$a_{k+2} = -\frac{a_k(k-3)}{(k+2)(k+1)}$$

- Solution for $r = -3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a_k(k-3)}{(k+2)(k+1)}, a_1 = 0\right]$$

- Recursion relation for $r = -2$; series terminates at $k = 2$

$$a_{k+2} = -\frac{a_k(k-2)}{(k+3)(k+2)}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k(k-2)}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right), a_{k+2} = -\frac{a_k(k-3)}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k(k-2)}{(k+3)(k+2)}, b_1 = 0\right]$$

1.666.3 Maple trace

```
Methods for second order ODEs:
```

1.666.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 35

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x^2+6)*diff(y(x),x)+6*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \,\mathrm{e}^{-\frac{x^2}{2}} \operatorname{hypergeom}\left([2], \left[\frac{1}{2}\right], \frac{x^2}{2}\right) + c_1 (x^2+3)\, x}{x^3}$$

1.666.5 Mathematica DSolve solution

Solving time : `0.195` (sec)
Leaf size : 65

```
DSolve[{x^2*D[y[x],{x,2}]+x*(6+x^2)*D[y[x],x]+6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{2\pi} c_2 x(x^2+3) \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) - 12c_1 x(x^2+3) + 2c_2 e^{-\frac{x^2}{2}}(x^2+2)}{12x^3}$$

1.667 problem 684

Internal problem ID [8805]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 684
Date solved : `Monday, October 21, 2024 at 05:22:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(1-x)\, y' - y = 0$$

1.667.1 Solved as second order ode using Kovacic algorithm

Time used: 0.243 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2 + x\right) y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 + x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 2x + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1274: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{2x^2} + \frac{1}{2x^3} + \frac{1}{4x^4} - \frac{1}{4x^5} - \frac{3}{4x^6} - \frac{3}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x + 3}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - (\alpha_{c_1}^{-}) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2x} \\ &= \frac{x-1}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2} - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(\frac{1}{2} - \frac{1}{2x}\right)^2 - \left(\frac{x^2 - 2x + 3}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{2x}\right) dx} \\ &= \frac{e^{\frac{x}{2}}}{\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2+x}{x^2}\, dx} \\ &= z_1 e^{\frac{x}{2} - \frac{\ln(x)}{2}} \\ &= z_1\left(\frac{e^{\frac{x}{2}}}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-(1+x)\, x\, e^{x-\ln(x)} e^{-2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^x}{x}\right) + c_2 \left(\frac{e^x}{x} \left(-(1+x)\, x\, e^{x-\ln(x)} e^{-2x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.667.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + x(1-x)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{y}{x^2} + \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x-1)y'}{x} - \frac{y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = -\frac{1}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -1$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2\left(\frac{d}{dx}y'\right) - x(x-1)\,y' - y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-1+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) - a_{k-1}(k+r-1)\right)x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 1\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-1)\left(a_k(k+r+1) - a_{k-1}\right) = 0$$

- Shift index using $k- >k+1$

  $$(k+r)\left(a_{k+1}(k+2+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+2+r}$
- Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+3}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

1.667.3 Maple trace

```
Methods for second order ODEs:
```

1.667.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(1-x)*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\,\mathrm{e}^{x} + c_1 x + c_1}{x}$$

1.667.5 Mathematica DSolve solution

Solving time : 0.019 (sec)
Leaf size : 23

```
DSolve[{x^2*D[y[x],{x,2}]+x*(1-x)*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 e^x - c_1(x+1)}{x}$$

1.668 problem 685

Internal problem ID [8806]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 685
Date solved : `Monday, October 21, 2024 at 05:22:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(x+3)\,y' + 4y = 0$$

1.668.1 Solved as second order ode using Kovacic algorithm

Time used: 0.283 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2 - 3x\right)y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 6x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1276: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4x^2} + \frac{3}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{2x} - \frac{5}{2x^2} + \frac{15}{2x^3} - \frac{115}{4x^4} + \frac{495}{4x^5} - \frac{2285}{4x^6} + \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{6x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 6. Dividing this by leading coefficient in t which is 4 gives $\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{3}{2}\right) - (0) \\ &= \frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2x} + \frac{1}{2} \\
&= \frac{1 + x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 6x - 1}{4x^2}\right)\right) = 0$$

$$\frac{1 - a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1 + x) \, e^{\int \left(\frac{1}{2x} + \frac{1}{2}\right) dx} \\
&= (1 + x) \, e^{\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= (1 + x) \sqrt{x} \, e^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{3\ln(x)}{2}} \\
&= z_1\left(x^{3/2}\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2\mathrm{e}^x(1+x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{-x}}{-1-x} - \mathrm{Ei}_1(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x^2\mathrm{e}^x(1+x)\right) + c_2\left(x^2\mathrm{e}^x(1+x)\left(-\frac{\mathrm{e}^{-x}}{-1-x} - \mathrm{Ei}_1(x)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.668.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x(x+3)\,y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = -\frac{4y}{x^2} + \frac{(x+3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{(x+3)y'}{x} + \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$\left[P_2(x) = -\frac{x+3}{x}, P_3(x) = \frac{4}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -3$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right) - x(x+3)\,y' + 4y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(-2+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(k+r-1)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 - a_{k-1}(k+r-1) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+r-1)^2 - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+r-1)^2}$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+2)}{(k+1)^2}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+2)}{(k+1)^2}\right]$

1.668.3 Maple trace

```
Methods for second order ODEs:
```

1.668.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 29

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(x+3)*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^2(c_2\,\mathrm{e}^x(1+x)\,\mathrm{Ei}_1(x) + c_1\,\mathrm{e}^x(1+x) - c_2)$$

1.668.5 Mathematica DSolve solution

Solving time : 0.093 (sec)
Leaf size : 34

```
DSolve[{x^2*D[y[x],{x,2}]-x*(x+3)*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x^2(c_2e^x(x+1)\,\text{ExpIntegralEi}(-x) + c_1e^x(x+1) + c_2)$$

1.669 problem 686

Internal problem ID [8807]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 686
Date solved : `Monday, October 21, 2024 at 05:22:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x^2y' - 2y = 0$$

1.669.1 Solved as second order ode using Kovacic algorithm

Time used: 0.249 (sec)

Writing the ode as

$$x^2y'' - x^2y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1278: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x^2} - \frac{4}{x^4} + \frac{16}{x^6} - \frac{80}{x^8} + \frac{448}{x^{10}} - \frac{2688}{x^{12}} + \frac{16896}{x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2}{x^2}\right) \\ &= \frac{1}{4} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{x} - \frac{1}{2} \\
&= -\frac{2 + x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{x} - \frac{1}{2}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 + 8}{4x^2}\right)\right) = 0
$$

$$
\frac{-2 + a_0}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 2 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2 + x)\, e^{\int \left(-\frac{1}{x} - \frac{1}{2}\right) dx} \\
&= (2 + x)\, e^{-\frac{x}{2} - \ln(x)} \\
&= \frac{(2 + x)\, e^{-\frac{x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2+x}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(-2+x)\,\mathrm{e}^{x}}{2+x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2+x}{x}\right) + c_2 \left(\frac{2+x}{x}\left(\frac{(-2+x)\,\mathrm{e}^{x}}{2+x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.669.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)-x^2y'-2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{2y}{x^2}+y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-y'-\frac{2y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-1, P_3(x)=-\frac{2}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-2$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right)-x^2y'-2y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^2\cdot y'$ to series expansion

    $x^2\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r+1}$

  ◦ Shift index using $k->k-1$

    $x^2\cdot y'=\sum_{k=1}^{\infty}a_{k-1}(k-1+r)\,x^{k+r}$

  ◦ Convert $x^2\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) - a_{k-1}(k-1+r) = 0$
- Shift index using $k- >k+1$

  $a_{k+1}(k+2+r)\,(k-1+r) - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k-1)}{(k+1)(k-2)}$
- Apply recursion relation for $k = 0$

  $a_1 = \frac{a_0}{2}$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 + \frac{x}{2}\right)$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+2)}{(k+4)(k+1)}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+2)}{(k+4)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 + \frac{x}{2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = \frac{b_k(k+2)}{(k+4)(k+1)}\right]$

1.669.3 Maple trace

```
Methods for second order ODEs:
```

1.669.4 Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)-x^2*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2(-2+x)\,\mathrm{e}^x + c_1(2+x)}{x}$$

1.669.5 Mathematica DSolve solution

Solving time : 0.071 (sec)
Leaf size : 72

```
DSolve[{x^2*D[y[x],{x,2}]-x^2*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{2e^{x/2}\left((c_1x + 2ic_2)\cosh\left(\frac{x}{2}\right) - (ic_2x + 2c_1)\sinh\left(\frac{x}{2}\right)\right)}{\sqrt{\pi}\sqrt{-ix}\sqrt{x}}$$

1.670 problem 687

Internal problem ID [8808]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 687
Date solved : `Monday, October 21, 2024 at 05:22:09 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x^2y' - (3x+2)\,y = 0$$

1.670.1 Solved as second order ode using Kovacic algorithm

Time used: 0.277 (sec)

Writing the ode as

$$x^2y'' - x^2y' + (-3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= -3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 12x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 12x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 12x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1280: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4} + \frac{2}{x^2} + \frac{3}{x}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned}
\tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{x} - \frac{7}{x^2} + \frac{42}{x^3} - \frac{301}{x^4} + \frac{2394}{x^5} - \frac{20342}{x^6} + \frac{180852}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 12x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{12x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{12x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 12. Dividing this by leading coefficient in t which is 4 gives 3. Now b can be found.

$$
\begin{aligned}
b &= (3) - (0) \\
&= 3
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{3}{\frac{1}{2}} - 0\right) &= 3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{3}{\frac{1}{2}} - 0\right) &= -3
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 12x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	3	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 3$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= 3 - (2) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{2}{x} + \left(\frac{1}{2}\right) \\
&= \frac{2}{x} + \frac{1}{2} \\
&= \frac{4+x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{2}{x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{2}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 12x + 8}{4x^2}\right)\right) &= 0 \\
\frac{4 - a_0}{x} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 4 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (4 + x) \, e^{\int \left(\frac{2}{x} + \frac{1}{2}\right) dx} \\
&= (4 + x) \, e^{\frac{x}{2} + 2 \ln(x)} \\
&= (4 + x) \, x^2 e^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}(4+x)\,x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{\mathrm{e}^{-x}(x^3+3x^2-2x+2)}{24\,(4+x)\,x^3}+\frac{\mathrm{Ei}_1(x)}{24}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{x}(4+x)\,x^2\right) + c_2\left(\mathrm{e}^{x}(4+x)\,x^2\left(-\frac{\mathrm{e}^{-x}(x^3+3x^2-2x+2)}{24\,(4+x)\,x^3}+\frac{\mathrm{Ei}_1(x)}{24}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.670.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x^2y' - (3x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = \frac{(3x+2)y}{x^2} + y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - y' - \frac{(3x+2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = -1, P_3(x) = -\frac{3x+2}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right) - x^2y' + (-3x-2)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$$

- Shift index using $k->k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+2+r) = 0$$

- Shift index using $k->k+1$

$$a_{k+1}(k+2+r)\,(k-1+r) - a_k(k+r+3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r+3)}{(k+2+r)(k-1+r)}$$

- Recursion relation for $r=-1$

$$a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$$

- Series not valid for $r=-1$, division by 0 in the recursion relation at $k=2$

$$a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$$

- Recursion relation for $r=2$

$$a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}$$

- Solution for $r=2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}\right]$$

1.670.3 Maple trace

```
Methods for second order ODEs:
```

1.670.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 48

```
dsolve(x^2*diff(diff(y(x),x),x)-x^2*diff(y(x),x)-(3*x+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-x^3 c_2 \,\mathrm{e}^{x} (4+x) \,\mathrm{Ei}_1(x) + x^3 c_1 (4+x) \,\mathrm{e}^{x} + c_2 (x^3+3x^2-2x+2)}{x}$$

1.670.5 Mathematica DSolve solution

Solving time : `0.096` (sec)
Leaf size : 59

```
DSolve[{x^2*D[y[x],{x,2}]-x^2*D[y[x],x]-(3*x+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{24} c_2 e^x (x+4) x^2 \,\text{ExpIntegralEi}(-x) + c_1 e^x (x+4) x^2 - \frac{c_2 (x^3+3x^2-2x+2)}{24x}$$

1.671 problem 688

Internal problem ID [8809]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 688
Date solved : `Monday, October 21, 2024 at 05:22:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2 y'' + x(5-x)\, y' + 4y = 0$$

1.671.1 Solved as second order ode using Kovacic algorithm

Time used: 0.353 (sec)

Writing the ode as

$$x^2 y'' + \left(-x^2 + 5x\right) y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 + 5x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 10x - 1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 10x - 1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 - 10x - 1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1282: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{4x^2} - \frac{5}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{5}{2x} - \frac{13}{2x^2} - \frac{65}{2x^3} - \frac{819}{4x^4} - \frac{5785}{4x^5} - \frac{43797}{4x^6} - \frac{347425}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-10x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-10x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -10. Dividing this by leading coefficient in t which is 4 gives $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &= -\frac{5}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &= \frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 10x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{5}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{5}{2} - \left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{-1+x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 10x - 1}{4x^2}\right)\right) = 0$$

$$\frac{(a_1 + 4)\,x + 2a_0 + a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2, a_1 = -4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 4x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 4x + 2\right) e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= \left(x^2 - 4x + 2\right) e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= \left(x^2 - 4x + 2\right) \sqrt{x}\, e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+5x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}-\frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{2}}}{x^{5/2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2-4x+2}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+5x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x-5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{x}(x-3)}{4\left(x^2-4x+2\right)} - \frac{\mathrm{Ei}_1\left(-x\right)}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2-4x+2}{x^2}\right) + c_2 \left(\frac{x^2-4x+2}{x^2}\left(-\frac{\mathrm{e}^{x}(x-3)}{4\left(x^2-4x+2\right)} - \frac{\mathrm{Ei}_1\left(-x\right)}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.671.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + x(5-x)\,y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{4y}{x^2} + \frac{(x-5)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x-5)y'}{x} + \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x-5}{x}, P_3(x) = \frac{4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 5$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - x(x-5)\,y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = -2$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)^2 - a_{k-1}(k+r-1) = 0$
- Shift index using $k- > k+1$

  $a_{k+1}(k+3+r)^2 - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+3+r)^2}$
- Recursion relation for $r = -2$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{(k+1)^2}$
- Apply recursion relation for $k = 0$

  $a_1 = -2a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{2}$
- Terminating series solution of the ODE for $r = -2$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 - 2x + \frac{1}{2}x^2\right)$

1.671.3 Maple trace

```
Methods for second order ODEs:
```

1.671.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 41

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(5-x)*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(x^2 - 4x + 2)\, c_2\ \mathrm{Ei}_1\left(-x\right) + c_2(x-3)\, \mathrm{e}^x + c_1(x^2 - 4x + 2)}{x^2}$$

1.671.5 Mathematica DSolve solution

Solving time : `0.151` (sec)
Leaf size : 48

```
DSolve[{x^2*D[y[x],{x,2}]+x*(5-x)*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2(x^2 - 4x + 2)\,\mathrm{ExpIntegralEi}(x) + 4c_1(x^2 - 4x + 2) - c_2 e^x (x-3)}{4x^2}$$

1.672 problem 689

Internal problem ID [8810]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 689
Date solved : `Monday, October 21, 2024 at 05:22:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 4x(1-x)\,y' + (2x-9)\,y = 0$$

1.672.1 Solved as second order ode using Kovacic algorithm

Time used: 0.246 (sec)

Writing the ode as

$$4x^2y'' + \left(-4x^2+4x\right)y' + (2x-9)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2+4x \\ C &= 2x-9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1284: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -1$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{x} \\ &= \frac{x - 2}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\ &= \frac{\mathrm{e}^{\frac{x}{2}}}{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4x^2 + 4x}{4x^2} \, dx} \\ &= z_1 e^{\frac{x}{2} - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}} \right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\left(x^2+2x+2\right)x\,e^{x-\ln(x)}e^{-2x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^x}{x^{3/2}}\right) + c_2\left(\frac{e^x}{x^{3/2}}\left(-\left(x^2+2x+2\right)x\,e^{x-\ln(x)}e^{-2x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.672.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) + 4x(1-x)\,y' + (2x-9)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(2x-9)y}{4x^2} + \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x-1)y'}{x} + \frac{(2x-9)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = \frac{2x-9}{4x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{9}{4}$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$4x^2\left(\frac{d}{dx}y'\right) - 4x(x-1)\,y' + (2x-9)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(3+2r)\,(-3+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+3)\,(2k+2r-3) - 2a_{k-1}(2k+2r-3)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(3+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{3}{2}, \frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $4\left(\left(k+r+\frac{3}{2}\right) a_{k}-a_{k-1}\right)\left(k+r-\frac{3}{2}\right)=0$
- Shift index using $k->k+1$

  $4\left(\left(k+\frac{5}{2}+r\right) a_{k+1}-a_{k}\right)\left(k-\frac{1}{2}+r\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2 a_{k}}{2 k+5+2 r}$
- Recursion relation for $r=-\frac{3}{2}$

  $a_{k+1}=\frac{2 a_{k}}{2 k+2}$
- Solution for $r=-\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k-\frac{3}{2}}, a_{k+1}=\frac{2 a_{k}}{2 k+2}\right]$
- Recursion relation for $r=\frac{3}{2}$

  $a_{k+1}=\frac{2 a_{k}}{2 k+8}$
- Solution for $r=\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k+\frac{3}{2}}, a_{k+1}=\frac{2 a_{k}}{2 k+8}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_{k} x^{k-\frac{3}{2}}\right)+\left(\sum_{k=0}^{\infty} b_{k} x^{k+\frac{3}{2}}\right), a_{k+1}=\frac{2 a_{k}}{2 k+2}, b_{k+1}=\frac{2 b_{k}}{2 k+8}\right]$

1.672.3 Maple trace

```
Methods for second order ODEs:
```

1.672.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 23

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*x*(1-x)*diff(y(x),x)+(2*x-9)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_{1} \mathrm{e}^{x}+c_{2}\left(x^{2}+2 x+2\right)}{x^{3/2}}$$

1.672.5 Mathematica DSolve solution

Solving time : 0.063 (sec)
Leaf size : 30

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x*(1-x)*D[y[x],x]+(2*x-9)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1 e^x - c_2(x^2 + 2x + 2)}{x^{3/2}}$$

1.673 problem 690

Internal problem ID [8811]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 690
Date solved : `Monday, October 21, 2024 at 05:22:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 2x(2+x)\,y' + 2(1+x)\,y = 0$$

1.673.1 Solved as second order ode using Kovacic algorithm

Time used: 0.266 (sec)

Writing the ode as

$$x^2y'' + \left(2x^2+4x\right)y' + (2x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2+4x \\ C &= 2x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2 + x}{x} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2 + x$$
$$t = x$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2 + x}{x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1286: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{2x^3} - \frac{5}{8x^4} + \frac{7}{8x^5} - \frac{21}{16x^6} + \frac{33}{16x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2+x}{x} \\ &= Q + \frac{R}{x} \\ &= (1) + \left(\frac{2}{x}\right) \\ &= 1 + \frac{2}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 2. Dividing this by leading coefficient in t which is 1 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{1} - 0\right) &= 1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{1} - 0\right) &= -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2+x}{x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} + (1) \\
&= 1 + \frac{1}{x} \\
&= 1 + \frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(1 + \frac{1}{x}\right)(0) + \left(\left(-\frac{1}{x^2}\right) + \left(1 + \frac{1}{x}\right)^2 - \left(\frac{2+x}{x}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(1+\frac{1}{x}\right) dx} \\ &= x\,\mathrm{e}^{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2+4x}{x^2}\, dx} \\ &= z_1 e^{-x-2\ln(x)} \\ &= z_1 \left(\frac{\mathrm{e}^{-x}}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{2x^2+4x}{x^2} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-2x-4\ln(x)}}{(y_1)^2} dx$$
$$= y_1 \left(-\frac{e^{-2x}}{3x} + \frac{e^{-2x}}{3} - \frac{2x\,e^{-2x}}{3} + \frac{4x^2 \operatorname{Ei}_1(2x)}{3} - \frac{4 \operatorname{Ei}_1(2x)\,x^3 - 2\,e^{-2x}x^2 + x\,e^{-2x} - 6x \operatorname{Ei}_1(2x) + 2\,e^{-2x}}{3x} \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x} \left(-\frac{e^{-2x}}{3x} + \frac{e^{-2x}}{3} - \frac{2x\,e^{-2x}}{3} + \frac{4x^2 \operatorname{Ei}_1(2x)}{3} - \frac{4 \operatorname{Ei}_1(2x)\,x^3 - 2\,e^{-2x}x^2 + x\,e^{-2x} - 6x \operatorname{Ei}_1(2x) + 2\,e^{-2x}}{3x} \right) \right)$$

Will add steps showing solving for IC soon.

1.673.2 Maple step by step solution

1.673.3 Maple trace

```
Methods for second order ODEs:
```

1.673.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 28

```
dsolve(x^2*diff(diff(y(x),x),x)+2*x*(2+x)*diff(y(x),x)+2*(1+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{2c_2 \operatorname{Ei}_1(2x)\,x - c_2\,e^{-2x} + c_1 x}{x^2}$$

1.673.5 Mathematica DSolve solution

Solving time : 0.067 (sec)
Leaf size : 32

```
DSolve[{x^2*D[y[x],{x,2}]+2*x*(2+x)*D[y[x],x]+2*(1+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-2c_2x\,\text{ExpIntegralEi}(-2x) + c_1x - c_2e^{-2x}}{x^2}$$

1.674 problem 691

Internal problem ID [8812]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 691
Date solved : `Monday, October 21, 2024 at 05:22:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x(1-x)\,y' + (1-x)\,y = 0$$

1.674.1 Solved as second order ode using Kovacic algorithm

Time used: 0.266 (sec)

Writing the ode as

$$x^2y'' + \left(x^2 - x\right)y' + (1-x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 - x \\ C &= 1 - x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 + 2x - 1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 2x - 1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 + 2x - 1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1287: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2x} - \frac{1}{2x^2} + \frac{1}{2x^3} - \frac{3}{4x^4} + \frac{5}{4x^5} - \frac{9}{4x^6} + \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{2x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{1}{2}\right) - (0) \\
&= \frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 2x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{2x} \\
&= \frac{x+1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2} + \frac{1}{2x}\right)^2 - \left(\frac{x^2+2x-1}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{1}{2x}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2-x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x+\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(-\operatorname{Ei}_1(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2(x(-\operatorname{Ei}_1(x))) \end{aligned}$$

Will add steps showing solving for IC soon.

1.674.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x(1-x)\,y' + (1-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(x-1)y}{x^2} - \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(x-1)y'}{x} - \frac{(x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x-1}{x}, P_3(x) = -\frac{x-1}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x=0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right)+x(x-1)\,y'+(1-x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m=0..1$

  $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m=1..2$

  $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r-1)^2+a_{k-1}(k-2+r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2=0$

- Values of r that satisfy the indicial equation

$r = 1$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)^2 + a_{k-1}(k-2+r) = 0$
- Shift index using $k- > k+1$

  $a_{k+1}(k+r)^2 + a_k(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-1)}{(k+r)^2}$
- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k k}{(k+1)^2}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k k}{(k+1)^2}\right]$

1.674.3 Maple trace

```
Methods for second order ODEs:
```

1.674.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 13

```
dsolve(x^2*diff(diff(y(x),x),x)-x*(1-x)*diff(y(x),x)+(1-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x(c_2 \operatorname{Ei}_1(x) + c_1)$$

1.674.5 Mathematica DSolve solution

Solving time : `0.047` (sec)
Leaf size : 17

```
DSolve[{x^2*D[y[x],{x,2}]-x*(1-x)*D[y[x],x]+(1-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x(c_2 \text{ExpIntegralEi}(-x) + c_1)$$

1.675 problem 692

Internal problem ID [8813]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 692
Date solved : `Monday, October 21, 2024 at 05:22:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 4x(1+2x)\,y' + (4x-1)\,y = 0$$

1.675.1 Solved as second order ode using Kovacic algorithm

Time used: 0.130 (sec)

Writing the ode as

$$4x^2y'' + \left(8x^2+4x\right)y' + (4x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 8x^2+4x \\ C &= 4x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1289: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^2+4x}{4x^2}\,dx}\\ &= z_1 e^{-x-\frac{\ln(x)}{2}}\\ &= z_1\left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-2x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-2x-\ln(x)}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\mathrm{e}^{-2x-\ln(x)}x\,\mathrm{e}^{4x}}{2}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-2x}}{\sqrt{x}}\right) + c_2 \left(\frac{e^{-2x}}{\sqrt{x}} \left(\frac{e^{-2x-\ln(x)} x\, e^{4x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.675.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) + 4x(1+2x)\,y' + (4x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x-1)y}{4x^2} - \frac{(1+2x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(1+2x)y'}{x} + \frac{(4x-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1+2x}{x}, P_3(x) = \frac{4x-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 4x(1+2x)\,y' + (4x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-1}(2k+2r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(\left(k+r+\frac{1}{2}\right)a_k + 2a_{k-1}\right) = 0$

- Shift index using $k- >k + 1$

  $4\left(k+r+\frac{1}{2}\right)\left(\left(k+\frac{3}{2}+r\right)a_{k+1} + 2a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k}{2k+3+2r}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{4a_k}{2k+2}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{4a_k}{2k+2}\right]$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+1} = -\frac{4a_k}{2k+4}$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{4a_k}{2k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{4a_k}{2k+2}, b_{k+1} = -\frac{4b_k}{2k+4}\right]$$

1.675.3 Maple trace

```
Methods for second order ODEs:
```

1.675.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 16

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*x*(1+2*x)*diff(y(x),x)+(4*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\,\mathrm{e}^{-2x} + c_1}{\sqrt{x}}$$

1.675.5 Mathematica DSolve solution

Solving time : `0.054` (sec)
Leaf size : 26

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x*(1+2*x)*D[y[x],x]+(4*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-2x} + c_2}{2\sqrt{x}}$$

1.676 problem 693

Internal problem ID [8814]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 693
Date solved : `Monday, October 21, 2024 at 05:22:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(4+x)\,y' + (2+x)\,y = 0$$

1.676.1 Solved as second order ode using Kovacic algorithm

Time used: 0.267 (sec)

Writing the ode as

$$x^2y'' + \left(x^2+4x\right) y' + (2+x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+4x \\ C &= 2+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4+x}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4 + x$$
$$t = 4x$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4+x}{4x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1291: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{5}{x^4} + \frac{14}{x^5} - \frac{42}{x^6} + \frac{132}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4 + x}{4x} \\
&= Q + \frac{R}{4x} \\
&= \left(\frac{1}{4}\right) + \left(\frac{1}{x}\right) \\
&= \frac{1}{4} + \frac{1}{x}
\end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned}
b &= (1) - (0) \\
&= 1
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{1}{\frac{1}{2}} - 0\right) &= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{1}{\frac{1}{2}} - 0\right) &= -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4+x}{4x}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{x} \\
&= \frac{1}{2} + \frac{1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{1}{x}\right)(0) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{2} + \frac{1}{x}\right)^2 - \left(\frac{4 + x}{4x}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{1}{x}\right) dx} \\
&= x \, \mathrm{e}^{\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 + 4x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - 2\ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+4x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x-4\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-x}}{3x} + \frac{\mathrm{e}^{-x}}{6} - \frac{x\,\mathrm{e}^{-x}}{6} + \frac{x^2\,\mathrm{Ei}_1(x)}{6} - \frac{\mathrm{Ei}_1(x)\,x^3 - \mathrm{e}^{-x}x^2 - 6x\,\mathrm{Ei}_1(x) + x\,\mathrm{e}^{-x} + 4\,\mathrm{e}^{-x}}{6x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(-\frac{\mathrm{e}^{-x}}{3x} + \frac{\mathrm{e}^{-x}}{6} - \frac{x\,\mathrm{e}^{-x}}{6} + \frac{x^2\,\mathrm{Ei}_1(x)}{6} \right.\right. \\ &\qquad \left.\left. - \frac{\mathrm{Ei}_1(x)\,x^3 - \mathrm{e}^{-x}x^2 - 6x\,\mathrm{Ei}_1(x) + x\,\mathrm{e}^{-x} + 4\,\mathrm{e}^{-x}}{6x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.676.2 Maple step by step solution

1.676.3 Maple trace

```
Methods for second order ODEs:
```

1.676.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 25

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(4+x)*diff(y(x),x)+(2+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-c_2 \,\mathrm{e}^{-x} + x(c_2 \;\mathrm{Ei}_1(x) + c_1)}{x^2}$$

1.676.5 Mathematica DSolve solution

Solving time : 0.066 (sec)
Leaf size : 32

```
DSolve[{x^2*D[y[x],{x,2}]+x*(4+x)*D[y[x],x]+(2+x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-c_2 x \,\mathrm{ExpIntegralEi}(-x) + c_1 x - c_2 e^{-x}}{x^2}$$

1.677 problem 694

Internal problem ID [8815]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 694
Date solved : `Monday, October 21, 2024 at 05:22:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right)y = 0$$

1.677.1 Solved as second order ode using Kovacic algorithm

Time used: 0.292 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{9}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1292: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{x^2} - \frac{i}{2x^4} - \frac{i}{2x^6} - \frac{5i}{8x^8} - \frac{7i}{8x^{10}} - \frac{21i}{16x^{12}} - \frac{33i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{2}{x^2}\right) \\ &= -1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(i) \\
&= -\frac{1}{x} - i \\
&= -\frac{1}{x} - i
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - i\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - i\right)^2 - \left(\frac{-x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{2ia_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -i\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - i
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x - i)\, e^{\int \left(-\frac{1}{x} - i\right) dx} \\
&= (x - i)\, e^{-\ln(x) - ix} \\
&= \frac{(x - i)\, e^{-ix}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x-i)\,\mathrm{e}^{-ix}}{x^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x^{3/2}}\right) + c_2 \left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x^{3/2}} \left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.677.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + xy' + \left(x^2 - \frac{9}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-9)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{x} + \frac{(4x^2-9)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-9}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{9}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 4xy' + (4x^2 - 9)y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(3+2r)\,(-3+2r)\, x^r + a_1(5+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+3)\,(2k+2r-3) + 4a_{\cdot}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(3+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{3}{2}, \frac{3}{2}\right\}$$

- Each term must be 0

  $$a_1(5+2r)\,(-1+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(4k^2 + 8kr + 4r^2 - 9) + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

  $$a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 9\right) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+7}$$

- Recursion relation for $r = -\frac{3}{2}$

  $$a_{k+2} = -\frac{4a_k}{4k^2+4k-8}$$

- Solution for $r = -\frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+4k-8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{3}{2}$

  $$a_{k+2} = -\frac{4a_k}{4k^2+28k+40}$$

- Solution for $r = \frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+28k+40}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{3}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+4k-8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+28k+40}, b_1=0\right]$$

1.677.3 Maple trace

```
Methods for second order ODEs:
```

1.677.4 Maple dsolve solution

Solving time : 0.014 (sec)
Leaf size : 30

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-9/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{-(-x+i)\,c_2\,\mathrm{e}^{-ix}+c_1(x+i)\,\mathrm{e}^{ix}}{x^{3/2}}$$

1.677.5 Mathematica DSolve solution

Solving time : 0.092 (sec)
Leaf size : 44

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-9/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\frac{\sqrt{\frac{2}{\pi}}((c_1x+c_2)\cos(x)+(c_2x-c_1)\sin(x))}{x^{3/2}}$$

1.678 problem 695

Internal problem ID [8816]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 695
Date solved : `Monday, October 21, 2024 at 05:22:16 PM`
CAS classification : `[_Lienard]`

Solve

$$xy'' + 2y' + xy = 0$$

1.678.1 Solved as second order ode using Kovacic algorithm

Time used: 0.146 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1294: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(x)}{x}\right) + c_2\left(\frac{\cos(x)}{x}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.678.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -y - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2y'}{x} + y = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k->k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k}{(k+1)(k+2)}, 0=0\right]$$

- Recursion relation for $r=0$

  $$a_{k+2}=-\frac{a_k}{(k+2)(k+3)}$$

- Solution for $r=0$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k}, a_{k+2}=-\frac{a_k}{(k+2)(k+3)}, 2a_1=0\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+2}=-\frac{a_k}{(k+1)(k+2)}, 0=0, b_{k+2}=-\frac{b_k}{(k+2)(k+3)}, 2b_1=0\right]$$

1.678.3 Maple trace

```
Methods for second order ODEs:
```

1.678.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1\sin\left(x\right)+c_2\cos\left(x\right)}{x}$$

1.678.5 Mathematica DSolve solution

Solving time : `0.041` (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{2c_1e^{-ix}-ic_2e^{ix}}{2x}$$

1.679 problem 696

Internal problem ID [8817]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 696
Date solved : `Monday, October 21, 2024 at 05:22:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2xy'' + 5(1-2x)\,y' - 5y = 0$$

1.679.1 Solved as second order ode using Kovacic algorithm

Time used: 0.375 (sec)

Writing the ode as

$$2xy'' + (-10x+5)\,y' - 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -10x+5 \\ C &= -5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{100x^2 - 60x + 5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 100x^2 - 60x + 5$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{100x^2 - 60x + 5}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1296: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{25}{4} + \frac{5}{16x^2} - \frac{15}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{5}{2} - \frac{3}{4x} - \frac{1}{20x^2} - \frac{3}{200x^3} - \frac{1}{200x^4} - \frac{9}{5000x^5} - \frac{137}{200000x^6} - \frac{543}{2000000x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{5}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{5}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{25}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{100x^2 - 60x + 5}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{25}{4}\right) + \left(\frac{-60x + 5}{16x^2}\right) \\ &= \frac{25}{4} + \frac{-60x + 5}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -60. Dividing this by leading coefficient in t which is 16 gives $-\frac{15}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{15}{4}\right) - (0) \\ &= -\frac{15}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{5}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{15}{4}}{\frac{5}{2}} - 0\right) &= -\frac{3}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{15}{4}}{\frac{5}{2}} - 0\right) &= \frac{3}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{100x^2 - 60x + 5}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{5}{2}$	$-\frac{3}{4}$	$\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{3}{4} - \left(-\frac{1}{4}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4x} + (-)\left(\frac{5}{2}\right) \\
&= -\frac{1}{4x} - \frac{5}{2} \\
&= -\frac{1}{4x} - \frac{5}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4x} - \frac{5}{2}\right)(1) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x} - \frac{5}{2}\right)^2 - \left(\frac{100x^2 - 60x + 5}{16x^2}\right) \right) = 0$$

$$\frac{-1 + 10a_0}{2x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{1}{10} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{10}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{10}\right) \mathrm{e}^{\int (-\frac{1}{4x} - \frac{5}{2}) dx} \\
&= \left(x + \frac{1}{10}\right) \mathrm{e}^{-\frac{5x}{2} - \frac{\ln(x)}{4}} \\
&= \frac{(1 + 10x) \, \mathrm{e}^{-\frac{5x}{2}}}{10x^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-10x+5}{2x} \, dx} \\
&= z_1 e^{\frac{5x}{2} - \frac{5 \ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{5x}{2}}}{x^{5/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1 + 10x}{10x^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x+5}{2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{5x - \frac{5 \ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{100 \, \mathrm{e}^{5x - \frac{5 \ln(x)}{2}} x^3}{(1 + 10x)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1+10x}{10x^{3/2}}\right) + c_2 \left(\frac{1+10x}{10x^{3/2}} \left(\int \frac{100\, \mathrm{e}^{5x-\frac{5\ln(x)}{2}} x^3}{(1+10x)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.679.2 Maple step by step solution

Let's solve

$2x\left(\frac{d}{dx}y'\right) + 5(1-2x)\,y' - 5y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{5y}{2x} + \frac{5(2x-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{5(2x-1)y'}{2x} - \frac{5y}{2x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{5(2x-1)}{2x}, P_3(x) = -\frac{5}{2x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x\left(\frac{d}{dx}y'\right) + (-10x+5)\,y' - 5y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+5+2r) - 5a_k(2k+2r+1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+1+r)\left(k+\frac{5}{2}+r\right) a_{k+1} - 10\left(k+r+\frac{1}{2}\right) a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{5(2k+2r+1)a_k}{(k+1+r)(2k+5+2r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{5(2k+1)a_k}{(k+1)(2k+5)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{5(2k+1)a_k}{(k+1)(2k+5)}\right]$

- Recursion relation for $r = -\frac{3}{2}$; series terminates at $k = 1$

  $a_{k+1} = \frac{5(2k-2)a_k}{\left(k-\frac{1}{2}\right)(2k+2)}$

- Apply recursion relation for $k = 0$

  $a_1 = 10a_0$

- Terminating series solution of the ODE for $r = -\frac{3}{2}$. Use reduction of order to find the second

  $y = a_0 \cdot (1+10x)$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+b_0\cdot(1+10x), a_{k+1}=\frac{5(2k+1)a_k}{(k+1)(2k+5)}\right]$$

1.679.3 Maple trace

```
Methods for second order ODEs:
```

1.679.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 44

```
dsolve(2*x*diff(diff(y(x),x),x)+5*(1-2*x)*diff(y(x),x)-5*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{10\sqrt{5}\,c_1\sqrt{\pi}\,\left(x+\frac{1}{10}\right)\operatorname{erfi}\left(\sqrt{5}\,\sqrt{x}\right)-10\,\mathrm{e}^{5x}c_1\sqrt{x}+10c_2\left(x+\frac{1}{10}\right)}{x^{3/2}}$$

1.679.5 Mathematica DSolve solution

Solving time : `0.047` (sec)
Leaf size : 40

```
DSolve[{2*x*D[y[x],{x,2}]+5*(1-2*x)*D[y[x],x]-5*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_2 L_{-\frac{1}{2}}^{\frac{3}{2}}(5x)+\frac{c_1(10x+1)}{10\sqrt{5}x^{3/2}}$$

1.680 problem 697

Internal problem ID [8818]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 697
Date solved : `Monday, October 21, 2024 at 05:22:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

1.680.1 Solved as second order ode using Kovacic algorithm

Time used: 0.162 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1298: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.680.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 4xy' + (4x^2 - 1)y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$$

1.680.3 Maple trace

```
Methods for second order ODEs:
```

1.680.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{\sqrt{x}}$$

1.680.5 Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

1.681 problem 698

Internal problem ID [8819]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 698
Date solved : `Monday, October 21, 2024 at 05:22:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (x+n)\,y' + (n+1)\,y = 0$$

1.681.1 Solved as second order ode using Kovacic algorithm

Time used: 0.444 (sec)

Writing the ode as

$$xy'' + (x+n)\,y' + (n+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x+n \\ C &= n+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = n^2 - 2xn + x^2 - 2n - 4x$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1300: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{\frac{1}{4}n^2 - \frac{1}{2}n}{x^2} + \frac{-\frac{n}{2} - 1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{1}{4}n^2 - \frac{1}{2}n$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{n}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = 1 - \frac{n}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} - \frac{n}{2x} - \frac{3n^6}{2x^7} - \frac{3n^5}{2x^6} - \frac{3n^4}{2x^5} - \frac{3n^3}{2x^4} - \frac{3n^2}{2x^3} - \frac{3n}{2x^2} - \frac{77n^5}{2x^7} - \frac{53n^4}{2x^6} - \frac{67n^3}{4x^5} - \frac{37n^2}{4x^4} - \frac{4n}{x^3} - \frac{1075n^4}{4x^7} - \frac{491n^3}{4x^6} - 9\ldots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{(-2n-4)\,x + n^2 - 2n}{4x^2}\right) \\ &= \frac{1}{4} + \frac{(-2n-4)\,x + n^2 - 2n}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is $-2n-4$. Dividing this by leading coefficient in t which is 4 gives $-\frac{n}{2}-1$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{n}{2}-1\right)-(0)\\&= -\frac{n}{2}-1\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2}\\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a}-v\right) &= \frac{1}{2}\left(\frac{-\frac{n}{2}-1}{\frac{1}{2}}-0\right) &= -\frac{n}{2}-1\\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &= \frac{1}{2}\left(-\frac{-\frac{n}{2}-1}{\frac{1}{2}}-0\right) &= \frac{n}{2}+1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{n^2-2xn+x^2-2n-4x}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{n}{2}$	$1-\frac{n}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{n}{2}-1$	$\frac{n}{2}+1$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{n}{2}+1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right)\\&= \frac{n}{2}+1-\left(\frac{n}{2}\right)\\&= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{n}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{n}{2x} - \frac{1}{2} \\
&= \frac{n-x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{n}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{n}{2x^2}\right) + \left(\frac{n}{2x} - \frac{1}{2}\right)^2 - \left(\frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2}\right)\right) = 0$$

$$\frac{n + a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -n\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - n$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-n)\, e^{\int \left(\frac{n}{2x} - \frac{1}{2}\right) dx} \\
&= (x-n)\, e^{-\frac{x}{2} + \frac{n \ln(x)}{2}} \\
&= -(n-x)\, x^{\frac{n}{2}} e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x+n}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{n\ln(x)}{2}} \\
&= z_1 \left(x^{-\frac{n}{2}} \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x-n)\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x+n}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-n\ln(x)-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-n\ln(x)-x}\mathrm{e}^{2x}}{(x-n)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x-n)\,\mathrm{e}^{-x}\right) + c_2\left((x-n)\,\mathrm{e}^{-x}\left(\int \frac{\mathrm{e}^{-n\ln(x)-x}\mathrm{e}^{2x}}{(x-n)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.681.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right)+(x+n)\,y'+(n+1)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(n+1)y}{x}-\frac{(x+n)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(x+n)y'}{x}+\frac{(n+1)y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

◦ Define functions

$\left[P_2(x)=\frac{x+n}{x},P_3(x)=\frac{n+1}{x}\right]$

◦ $x\cdot P_2(x)$ is analytic at $x=0$

$$(x\cdot P_2(x))\bigg|_{x=0}=n$$

◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$(x^2\cdot P_3(x))\bigg|_{x=0}=0$$

◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$x\left(\frac{d}{dx}y'\right)+(x+n)\,y'+(n+1)\,y=0$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty}a_k x^{k+r}$$

□ Rewrite ODE with series expansions

◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

$$x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$$

◦ Shift index using $k->k+1-m$

$$x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

◦ Convert $x\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+r+n)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r+n) + a_k(k+r+n+1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-1+r+n) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, -n+1\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k+r+n) + a_k(k+r+n+1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k(k+r+n+1)}{(k+1+r)(k+r+n)}$
- Recursion relation for $r = 0$
  $a_{k+1} = -\frac{a_k(k+1+n)}{(k+1)(k+n)}$
- Solution for $r = 0$
  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k(k+1+n)}{(k+1)(k+n)}\right]$$
- Recursion relation for $r = -n+1$
  $a_{k+1} = -\frac{a_k(k+2)}{(k+2-n)(k+1)}$
- Solution for $r = -n+1$
  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-n+1}, a_{k+1} = -\frac{a_k(k+2)}{(k+2-n)(k+1)}\right]$$
- Combine solutions and rename parameters
  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-n+1}\right), a_{k+1} = -\frac{a_k(k+1+n)}{(k+1)(k+n)}, b_{k+1} = -\frac{b_k(k+2)}{(k+2-n)(k+1)}\right]$$

1.681.3 Maple trace

```
Methods for second order ODEs:
```

1.681.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 42

```
dsolve(x*diff(diff(y(x),x),x)+(x+n)*diff(y(x),x)+(n+1)*y(x) = 0,
       y(x),singsol=all)
```

$$
y = \frac{\left(c_2\, x^{-n+1} \operatorname{hypergeom}\left(\left[-n\right], \left[-n+2\right], x\right) n + c_1 (n-x)\right) \mathrm{e}^{-x}}{n}
$$

1.681.5 Mathematica DSolve solution

Solving time : 0.724 (sec)
Leaf size : 48

```
DSolve[{x*D[y[x],{x,2}]+(x+n)*D[y[x],x]+(n+1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to e^{-x}(n-x)\left(c_2 \int_1^x \frac{e^{K[1]}K[1]^{-n}}{(n-K[1])^2}dK[1] + c_1\right)
$$

1.682 problem 699

Internal problem ID [8820]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 699
Date solved : `Monday, October 21, 2024 at 05:22:20 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^4y'' + xy' + y = 0$$

1.682.1 Solved as second order ode using Kovacic algorithm

Time used: 0.354 (sec)

Writing the ode as

$$x^4y'' + xy' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-10x^2 + 1}{4x^6} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -10x^2 + 1 \\
t &= 4x^6
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-10x^2 + 1}{4x^6}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1302: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^6$. There is a pole at $x = 0$ of order 6. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{1}{4x^6} - \frac{5}{2x^4}$$

There is pole in r at $x = 0$ of order 6, hence $v = 3$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^3} - \frac{5}{2x} - \frac{25x}{4} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 3$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^3} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^3}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^4}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $-\frac{5}{2}$. Therefore

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^3} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} + 3\right) &&= -1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} + 3\right) &&= 4 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-10x^2 + 1}{4x^6}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	6	$\frac{1}{2x^3}$	-1	4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to

determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= 1 - (-1) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x^3} - \frac{1}{x} + (-)\,(0) \\
&= \frac{1}{2x^3} - \frac{1}{x} \\
&= \frac{1}{2x^3} - \frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2x^3} - \frac{1}{x}\right)(2x + a_1) + \left(\left(-\frac{3}{2x^4} + \frac{1}{x^2}\right) + \left(\frac{1}{2x^3} - \frac{1}{x}\right)^2 - \left(\frac{-10x^2+1}{4x^6}\right)\right) = 0$$

$$\frac{(2a_0+2)\,x + a_1}{x^3} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{\int \left(\frac{1}{2x^3} - \frac{1}{x}\right) dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{-\frac{1}{4x^2} - \ln(x)} \\
&= \frac{\left(x^2 - 1\right) \mathrm{e}^{-\frac{1}{4x^2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^4} \, dx} \\
&= z_1 e^{\frac{1}{4x^2}} \\
&= z_1 \left(\mathrm{e}^{\frac{1}{4x^2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 - 1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^4} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{1}{2x^2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{1}{2x^2}} x^2}{\left(x^2 - 1\right)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2-1}{x}\right) + c_2\left(\frac{x^2-1}{x}\left(\int \frac{e^{\frac{1}{2x^2}}x^2}{(x^2-1)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.682.2 Maple step by step solution

1.682.3 Maple trace

```
Methods for second order ODEs:
```

1.682.4 Maple dsolve solution

Solving time : `0.012` (sec)
Leaf size : 50

```
dsolve(x^4*diff(diff(y(x),x),x)+x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$
y = \frac{c_1\sqrt{2}\,\sqrt{\pi}\,(x-1)\,(x+1)\,\mathrm{erfi}\left(\frac{\sqrt{2}}{2x}\right) + c_2\,x^2 + 2\,e^{\frac{1}{2x^2}}c_1 x - c_2}{x}
$$

1.682.5 Mathematica DSolve solution

Solving time : `0.176` (sec)
Leaf size : 61

```
DSolve[{x^4*D[y[x],{x,2}]+x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to -\frac{\sqrt{2\pi}c_2(x^2-1)\,\mathrm{erfi}\left(\frac{1}{\sqrt{2}x}\right) - 4c_1(x^2-1) + 2c_2e^{\frac{1}{2x^2}}x}{4x}
$$

1.683 problem 700

Internal problem ID [8821]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 700
Date solved : `Monday, October 21, 2024 at 05:22:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + (2x^2 + x)\, y' - 4y = 0$$

1.683.1 Solved as second order ode using Kovacic algorithm

Time used: 0.277 (sec)

Writing the ode as

$$x^2y'' + (2x^2 + x)\, y' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 + x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 + 4x + 15$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1303: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{15}{4x^2} + \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} + \frac{7}{4x^2} - \frac{7}{8x^3} - \frac{35}{32x^4} + \frac{133}{64x^5} + \frac{63}{128x^6} - \frac{1239}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 4x + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{4x + 15}{4x^2}\right) \\ &= 1 + \frac{4x + 15}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 + 4x + 15}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{2x} + (-)\,(1) \\
&= -\frac{3}{2x} - 1 \\
&= -\frac{3}{2x} - 1
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{3}{2x} - 1\right)(1) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x} - 1\right)^2 - \left(\frac{4x^2 + 4x + 15}{4x^2}\right)\right) &= 0 \\
\frac{-3 + 2a_0}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{3}{2}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{3}{2}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{3}{2}\right) e^{\int \left(-\frac{3}{2x} - 1\right) dx} \\
&= \left(x + \frac{3}{2}\right) e^{-x - \frac{3\ln(x)}{2}} \\
&= \frac{(3 + 2x)\, e^{-x}}{2x^{3/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2+x}{x^2}\,dx} \\
&= z_1 e^{-x-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-2x}(3+2x)}{2x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(2x^2-4x+3)\,x\,\mathrm{e}^{-2x-\ln(x)}\mathrm{e}^{4x}}{6+4x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-2x}(3+2x)}{2x^2}\right) + c_2 \left(\frac{\mathrm{e}^{-2x}(3+2x)}{2x^2}\left(\frac{(2x^2-4x+3)\,x\,\mathrm{e}^{-2x-\ln(x)}\mathrm{e}^{4x}}{6+4x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.683.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+\left(2x^2+x\right)y'-4y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{4y}{x^2}-\frac{(2x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(2x+1)y'}{x}-\frac{4y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

 ◦ Define functions

 $\left[P_2(x)=\frac{2x+1}{x}, P_3(x)=-\frac{4}{x^2}\right]$

 ◦ $x\cdot P_2(x)$ is analytic at $x=0$

 $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$

 ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

 $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-4$

 ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right)+x(2x+1)y'-4y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

 ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

 $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

 ◦ Shift index using $k->k+1-m$

 $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

 ◦ Convert $x^2\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(2+r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-2) + 2a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(2+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 2\}$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+2)\,(k+r-2) + 2a_{k-1}(k+r-1) = 0$
- Shift index using $k->k+1$
  $a_{k+1}(k+3+r)\,(k+r-1) + 2a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{2a_k(k+r)}{(k+3+r)(k+r-1)}$
- Recursion relation for $r=-2$; series terminates at $k=2$
  $a_{k+1} = -\frac{2a_k(k-2)}{(k+1)(k-3)}$
- Apply recursion relation for $k=0$
  $a_1 = -\frac{4a_0}{3}$
- Apply recursion relation for $k=1$
  $a_2 = -\frac{a_1}{2}$
- Express in terms of a_0
  $a_2 = \frac{2a_0}{3}$
- Terminating series solution of the ODE for $r=-2$. Use reduction of order to find the second
  $y = a_0 \cdot \left(1 - \frac{4}{3}x + \frac{2}{3}x^2\right)$
- Recursion relation for $r=2$
  $a_{k+1} = -\frac{2a_k(k+2)}{(k+5)(k+1)}$
- Solution for $r=2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{2a_k(k+2)}{(k+5)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y = a_0 \cdot \left(1 - \frac{4}{3}x + \frac{2}{3}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = -\frac{2b_k(k+2)}{(k+5)(k+1)}\right]$

1.683.3 Maple trace

```
Methods for second order ODEs:
```

1.683.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 31

```
dsolve(x^2*diff(diff(y(x),x),x)+(2*x^2+x)*diff(y(x),x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \, \mathrm{e}^{-2x}(3+2x)+2c_1\left(x^2-2x+\frac{3}{2}\right)}{x^2}$$

1.683.5 Mathematica DSolve solution

Solving time : 0.105 (sec)
Leaf size : 44

```
DSolve[{x^2*D[y[x],{x,2}]+(x+2*x^2)*D[y[x],x]-4*y[x]==2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}\left(\frac{2c_1e^{-2x}(2x+3)}{x^2}+\frac{c_2(2x^2-4x+3)}{x^2}-2\right)$$

1.684 problem 701

Internal problem ID [8822]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 701
Date solved : `Monday, October 21, 2024 at 05:22:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(4x^3-14x^2-2x\right)y''-\left(6x^2-7x+1\right)y'+\left(6x-1\right)y=0$$

1.684.1 Solved as second order ode using Kovacic algorithm

Time used: 0.463 (sec)

Writing the ode as

$$\left(4x^3-14x^2-2x\right)y''+\left(-6x^2+7x-1\right)y'+\left(6x-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^3-14x^2-2x\\B&=-6x^2+7x-1\\C&=6x-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16\left(2x^3 - 7x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -12x^4 + 156x^3 + 297x^2 - 78x - 3 \\ t &= 16\left(2x^3 - 7x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16\left(2x^3 - 7x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1305: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 6 - 4\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^3 - 7x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{7}{4} + \frac{\sqrt{57}}{4}$ of order 2. There is a pole at $x = \frac{7}{4} - \frac{\sqrt{57}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{9}{4x} - \frac{3}{16x^2} + \frac{3}{4\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)^2} + \frac{3}{4\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)^2} + \frac{\frac{9}{8} - \frac{29\sqrt{57}}{152}}{x - \frac{7}{4} - \frac{\sqrt{57}}{4}} + \frac{\frac{9}{8} + \frac{29\sqrt{57}}{152}}{x - \frac{7}{4} + \frac{\sqrt{57}}{4}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4}\end{aligned}$$

For the pole at $x = \frac{7}{4} + \frac{\sqrt{57}}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = \frac{7}{4} - \frac{\sqrt{57}}{4}$ let b be the coefficient of $\frac{1}{\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16\left(2x^3 - 7x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16\left(2x^3 - 7x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$\frac{7}{4} + \frac{\sqrt{57}}{4}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$\frac{7}{4} - \frac{\sqrt{57}}{4}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{1}{4} - \left(-\frac{3}{4}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)} + (-)(0) \\ &= \frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)} \\ &= \frac{-6x^2+7x-1}{8x^3-28x^2-4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)}\right)(1) + \left(\left(-\frac{1}{4x^2} + \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)^2} + \frac{1}{2\left(x-\frac{7}{4}+\right.}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-1)\, \mathrm{e}^{\int \left(\frac{1}{4x} - \frac{1}{2\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)}\right) dx} \\
&= (x-1)\, \mathrm{e}^{-\frac{\left(57+7\sqrt{57}\right)\sqrt{57}\, \ln\left(4x-7-\sqrt{57}\right)}{2\left(399+57\sqrt{57}\right)} + \frac{\left(-57+7\sqrt{57}\right)\sqrt{57}\, \ln\left(4x-7+\sqrt{57}\right)}{-798+114\sqrt{57}} + \frac{2\ln(x)}{\left(7+\sqrt{57}\right)\left(-7+\sqrt{57}\right)}} \\
&= \frac{(x-1)\, x^{1/4}}{\sqrt{4x-7-\sqrt{57}}\, \sqrt{4x-7+\sqrt{57}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-6x^2+7x-1}{4x^3-14x^2-2x} \, dx} \\
&= z_1 e^{\frac{\ln\left(2x^2-7x-1\right)}{2} - \frac{\ln(x)}{4}} \\
&= z_1 \left(\frac{\sqrt{2x^2-7x-1}}{x^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x-1)\sqrt{2}}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x^2+7x-1}{4x^3-14x^2-2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(2x^2-7x-1)-\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{16x(2x+1)\,\mathrm{e}^{\ln(2x^2-7x-1)-\frac{\ln(x)}{2}}}{(x-1)(2x^2-7x-1)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x-1)\sqrt{2}}{4}\right) + c_2 \left(\frac{(x-1)\sqrt{2}}{4}\left(\frac{16x(2x+1)\,\mathrm{e}^{\ln(2x^2-7x-1)-\frac{\ln(x)}{2}}}{(x-1)(2x^2-7x-1)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.684.2 Maple step by step solution

Let's solve

$(4x^3-14x^2-2x)\left(\frac{d}{dx}y'\right)-(6x^2-7x+1)\,y'+(6x-1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(6x-1)y}{2x(2x^2-7x-1)}+\frac{(6x^2-7x+1)y'}{2x(2x^2-7x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(6x^2-7x+1)y'}{2x(2x^2-7x-1)}+\frac{(6x-1)y}{2x(2x^2-7x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{6x^2-7x+1}{2x(2x^2-7x-1)}, P_3(x) = \frac{6x-1}{2x(2x^2-7x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$2x(2x^2 - 7x - 1)\left(\frac{d}{dx}y'\right) + (-6x^2 + 7x - 1)\, y' + (6x - 1)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..3$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r-2+m}$$

- Shift index using $k- > k + 2 - m$

  $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-1 + 2r)\, x^{-1+r} + \left(-a_1(1 + r)\,(1 + 2r) - a_0(14r^2 - 21r + 1)\right) x^r + \left(\sum_{k=1}^{\infty} (-a_{k+1}(k + 1 + r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-1 + 2r) = 0$$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{2}\right\}$

- Each term must be 0

  $-a_1(1+r)(1+2r) - a_0(14r^2 - 21r + 1) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-14a_k + 4a_{k-1} - 2a_{k+1}) k^2 + ((-28a_k + 8a_{k-1} - 4a_{k+1}) r + 21a_k - 18a_{k-1} - 3a_{k+1}) k + (-14a$

- Shift index using $k- >k+1$

  $(-14a_{k+1} + 4a_k - 2a_{k+2}) (k+1)^2 + ((-28a_{k+1} + 8a_k - 4a_{k+2}) r + 21a_{k+1} - 18a_k - 3a_{k+2}) (k +$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} + 8kra_k - 28kra_{k+1} + 4r^2a_k - 14r^2a_{k+1} - 10ka_k - 7ka_{k+1} - 10ra_k - 7ra_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 4kr + 2r^2 + 7k + 7r + 6}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 10ka_k - 7ka_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 10ka_k - 7ka_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}, -a_1 - a_0 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 6ka_k - 21ka_{k+1} + 2a_k - a_{k+1}}{2k^2 + 9k + 10}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 6ka_k - 21ka_{k+1} + 2a_k - a_{k+1}}{2k^2 + 9k + 10}, -3a_1 + 6a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 10ka_k - 7ka_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}, -a_1 - a_0 = 0, b_k\right.$

1.684.3 Maple trace

```
Methods for second order ODEs:
```

1.684.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 21

```
dsolve((4*x^3-14*x^2-2*x)*diff(diff(y(x),x),x)-(6*x^2-7*x+1)*diff(y(x),x)+(6*x-1)*y(x)
       y(x),singsol=all)
```

$$y = c_2\sqrt{x} + c_1(x-1) + 2c_2\, x^{3/2}$$

1.684.5 Mathematica DSolve solution

Solving time : 9.894 (sec)
Leaf size : 26

```
DSolve[{(4*x^3-14*x^2-2*x)*D[y[x],{x,2}]-(6*x^2-7*x+1)*D[y[x],x]+(6*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1(x-1) - 2c_2\sqrt{x}(2x+1)$$

1.685 problem 702

Internal problem ID [8823]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 702
Date solved : `Monday, October 21, 2024 at 05:22:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x^2y' + (x-2)\, y = 0$$

1.685.1 Solved as second order ode using Kovacic algorithm

Time used: 0.225 (sec)

Writing the ode as

$$x^2y'' + x^2y' + (x-2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1307: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{x} \\
&= \frac{x - 2}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\
&= \frac{e^{\frac{x}{2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(e^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1\left(-\left(x^2+2x+2\right) \mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(-\left(x^2+2x+2\right) \mathrm{e}^{-x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.685.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + x^2 y' + (x-2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x-2)y}{x^2} - y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + y' + \frac{(x-2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$\left[P_2(x) = 1, P_3(x) = \frac{x-2}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2\left(\frac{d}{dx}y'\right) + x^2y' + (x-2)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y'$ to series expansion

  $$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$$

- Shift index using $k->k-1$

  $$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-2+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r)\right)x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-2) + a_{k-1}(k+r) = 0$
- Shift index using $k- > k+1$

  $a_{k+1}(k+2+r)(k-1+r) + a_k(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+1)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k k}{(k+1)(k-2)}$
- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = -\frac{a_k k}{(k+1)(k-2)}$
- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{a_k(k+3)}{(k+4)(k+1)}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k(k+3)}{(k+4)(k+1)}\right]$

1.685.3 Maple trace

```
Methods for second order ODEs:
```

1.685.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 24

```
dsolve(x^2*diff(diff(y(x),x),x)+x^2*diff(y(x),x)+(x-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2(x^2 + 2x + 2)\,e^{-x} + c_1}{x}$$

1.685.5 Mathematica DSolve solution

Solving time : 0.054 (sec)
Leaf size : 29

```
DSolve[{x^2*D[y[x],{x,2}]+x^2*D[y[x],x]+(x-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1 - c_2 e^{-x}(x^2 + 2x + 2)}{x}$$

1.686 problem 703

Internal problem ID [8824]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 703
Date solved : `Monday, October 21, 2024 at 05:22:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - x^2y' + (x-2)\, y = 0$$

1.686.1 Solved as second order ode using Kovacic algorithm

Time used: 0.226 (sec)

Writing the ode as

$$x^2y'' - x^2y' + (x-2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1309: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{x} \\ &= \frac{x - 2}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\ &= \frac{\mathrm{e}^{\frac{x}{2}}}{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x^2}{x^2} \, dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^x}{(y_1)^2}\, dx \\
&= y_1 \left(-\left(x^2+2x+2\right) e^{-x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{x}\right) + c_2 \left(\frac{e^x}{x} \left(-\left(x^2+2x+2\right) e^{-x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.686.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - x^2 y' + (x-2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x-2)y}{x^2} + y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - y' + \frac{(x-2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -1, P_3(x) = \frac{x-2}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - x^2 y' + (x-2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$

- Shift index using $k- >k - 1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left(a_k(k+r+1)-a_{k-1}\right)=0$
- Shift index using $k- >k+1$

  $(k-1+r)\left(a_{k+1}(k+2+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+2+r}$
- Recursion relation for $r=-1$

  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+4}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=\frac{a_k}{k+4}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+4}\right]$

1.686.3 Maple trace

```
Methods for second order ODEs:
```

1.686.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 23

```
dsolve(x^2*diff(diff(y(x),x),x)-x^2*diff(y(x),x)+(x-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1 \, \mathrm{e}^x+c_2(x^2+2x+2)}{x}$$

1.686.5 Mathematica DSolve solution

Solving time : 0.051 (sec)
Leaf size : 28

```
DSolve[{x^2*D[y[x],{x,2}]-x^2*D[y[x],x]+(x-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1 e^x - c_2(x^2 + 2x + 2)}{x}$$

1.687 problem 704

Internal problem ID [8825]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 704
Date solved : `Monday, October 21, 2024 at 05:22:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1-4x)\,y''+\left(-\frac{1}{4}x-x^2\right)y'-\frac{5xy}{16}=0$$

1.687.1 Solved as second order ode using Kovacic algorithm

Time used: 0.375 (sec)

Writing the ode as

$$\left(-4x^3+x^2\right)y''+\left(-\frac{1}{4}x-x^2\right)y'-\frac{5xy}{16}=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-4x^3+x^2\\B&=-\frac{1}{4}x-x^2\\C&=-\frac{5x}{16}\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-192x^2 - 36x + 9}{64\left(4x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -192x^2 - 36x + 9 \\ t &= 64\left(4x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-192x^2 - 36x + 9}{64\left(4x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1311: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(4x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{16x} + \frac{9}{64x^2} - \frac{3}{16\left(x - \frac{1}{4}\right)^2} - \frac{9}{16\left(x - \frac{1}{4}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \left\{2, -\frac{1}{2}, \frac{9}{2}\right\} \end{aligned}$$

For the pole at $x = \frac{1}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-192x^2 - 36x + 9}{64\left(4x^2 - x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$\begin{aligned}
E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\
&= \{1,2,3\}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{2, -\frac{1}{2}, \frac{9}{2}\}$
$\frac{1}{4}$	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{1,2,3\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1=2,\ e_2=1,\ e_\infty=3$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\
&= \frac{1}{2}(3-(2+(1)))\\
&= 0
\end{aligned}$$

We now form the following rational function

$$\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\
&= \frac{1}{2}\left(\frac{2}{(x-(0))}+\frac{1}{(x-(\frac{1}{4}))}\right)\\
&= \frac{1}{x}+\frac{1}{2x-\frac{1}{2}}
\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p'''+3\theta p''+\left(3\theta^2+3\theta'-4r\right)p'+\left(\theta''+3\theta\theta'+\theta^3-4r\theta-2r'\right)p=0 \tag{1A}$$

Since $d=0$, then letting

$$p=1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x} + \frac{1}{2x - \frac{1}{2}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x} + \frac{1}{2x - \frac{1}{2}}\right) w + \frac{576x^2 - 92x - 9}{64x^2 (-1 + 4x)^2} = 0$$

Solving for ω gives

$$\omega = \frac{24x - 4 + 5\sqrt{1 - 4x}}{8x (-1 + 4x)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= e^{\int \frac{24x - 4 + 5\sqrt{1-4x}}{8x(-1+4x)} dx} \\ &= \frac{\sqrt{x}\,(-1 + 4x)^{1/4}\, 2^{3/4} \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{5/4}}{4}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-\frac{1}{4}x - x^2}{-4x^3 + x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{8} - \frac{\ln(-1+4x)}{4}} \\ &= z_1 \left(\frac{x^{1/8}}{(-1 + 4x)^{1/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/8} 2^{3/4} \left(\sqrt{1-4x}+1\right) \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{1/4}}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{1}{4}x - x^2}{-4x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{4} - \frac{\ln(-1+4x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{4\, \mathrm{e}^{\frac{\ln(x)}{4} - \frac{\ln(-1+4x)}{2}} \sqrt{2}}{x^{1/4} \left(\sqrt{1-4x}+1\right)^2 \sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/8} 2^{3/4} \left(\sqrt{1-4x}+1\right) \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{1/4}}{4} \right) + c_2 \left(\frac{x^{1/8} 2^{3/4} \left(\sqrt{1-4x}+1\right) \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{1/4}}{4} \left(\int \frac{4\, \mathrm{e}^{\ln}}{x^{1/4} \left(\sqrt{1-\right.}} \right.\right.
\end{aligned}$$

Will add steps showing solving for IC soon.

1.687.2 Maple step by step solution

Let's solve

$x^2(1-4x)\left(\frac{d}{dx}y'\right)+\left(-\frac{1}{4}x-x^2\right)y'-\frac{5xy}{16}=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{5y}{16x(-1+4x)}-\frac{(1+4x)y'}{4x(-1+4x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(1+4x)y'}{4x(-1+4x)}+\frac{5y}{16x(-1+4x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{1+4x}{4x(-1+4x)}, P_3(x)=\frac{5}{16x(-1+4x)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-\frac{1}{4}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $16x(-1+4x)\left(\frac{d}{dx}y'\right)+(16x+4)y'+5y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

    $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=1..2$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- ◦ Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-4a_0r(-5+4r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (-4a_{k+1}(k+1+r)(4k-1+4r) + a_k(8k+8r-1)(8k+8r-5$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-4r(-5+4r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{5}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-16(k+1+r)\left(k-\frac{1}{4}+r\right) a_{k+1} + 64\left(k+r-\frac{5}{8}\right) a_k \left(k+r-\frac{1}{8}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(8k+8r-5)a_k(8k+8r-1)}{4(k+1+r)(4k-1+4r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{(8k-5)a_k(8k-1)}{4(k+1)(4k-1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(8k-5)a_k(8k-1)}{4(k+1)(4k-1)}\right]$$

- Recursion relation for $r = \frac{5}{4}$

$$a_{k+1} = \frac{(8k+5)a_k(8k+9)}{4\left(k+\frac{9}{4}\right)(4k+4)}$$

- Solution for $r = \frac{5}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{4}}, a_{k+1} = \frac{(8k+5)a_k(8k+9)}{4\left(k+\frac{9}{4}\right)(4k+4)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{4}}\right), a_{k+1} = \frac{(8k-5)a_k(8k-1)}{4(k+1)(4k-1)}, b_{k+1} = \frac{(8k+5)b_k(8k+9)}{4\left(k+\frac{9}{4}\right)(4k+4)}\right]$$

1.687.3 Maple trace

```
Methods for second order ODEs:
```

1.687.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 55

```
dsolve(x^2*(1-4*x)*diff(diff(y(x),x),x)+(-1/4*x-x^2)*diff(y(x),x)-5/16*x*y(x) = 0,
       y(x),singsol=all)
```

$$
y = -\frac{2^{1/4}\left(c_1\sqrt{2}\left(x-\frac{\sqrt{1-4x}}{2}-\frac{1}{2}\right)\sqrt{\sqrt{1-4x}+1}-2c_2\,x^{5/4}\right)}{\left(\sqrt{1-4x}+1\right)^{5/4}}
$$

1.687.5 Mathematica DSolve solution

Solving time : `4.062` (sec)
Leaf size : 111

```
DSolve[{x^2*(1-4*x)*D[y[x],{x,2}]+((1-(5/4))*x-(6-4*(5/4))*x^2)*D[y[x],x]+(5/4)*(1-(5/4))*x*
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x)\to \frac{\sqrt[8]{x}\sqrt[4]{4x-1}\left(5c_1\left(\sqrt{4x-1}-i\right)^{5/4}+ic_2\left(\sqrt{4x-1}+i\right)^{5/4}\right)}{5\sqrt[4]{1-4x}\sqrt[8]{\sqrt{4x-1}-i}\sqrt[8]{\sqrt{4x-1}+i}}
$$

1.688 problem 705

Internal problem ID [8826]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 705
Date solved : `Monday, October 21, 2024 at 05:22:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \left(x^2 + x\right) y' + (x - 9)\, y = 0$$

1.688.1 Solved as second order ode using Kovacic algorithm

Time used: 0.279 (sec)

Writing the ode as

$$x^2y'' + \left(x^2 + x\right) y' + (x - 9)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 + x \\ C &= x - 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 2x + 35$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 35}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1313: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} + \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{17}{2x^2} + \frac{17}{2x^3} - \frac{255}{4x^4} - \frac{833}{4x^5} + \frac{3213}{4x^6} + \frac{21709}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 35}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 35}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x + 35}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 2x + 35}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{5}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + \left(\frac{1}{2}\right) \\
&= -\frac{5}{2x} + \frac{1}{2} \\
&= \frac{-5+x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{5}{2x} + \frac{1}{2}\right)(2x + a_1) + \left(\left(\frac{5}{2x^2}\right) + \left(-\frac{5}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 2x + 35}{4x^2}\right) \right) = 0$$

$$\frac{(-a_1 - 8)\,x - 2a_0 - 5a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 20, a_1 = -8\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 8x + 20$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 8x + 20\right) e^{\int \left(-\frac{5}{2x} + \frac{1}{2}\right) dx} \\
&= \left(x^2 - 8x + 20\right) e^{\frac{x}{2} - \frac{5\ln(x)}{2}} \\
&= \frac{\left(x^2 - 8x + 20\right) e^{\frac{x}{2}}}{x^{5/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2-8x+20}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(x^3+9x^2+36x+60)\,x\,\mathrm{e}^{-x-\ln(x)}}{x^2-8x+20}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2-8x+20}{x^3}\right) + c_2 \left(\frac{x^2-8x+20}{x^3}\left(-\frac{(x^3+9x^2+36x+60)\,x\,\mathrm{e}^{-x-\ln(x)}}{x^2-8x+20}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.688.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+\left(x^2+x\right)y'+(x-9)\,y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(x-9)y}{x^2}-\frac{(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(x+1)y'}{x}+\frac{(x-9)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

  $\left[P_2(x)=\frac{x+1}{x},P_3(x)=\frac{x-9}{x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-9$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right)+x(x+1)\,y'+(x-9)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  - Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(-3+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+3)\,(k+r-3) + a_{k-1}(k+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, 3\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)\,(k+r-3) + a_{k-1}(k+r) = 0$
- Shift index using $k->k+1$

  $a_{k+1}(k+4+r)\,(k-2+r) + a_k(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+1)}{(k+4+r)(k-2+r)}$
- Recursion relation for $r=-3$; series terminates at $k=2$

  $a_{k+1} = -\frac{a_k(k-2)}{(k+1)(k-5)}$
- Apply recursion relation for $k=0$

  $a_1 = -\frac{2a_0}{5}$
- Apply recursion relation for $k=1$

  $a_2 = -\frac{a_1}{8}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{20}$
- Terminating series solution of the ODE for $r=-3$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 - \frac{2}{5}x + \frac{1}{20}x^2\right)$
- Recursion relation for $r=3$

  $a_{k+1} = -\frac{a_k(k+4)}{(k+7)(k+1)}$
- Solution for $r=3$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1}=-\frac{a_k(k+4)}{(k+7)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y=a_0\cdot\left(1-\frac{2}{5}x+\frac{1}{20}x^2\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), b_{k+1}=-\frac{b_k(k+4)}{(k+7)(k+1)}\right]$$

1.688.3 Maple trace

```
Methods for second order ODEs:
```

1.688.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 38

```
dsolve(x^2*diff(diff(y(x),x),x)+(x^2+x)*diff(y(x),x)+(x-9)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_2(x^3+9x^2+36x+60)\,\mathrm{e}^{-x}+c_1(x^2-8x+20)}{x^3}$$

1.688.5 Mathematica DSolve solution

Solving time : `0.096` (sec)
Leaf size : 42

```
DSolve[{x^2*D[y[x],{x,2}]+(x+x^2)*D[y[x],x]+(x-9)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{c_1((x-8)x+20)-c_2e^{-x}(x^3+9x^2+36x+60)}{x^3}$$

1.689 problem 706

Internal problem ID [8827]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 706
Date solved : `Monday, October 21, 2024 at 05:22:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + x(x+1)\,y' + (3x-1)\,y = 0$$

1.689.1 Solved as second order ode using Kovacic algorithm

Time used: 0.306 (sec)

Writing the ode as

$$x^2y'' + \left(x^2+x\right)y' + (3x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 + x \\ C &= 3x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 10x + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 10x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1315: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4x^2} - \frac{5}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{5}{2x} - \frac{11}{2x^2} - \frac{55}{2x^3} - \frac{671}{4x^4} - \frac{4565}{4x^5} - \frac{33231}{4x^6} - \frac{253275}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-10x + 3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-10x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -10. Dividing this by leading coefficient in t which is 4 gives $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &= -\frac{5}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &= \frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 10x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{5}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{5}{2} - \left(\frac{3}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{3}{2x} - \frac{1}{2} \\
&= -\frac{-3+x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{3}{2x^2}\right) + \left(\frac{3}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 10x + 3}{4x^2}\right) \right) = 0$$

$$\frac{3 + a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -3 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (-3 + x)\, e^{\int \left(\frac{3}{2x} - \frac{1}{2}\right) dx} \\
&= (-3 + x)\, e^{-\frac{x}{2} + \frac{3\ln(x)}{2}} \\
&= (-3 + x)\, x^{3/2} e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-x}(-3+x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{7\,\mathrm{e}^{x}}{54x} - \frac{\mathrm{Ei}_1(-x)}{6} - \frac{\mathrm{e}^{x}}{27(-3+x)} - \frac{\mathrm{e}^{x}}{18x^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\big(x\,\mathrm{e}^{-x}(-3+x)\big) + c_2\left(x\,\mathrm{e}^{-x}(-3+x)\left(-\frac{7\,\mathrm{e}^{x}}{54x} - \frac{\mathrm{Ei}_1(-x)}{6} - \frac{\mathrm{e}^{x}}{27(-3+x)} - \frac{\mathrm{e}^{x}}{18x^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.689.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+x(x+1)\,y'+(3x-1)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(3x-1)y}{x^2}-\frac{(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(x+1)y'}{x}+\frac{(3x-1)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{x+1}{x},P_3(x)=\frac{3x-1}{x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-1$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right)+x(x+1)\,y'+(3x-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) + a_{k-1}(k+2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 1\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+1)\,(k+r-1) + a_{k-1}(k+2+r) = 0$$

- Shift index using $k- >k+1$

$$a_{k+1}(k+2+r)\,(k+r) + a_k(k+r+3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r+3)}{(k+2+r)(k+r)}$$

- Recursion relation for $r = -1$

$$a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-1)}$$

- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 1$

$$a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-1)}$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k(k+4)}{(k+3)(k+1)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k(k+4)}{(k+3)(k+1)}\right]$$

1.689.3 Maple trace

```
Methods for second order ODEs:
```

1.689.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 48

```
dsolve(x^2*diff(diff(y(x),x),x)+x*(x+1)*diff(y(x),x)+(3*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^2 c_2 \,\mathrm{e}^{-x}(-3+x)\,\mathrm{Ei}_1\left(-x\right)+x^2 c_1(-3+x)\,\mathrm{e}^{-x}+c_2(x^2-2x-1)}{x}$$

1.689.5 Mathematica DSolve solution

Solving time : `0.119` (sec)
Leaf size : 66

```
DSolve[{x^2*D[y[x],{x,2}]+x*(x+1)*D[y[x],x]+(3*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x}(c_2(x-3)x^2 \,\text{ExpIntegralEi}(x)+6c_1x^3-x^2(c_2e^x+18c_1)+2c_2e^x x+c_2e^x)}{6x}$$

1.690 problem 707

Internal problem ID [8828]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 707
Date solved : `Monday, October 21, 2024 at 05:22:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - \left(x^2+4x\right)y' + 4y = 0$$

1.690.1 Solved as second order ode using Kovacic algorithm

Time used: 0.265 (sec)

Writing the ode as

$$x^2y'' + \left(-x^2-4x\right)y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2-4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 8x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 8x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 8x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1317: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x} - \frac{2}{x^2} + \frac{8}{x^3} - \frac{36}{x^4} + \frac{176}{x^5} - \frac{912}{x^6} + \frac{4928}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 8x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{8x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{8x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 8. Dividing this by leading coefficient in t which is 4 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 0\right) &&= 2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 0\right) &&= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 8x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	2	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 2$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\ &= 2 - (2) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{2}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{2}{x} \\
&= \frac{x+4}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{2}{x}\right)(0) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{1}{2} + \frac{2}{x}\right)^2 - \left(\frac{x^2 + 8x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{2}{x}\right) dx} \\
&= x^2 \mathrm{e}^{\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2 - 4x}{x^2} \, dx} \\
&= z_1 e^{\frac{x}{2} + 2\ln(x)} \\
&= z_1 \left(x^2 \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^4 \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-4x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+4\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-x}}{3x^3} + \frac{\mathrm{e}^{-x}}{6x^2} - \frac{\mathrm{e}^{-x}}{6x} + \frac{\mathrm{Ei}_1(x)}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^4 \mathrm{e}^x\right) + c_2 \left(x^4 \mathrm{e}^x \left(-\frac{\mathrm{e}^{-x}}{3x^3} + \frac{\mathrm{e}^{-x}}{6x^2} - \frac{\mathrm{e}^{-x}}{6x} + \frac{\mathrm{Ei}_1(x)}{6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.690.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) - \left(x^2 + 4x\right) y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{4y}{x^2} + \frac{(x+4)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' - \frac{(x+4)y'}{x} + \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{x+4}{x}, P_3(x) = \frac{4}{x^2}\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -4$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

• Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right) - x(x+4)\,y' + 4y = 0$

• Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-4+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-4) - a_{k-1}(k+r-1)\right)x^{k+r}\right) = 0$

• $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)\,(-4+r) = 0$

• Values of r that satisfy the indicial equation

$r \in \{1, 4\}$

• Each term in the series must be 0, giving the recursion relation

$(k+r-1)\left(a_k(k+r-4) - a_{k-1}\right) = 0$

• Shift index using $k- >k+1$

$(k+r)\left(a_{k+1}(k-3+r)-a_k\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k-3+r}$
- Recursion relation for $r=1$

  $a_{k+1}=\frac{a_k}{k-2}$
- Series not valid for $r=1$, division by 0 in the recursion relation at $k=2$

  $a_{k+1}=\frac{a_k}{k-2}$
- Recursion relation for $r=4$

  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=4$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+1}=\frac{a_k}{k+1}\right]$

1.690.3 Maple trace

```
Methods for second order ODEs:
```

1.690.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 35

```
dsolve(x^2*diff(diff(y(x),x),x)-(x^2+4*x)*diff(y(x),x)+4*y(x) = 0,
       y(x),singsol=all)
```

$$y=x\left(\mathrm{Ei}_1(x)\,\mathrm{e}^x c_2\,x^3+c_1\,x^3\mathrm{e}^x-c_2\left(x^2-x+2\right)\right)$$

1.690.5 Mathematica DSolve solution

Solving time : `0.024` (sec)
Leaf size : 41

```
DSolve[{x^2*D[y[x],{x,2}]-(x^2+4*x)*D[y[x],x]+4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_2e^x x^4-\frac{1}{6}c_1x\left(e^x x^3\,\text{ExpIntegralEi}(-x)+x^2-x+2\right)$$

1.691 problem 708

Internal problem ID [8829]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 708
Date solved : `Monday, October 21, 2024 at 05:22:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' - (3x+2)\,y' + \frac{(2x-1)\,y}{x} = 0$$

1.691.1 Solved as second order ode using Kovacic algorithm

Time used: 0.491 (sec)

Writing the ode as

$$2x^2y'' + (-3x-2)\,y' + \left(2 - \frac{1}{x}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -3x-2 \\ C &= 2 - \frac{1}{x} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 36x + 4}{16x^4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 + 36x + 4$$
$$t = 16x^4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 36x + 4}{16x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1319: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned}\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right)\\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right)\end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{5}{16x^2} + \frac{1}{4x^4} + \frac{9}{4x^3}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^2} + \frac{9}{4x} - \frac{19}{4} + \frac{171x}{8} - \frac{475x^2}{4} + \frac{11799x^3}{16} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $\frac{9}{4}$. Therefore

$$\begin{aligned} b &= \left(\frac{9}{4}\right) - (0) \\ &= \frac{9}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{\frac{9}{4}}{\frac{1}{2}} + 2\right) &= \frac{13}{4} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{\frac{9}{4}}{\frac{1}{2}} + 2\right) &= -\frac{5}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 + 36x + 4}{16x^4}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 + 36x + 4}{16x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{1}{2x^2}$	$\frac{13}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{4} - \left(-\frac{5}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x^2} - \frac{5}{4x} + (-)(0) \\
&= -\frac{1}{2x^2} - \frac{5}{4x} \\
&= \frac{-2-5x}{4x^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x^2} - \frac{5}{4x}\right)(1) + \left(\left(\frac{1}{x^3} + \frac{5}{4x^2}\right) + \left(-\frac{1}{2x^2} - \frac{5}{4x}\right)^2 - \left(\frac{5x^2 + 36x + 4}{16x^4}\right)\right) = 0$$

$$\frac{-2 + 5a_0}{2x^2} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{2}{5}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{2}{5}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{2}{5}\right) e^{\int \left(-\frac{1}{2x^2} - \frac{5}{4x}\right) dx} \\
&= \left(x + \frac{2}{5}\right) e^{\frac{1}{2x} - \frac{5 \ln(x)}{4}} \\
&= \frac{(2 + 5x) e^{\frac{1}{2x}}}{5x^{5/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3x-2}{2x^2} \, dx} \\
&= z_1 e^{-\frac{1}{2x} + \frac{3 \ln(x)}{4}} \\
&= z_1 \left(x^{3/4} e^{-\frac{1}{2x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{2 + 5x}{5\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-2}{2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{1}{x}+\frac{3\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{25\, \mathrm{e}^{-\frac{1}{x}+\frac{3\ln(x)}{2}} x}{(2+5x)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2+5x}{5\sqrt{x}}\right) + c_2 \left(\frac{2+5x}{5\sqrt{x}}\left(\int \frac{25\, \mathrm{e}^{-\frac{1}{x}+\frac{3\ln(x)}{2}} x}{(2+5x)^2} dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.691.2 Maple step by step solution

1.691.3 Maple trace

```
Methods for second order ODEs:
```

1.691.4 Maple dsolve solution

Solving time : `0.019` (sec)
Leaf size : 35

```
dsolve(2*x^2*diff(diff(y(x),x),x)-(3*x+2)*diff(y(x),x)+(2*x-1)/x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\, \mathrm{e}^{-\frac{1}{x}} \operatorname{hypergeom}\left([2], \left[-\frac{1}{2}\right], \frac{1}{x}\right) x^{5/2} + 5c_1 x + 2c_1}{\sqrt{x}}$$

1.691.5 Mathematica DSolve solution

Solving time : 0.199 (sec)
Leaf size : 70

```
DSolve[{2*x^2*D[y[x],{x,2}]-(3*x+2)*D[y[x],x]+(2*x-1)/x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\frac{2\sqrt{\pi}c_2(5x+2)\operatorname{erf}\left(\frac{1}{\sqrt{x}}\right)}{3\sqrt{x}}+\frac{2}{3}c_2e^{-1/x}\left(x^2-4x-2\right)+\frac{c_1(5x+2)}{5\sqrt{x}}$$

1.692 problem 709

Internal problem ID [8830]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 709
Date solved : `Monday, October 21, 2024 at 05:22:30 PM`
CAS classification : `[_Jacobi]`

Solve

$$x(1-x)\, y'' + \left(\frac{3}{2}-2x\right) y' - \frac{y}{4} = 0$$

1.692.1 Solved as second order ode using Kovacic algorithm

Time used: 0.246 (sec)

Writing the ode as

$$\left(-x^2+x\right) y'' + \left(\frac{3}{2}-2x\right) y' - \frac{y}{4} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+x \\ B &= \frac{3}{2}-2x \\ C &= -\frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x^2 + 4x - 3}{16(x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4x^2 + 4x - 3$$
$$t = 16(x^2 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^2 + 4x - 3}{16(x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1320: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(-1+x)^2} - \frac{3}{16x^2} - \frac{1}{8x} + \frac{1}{-8+8x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4x^2 + 4x - 3}{16(x^2 - x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-4x^2+4x-3}{16\left(x^2-x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4x} + \frac{1}{-4+4x} + (-)(0) \\
&= \frac{1}{4x} + \frac{1}{-4+4x} \\
&= \frac{2x-1}{4x(-1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{4x} + \frac{1}{-4+4x}\right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{1}{4(-1+x)^2}\right) + \left(\frac{1}{4x} + \frac{1}{-4+4x}\right)^2 - \left(\frac{-4x^2+4x-3}{16(x^2-x)^2}\right)\right) =
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{4x} + \frac{1}{-4+4x}\right) dx} \\
&= (x(-1+x))^{1/4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{3}{2}-2x}{-x^2+x} \, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4} - \frac{\ln(-1+x)}{4}} \\
&= z_1 \left(\frac{1}{x^{3/4}(-1+x)^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{(x(-1+x))^{1/4}}{x^{3/4}(-1+x)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{3}{2}-2x}{-x^2+x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}-\frac{\ln(-1+x)}{2}}}{(y_1)^2}\,dx \\ &= y_1\left(\ln\left(-\frac{1}{2}+x+\sqrt{x^2-x}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(x(-1+x))^{1/4}}{x^{3/4}(-1+x)^{1/4}}\right) + c_2\left(\frac{(x(-1+x))^{1/4}}{x^{3/4}(-1+x)^{1/4}}\left(\ln\left(-\frac{1}{2}+x+\sqrt{x^2-x}\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.692.2 Maple step by step solution

Let's solve

$x(1-x)\left(\frac{d}{dx}y'\right)+\left(\frac{3}{2}-2x\right)y'-\frac{y}{4}=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{4x(-1+x)} - \frac{(-3+4x)y'}{2x(-1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + \frac{(-3+4x)y'}{2x(-1+x)} + \frac{y}{4x(-1+x)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{-3+4x}{2x(-1+x)}, P_3(x) = \frac{1}{4x(-1+x)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{3}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x(-1+x)\left(\frac{d}{dx}y'\right) + (8x-6)\,y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(1+2r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(2k+3+2r) + a_k(2k+2r+1)^2\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(1+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(2k+2r+1)^2-4\left(k+\frac{3}{2}+r\right)(k+1+r)\,a_{k+1}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(2k+2r+1)^2}{2(2k+3+2r)(k+1+r)}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{a_k(2k+1)^2}{2(2k+3)(k+1)}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty}a_kx^k, a_{k+1}=\frac{a_k(2k+1)^2}{2(2k+3)(k+1)}\right]$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+1}=\frac{2a_kk^2}{(2k+2)\left(k+\frac{1}{2}\right)}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k-\frac{1}{2}}, a_{k+1}=\frac{2a_kk^2}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k-\frac{1}{2}}\right), a_{k+1}=\frac{a_k(2k+1)^2}{2(2k+3)(k+1)}, b_{k+1}=\frac{2b_kk^2}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

1.692.3 Maple trace

```
Methods for second order ODEs:
```

1.692.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 32

```
dsolve(x*(1-x)*diff(diff(y(x),x),x)+(3/2-2*x)*diff(y(x),x)-1/4*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_2\ln\left(-1+2x+2\sqrt{x\,(-1+x)}\right)-c_2\ln(2)+c_1}{\sqrt{x}}$$

1.692.5 Mathematica DSolve solution

Solving time : 0.096 (sec)
Leaf size : 53

```
DSolve[{x*(1-x)*D[y[x],{x,2}]+(3/2-2*x)*D[y[x],x]-1/4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_2\sqrt{x-1}\operatorname{arctanh}\left(\frac{\sqrt{x-1}}{\sqrt{x+1}}\right)}{\sqrt{-((x-1)x)}} + \frac{c_1}{\sqrt{x}}$$

1.693 problem 710

Internal problem ID [8831]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 710
Date solved : `Monday, October 21, 2024 at 05:22:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x(1-x)\,y''+xy'-y=0$$

1.693.1 Solved as second order ode using Kovacic algorithm

Time used: 0.245 (sec)

Writing the ode as

$$\left(-2x^2+2x\right)y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^2+2x \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-3x + 8}{16x\left(-1 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -3x + 8 \\
t &= 16x(-1 + x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-3x + 8}{16x\left(-1 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1322: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x(-1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{2x} + \frac{5}{16\,(-1+x)^2} - \frac{1}{2\,(-1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x+8}{16x\,(-1+x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-3x+8}{16x\left(-1+x\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{3}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{3}{4} - \left(\frac{3}{4}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} - \frac{1}{4(-1+x)} + (0) \\
&= \frac{1}{x} - \frac{1}{4(-1+x)} \\
&= \frac{1}{x} - \frac{1}{-4+4x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{x} - \frac{1}{4(-1+x)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{4(-1+x)^2}\right) + \left(\frac{1}{x} - \frac{1}{4(-1+x)}\right)^2 - \left(\frac{-3x+8}{16x(-1+x)^2}\right)\right) &= \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{x} - \frac{1}{4(-1+x)}\right) dx} \\
&= \frac{x}{(-1+x)^{1/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{-2x^2+2x}\, dx} \\
&= z_1 e^{\frac{\ln(-1+x)}{4}} \\
&= z_1\left((-1+x)^{1/4}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{-2x^2+2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln(-1+x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\sqrt{-1+x}}{x} + \arctan\left(\sqrt{-1+x}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(-\frac{\sqrt{-1+x}}{x} + \arctan\left(\sqrt{-1+x}\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.693.2 Maple step by step solution

1.693.3 Maple trace

```
Methods for second order ODEs:
```

1.693.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 25

```
dsolve(2*x*(1-x)*diff(diff(y(x),x),x)+x*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + \arctan\left(\sqrt{-1+x}\right) x c_2 - \sqrt{-1+x}\, c_2$$

1.693.5 Mathematica DSolve solution

Solving time : 0.092 (sec)
Leaf size : 43

```
DSolve[{2*x*(1-x)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt[4]{2}\left(c_2 x \operatorname{arctanh}\left(\sqrt{1-x}\right) + c_1 x - c_2\sqrt{1-x}\right)$$

1.694 problem 711

Internal problem ID [8832]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 711
Date solved : `Monday, October 21, 2024 at 05:22:32 PM`
CAS classification : `[_Jacobi]`

Solve

$$2x(1-x)\,y''+(1-11x)\,y'-10y=0$$

1.694.1 Solved as second order ode using Kovacic algorithm

Time used: 0.241 (sec)

Writing the ode as

$$\left(-2x^2+2x\right)y''+(1-11x)\,y'-10y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^2+2x \\ B &= 1-11x \\ C &= -10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -3x^2 + 66x - 3 \\
t &= 16\left(x^2 - x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1323: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} - \frac{15}{4(-1+x)} + \frac{15}{4x} + \frac{15}{4(-1+x)^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 66x - 3}{16(x^2 - x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-3x^2+66x-3}{16\left(x^2-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{4}-\left(-\frac{3}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4x} - \frac{3}{2\left(-1+x\right)} + (-)\left(0\right) \\
&= \frac{3}{4x} - \frac{3}{2\left(-1+x\right)} \\
&= -\frac{3(x+1)}{4x\left(-1+x\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4x} - \frac{3}{2\left(-1+x\right)}\right)(1) + \left(\left(-\frac{3}{4x^2} + \frac{3}{2\left(-1+x\right)^2}\right) + \left(\frac{3}{4x} - \frac{3}{2\left(-1+x\right)}\right)^2 - \left(\frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2}\right.\right.
$$

$$
\frac{-3+3a_0}{2x\left(-1+x\right)}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x+1\right) \mathrm{e}^{\int \left(\frac{3}{4x} - \frac{3}{2(-1+x)}\right) dx} \\
&= \left(x+1\right) \mathrm{e}^{-\frac{3\ln(-1+x)}{2} + \frac{3\ln(x)}{4}} \\
&= \frac{\left(x+1\right) x^{3/4}}{\left(-1+x\right)^{3/2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-11x}{-2x^2+2x}\,dx} \\
&= z_1 e^{-\frac{5\ln(-1+x)}{2}-\frac{\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{(-1+x)^{5/2}\,x^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x}\,(x+1)}{(-1+x)^4}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-11x}{-2x^2+2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-5\ln(-1+x)-\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2(x^2+6x+1)\,(-1+x)^5\,\mathrm{e}^{-5\ln(-1+x)-\frac{\ln(x)}{2}}}{x+1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}\,(x+1)}{(-1+x)^4}\right) + c_2 \left(\frac{\sqrt{x}\,(x+1)}{(-1+x)^4}\left(\frac{2(x^2+6x+1)\,(-1+x)^5\,\mathrm{e}^{-5\ln(-1+x)-\frac{\ln(x)}{2}}}{x+1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.694.2 Maple step by step solution

Let's solve

$2x(1-x)\left(\frac{d}{dx}y'\right)+(1-11x)\,y'-10y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{5y}{x(-1+x)}-\frac{(-1+11x)y'}{2x(-1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(-1+11x)y'}{2x(-1+x)}+\frac{5y}{x(-1+x)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{-1+11x}{2x(-1+x)},P_3(x)=\frac{5}{x(-1+x)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$(x\cdot P_2(x))\bigg|_{x=0}=\frac{1}{2}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2\cdot P_3(x))\bigg|_{x=0}=0$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$2x(-1+x)\left(\frac{d}{dx}y'\right)+(-1+11x)\,y'+10y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

$x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=1..2$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(2k+1+2r) + a_k(2k+2r+5)(k+r+2)\right) x^{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 2\left(k+r+\frac{5}{2}\right) a_k(k+r+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(2k+2r+5)a_k(k+r+2)}{(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r=0$

$$a_{k+1} = \frac{(2k+5)a_k(k+2)}{(2k+1)(k+1)}$$

- Solution for $r=0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(2k+5)a_k(k+2)}{(2k+1)(k+1)}\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1} = \frac{(2k+6)a_k\left(k+\frac{5}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{(2k+6)a_k\left(k+\frac{5}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = \frac{(2k+5)a_k(k+2)}{(2k+1)(k+1)}, b_{k+1} = \frac{(2k+6)b_k\left(k+\frac{5}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

1.694.3 Maple trace

```
Methods for second order ODEs:
```

1.694.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 29

```
dsolve(2*x*(1-x)*diff(diff(y(x),x),x)+(1-11*x)*diff(y(x),x)-10*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1(x^2+6x+1)+c_2\sqrt{x}\,(x+1)}{(-1+x)^4}$$

1.694.5 Mathematica DSolve solution

Solving time : 0.096 (sec)
Leaf size : 35

```
DSolve[{2*x*(1-x)*D[y[x],{x,2}]+(1-11*x)*D[y[x],x]-10*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1\sqrt{x}(x+1)-2c_2(x^2+6x+1)}{(x-1)^4}$$

1.695 problem 712

Internal problem ID [8833]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 712
Date solved : `Monday, October 21, 2024 at 05:22:33 PM`
CAS classification : `[_Jacobi]`

Solve

$$x(1-x)\,y''+\frac{(1-2x)\,y'}{3}+\frac{20y}{9}=0$$

1.695.1 Solved as second order ode using Kovacic algorithm

Time used: 0.254 (sec)

Writing the ode as

$$\left(-x^{2}+x\right)y''+\left(-\frac{2x}{3}+\frac{1}{3}\right)y'+\frac{20y}{9}=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^{2}+x \\ B &= -\frac{2x}{3}+\frac{1}{3} \\ C &= \frac{20}{9} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{72x^2 - 72x - 5}{36\,(x^2 - x)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 72x^2 - 72x - 5 \\
t &= 36\big(x^2 - x\big)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{72x^2 - 72x - 5}{36\,(x^2 - x)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1325: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36(-1+x)^2} - \frac{5}{36x^2} - \frac{41}{18x} + \frac{41}{18(-1+x)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{72x^2 - 72x - 5}{36(x^2 - x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{72x^2 - 72x - 5}{36\left(x^2 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$
1	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{6x} + \frac{5}{6\left(-1+x\right)} + (0) \\ &= \frac{1}{6x} + \frac{5}{6\left(-1+x\right)} \\ &= \frac{-1+6x}{6x\left(-1+x\right)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{6x} + \frac{5}{6\left(-1+x\right)}\right)(1) + \left(\left(-\frac{1}{6x^2} - \frac{5}{6\left(-1+x\right)^2}\right) + \left(\frac{1}{6x} + \frac{5}{6\left(-1+x\right)}\right)^2 - \left(\frac{72x^2-72x-5}{36\left(x^2-x\right)^2}\right.\right.$$
$$\frac{-1-6a_0}{3x\left(-1+x\right)}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{1}{6}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -\frac{1}{6} + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(-\frac{1}{6} + x\right) e^{\int \left(\frac{1}{6x} + \frac{5}{6(-1+x)}\right) dx} \\ &= \left(-\frac{1}{6} + x\right) e^{\frac{\ln(x)}{6} + \frac{5\ln(-1+x)}{6}} \\ &= \left(-\frac{1}{6} + x\right) x^{1/6}(-1+x)^{5/6}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-\frac{2x}{3}+\frac{1}{3}}{-x^2+x}\,dx} \\
&= z_1 e^{-\frac{\ln(x(-1+x))}{6}} \\
&= z_1\left(\frac{1}{\left(x\left(-1+x\right)\right)^{1/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(-1+6x\right)x^{1/6}\left(-1+x\right)^{5/6}}{6\left(x\left(-1+x\right)\right)^{1/6}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{2x}{3}+\frac{1}{3}}{-x^2+x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x(-1+x))}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{54x^{2/3}\left(-5+6x\right)}{5\left(-1+6x\right)\left(-1+x\right)^{2/3}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(-1+6x\right)x^{1/6}\left(-1+x\right)^{5/6}}{6\left(x\left(-1+x\right)\right)^{1/6}}\right) + c_2\left(\frac{\left(-1+6x\right)x^{1/6}\left(-1+x\right)^{5/6}}{6\left(x\left(-1+x\right)\right)^{1/6}}\left(-\frac{54x^{2/3}\left(-5+6x\right)}{5\left(-1+6x\right)\left(-1+x\right)^{2/3}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.695.2 Maple step by step solution

Let's solve

$x(1-x)\left(\frac{d}{dx}y'\right)+\frac{(1-2x)y'}{3}+\frac{20y}{9}=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{20y}{9x(-1+x)}-\frac{(2x-1)y'}{3x(-1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(2x-1)y'}{3x(-1+x)}-\frac{20y}{9x(-1+x)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

  $\left[P_2(x)=\frac{2x-1}{3x(-1+x)}, P_3(x)=-\frac{20}{9x(-1+x)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=\frac{1}{3}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

$9x(-1+x)\left(\frac{d}{dx}y'\right)+(6x-3)y'-20y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=1..2$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-3a_0 r(-2+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (-3a_{k+1}(k+1+r)(3k+1+3r) + a_k(3k+3r+4)(3k+3r-5)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-3r(-2+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{2}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-9(k+1+r)\left(k+\frac{1}{3}+r\right) a_{k+1} + 9\left(k+r-\frac{5}{3}\right) a_k \left(k+r+\frac{4}{3}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(3k+3r-5)a_k(3k+3r+4)}{3(k+1+r)(3k+1+3r)}$$

- Recursion relation for $r=0$

$$a_{k+1} = \frac{(3k-5)a_k(3k+4)}{3(k+1)(3k+1)}$$

- Solution for $r=0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(3k-5)a_k(3k+4)}{3(k+1)(3k+1)}\right]$$

- Recursion relation for $r=\frac{2}{3}$; series terminates at $k=1$

$$a_{k+1} = \frac{(3k-3)a_k(3k+6)}{3\left(k+\frac{5}{3}\right)(3k+3)}$$

- Apply recursion relation for $k=0$

$$a_1 = -\frac{6a_0}{5}$$

- Terminating series solution of the ODE for $r=\frac{2}{3}$. Use reduction of order to find the second li

$$y = a_0 \cdot \left(-\frac{6x}{5}+1\right)$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + b_0 \cdot \left(-\frac{6x}{5}+1\right), a_{k+1} = \frac{(3k-5)a_k(3k+4)}{3(k+1)(3k+1)}\right]$$

1.695.3 Maple trace

```
Methods for second order ODEs:
```

1.695.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 27

```
dsolve(x*(1-x)*diff(diff(y(x),x),x)+1/3*(1-2*x)*diff(y(x),x)+20/9*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x^{2/3}(-5 + 6x) + c_2(-1 + 6x)(-1 + x)^{2/3}$$

1.695.5 Mathematica DSolve solution

Solving time : 0.08 (sec)
Leaf size : 51

```
DSolve[{x*(1-x)*D[y[x],{x,2}]+1/3*(1-2*x)*D[y[x],x]+20/9*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 \sqrt[3]{-((x-1)x)}Q_1^{\frac{2}{3}}(2x-1) + \frac{c_1 x^{2/3}(6x-5)}{3\,\text{Gamma}\left(\frac{4}{3}\right)}$$

1.696 problem 713

Internal problem ID [8834]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 713
Date solved : `Monday, October 21, 2024 at 05:22:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4y''+\frac{3(-x^2+2)\,y}{(-x^2+1)^2}=0$$

1.696.1 Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$4y''+\frac{(-3x^2+6)\,y}{(x^2-1)^2}=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4\\ B &= 0 \\ C &= \frac{-3x^2+6}{(x^2-1)^2}\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3x^2 - 6}{4\left(x^2 - 1\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3x^2 - 6 \\
t &= 4\left(x^2 - 1\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3x^2 - 6}{4\left(x^2 - 1\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1327: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{9}{16(x+1)} - \frac{3}{16(x+1)^2} + \frac{9}{16(x-1)} - \frac{3}{16(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 6}{4(x^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{3x^2-6}{4\left(x^2-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{+}\right) \\
&= \frac{3}{2}-\left(\frac{3}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\ &= \frac{3}{4(x-1)} + \frac{3}{4(x+1)} + (0) \\ &= \frac{3}{4(x-1)} + \frac{3}{4(x+1)} \\ &= \frac{3x}{2x^2-2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4(x-1)} + \frac{3}{4(x+1)}\right)(0) + \left(\left(-\frac{3}{4(x-1)^2} - \frac{3}{4(x+1)^2}\right) + \left(\frac{3}{4(x-1)} + \frac{3}{4(x+1)}\right)^2 - \left(\frac{3}{4(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{3}{4(x-1)} + \frac{3}{4(x+1)}\right) dx} \\ &= (x^2-1)^{3/4}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= (x^2-1)^{3/4}\end{aligned}$$

Which simplifies to

$$y_1 = (x^2 - 1)^{3/4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= (x^2 - 1)^{3/4} \int \frac{1}{(x^2 - 1)^{3/2}}\,dx \\
&= (x^2 - 1)^{3/4} \left(-\frac{(x-1)(x+1)\,x}{(x^2 - 1)^{3/2}} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left((x^2 - 1)^{3/4} \right) + c_2 \left((x^2 - 1)^{3/4} \left(-\frac{(x-1)(x+1)\,x}{(x^2 - 1)^{3/2}} \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.696.2 Maple step by step solution

Let's solve

$4 \frac{d}{dx} y' + \frac{3(-x^2+2)y}{(-x^2+1)^2} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{3(x^2-2)y}{4(x^2-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' - \frac{3(x^2-2)y}{4(x^2-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = 0, P_3(x) = -\frac{3(x^2-2)}{4(x^2-1)^2}\right]$

- $(x+1)\cdot P_2(x)$ is analytic at $x = -1$

  $((x+1)\cdot P_2(x))\bigg|_{x=-1} = 0$

- $(x+1)^2\cdot P_3(x)$ is analytic at $x = -1$

  $((x+1)^2\cdot P_3(x))\bigg|_{x=-1} = \frac{3}{16}$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4(x^2-1)^2\left(\frac{d}{dx}y'\right) + (-3x^2+6)\,y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^4 - 16u^3 + 16u^2)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-3u^2+6u+3)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m = 0..2$

  $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 2..4$

  $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-1+4r)(-3+4r)u^r + (a_1(3+4r)(1+4r) - 2a_0(8r^2-8r-3))u^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(4k+4r-\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $$(-1+4r)(-3+4r) = 0$$
- Values of r that satisfy the indicial equation
  $$r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$$
- Each term must be 0
  $$a_1(3+4r)(1+4r) - 2a_0(8r^2-8r-3) = 0$$
- Solve for the dependent coefficient(s)
  $$a_1 = \frac{2a_0(8r^2-8r-3)}{16r^2+16r+3}$$
- Each term in the series must be 0, giving the recursion relation
  $$4(4a_k + a_{k-2} - 4a_{k-1})k^2 + 4(2(4a_k + a_{k-2} - 4a_{k-1})r - 4a_k - 5a_{k-2} + 12a_{k-1})k + 4(4a_k + a_{k-2}$$
- Shift index using $k->k+2$
  $$4(4a_{k+2} + a_k - 4a_{k+1})(k+2)^2 + 4(2(4a_{k+2} + a_k - 4a_{k+1})r - 4a_{k+2} - 5a_k + 12a_{k+1})(k+2) + 4$$
- Recursion relation that defines series solution to ODE
  $$a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 8kra_k - 32kra_{k+1} + 4r^2a_k - 16r^2a_{k+1} - 4ka_k - 16ka_{k+1} - 4ra_k - 16ra_{k+1} - 3a_k + 6a_{k+1}}{16k^2 + 32kr + 16r^2 + 48k + 48r + 35}$$
- Recursion relation for $r = \frac{1}{4}$
  $$a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_{k+1}}{16k^2 + 56k + 48}$$
- Solution for $r = \frac{1}{4}$
  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_{k+1}}{16k^2 + 56k + 48}, a_1 = -\frac{9a_0}{8}\right]$$
- Revert the change of variables $u = x+1$
  $$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{1}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_{k+1}}{16k^2 + 56k + 48}, a_1 = -\frac{9a_0}{8}\right]$$
- Recursion relation for $r = \frac{3}{4}$
  $$a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 2ka_k - 40ka_{k+1} - \frac{15}{4}a_k - 15a_{k+1}}{16k^2 + 72k + 80}$$
- Solution for $r = \frac{3}{4}$
  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 2ka_k - 40ka_{k+1} - \frac{15}{4}a_k - 15a_{k+1}}{16k^2 + 72k + 80}, a_1 = -\frac{3a_0}{8}\right]$$
- Revert the change of variables $u = x+1$
  $$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{3}{4}}, a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 2ka_k - 40ka_{k+1} - \frac{15}{4}a_k - 15a_{k+1}}{16k^2 + 72k + 80}, a_1 = -\frac{3a_0}{8}\right]$$
- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+\frac{3}{4}}\right), a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} - 2k a_k - 24k a_{k+1} - \frac{15}{4} a_k + a_{k+1}}{16k^2 + 56k + 48},\right.$$

1.696.3 Maple trace

```
Methods for second order ODEs:
```

1.696.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 24

```
dsolve(4*diff(diff(y(x),x),x)+3*(-x^2+2)/(-x^2+1)^2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \left(x^2 - 1\right)^{3/4} + c_2 x \left(x^2 - 1\right)^{1/4}$$

1.696.5 Mathematica DSolve solution

Solving time : 0.063 (sec)
Leaf size : 51

```
DSolve[{4*D[y[x],{x,2}]+3*(2-x^2)/(1-x^2)^2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{x^2 - 1} \left(c_2 Q_{\frac{1}{2}}^{\frac{1}{2}}(x) + \frac{\sqrt{\frac{2}{\pi}} c_1 x}{\sqrt[4]{1 - x^2}}\right)$$

1.697 problem 714

Internal problem ID [8835]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 714
Date solved : `Monday, October 21, 2024 at 05:22:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' - \frac{2u'}{x} - a^2 u = 0$$

1.697.1 Solved as second order ode using Kovacic algorithm

Time used: 0.292 (sec)

Writing the ode as

$$u'' - \frac{2u'}{x} - a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{2}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{a^2x^2 + 2}{x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= a^2x^2 + 2 \\
t &= x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{a^2x^2 + 2}{x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$
u = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1329: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + a^2$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \quad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \qquad (10)\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{2}{x^2}\right) \\ &= \frac{2}{x^2} + a^2\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned}b &= (0) - (0) \\ &= 0\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{a^2x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(a) \\
&= -\frac{1}{x} - a \\
&= \frac{-ax - 1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{x} - a\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - a\right)^2 - \left(\frac{a^2x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{2aa_0 - 2}{x} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{a}\right) e^{\int \left(-\frac{1}{x} - a\right) dx} \\
&= \left(x + \frac{1}{a}\right) e^{-ax - \ln(x)} \\
&= \frac{(ax + 1)\, e^{-ax}}{ax}
\end{aligned}$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-\frac{2}{x}}{1}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{(ax+1)\,\mathrm{e}^{-ax}}{a}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{-\frac{2}{x}}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{2\ln(x)}}{(u_1)^2}\,dx \\
&= u_1 \left(\frac{(ax-1)\,\mathrm{e}^{2ax}}{2a\,(ax+1)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a}\right) + c_2 \left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{a}\left(\frac{(ax-1)\,\mathrm{e}^{2ax}}{2a\,(ax+1)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.697.2 Maple step by step solution

Let's solve

$\frac{d}{dx}u' - \frac{2u'}{x} - a^2u = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}u'$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = -a^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $-a^2ux + \left(\frac{d}{dx}u'\right)x - 2u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  ◦ Convert u' to series expansion

    $u' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  ◦ Shift index using $k- >k+1$

    $u' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}u'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$
- Shift index using $k->k+1$

  $x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\, x^{-1+r} + a_1(1+r)(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(k-2+r) - a^2 a_{k-1}\right) x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$
- Each term must be 0

  $a_1(1+r)(-2+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k-2+r) - a^2 a_{k-1} = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+r-1) - a^2 a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+r-1)}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a^2 a_k}{(k+2)(k-1)}$
- Solution for $r = 0$

  $\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k-1)}, -2a_1 = 0\right]$
- Recursion relation for $r = 3$

  $a_{k+2} = \frac{a^2 a_k}{(k+5)(k+2)}$
- Solution for $r = 3$

  $\left[u = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{a^2 a_k}{(k+5)(k+2)}, 4a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[u = \left(\sum_{k=0}^{\infty} b_k x^k\right) + \left(\sum_{k=0}^{\infty} c_k x^{k+3}\right), b_{k+2} = \frac{a^2 b_k}{(k+2)(k-1)}, -2b_1 = 0, c_{k+2} = \frac{a^2 c_k}{(k+5)(k+2)}, 4c_1 = 0\right]$

1.697.3 Maple trace

```
Methods for second order ODEs:
```

1.697.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 28

```
dsolve(diff(diff(u(x),x),x)-2/x*diff(u(x),x)-a^2*u(x) = 0,
       u(x),singsol=all)
```

$$u = c_1 \, e^{ax}(ax - 1) + c_2(ax + 1) \, e^{-ax}$$

1.697.5 Mathematica DSolve solution

Solving time : 0.147 (sec)
Leaf size : 68

```
DSolve[{D[u[x],{x,2}]-2/x*D[u[x],x]-a^2*u[x]==0,{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{\sqrt{\frac{2}{\pi}}\sqrt{x}((iac_2x + c_1)\sinh(ax) - (ac_1x + ic_2)\cosh(ax))}{a\sqrt{-iax}}$$

1.698 problem 715

Internal problem ID [8836]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 715
Date solved : `Monday, October 21, 2024 at 05:22:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' + \frac{2u'}{x} - a^2 u = 0$$

1.698.1 Solved as second order ode using Kovacic algorithm

Time used: 0.130 (sec)

Writing the ode as

$$u'' + \frac{2u'}{x} - a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{a^2}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= a^2 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = (a^2)\, z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$
u = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1331: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{\sqrt{a^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \frac{e^{\operatorname{csgn}(a)ax}}{x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\, dx}}{\left(u_1\right)^2}\, dx \\ &= u_1 \int \frac{e^{-2\ln(x)}}{\left(u_1\right)^2}\, dx \\ &= u_1 \left(-\frac{e^{-2\operatorname{csgn}(a)ax}}{2\operatorname{csgn}(a)\, a}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{e^{\operatorname{csgn}(a) a x}}{x}\right) + c_2 \left(\frac{e^{\operatorname{csgn}(a) a x}}{x} \left(-\frac{e^{-2 \operatorname{csgn}(a) a x}}{2 \operatorname{csgn}(a) a}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.698.2 Maple step by step solution

Let's solve

$\frac{d}{dx}u' + \frac{2u'}{x} - a^2 u = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}u'$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -a^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $-a^2 u x + \left(\frac{d}{dx}u'\right) x + 2u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  ◦ Shift index using $k- >k - 1$

$$x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert u' to series expansion

$$u' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$u' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}u'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) - a^2 a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 0\}$
- Each term must be 0
  $a_1(1+r)\,(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+r+1)\,(k+2+r) - a^2 a_{k-1} = 0$
- Shift index using $k->k+1$
  $a_{k+2}(k+2+r)\,(k+3+r) - a^2 a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$
  $a_{k+2} = \frac{a^2 a_k}{(k+1)(k+2)}$
- Solution for $r = -1$
  $\left[u = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{a^2 a_k}{(k+1)(k+2)}, 0 = 0\right]$
- Recursion relation for $r = 0$

$a_{k+2} = \frac{a^2 a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = \frac{a^2 b_k}{(k+1)(k+2)}, 0 = 0, c_{k+2} = \frac{a^2 c_k}{(k+2)(k+3)}, 2c_1 = 0\right]$

1.698.3 Maple trace

```
Methods for second order ODEs:
```

1.698.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 21

```
dsolve(diff(diff(u(x),x),x)+2/x*diff(u(x),x)-a^2*u(x) = 0,
       u(x),singsol=all)
```

$$u = \frac{c_1 \sinh(ax) + c_2 \cosh(ax)}{x}$$

1.698.5 Mathematica DSolve solution

Solving time : `0.047` (sec)
Leaf size : 35

```
DSolve[{D[u[x],{x,2}]+2/x*D[u[x],x]-a^2*u[x]==0,{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{2ac_1 e^{-ax} + c_2 e^{ax}}{2ax}$$

1.699 problem 716

Internal problem ID [8837]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 716
Date solved : `Monday, October 21, 2024 at 05:22:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' + \frac{2u'}{x} + a^2 u = 0$$

1.699.1 Solved as second order ode using Kovacic algorithm

Time used: 0.142 (sec)

Writing the ode as

$$u'' + \frac{2u'}{x} + a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-a^2}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -a^2 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = (-a^2)\, z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$
u = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1333: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{-a^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1}\, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$u_1 = \frac{\mathrm{e}^{\sqrt{-a^2}\, x}}{x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\, dx}}{(u_1)^2}\, dx \\
&= u_1 \int \frac{e^{-2\ln(x)}}{(u_1)^2}\, dx \\
&= u_1 \left(\frac{\sqrt{-a^2}\, \mathrm{e}^{-2\sqrt{-a^2}\, x}}{2a^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{\mathrm{e}^{\sqrt{-a^2}\, x}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{\sqrt{-a^2}\, x}}{x} \left(\frac{\sqrt{-a^2}\, \mathrm{e}^{-2\sqrt{-a^2}\, x}}{2a^2}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.699.2 Maple step by step solution

Let's solve

$\frac{d}{dx}u' + \frac{2u'}{x} + a^2 u = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}u'$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = a^2\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators
  $a^2ux + \left(\frac{d}{dx}u'\right)x + 2u' = 0$
- Assume series solution for u
  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ○ Convert $x \cdot u$ to series expansion
    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - ○ Shift index using $k->k-1$
    $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - ○ Convert u' to series expansion
    $u' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$
  - ○ Shift index using $k->k+1$
    $u' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
  - ○ Convert $x \cdot \left(\frac{d}{dx}u'\right)$ to series expansion
    $x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$
  - ○ Shift index using $k->k+1$
    $x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions
  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a^2 a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 0\}$
- Each term must be 0
  $a_1(1+r)\,(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+r+1)\,(k+2+r) + a^2 a_{k-1} = 0$
- Shift index using $k->k+1$

$a_{k+2}(k+2+r)(k+3+r)+a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a^2 a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a^2 a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[u = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a^2 a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[u = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} c_k x^{k}\right), b_{k+2} = -\frac{a^2 b_k}{(k+1)(k+2)}, 0 = 0, c_{k+2} = -\frac{a^2 c_k}{(k+2)(k+3)}, 2c_1 = 0\right]$

1.699.3 Maple trace

```
Methods for second order ODEs:
```

1.699.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 21

```
dsolve(diff(diff(u(x),x),x)+2/x*diff(u(x),x)+a^2*u(x) = 0,
       u(x),singsol=all)
```

$$u = \frac{c_1 \sin(ax) + c_2 \cos(ax)}{x}$$

1.699.5 Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 42

```
DSolve[{D[u[x],{x,2}]+2/x*D[u[x],x]+a^2*u[x]==0,{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{e^{-iax}\left(2c_1 - \frac{ic_2e^{2iax}}{a}\right)}{2x}$$

1.700 problem 717

Internal problem ID [8838]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 717
Date solved : `Monday, October 21, 2024 at 05:22:37 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' + \frac{4u'}{x} - a^2 u = 0$$

1.700.1 Solved as second order ode using Kovacic algorithm

Time used: 0.290 (sec)

Writing the ode as

$$u'' + \frac{4u'}{x} - a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{4}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = a^2x^2 + 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1335: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + a^2$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned}
\tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \quad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \quad (10)$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{2}{x^2}\right) \\ &= \frac{2}{x^2} + a^2 \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{a^2 x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(a) \\
&= -\frac{1}{x} - a \\
&= \frac{-ax - 1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - a\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - a\right)^2 - \left(\frac{a^2x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{2aa_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{1}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{1}{a}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{a}\right) e^{\int \left(-\frac{1}{x} - a\right) dx} \\
&= \left(x + \frac{1}{a}\right) e^{-ax - \ln(x)} \\
&= \frac{(ax + 1)\, e^{-ax}}{ax}
\end{aligned}
$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{4}{x}}{1}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1\left(\frac{1}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{(ax+1)\,\mathrm{e}^{-ax}}{x^3 a}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{4}{x}}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{-4\ln(x)}}{(u_1)^2}\,dx \\
&= u_1\left(\frac{(ax-1)\,\mathrm{e}^{2ax}}{2a\,(ax+1)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1\left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{x^3 a}\right) + c_2\left(\frac{(ax+1)\,\mathrm{e}^{-ax}}{x^3 a}\left(\frac{(ax-1)\,\mathrm{e}^{2ax}}{2a\,(ax+1)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.700.2 Maple step by step solution

Let's solve

$\frac{d}{dx}u' + \frac{4u'}{x} - a^2u = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}u'$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = -a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $-a^2ux + \left(\frac{d}{dx}u'\right)x + 4u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k - 1$

    $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert u' to series expansion

    $u' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1}$
  - Shift index using $k- >k + 1$

    $u' = \sum_{k=-1}^{\infty} a_{k+1}(k + r + 1)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}u'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(3+r)\, x^{-1+r} + a_1(1+r)(4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(k+4+r) - a^2a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$

- Each term must be 0

  $a_1(1+r)(4+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+4+r) - a^2a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+5+r) - a^2a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2a_k}{(k+2+r)(k+5+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = \frac{a^2a_k}{(k-1)(k+2)}$

- Solution for $r = -3$

  $\left[u = \sum_{k=0}^{\infty} a_kx^{k-3}, a_{k+2} = \frac{a^2a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a^2a_k}{(k+2)(k+5)}$

- Solution for $r = 0$

  $\left[u = \sum_{k=0}^{\infty} a_kx^k, a_{k+2} = \frac{a^2a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[u = \left(\sum_{k=0}^{\infty} b_kx^{k-3}\right) + \left(\sum_{k=0}^{\infty} c_kx^k\right), b_{k+2} = \frac{a^2b_k}{(k-1)(k+2)}, -2b_1 = 0, c_{k+2} = \frac{a^2c_k}{(k+2)(k+5)}, 4c_1 = 0\right]$

1.700.3 Maple trace

```
Methods for second order ODEs:
```

1.700.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 32

```
dsolve(diff(diff(u(x),x),x)+4/x*diff(u(x),x)-a^2*u(x) = 0,
       u(x),singsol=all)
```

$$u = \frac{c_2(ax+1)\,\mathrm{e}^{-ax} + c_1(ax-1)\,\mathrm{e}^{ax}}{x^3}$$

1.700.5 Mathematica DSolve solution

Solving time : 0.143 (sec)
Leaf size : 68

```
DSolve[{D[u[x],{x,2}]+4/x*D[u[x],x]-a^2*u[x]==0,{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{\sqrt{\frac{2}{\pi}}((iac_2x + c_1)\sinh(ax) - (ac_1x + ic_2)\cosh(ax))}{ax^{5/2}\sqrt{-iax}}$$

1.701 problem 718

Internal problem ID [8839]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 718
Date solved : `Monday, October 21, 2024 at 05:22:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' + \frac{4u'}{x} + a^2 u = 0$$

1.701.1 Solved as second order ode using Kovacic algorithm

Time used: 0.330 (sec)

Writing the ode as

$$u'' + \frac{4u'}{x} + a^2 u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{4}{x} \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -a^2x^2 + 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-a^2x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1337: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} - a^2$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx ia - \frac{i}{a\,x^2} - \frac{i}{2a^3x^4} - \frac{i}{2a^5x^6} - \frac{5i}{8a^7x^8} - \frac{7i}{8a^9x^{10}} - \frac{21i}{16a^{11}x^{12}} - \frac{33i}{16a^{13}x^{14}} + \dots \quad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = ia$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= ia \end{aligned} \quad (10)$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= \left(-a^2\right) + \left(\frac{2}{x^2}\right) \\ &= \frac{2}{x^2} - a^2 \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= ia \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{ia} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{ia} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-a^2x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	ia	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(ia) \\
&= -\frac{1}{x} - ia \\
&= -\frac{1}{x} - ia
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - ia\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - ia\right)^2 - \left(\frac{-a^2x^2 + 2}{x^2}\right)\right) = 0$$

$$\frac{2iaa_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{i}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{i}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{i}{a}\right) e^{\int \left(-\frac{1}{x} - ia\right) dx} \\
&= \left(x - \frac{i}{a}\right) e^{-\ln(x) - iax} \\
&= \frac{(ax - i)\, e^{-iax}}{xa}
\end{aligned}$$

The first solution to the original ode in u is found from

$$\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{4}{x}}{1}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1\left(\frac{1}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$u_1 = \frac{(ax-i)\,\mathrm{e}^{-iax}}{x^3 a}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{4}{x}}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{-4\ln(x)}}{(u_1)^2}\,dx \\
&= u_1\left(\frac{(iax-1)\,\mathrm{e}^{2iax}}{2a\,(-ax+i)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1\left(\frac{(ax-i)\,\mathrm{e}^{-iax}}{x^3 a}\right) + c_2\left(\frac{(ax-i)\,\mathrm{e}^{-iax}}{x^3 a}\left(\frac{(iax-1)\,\mathrm{e}^{2iax}}{2a\,(-ax+i)}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.701.2 Maple step by step solution

Let's solve

$\frac{d}{dx}u' + \frac{4u'}{x} + a^2u = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}u'$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 4$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $a^2ux + \left(\frac{d}{dx}u'\right)x + 4u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

    $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert u' to series expansion

    $u' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $u' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$

- ○ Convert $x \cdot \left(\frac{d}{dx}u'\right)$ to series expansion

  $$x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-1}$$

- ○ Shift index using $k->k+1$

  $$x \cdot \left(\frac{d}{dx}u'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(3+r)\, x^{-1+r} + a_1(1+r)(4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(k+4+r) + a^2 a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-3, 0\}$$

- Each term must be 0

  $$a_1(1+r)(4+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+r+1)(k+4+r) + a^2 a_{k-1} = 0$$

- Shift index using $k->k+1$

  $$a_{k+2}(k+2+r)(k+5+r) + a^2 a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a^2 a_k}{(k+2+r)(k+5+r)}$$

- Recursion relation for $r = -3$

  $$a_{k+2} = -\frac{a^2 a_k}{(k-1)(k+2)}$$

- Solution for $r = -3$

  $$\left[u = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a^2 a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+5)}$$

- Solution for $r = 0$

  $$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = -\frac{a^2 b_k}{(k-1)(k+2)}, -2b_1 = 0, c_{k+2} = -\frac{a^2 c_k}{(k+2)(k+5)}, 4c_1 = 0\right]$$

1.701.3 Maple trace

```
Methods for second order ODEs:
```

1.701.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 33

```
dsolve(diff(diff(u(x),x),x)+4/x*diff(u(x),x)+a^2*u(x) = 0,
       u(x),singsol=all)
```

$$u = \frac{(c_1 ax + c_2)\cos(ax) + \sin(ax)(c_2 ax - c_1)}{x^3}$$

1.701.5 Mathematica DSolve solution

Solving time : `0.13` (sec)
Leaf size : 57

```
DSolve[{D[u[x],{x,2}]+4/x*D[u[x],x]+a^2*u[x]==0,{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to -\frac{\sqrt{\frac{2}{\pi}}((ac_1 x + c_2)\cos(ax) + (ac_2 x - c_1)\sin(ax))}{x^{3/2}(ax)^{3/2}}$$

1.702 problem 719

Internal problem ID [8840]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 719
Date solved : `Monday, October 21, 2024 at 05:22:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - a^2 y = \frac{6y}{x^2}$$

1.702.1 Solved as second order ode using Kovacic algorithm

Time used: 0.309 (sec)

Writing the ode as

$$y'' + \left(-a^2 - \frac{6}{x^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -a^2 - \frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = a^2x^2 + 6$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1339: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = a^2 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{3}{a\,x^2} - \frac{9}{2a^3x^4} + \frac{27}{2a^5x^6} - \frac{405}{8a^7x^8} + \frac{1701}{8a^9x^{10}} - \frac{15309}{16a^{11}x^{12}} + \frac{72171}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{6}{x^2}\right) \\ &= a^2 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{a^2 x^2 + 6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c) [\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty) [\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + (-)\,(a) \\
&= -\frac{2}{x} - a \\
&= \frac{-ax - 2}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(2) + 2\left(-\frac{2}{x} - a\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - a\right)^2 - \left(\frac{a^2 x^2 + 6}{x^2}\right)\right) &= 0 \\
\frac{2axa_1 + 4aa_0 - 6x - 4a_1}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{3}{a^2}, a_1 = \frac{3}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + \frac{3x}{a} + \frac{3}{a^2}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + \frac{3x}{a} + \frac{3}{a^2}\right) e^{\int \left(-\frac{2}{x} - a\right) dx} \\
&= \left(x^2 + \frac{3x}{a} + \frac{3}{a^2}\right) e^{-ax - 2\ln(x)} \\
&= \frac{(a^2 x^2 + 3ax + 3)\, e^{-ax}}{a^2 x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax}}{a^2x^2} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax}}{a^2x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax}}{a^2x^2} \int \frac{1}{\frac{(a^2x^2+3ax+3)^2\mathrm{e}^{-2ax}}{a^4x^4}}\,dx \\ &= \frac{(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax}}{a^2x^2} \left(\frac{(a^2x^2 - 3ax + 3)\,\mathrm{e}^{2ax}}{2a\,(a^2x^2 + 3ax + 3)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\left(\frac{(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax}}{a^2x^2}\right) + c_2\left(\frac{(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax}}{a^2x^2}\left(\frac{(a^2x^2 - 3ax + 3)\,\mathrm{e}^{2ax}}{2a\,(a^2x^2 + 3ax + 3)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.702.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - a^2 y = \frac{6y}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{y(a^2x^2+6)}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y(a^2x^2+6)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{a^2x^2+6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -6$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d}{dx}y'\right)x^2 + (-a^2x^2 - 6)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^{k+r}$

Rewrite ODE with series expansions

$a_0(2+r)(-3+r)x^r + a_1(3+r)(-2+r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)(k+r-3) - a^2a_{k-2})x^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(2+r)(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 3\}$
- Each term must be 0
  $a_1(3+r)(-2+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+2)(k+r-3) - a^2a_{k-2} = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(k+4+r)(k+r-1) - a^2a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a^2a_k}{(k+4+r)(k+r-1)}$
- Recursion relation for $r = -2$
  $a_{k+2} = \frac{a^2a_k}{(k+2)(k-3)}$
- Solution for $r = -2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = \frac{a^2a_k}{(k+2)(k-3)}, a_1 = 0\right]$
- Recursion relation for $r = 3$
  $a_{k+2} = \frac{a^2a_k}{(k+7)(k+2)}$
- Solution for $r = 3$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{a^2a_k}{(k+7)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} c_k x^{k+3}\right), b_{k+2} = \frac{a^2b_k}{(k+2)(k-3)}, b_1 = 0, c_{k+2} = \frac{a^2c_k}{(k+7)(k+2)}, c_1 = 0\right]$

1.702.3 Maple trace

```
Methods for second order ODEs:
```

1.702.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 48

```
dsolve(diff(diff(y(x),x),x)-a^2*y(x) = 6*y(x)/x^2,
       y(x),singsol=all)
```

$$y = \frac{c_2(a^2x^2 + 3ax + 3)\,\mathrm{e}^{-ax} + c_1\,\mathrm{e}^{ax}(a^2x^2 - 3ax + 3)}{x^2}$$

1.702.5 Mathematica DSolve solution

Solving time : `0.199` (sec)
Leaf size : 90

```
DSolve[{D[y[x],{x,2}]-a^2*y[x]==6*y[x]/x^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{\frac{2}{\pi}}((a^2c_2x^2 - 3iac_1x + 3c_2)\cosh(ax) + i(c_1(a^2x^2 + 3) + 3iac_2x)\sinh(ax))}{a^2x^{3/2}\sqrt{-iax}}$$

1.703 problem 720

Internal problem ID [8841]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 720
Date solved : `Monday, October 21, 2024 at 05:22:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + n^2 y = \frac{6y}{x^2}$$

1.703.1 Solved as second order ode using Kovacic algorithm

Time used: 0.372 (sec)

Writing the ode as

$$y'' + \left(n^2 - \frac{6}{x^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= n^2 - \frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-n^2x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -n^2x^2 + 6$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-n^2x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1341: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -n^2 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx in - \frac{3i}{n\,x^2} - \frac{9i}{2n^3x^4} - \frac{27i}{2n^5x^6} - \frac{405i}{8n^7x^8} - \frac{1701i}{8n^9x^{10}} - \frac{15309i}{16n^{11}x^{12}} - \frac{72171i}{16n^{13}x^{14}} + \dots \quad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = in$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= in \end{aligned} \quad (10)$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -n^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-n^2x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= \left(-n^2\right) + \left(\frac{6}{x^2}\right) \\ &= -n^2 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= in \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{in} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{in} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-n^2x^2 + 6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	in	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x} + (-)\,(in) \\
&= -\frac{2}{x} - in \\
&= -\frac{2}{x} - in
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{2}{x} - in\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - in\right)^2 - \left(\frac{-n^2x^2 + 6}{x^2}\right)\right) = 0
$$

$$
\frac{(2ina_1 - 6)\,x + 4ina_0 - 4a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{3}{n^2}, a_1 = -\frac{3i}{n}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - \frac{3ix}{n} - \frac{3}{n^2}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - \frac{3ix}{n} - \frac{3}{n^2}\right) e^{\int \left(-\frac{2}{x} - in\right) dx} \\
&= \left(x^2 - \frac{3ix}{n} - \frac{3}{n^2}\right) e^{-2\ln(x) - inx} \\
&= \frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2 n^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} \int \frac{1}{\frac{(n^2x^2-3inx-3)^2\mathrm{e}^{-2inx}}{x^4n^4}}\,dx \\ &= \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} \left(\frac{(in^2x^2 - 3nx - 3i)\,\mathrm{e}^{2inx}}{2n\,(-n^2x^2 + 3inx + 3)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\left(\frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2}\right) + c_2\left(\frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2}\left(\frac{(in^2x^2 - 3nx - 3i)\,\mathrm{e}^{2inx}}{2n\,(-n^2x^2 + 3inx + 3)}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.703.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + n^2 y = \frac{6y}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y(n^2x^2-6)}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y(n^2x^2-6)}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{n^2x^2-6}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -6$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d}{dx}y'\right)x^2 + (n^2x^2 - 6)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(2+r)(-3+r)\, x^r + a_1(3+r)(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)(k+r-3) + n^2 a_{k-2}\right) x^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, 3\}$
- Each term must be 0

  $a_1(3+r)(-2+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-3) + n^2 a_{k-2} = 0$
- Shift index using $k- > k+2$

  $a_{k+2}(k+4+r)(k+r-1) + n^2 a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{n^2 a_k}{(k+4+r)(k+r-1)}$
- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{n^2 a_k}{(k+2)(k-3)}$
- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{n^2 a_k}{(k+2)(k-3)}, a_1 = 0\right]$
- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{n^2 a_k}{(k+7)(k+2)}$
- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{n^2 a_k}{(k+7)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{n^2 a_k}{(k+2)(k-3)}, a_1 = 0, b_{k+2} = -\frac{n^2 b_k}{(k+7)(k+2)}, b_1 = 0\right]$

1.703.3 Maple trace

```
Methods for second order ODEs:
```

1.703.4 Maple dsolve solution

Solving time : 0.014 (sec)
Leaf size : 53

```
dsolve(diff(diff(y(x),x),x)+n^2*y(x) = 6*y(x)/x^2,
       y(x),singsol=all)
```

$$y = \frac{(c_1 n^2 x^2 + 3c_2 nx - 3c_1) \cos (nx) + \sin (nx) (c_2 n^2 x^2 - 3c_1 nx - 3c_2)}{x^2}$$

1.703.5 Mathematica DSolve solution

Solving time : 0.201 (sec)
Leaf size : 79

```
DSolve[{D[y[x],{x,2}]+n^2*y[x]==6*y[x]/x^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\sqrt{x}((c_2(-n^2)\, x^2 + 3c_1 nx + 3c_2) \cos(nx) + (c_1(n^2x^2 - 3) + 3c_2 nx) \sin(nx))}{(nx)^{5/2}}$$

1.704 problem 721

Internal problem ID [8842]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 721
Date solved : `Monday, October 21, 2024 at 05:22:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' - \left(x^2 + \frac{1}{4}\right)y = 0$$

1.704.1 Solved as second order ode using Kovacic algorithm

Time used: 0.108 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(-x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1343: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{e^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-x}}{\sqrt{x}}\right) + c_2 \left(\frac{e^{-x}}{\sqrt{x}} \left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.704.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + xy' - \left(x^2 + \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{(4x^2+1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{y'}{x} - \frac{(4x^2+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{4x^2+1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 \left(\frac{d}{dx} y'\right) + 4xy' + (-4x^2 - 1) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) - 4a_{k-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

$$a_1(3+2r)\,(1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(4k^2 + 8kr + 4r^2 - 1) - 4a_{k-2} = 0$$

- Shift index using $k- >k + 2$

$$a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) - 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+2} = \frac{4a_k}{4k^2+12k+8}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

1.704.3 Maple trace

```
Methods for second order ODEs:
```

1.704.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)-(x^2+1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1\sinh(x)+c_2\cosh(x)}{\sqrt{x}}$$

1.704.5 Mathematica DSolve solution

Solving time : `0.047` (sec)
Leaf size : 32

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]-(x^2+1/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{e^{-x}(c_2e^{2x}+2c_1)}{2\sqrt{x}}$$

1.705 problem 722

Internal problem ID [8843]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 722
Date solved : `Monday, October 21, 2024 at 05:22:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \frac{(-9a^2 + 4x^2)\, y}{4a^2} = 0$$

1.705.1 Solved as second order ode using Kovacic algorithm

Time used: 0.334 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(-\frac{9}{4} + \frac{x^2}{a^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -\frac{9}{4} + \frac{x^2}{a^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2a^2 - x^2}{x^2a^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2a^2 - x^2$$
$$t = x^2a^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2a^2 - x^2}{x^2a^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1345: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2a^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{a^2} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx -\frac{33ia^{13}}{16x^{14}} - \frac{21ia^{11}}{16x^{12}} - \frac{7ia^9}{8x^{10}} - \frac{5ia^7}{8x^8} - \frac{ia^5}{2x^6} - \frac{ia^3}{2x^4} - \frac{ia}{x^2} + \frac{i}{a} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i}{a}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{i}{a} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{1}{a^2}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2a^2 - x^2}{x^2 a^2} \\ &= Q + \frac{R}{x^2 a^2} \\ &= \left(-\frac{1}{a^2}\right) + \left(\frac{2}{x^2}\right) \\ &= -\frac{1}{a^2} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{i}{a} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{i}{a}} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{i}{a}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2a^2 - x^2}{x^2 a^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{i}{a}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{i}{a}\right) \\
&= -\frac{1}{x} - \frac{i}{a} \\
&= -\frac{ix + a}{xa}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{i}{a}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{i}{a}\right)^2 - \left(\frac{2a^2 - x^2}{x^2 a^2}\right)\right) &= 0 \\
\frac{2ia_0 - 2a}{xa} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -ia\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = -ia + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (-ia + x) \, \mathrm{e}^{\int \left(-\frac{1}{x} - \frac{i}{a}\right) dx} \\
&= (-ia + x) \, \mathrm{e}^{-\ln(x) - \frac{ix}{a}} \\
&= \frac{(-ia + x) \, \mathrm{e}^{-\frac{ix}{a}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(-ia+x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{3/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(ix+a)\,a(ia+x)\,\mathrm{e}^{\frac{2ix}{a}}}{2\,(ia-x)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(-ia+x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{3/2}}\right) + c_2\left(\frac{(-ia+x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{3/2}}\left(-\frac{(ix+a)\,a(ia+x)\,\mathrm{e}^{\frac{2ix}{a}}}{2\,(ia-x)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.705.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+xy'+\frac{(-9a^2+4x^2)y}{4a^2}=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{(9a^2-4x^2)y}{4x^2a^2}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{y'}{x}-\frac{(9a^2-4x^2)y}{4x^2a^2}=0$

- Multiply by denominators of the ODE

  $4x^2\left(\frac{d}{dx}y'\right)a^2+4xy'a^2-(9a^2-4x^2)y=0$

- Make a change of variables

  $t=\ln(x)$

- ☐ Substitute the change of variables back into the ODE
  - ○ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y'=\left(\frac{d}{dt}y(t)\right)t'(x)$

  - ○ Compute derivative

    $y'=\frac{\frac{d}{dt}y(t)}{x}$

  - ○ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y'=\left(\frac{d}{dt}\frac{d}{dt}y(t)\right)t'(x)^2+\left(\frac{d}{dx}t'(x)\right)\left(\frac{d}{dt}y(t)\right)$

  - ○ Compute derivative

    $\frac{d}{dx}y'=\frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $4x^2\left(\frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)a^2+4\left(\frac{d}{dt}y(t)\right)a^2-(9a^2-4x^2)y(t)=0$

- Simplify

  $4a^2\left(\frac{d}{dt}\frac{d}{dt}y(t)\right)-9y(t)a^2+4y(t)x^2=0$

- Isolate 2nd derivative

  $\frac{d}{dt}\frac{d}{dt}y(t)=\frac{(9a^2-4x^2)y(t)}{4a^2}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d}{dt}\frac{d}{dt}y(t)-\frac{(9a^2-4x^2)y(t)}{4a^2}=0$

- Characteristic polynomial of ODE

$$r^2 - \frac{9a^2-4x^2}{4a^2} = 0$$

- Factor the characteristic polynomial

$$\frac{4r^2a^2-9a^2+4x^2}{4a^2} = 0$$

- Roots of the characteristic polynomial

$$r = \left(\frac{\sqrt{9a^2-4x^2}}{2a}, -\frac{\sqrt{9a^2-4x^2}}{2a}\right)$$

- 1st solution of the ODE

$$y_1(t) = e^{\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$$

- 2nd solution of the ODE

$$y_2(t) = e^{-\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$$

- General solution of the ODE

$$y(t) = C1y_1(t) + C2y_2(t)$$

- Substitute in solutions

$$y(t) = C1\,e^{\frac{\sqrt{9a^2-4x^2}\,t}{2a}} + C2\,e^{-\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$$

- Change variables back using $t = \ln(x)$

$$y = C1\,e^{\frac{\sqrt{9a^2-4x^2}\,\ln(x)}{2a}} + C2\,e^{-\frac{\sqrt{9a^2-4x^2}\,\ln(x)}{2a}}$$

- Simplify

$$y = C1\,x^{\frac{\sqrt{9a^2-4x^2}}{2a}} + C2\,x^{-\frac{\sqrt{9a^2-4x^2}}{2a}}$$

1.705.3 Maple trace

```
Methods for second order ODEs:
```

1.705.4 Maple dsolve solution

Solving time : `0.017` (sec)
Leaf size : 37

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+1/4*(-9*a^2+4*x^2)/a^2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2(ix+a)\,e^{-\frac{ix}{a}} + (-ix+a)\,e^{\frac{ix}{a}}c_1}{x^{3/2}}$$

1.705.5 Mathematica DSolve solution

Solving time : 0.137 (sec)
Leaf size : 62

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(4*x^2-9*a^2)/(4*a^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\left((ac_2 + c_1 x)\cos\left(\frac{x}{a}\right) + (c_2 x - ac_1)\sin\left(\frac{x}{a}\right)\right)}{x\sqrt{\frac{x}{a}}}$$

1.706 problem 723

Internal problem ID [8844]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 723
Date solved : `Monday, October 21, 2024 at 05:22:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right)y = 0$$

1.706.1 Solved as second order ode using Kovacic algorithm

Time used: 0.310 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{25}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 6$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1347: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= -1 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 + 6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + (-)\,(i) \\
&= -\frac{2}{x} - i \\
&= -\frac{2}{x} - i
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2 + 6}{x^2}\right)\right) = 0
$$

$$
\frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -3, a_1 = -3i\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 3ix - 3
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \left(x^2 - 3ix - 3\right) \mathrm{e}^{\int \left(-\frac{2}{x} - i\right) dx} \\
&= \left(x^2 - 3ix - 3\right) \mathrm{e}^{-2\ln(x) - ix} \\
&= \frac{\left(x^2 - 3ix - 3\right) \mathrm{e}^{-ix}}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}\right) + c_2\left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}\left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.706.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+xy'+\left(x^2-\frac{25}{4}\right)y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{(4x^2-25)y}{4x^2}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{y'}{x}+\frac{(4x^2-25)y}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{4x^2-25}{4x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{25}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right)+4xy'+(4x^2-25)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x\cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(5+2r)\,(-5+2r)\, x^r + a_1(7+2r)\,(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+5)\,(2k+2r-5) + 4a_{\cdot}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(5+2r)\,(-5+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-\frac{5}{2}, \frac{5}{2}\right\}$
- Each term must be 0
  $a_1(7+2r)\,(-3+2r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(2k+2r+5)\,(2k+2r-5) + 4a_{k-2} = 0$
- Shift index using $k->k+2$
  $a_{k+2}(2k+9+2r)\,(2k-1+2r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$
- Recursion relation for $r = -\frac{5}{2}$
  $a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}$
- Solution for $r = -\frac{5}{2}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0\right]$
- Recursion relation for $r = \frac{5}{2}$
  $a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}$
- Solution for $r = \frac{5}{2}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}, a_1 = 0\right]$
- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(2k+14)(2k+4)}, b_1 = 0\right]$$

1.706.3 Maple trace

```
Methods for second order ODEs:
```

1.706.4 Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 43

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-25/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{-3\left(ix - \frac{1}{3}x^2 + 1\right) c_2\, \mathrm{e}^{-ix} + 3\left(ix + \frac{1}{3}x^2 - 1\right) \mathrm{e}^{ix} c_1}{x^{5/2}}$$

1.706.5 Mathematica DSolve solution

Solving time : 0.138 (sec)
Leaf size : 59

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-25/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}((-c_2x^2 + 3c_1x + 3c_2)\cos(x) + (c_1(x^2 - 3) + 3c_2x)\sin(x))}{x^{5/2}}$$

1.707 problem 724

Internal problem ID [8845]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 724
Date solved : `Monday, October 21, 2024 at 05:22:44 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + qy' = \frac{2y}{x^2}$$

1.707.1 Solved as second order ode using Kovacic algorithm

Time used: 0.293 (sec)

Writing the ode as

$$y'' + qy' - \frac{2y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= q \\ C &= -\frac{2}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{q^2x^2 + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = q^2x^2 + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{q^2x^2 + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1349: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{q^2}{4} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{q}{2} + \frac{2}{q\,x^2} - \frac{4}{q^3x^4} + \frac{16}{q^5x^6} - \frac{80}{q^7x^8} + \frac{448}{q^9x^{10}} - \frac{2688}{q^{11}x^{12}} + \frac{16896}{q^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{q}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{q}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{q^2}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{q^2x^2 + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{q^2}{4}\right) + \left(\frac{2}{x^2}\right) \\ &= \frac{q^2}{4} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{q}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{q}{2}} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{q}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{q^2x^2 + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{q}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{q}{2}\right) \\
&= -\frac{1}{x} - \frac{q}{2} \\
&= -\frac{qx + 2}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{q}{2}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{q}{2}\right)^2 - \left(\frac{q^2 x^2 + 8}{4x^2}\right)\right) &= 0 \\
\frac{q a_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{2}{q}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{2}{q}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{2}{q}\right) e^{\int \left(-\frac{1}{x} - \frac{q}{2}\right) dx} \\
&= \left(x + \frac{2}{q}\right) e^{-\frac{qx}{2} - \ln(x)} \\
&= \frac{(qx + 2)\, e^{-\frac{qx}{2}}}{qx}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{q}{1}\,dx} \\
&= z_1 e^{-\frac{qx}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{qx}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-qx}(qx+2)}{qx}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{q}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-qx}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(qx-2)\,\mathrm{e}^{qx}}{q\,(qx+2)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-qx}(qx+2)}{qx}\right) + c_2 \left(\frac{\mathrm{e}^{-qx}(qx+2)}{qx}\left(\frac{(qx-2)\,\mathrm{e}^{qx}}{q\,(qx+2)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.707.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + qy' = \frac{2y}{x^2}$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y' = -qy' + \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y' + qy' - \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = q, P_3(x) = -\frac{2}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$qy'x^2 + \left(\frac{d}{dx}y'\right)x^2 - 2y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^2 \cdot y'$ to series expansion

$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

- ◦ Shift index using $k- > k-1$

$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$

- ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$x^{2}\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\left(k-1+r\right)x^{k+r}$

Rewrite ODE with series expansions

$a_{0}(1+r)\left(-2+r\right)x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k}(k+r+1)\left(k+r-2\right)+q a_{k-1}(k-1+r)\right)x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+r)\left(-2+r\right)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-1,2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r+1)\left(k+r-2\right)+q a_{k-1}(k-1+r)=0$
- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)\left(k-1+r\right)+q a_{k}(k+r)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{q a_{k}(k+r)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r=-1$; series terminates at $k=1$

  $a_{k+1}=-\frac{q a_{k}(k-1)}{(k+1)(k-2)}$
- Apply recursion relation for $k=0$

  $a_{1}=-\frac{q a_{0}}{2}$
- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second

  $y=a_{0}\cdot\left(-\frac{qx}{2}+1\right)$
- Recursion relation for $r=2$

  $a_{k+1}=-\frac{q a_{k}(k+2)}{(k+4)(k+1)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty}a_{k}x^{k+2}, a_{k+1}=-\frac{q a_{k}(k+2)}{(k+4)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y=a_{0}\cdot\left(-\frac{qx}{2}+1\right)+\left(\sum_{k=0}^{\infty}b_{k}x^{k+2}\right), b_{k+1}=-\frac{q b_{k}(k+2)}{(k+4)(k+1)}\right]$

1.707.3 Maple trace

```
Methods for second order ODEs:
```

1.707.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 28

```
dsolve(diff(diff(y(x),x),x)+q*diff(y(x),x) = 2*y(x)/x^2,
       y(x),singsol=all)
```

$$y = \frac{c_2 \, \mathrm{e}^{-qx}(qx+2)+c_1(qx-2)}{x}$$

1.707.5 Mathematica DSolve solution

Solving time : 0.087 (sec)
Leaf size : 80

```
DSolve[{D[y[x],{x,2}]+q*D[y[x],x]==2*y[x]/x^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{qx^{3/2}e^{-\frac{qx}{2}}\left(2(ic_2qx+2c_1)\sinh\left(\frac{qx}{2}\right)-2(c_1qx+2ic_2)\cosh\left(\frac{qx}{2}\right)\right)}{\sqrt{\pi}(-iqx)^{5/2}}$$

1.708 problem 725

Internal problem ID [8846]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 725
Date solved : `Monday, October 21, 2024 at 05:22:45 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$xy'' + 3y' + 4x^3y = 0$$

1.708.1 Solved as second order ode using Kovacic algorithm

Time used: 0.288 (sec)

Writing the ode as

$$xy'' + 3y' + 4x^3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= 4x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-16x^4 + 3}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -16x^4 + 3 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1351: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -4x^2 + \frac{3}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-4x^2) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 2ix \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-16x^4 + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)(2ix) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4 + 3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\
&= \frac{\mathrm{e}^{-ix^2}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3}{x} \, dx} \\
&= z_1 e^{-\frac{3 \ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-i x^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{i\mathrm{e}^{2 i x^2}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-i x^2}}{x^2}\right) + c_2 \left(\frac{\mathrm{e}^{-i x^2}}{x^2} \left(-\frac{i\mathrm{e}^{2 i x^2}}{4}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.708.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 3y' + 4x^3 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -4x^2 y - \frac{3y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{3y'}{x} + 4x^2 y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + 3y' + 4x^3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

  $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

- Shift index using $k- >k - 3$

  $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\,(k + r)\, x^{k+r}$

Rewrite ODE with series expansions

$$a_0 r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^{r} + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 0\}$
- The coefficients of each power of x must be 0
  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$
- Solve for the dependent coefficient(s)
  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$
- Shift index using $k- > k+3$
  $a_{k+4}(k+4+r)\,(k+6+r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$
- Recursion relation for $r = -2$
  $a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$
- Solution for $r = -2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Recursion relation for $r = 0$
  $a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{4b_k}{(k+4)(k+6)}\right.$

1.708.3 Maple trace

```
Methods for second order ODEs:
```

1.708.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 21

```
dsolve(x*diff(diff(y(x),x),x)+3*diff(y(x),x)+4*x^3*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x^2) + c_2 \cos(x^2)}{x^2}$$

1.708.5 Mathematica DSolve solution

Solving time : 0.072 (sec)
Leaf size : 41

```
DSolve[{x*D[y[x],{x,2}]+3*D[y[x],x]+4*x^3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1 e^{-ix^2} - ic_2 e^{ix^2}}{4x^2}$$

1.709 problem 726

Internal problem ID [8847]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 726
Date solved : `Monday, October 21, 2024 at 05:22:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(2-x)\,y''+2xy'-2y=0$$

1.709.1 Solved as second order ode using Kovacic algorithm

Time used: 0.216 (sec)

Writing the ode as

$$\left(-x^3+2x^2\right)y''+2xy'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= -x^3+2x^2\\ B &= 2x\\ C &= -2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(x^2 - 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = (x^2 - 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(x^2 - 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1353: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2} - \frac{3}{4(-2+x)} + \frac{3}{4x} + \frac{3}{4(-2+x)^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(-2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{\left(x^2 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + \frac{3}{2(-2 + x)} + (-)(0) \\ &= -\frac{1}{2x} + \frac{3}{2(-2 + x)} \\ &= \frac{1 + x}{x(-2 + x)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} + \frac{3}{2(-2+x)}\right)(0) + \left(\left(\frac{1}{2x^2} - \frac{3}{2(-2+x)^2}\right) + \left(-\frac{1}{2x} + \frac{3}{2(-2+x)}\right)^2 - \left(\frac{3}{(x^2-2x)^2}\right)\right)$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2x} + \frac{3}{2(-2+x)}\right) dx} \\ &= \frac{(-2+x)^{3/2}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x}{-x^3+2x^2}\, dx} \\ &= z_1 e^{\frac{\ln(-2+x)}{2} - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\sqrt{-2+x}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(-2+x)^2}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{-x^3+2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(-2+x)-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{(x-1)\, x\, \mathrm{e}^{\ln(-2+x)-\ln(x)}}{(-2+x)^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(-2+x)^2}{x}\right) + c_2 \left(\frac{(-2+x)^2}{x} \left(-\frac{(x-1)\, x\, \mathrm{e}^{\ln(-2+x)-\ln(x)}}{(-2+x)^3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.709.2 Maple step by step solution

Let's solve

$x^2(2-x)\left(\frac{d}{dx}y'\right) + 2xy' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2y}{x^2(-2+x)} + \frac{2y'}{x(-2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y'}{x(-2+x)} + \frac{2y}{x^2(-2+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x(-2+x)}, P_3(x) = \frac{2}{x^2(-2+x)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = -1$$

- $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2(-2+x)\left(\frac{d}{dx}y'\right) - 2xy' + 2y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0(1+r)(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(-2a_k(k+r+1)(k+r-1) + a_{k-1}(k+r-1)(k-2+r)\right) x^{k+}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2(1+r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 1\}$

- Each term in the series must be 0, giving the recursion relation

$-2(k+r-1)\left(\left(-\frac{k}{2} - \frac{r}{2} + 1\right) a_{k-1} + a_k(k+r+1)\right) = 0$

- Shift index using $k->k+1$

$-2(k+r)\left(\left(-\frac{k}{2} + \frac{1}{2} - \frac{r}{2}\right) a_k + a_{k+1}(k+2+r)\right) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{(k+r-1)a_k}{2(k+2+r)}$

- Recursion relation for $r = -1$; series terminates at $k = 2$

$a_{k+1} = \frac{(k-2)a_k}{2(k+1)}$

- Apply recursion relation for $k=0$
  $a_1 = -a_0$
- Apply recursion relation for $k=1$
  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0
  $a_2 = \frac{a_0}{4}$
- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second
  $y = a_0 \cdot \left(1 - x + \frac{1}{4}x^2\right)$
- Recursion relation for $r=1$
  $a_{k+1} = \frac{k a_k}{2(k+3)}$
- Solution for $r=1$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{k a_k}{2(k+3)}\right]$
- Combine solutions and rename parameters
  $\left[y = a_0 \cdot \left(1 - x + \frac{1}{4}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), b_{k+1} = \frac{k b_k}{2(k+3)}\right]$

1.709.3 Maple trace

```
Methods for second order ODEs:
```

1.709.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 19

```
dsolve(x^2*(2-x)*diff(diff(y(x),x),x)+2*x*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 x^2 + c_2(x-1)}{x}$$

1.709.5 Mathematica DSolve solution

Solving time : 0.067 (sec)
Leaf size : 24

```
DSolve[{x^2*(2-x)*D[y[x],{x,2}]+2*x*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1(x-2)^2 + c_2(x-1)}{x}$$

1.710 problem 727

Internal problem ID [8848]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 727
Date solved : `Monday, October 21, 2024 at 05:22:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-2xy'+2y=0$$

1.710.1 Solved as second order ode using Kovacic algorithm

Time used: 0.284 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (x^2+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1355: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\ &= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\ &= \frac{x-2i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{3/2}}{(ix+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1} \, dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1 \left(\sqrt{x^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{x}{(x+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2\left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.710.2 Maple step by step solution

1.710.3 Maple trace

```
Methods for second order ODEs:
```

1.710.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 16

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$
y = c_2\,x^2 + c_1 x - c_2
$$

1.710.5 Mathematica DSolve solution

Solving time : 0.07 (sec)
Leaf size : 21

```
DSolve[{(x^2+1)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x - c_1 (x - i)^2$$

1.711 problem 728

Internal problem ID [8849]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 728
Date solved : `Monday, October 21, 2024 at 05:22:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - 2(x+1)\, y' + (x+2)\, y = 0$$

1.711.1 Solved as second order ode using Kovacic algorithm

Time used: 0.151 (sec)

Writing the ode as

$$xy'' + (-2x-2)\, y' + (x+2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1356: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(0) \\ &= -\frac{1}{x} \\ &= -\frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x \, e^x) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x \, e^{2x+2\ln(x)} e^{-2x}}{3} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x\,\mathrm{e}^{2x+2\ln(x)}\mathrm{e}^{-2x}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.711.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) - 2(x+1)\,y' + (x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x+2)y}{x} + \frac{2(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(x+1)y'}{x} + \frac{(x+2)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{x+2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + (-2x-2)\,y' + (x+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\, x^{-1+r} + (a_1(1+r)\,(-2+r) - 2a_0(-1+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-2+r)\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$
- Each term must be 0

  $a_1(1+r)\,(-2+r) - 2a_0(-1+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k-2+r) - 2a_k k - 2a_k r + 2a_k + a_{k-1} = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+r-1) - 2a_{k+1}(k+1) - 2r a_{k+1} + 2a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2k a_{k+1} + 2r a_{k+1} - a_k}{(k+2+r)(k+r-1)}$
- Recursion relation for $r = 0$

$a_{k+2} = \frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Recursion relation for $r = 3$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}, 4a_1 - 4a_0 = 0\right]$

1.711.3 Maple trace

```
Methods for second order ODEs:
```

1.711.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 14

```
dsolve(x*diff(diff(y(x),x),x)-2*(x+1)*diff(y(x),x)+(x+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{x} \left(c_2\, x^3 + c_1\right)$$

1.711.5 Mathematica DSolve solution

Solving time : `0.041` (sec)
Leaf size : 23

```
DSolve[{x*D[y[x],{x,2}]-2*(x+1)*D[y[x],x]+(x+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3} e^x \left(c_2 x^3 + 3c_1\right)$$

1.712 problem 729

Internal problem ID [8850]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 729
Date solved : `Monday, October 21, 2024 at 05:22:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3xy'' - 2(3x-1)\,y' + (3x-2)\,y = 0$$

1.712.1 Solved as second order ode using Kovacic algorithm

Time used: 0.154 (sec)

Writing the ode as

$$3xy'' + (-6x+2)\,y' + (3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x \\ B &= -6x+2 \\ C &= 3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{9x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -2$$
$$t = 9x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{9x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1358: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 9x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{2}{9x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{9x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{3}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{3x} + (-)(0) \\
&= \frac{1}{3x} \\
&= \frac{1}{3x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{3x}\right)(0) + \left(\left(-\frac{1}{3x^2}\right) + \left(\frac{1}{3x}\right)^2 - \left(-\frac{2}{9x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx}\\ &= \mathrm{e}^{\int \frac{1}{3x} dx}\\ &= x^{1/3}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-6x+2}{3x} \, dx}\\ &= z_1 e^{x - \frac{\ln(x)}{3}}\\ &= z_1\left(\frac{\mathrm{e}^x}{x^{1/3}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x+2}{3x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x-\frac{2\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(3x\,\mathrm{e}^{2x-\frac{2\ln(x)}{3}}\,\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(3x\,\mathrm{e}^{2x-\frac{2\ln(x)}{3}}\,\mathrm{e}^{-2x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.712.2 Maple step by step solution

Let's solve

$3x\left(\frac{d}{dx}y'\right) - 2(3x-1)\,y' + (3x-2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(3x-2)y}{3x} + \frac{2(3x-1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(3x-1)y'}{3x} + \frac{(3x-2)y}{3x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2(3x-1)}{3x}, P_3(x) = \frac{3x-2}{3x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{2}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x\left(\frac{d}{dx}y'\right) + (-6x+2)\,y' + (3x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+3r)\, x^{-1+r} + \left(a_1(1+r)\,(2+3r) - 2a_0(1+3r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(3k+2+3r\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(1+r)\,(2+3r) - 2a_0(1+3r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $3\left(k+\frac{2}{3}+r\right)(k+1+r)a_{k+1}-6a_k k-6a_k r-2a_k+3a_{k-1}=0$
- Shift index using $k->k+1$

  $3\left(k+\frac{5}{3}+r\right)(k+2+r)a_{k+2}-6a_{k+1}(k+1)-6ra_{k+1}-2a_{k+1}+3a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{6ka_{k+1}+6ra_{k+1}-3a_k+8a_{k+1}}{(3k+5+3r)(k+2+r)}$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{6ka_{k+1}-3a_k+8a_{k+1}}{(3k+5)(k+2)}$
- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+2}=\frac{6ka_{k+1}-3a_k+8a_{k+1}}{(3k+5)(k+2)}, 2a_1-2a_0=0\right]$
- Recursion relation for $r=\frac{1}{3}$

  $a_{k+2}=\frac{6ka_{k+1}-3a_k+10a_{k+1}}{(3k+6)\left(k+\frac{7}{3}\right)}$
- Solution for $r=\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{3}}, a_{k+2}=\frac{6ka_{k+1}-3a_k+10a_{k+1}}{(3k+6)\left(k+\frac{7}{3}\right)}, 4a_1-4a_0=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^k\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{3}}\right), a_{k+2}=\frac{6ka_{k+1}-3a_k+8a_{k+1}}{(3k+5)(k+2)}, 2a_1-2a_0=0, b_{k+2}=\frac{6kb_{k+1}-3b_k+10b_k}{(3k+6)\left(k+\frac{7}{3}\right)}\right.$

1.712.3 Maple trace

```
Methods for second order ODEs:
```

1.712.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 14

```
dsolve(3*x*diff(diff(y(x),x),x)-2*(3*x-1)*diff(y(x),x)+(3*x-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y=e^x\left(c_1+x^{1/3}c_2\right)$$

1.712.5 Mathematica DSolve solution

Solving time : 0.043 (sec)
Leaf size : 21

```
DSolve[{3*x*D[y[x],{x,2}]-2*(3*x-1)*D[y[x],x]+(3*x-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^x \left(3c_2 \sqrt[3]{x} + c_1\right)$$

1.713 problem 730

Internal problem ID [8851]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 730
Date solved : `Monday, October 21, 2024 at 05:22:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(x+1)\, y'' - (x-1)\, y' + y = 0$$

1.713.1 Solved as second order ode using Kovacic algorithm

Time used: 0.242 (sec)

Writing the ode as

$$\left(x^2+x\right) y'' + (1-x)\, y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+x \\ B &= 1-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 10x - 1$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1360: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{2}{x+1} + \frac{2}{(x+1)^2} - \frac{2}{x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4(x^2 + x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^2-10x-1}{4\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x+1} + \frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{x+1} + \frac{1}{2x} \\
&= -\frac{x-1}{2x\,(x+1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{x+1} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(x+1)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{x+1} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2-10x-1}{4\,(x^2+x)^2}\right)\right) = 0
$$

$$
\frac{1+a_0}{x\,(x+1)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x-1)\,\mathrm{e}^{\int \left(-\frac{1}{x+1} + \frac{1}{2x}\right) dx} \\
&= (x-1)\,\mathrm{e}^{-\ln(x+1) + \frac{\ln(x)}{2}} \\
&= \frac{(x-1)\sqrt{x}}{x+1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-x}{x^2+x}\,dx} \\
&= z_1 e^{\ln(x+1)-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{x+1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x - 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-x}{x^2+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x+1)-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\ln(x) - \frac{4}{x-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x-1) + c_2\left(x - 1\left(\ln(x) - \frac{4}{x-1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.713.2 Maple step by step solution

Let's solve

$x(x+1)\left(\frac{d}{dx}y'\right)-(x-1)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{y}{x(x+1)}+\frac{(x-1)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'-\frac{(x-1)y'}{x(x+1)}+\frac{y}{x(x+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{x-1}{x(x+1)},P_3(x)=\frac{1}{x(x+1)}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=-2$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$x(x+1)\left(\frac{d}{dx}y'\right)+(1-x)\,y'+y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2-u)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty}a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)$ to series expansion for $m =1..2$

$$u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- Shift index using $k - >k +2-m$

$$u^{m}\cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$-a_{0} r \left(-3+r \right) u^{-1+r}+\left(\sum_{k =0}^{\infty}\left(-a_{k +1} \left(k +1+r \right) \left(k -2+r \right)+a_{k} \left(k +r -1\right)^{2}\right) u^{k +r}\right)=0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$-r \left(-3+r \right)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, 3\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-a_{k +1} \left(k +1+r \right) \left(k -2+r \right)+a_{k} \left(k +r -1\right)^{2}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k +1}=\frac{a_{k} \left(k +r -1\right)^{2}}{\left(k +1+r \right) \left(k -2+r \right)}$$

- Recursion relation for $r =0$; series terminates at $k =1$

$$a_{k +1}=\frac{a_{k} \left(k -1\right)^{2}}{\left(k +1\right) \left(k -2\right)}$$

- Apply recursion relation for $k =0$

$$a_{1}=-\frac{a_{0}}{2}$$

- Terminating series solution of the ODE for $r =0$. Use reduction of order to find the second li

$$y \left(u \right)=a_{0}\cdot \left(1-\frac{u}{2}\right)$$

- Revert the change of variables $u =x +1$

$$\left[y =a_{0} \left(-\frac{x}{2}+\frac{1}{2}\right)\right]$$

- Recursion relation for $r =3$

$$a_{k +1}=\frac{a_{k} \left(k +2\right)^{2}}{\left(k +4\right) \left(k +1\right)}$$

- Solution for $r =3$

$$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k +3}, a_{k +1}=\frac{a_{k} \left(k +2\right)^{2}}{\left(k +4\right) \left(k +1\right)}\right]$$

- Revert the change of variables $u =x +1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+3}, a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0\left(-\frac{x}{2} + \frac{1}{2}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+3}\right), b_{k+1} = \frac{b_k (k+2)^2}{(k+4)(k+1)}\right]$$

1.713.3 Maple trace

```
Methods for second order ODEs:
```

1.713.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 20

```
dsolve(x*(x+1)*diff(diff(y(x),x),x)-(x-1)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2(x-1)\ln(x) - 4c_2 + c_1(x-1)$$

1.713.5 Mathematica DSolve solution

Solving time : 0.073 (sec)
Leaf size : 23

```
DSolve[{x*(x+1)*D[y[x],{x,2}]-(x-1)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1(x-1) + c_2((x-1)\log(x) - 4)$$

1.714 problem 731

Internal problem ID [8852]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 731
Date solved : `Monday, October 21, 2024 at 05:22:50 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+2x\right) y''-2(x+1) y'+2y=0$$

1.714.1 Solved as second order ode using Kovacic algorithm

Time used: 0.233 (sec)

Writing the ode as

$$\left(x^2+2x\right) y''+(-2x-2) y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+2x \\ B &= -2x-2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = \left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1362: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4(x+2)} + \frac{3}{4x^2} + \frac{3}{4(x+2)^2} - \frac{3}{4x}
$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{\left(x^2+2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x+2\right)} + \frac{3}{2x} + (-)\left(0\right) \\
&= -\frac{1}{2\left(x+2\right)} + \frac{3}{2x} \\
&= \frac{x+3}{x\left(x+2\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)(0) + \left(\left(\frac{1}{2(x+2)^2} - \frac{3}{2x^2}\right) + \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)^2 - \left(\frac{3}{(x^2+2x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right) dx} \\ &= \frac{x^{3/2}}{\sqrt{x+2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x^2+2x}\, dx} \\ &= z_1 e^{\frac{\ln(x(x+2))}{2}} \\ &= z_1\left(\sqrt{x(x+2)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x(x+2)}\, x^{3/2}}{\sqrt{x+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x^2+2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x(x+2))}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{1}{x^2} - \frac{1}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x(x+2)}\, x^{3/2}}{\sqrt{x+2}}\right) + c_2 \left(\frac{\sqrt{x(x+2)}\, x^{3/2}}{\sqrt{x+2}} \left(-\frac{1}{x^2} - \frac{1}{x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.714.2 Maple step by step solution

Let's solve

$(x^2+2x)\left(\frac{d}{dx}y'\right) - 2(x+1)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2y}{x(x+2)} + \frac{2(x+1)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(x+1)y'}{x(x+2)} + \frac{2y}{x(x+2)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(x+1)}{x(x+2)}, P_3(x) = \frac{2}{x(x+2)}\right]$

  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2} = -1$

  - $(x+2)^2 \cdot P_3(x)$ is analytic at $x=-2$

$$\left((x+2)^2 \cdot P_3(x)\right)\Bigg|_{x=-2} = 0$$

- ◦ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $x(x+2)\left(\frac{d}{dx}y'\right) + (-2x-2)\,y' + 2y = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- ◦ Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- ◦ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- ◦ Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(k+r-1) + a_k(k+r-1)\,(k+r-2)\right)u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left((-2k-2r-2)\,a_{k+1} + a_k(k+r-2)\right) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r-2)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{2(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{4}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - u + \frac{1}{4}u^2\right)$
- Revert the change of variables $u = x + 2$

  $\left[y = \frac{a_0 x^2}{4}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k}{2(k+3)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$
- Revert the change of variables $u = x + 2$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$
- Combine solutions and rename parameters

  $\left[y = \frac{a_0 x^2}{4} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$

1.714.3 Maple trace

```
Methods for second order ODEs:
```

1.714.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve((x^2+2*x)*diff(diff(y(x),x),x)-2*(x+1)*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x^2 + c_2 x + c_2$$

1.714.5 Mathematica DSolve solution

Solving time : 0.055 (sec)
Leaf size : 19

```
DSolve[{(x^2+2*x)*D[y[x],{x,2}]-2*(x+1)*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x^2 - c_2(x+1)$$

1.715 problem 732

Internal problem ID [8853]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 732
Date solved : `Monday, October 21, 2024 at 05:22:51 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x^2+2x)\,y''-2(x+1)\,y'+2y=0$$

1.715.1 Solved as second order ode using Kovacic algorithm

Time used: 0.240 (sec)

Writing the ode as

$$(x^2+2x)\,y''+(-2x-2)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+2x\\B&=-2x-2\\C&=2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(x^2 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = (x^2 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(x^2 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1364: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\,(x+2)} + \frac{3}{4x^2} - \frac{3}{4x} + \frac{3}{4\,(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(x^2+2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(x+2)} + \frac{3}{2x} + (-)(0) \\ &= -\frac{1}{2(x+2)} + \frac{3}{2x} \\ &= \frac{x+3}{x(x+2)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)(0) + \left(\left(\frac{1}{2(x+2)^2} - \frac{3}{2x^2}\right) + \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)^2 - \left(\frac{3}{(x^2+2x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right) dx} \\
&= \frac{x^{3/2}}{\sqrt{x+2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x^2+2x}\, dx} \\
&= z_1 e^{\frac{\ln(x(x+2))}{2}} \\
&= z_1 \left(\sqrt{x(x+2)}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x(x+2)}\, x^{3/2}}{\sqrt{x+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x^2+2x}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x(x+2))}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(-\frac{1}{x^2} - \frac{1}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x\,(x+2)}\, x^{3/2}}{\sqrt{x+2}}\right) + c_2 \left(\frac{\sqrt{x\,(x+2)}\, x^{3/2}}{\sqrt{x+2}} \left(-\frac{1}{x^2} - \frac{1}{x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.715.2 Maple step by step solution

Let's solve

$(x^2+2x)\left(\frac{d}{dx}y'\right) - 2(x+1)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$
- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2y}{x(x+2)} + \frac{2(x+1)y'}{x(x+2)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(x+1)y'}{x(x+2)} + \frac{2y}{x(x+2)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{2(x+1)}{x(x+2)}, P_3(x) = \frac{2}{x(x+2)}\right]$
- ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2} = -1$
- ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

$$\left((x+2)^2 \cdot P_3(x)\right)\Big|_{x=-2} = 0$$

- $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $x(x+2)\left(\frac{d}{dx}y'\right) + (-2x-2)\,y' + 2y = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

    $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

    $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(k+r-1) + a_k(k+r-1)\,(k+r-2)\right)u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $\left((-2k-2r-2)\,a_{k+1} + a_k(k+r-2)\right)(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r-2)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{2(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{4}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - u + \frac{1}{4}u^2\right)$
- Revert the change of variables $u = x + 2$

  $\left[y = \frac{a_0 x^2}{4}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k}{2(k+3)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$
- Revert the change of variables $u = x + 2$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$
- Combine solutions and rename parameters

  $\left[y = \frac{a_0 x^2}{4} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$

1.715.3 Maple trace

```
Methods for second order ODEs:
```

1.715.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve((x^2+2*x)*diff(diff(y(x),x),x)-2*(x+1)*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x^2 + c_2 x + c_2$$

1.715.5 Mathematica DSolve solution

Solving time : 0.051 (sec)
Leaf size : 19

```
DSolve[{(x^2+2*x)*D[y[x],{x,2}]-2*(x+1)*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x^2 - c_2(x + 1)$$

1.716 problem 733

Internal problem ID [8854]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 733
Date solved : `Monday, October 21, 2024 at 05:22:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x^2+1)\,y''-2xy'+2y=0$$

1.716.1 Solved as second order ode using Kovacic algorithm

Time used: 0.286 (sec)

Writing the ode as

$$(x^2+1)\,y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= (x^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1366: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{3/2}}{(ix+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1 \left(\sqrt{x^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx$$

$$= y_1\left(-\frac{x}{(x+i)^2}\right)$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$

$$= c_1\left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2\left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)$$

Will add steps showing solving for IC soon.

1.716.2 Maple step by step solution

1.716.3 Maple trace

```
Methods for second order ODEs:
```

1.716.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 16

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2\,x^2 + c_1x - c_2$$

1.716.5 Mathematica DSolve solution

Solving time : 0.064 (sec)
Leaf size : 21

```
DSolve[{(x^2+1)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x - c_1 (x - i)^2$$

1.717 problem 734

Internal problem ID [8855]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 734
Date solved : `Monday, October 21, 2024 at 05:22:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right)y''-2xy'+2y=0$$

1.717.1 Solved as second order ode using Kovacic algorithm

Time used: 0.289 (sec)

Writing the ode as

$$\left(x^2+1\right)y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2+1\\ B&=-2x\\ C&=2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= (x^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1367: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 0\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{3/2}}{(ix+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1\left(\sqrt{x^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x}{(x+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2 \left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.717.2 Maple step by step solution

1.717.3 Maple trace

```
Methods for second order ODEs:
```

1.717.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 16

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$
y = c_2\, x^2 + c_1 x - c_2
$$

1.717.5 Mathematica DSolve solution

Solving time : 0.058 (sec)
Leaf size : 21

```
DSolve[{(x^2+1)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x - c_1 (x - i)^2$$

1.718 problem 735

Internal problem ID [8856]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 735
Date solved : `Monday, October 21, 2024 at 05:22:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0$$

1.718.1 Solved as second order ode using Kovacic algorithm

Time used: 0.083 (sec)

Writing the ode as

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1368: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\
&= z_1 e^{x^2} \\
&= z_1\left(\mathrm{e}^{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{x^2}\right) + c_2\left(\mathrm{e}^{x^2}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.718.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - 4xy' + (4x^2 - 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - ○ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - ○ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - ○ Shift index using $k- >k + 2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 - 2a_0 + (6a_3 - 6a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = a_0, a_3 = a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$
- Shift index using $k- > k+2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2ka_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$

1.718.3 Maple trace

```
Methods for second order ODEs:
```

1.718.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(4*x^2-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{x^2}(c_2 x + c_1)$$

1.718.5 Mathematica DSolve solution

Solving time : `0.032` (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x^2}(c_2 x + c_1)$$

1.719 problem 736

Internal problem ID [8857]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 736
Date solved : `Monday, October 21, 2024 at 05:22:54 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4xy' + (4x^2 - 2)\, y = 0$$

1.719.1 Solved as second order ode using Kovacic algorithm

Time used: 0.081 (sec)

Writing the ode as

$$y'' - 4xy' + (4x^2 - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1370: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\ &= z_1 e^{x^2} \\ &= z_1 \left(\mathrm{e}^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{x^2}\right) + c_2\left(e^{x^2}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.719.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - 4xy' + (4x^2 - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ∘ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ∘ Shift index using $k- > k - m$

$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ∘ Convert $x \cdot y'$ to series expansion

$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ∘ Convert $\frac{d}{dx}y'$ to series expansion

$\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ∘ Shift index using $k- > k + 2$

$\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 - 2a_0 + (6a_3 - 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = a_0, a_3 = a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2k a_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$

1.719.3 Maple trace

```
Methods for second order ODEs:
```

1.719.4 Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(4*x^2-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{x^2}(c_2 x + c_1)$$

1.719.5 Mathematica DSolve solution

Solving time : `0.028` (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x^2}(c_2 x + c_1)$$

1.720 problem 737

Internal problem ID [8858]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 737
Date solved : `Monday, October 21, 2024 at 05:22:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(2x-3)\,y''-xy'+y=0$$

1.720.1 Solved as second order ode using Kovacic algorithm

Time used: 0.368 (sec)

Writing the ode as

$$(2x-3)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x-3\\B&=-x\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 8x + 18}{4\left(2x - 3\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 8x + 18 \\
t &= 4(2x - 3)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 - 8x + 18}{4\left(2x - 3\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1372: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x-3)^2$. There is a pole at $x = \frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{16} + \frac{33}{64\left(x - \frac{3}{2}\right)^2} - \frac{5}{16\left(x - \frac{3}{2}\right)}
$$

For the pole at $x = \frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{5}{8x} - \frac{11}{16x^2} - \frac{1}{32x^3} + \frac{245}{64x^4} + \frac{2591}{128x^5} + \frac{21117}{256x^6} + \frac{154743}{512x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 8x + 18}{16x^2 - 48x + 36} \\ &= Q + \frac{R}{16x^2 - 48x + 36} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-5x + \frac{63}{4}}{16x^2 - 48x + 36}\right) \\ &= \frac{1}{16} + \frac{-5x + \frac{63}{4}}{16x^2 - 48x + 36}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -5. Dividing this by leading coefficient in t which is 16 gives $-\frac{5}{16}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{16}\right) - (0) \\ &= -\frac{5}{16} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{16}}{\frac{1}{4}} - 0\right) &= -\frac{5}{8} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{16}}{\frac{1}{4}} - 0\right) &= \frac{5}{8} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 8x + 18}{4(2x-3)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{3}{2}$	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$-\frac{5}{8}$	$\frac{5}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{5}{8} - \left(-\frac{3}{8}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{8\left(x-\frac{3}{2}\right)} + (-)\left(\frac{1}{4}\right) \\
&= -\frac{3}{8\left(x-\frac{3}{2}\right)} - \frac{1}{4} \\
&= -\frac{x}{4x-6}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{8\left(x-\frac{3}{2}\right)} - \frac{1}{4}\right)(1) + \left(\left(\frac{3}{8\left(x-\frac{3}{2}\right)^2}\right) + \left(-\frac{3}{8\left(x-\frac{3}{2}\right)} - \frac{1}{4}\right)^2 - \left(\frac{x^2-8x+18}{4\left(2x-3\right)^2}\right)\right) = 0$$

$$\frac{a_0}{2x-3} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x)\, e^{\int \left(-\frac{3}{8\left(x-\frac{3}{2}\right)} - \frac{1}{4}\right) dx} \\
&= (x)\, e^{-\frac{x}{4} - \frac{3\ln(2x-3)}{8}} \\
&= \frac{x\, e^{-\frac{x}{4}}}{(2x-3)^{3/8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{2x-3}\,dx} \\
&= z_1 e^{\frac{x}{4}+\frac{3\ln(2x-3)}{8}} \\
&= z_1\left((2x-3)^{3/8}\,\mathrm{e}^{\frac{x}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{2x-3}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x}{2}+\frac{3\ln(2x-3)}{4}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{\frac{x}{2}+\frac{3\ln(2x-3)}{4}}}{x^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\int \frac{\mathrm{e}^{\frac{x}{2}+\frac{3\ln(2x-3)}{4}}}{x^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.720.2 Maple step by step solution

Let's solve

$(2x-3)\left(\frac{d}{dx}y'\right)-xy'+y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{y}{2x-3}+\frac{xy'}{2x-3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{xy'}{2x-3}+\frac{y}{2x-3}=0$

□ Check to see if $x_0=\frac{3}{2}$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x}{2x-3}, P_3(x)=\frac{1}{2x-3}\right]$

  ◦ $\left(x-\frac{3}{2}\right)\cdot P_2(x)$ is analytic at $x=\frac{3}{2}$

  $\left(\left(x-\frac{3}{2}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{3}{2}}=-\frac{3}{4}$

  ◦ $\left(x-\frac{3}{2}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{3}{2}$

  $\left(\left(x-\frac{3}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{3}{2}}=0$

  ◦ $x=\frac{3}{2}$is a regular singular point

  Check to see if $x_0=\frac{3}{2}$ is a regular singular point

  $x_0=\frac{3}{2}$

- Multiply by denominators

  $(2x-3)\left(\frac{d}{dx}y'\right)-xy'+y=0$

- Change variables using $x=u+\frac{3}{2}$ so that the regular singular point is at $u=0$

  $2u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\left(-u-\frac{3}{2}\right)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$u^{m}\cdot \left(\frac{d}{du}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty}a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u \cdot \left(\frac{d}{du}\frac{d}{du}y \left(u \right)\right)$ to series expansion

  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -1}$$

- Shift index using $k->k+1$

  $$u \cdot \left(\frac{d}{du}\frac{d}{du}y \left(u \right)\right)=\sum_{k =-1}^{\infty}a_{k +1} \left(k +1+r \right) \left(k +r \right) u^{k +r}$$

  Rewrite ODE with series expansions

  $$\frac{a_{0} r \left(-7+4 r \right) u^{-1+r}}{2}+\left(\sum_{k =0}^{\infty}\left(\frac{a_{k +1} \left(k +1+r \right) \left(4 k -3+4 r \right)}{2}-a_{k} \left(k +r -1\right)\right) u^{k +r}\right)=0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$\frac{r \left(-7+4 r \right)}{2}=0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{7}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2 \left(k +1+r \right) \left(k -\frac{3}{4}+r \right) a_{k +1}-a_{k} \left(k +r -1\right)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k +1}=\frac{2 a_{k} \left(k +r -1\right)}{\left(k +1+r \right) \left(4 k -3+4 r \right)}$$

- Recursion relation for $r =0$; series terminates at $k =1$

  $$a_{k +1}=\frac{2 a_{k} \left(k -1\right)}{\left(k +1\right) \left(4 k -3\right)}$$

- Apply recursion relation for $k =0$

  $$a_{1}=\frac{2 a_{0}}{3}$$

- Terminating series solution of the ODE for $r =0$. Use reduction of order to find the second li

  $$y \left(u \right)=a_{0}\cdot \left(1+\frac{2 u}{3}\right)$$

- Revert the change of variables $u =x -\frac{3}{2}$

  $$\left[y=\frac{2 a_{0} x}{3}\right]$$

- Recursion relation for $r =\frac{7}{4}$

  $$a_{k +1}=\frac{2 a_{k} \left(k +\frac{3}{4}\right)}{\left(k +\frac{11}{4}\right) \left(4 k +4\right)}$$

- Solution for $r =\frac{7}{4}$

  $$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k +\frac{7}{4}}, a_{k +1}=\frac{2 a_{k} \left(k +\frac{3}{4}\right)}{\left(k +\frac{11}{4}\right) \left(4 k +4\right)}\right]$$

- Revert the change of variables $u = x - \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{3}{2}\right)^{k+\frac{7}{4}}, a_{k+1} = \frac{2a_k\left(k+\frac{3}{4}\right)}{\left(k+\frac{11}{4}\right)(4k+4)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \frac{2a_0 x}{3} + \left(\sum_{k=0}^{\infty} b_k \left(x - \frac{3}{2}\right)^{k+\frac{7}{4}}\right), b_{k+1} = \frac{2b_k\left(k+\frac{3}{4}\right)}{\left(k+\frac{11}{4}\right)(4k+4)}\right]$$

1.720.3 Maple trace

```
Methods for second order ODEs:
```

1.720.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 29

```
dsolve((2*x-3)*diff(diff(y(x),x),x)-x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = 2c_1\left(x - \frac{3}{2}\right)(2x-3)^{3/4}\,\mathrm{KummerM}\left(\frac{3}{4}, \frac{11}{4}, \frac{x}{2} - \frac{3}{4}\right) + c_2 x$$

1.720.5 Mathematica DSolve solution

Solving time : 0.135 (sec)
Leaf size : 63

```
DSolve[{(2*x-3)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 2\ 2^{3/4}(2x-3)\left(c_2(2x-3)^{3/4} L_{-\frac{3}{4}}^{\frac{7}{4}}\left(\frac{x}{2} - \frac{3}{4}\right) + \frac{4\sqrt{2}c_1 x}{2x-3}\right)$$

1.721 problem 738

Internal problem ID [8859]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 738
Date solved : `Monday, October 21, 2024 at 05:22:56 PM`
CAS classification : `[_Hermite]`

Solve

$$y'' - xy' - 3y = 0$$

1.721.1 Solved as second order ode using Kovacic algorithm

Time used: 0.242 (sec)

Writing the ode as

$$y'' - xy' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 10$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1374: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{5}{2x} - \frac{25}{4x^3} + \frac{125}{4x^5} - \frac{3125}{16x^7} + \frac{21875}{16x^9} - \frac{328125}{32x^{11}} + \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} + \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} + \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{2}\right) - (0) \\ &= \frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	2	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(\frac{x}{2}\right)(2x + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{5}{2}\right)\right) &= 0 \\ -a_1 x - 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= (x^2 + 1)\, e^{\int \frac{x}{2} dx} \\
&= (x^2 + 1)\, e^{\frac{x^2}{4}} \\
&= (x^2 + 1)\, e^{\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(e^{\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{x^2}{2}} (x^2 + 1)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2 + 1)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{x^2}{2}}\left(x^2+1\right)\right) + c_2\left(\mathrm{e}^{\frac{x^2}{2}}\left(x^2+1\right)\left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^2+1\right)^2}dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.721.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(k+3)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} - a_k(k+3) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k(k+3)}{k^2+3k+2}\right]$

1.721.3 Maple trace

```
Methods for second order ODEs:
```

1.721.4 Maple dsolve solution

Solving time : 0.015 (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-3*y(x) = 0,
       y(x),singsol=all)
```

$$y = (x^2 + 1)\left(c_1 \operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{\pi} + c_2\right)\mathrm{e}^{\frac{x^2}{2}} + \sqrt{2}\,c_1 x$$

1.721.5 Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 35

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 \operatorname{HermiteH}\left(-3, \frac{x}{\sqrt{2}}\right) + c_2 e^{\frac{x^2}{2}}(x^2 + 1)$$

1.722 problem 739

Internal problem ID [8860]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 739
Date solved : `Monday, October 21, 2024 at 05:22:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right) y''-x y'+y=0$$

1.722.1 Solved as second order ode using Kovacic algorithm

Time used: 0.269 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-x y'+y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-x^2 - 6}{4\left(x^2+1\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -x^2 - 6 \\
t &= 4\left(x^2+1\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-x^2 - 6}{4\left(x^2+1\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1376: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16(x-i)^2} + \frac{5}{16(x+i)^2} + \frac{7i}{16(x-i)} - \frac{7i}{16(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2-6}{4(x^2+1)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 - 6}{4\left(x^2+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} \\
&= -\frac{x}{2x^2+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)}\right)(1) + \left(\left(\frac{1}{4\left(x-i\right)^2} + \frac{1}{4\left(x+i\right)^2}\right) + \left(-\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)}\right)^2 - \left(\frac{-}{4(}\right.\right. \cdots$$

$$\frac{(x^2+1 \cdots}{(-x+i)^2 (\cdots}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\
&= (x)\, \frac{1}{\left(\left(-x+i\right)\left(x+i\right)\right)^{1/4}} \\
&= \frac{x}{\left(-x^2-1\right)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2+1}\,dx} \\
&= z_1 e^{\frac{\ln\left(x^2+1\right)}{4}} \\
&= z_1\left(\left(x^2+1\right)^{1/4}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(\frac{1}{2}-\frac{i}{2}\right) x\sqrt{2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln\left(x^2+1\right)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(i\left(-\frac{\left(x^2+1\right)^{3/2}}{x}+x\sqrt{x^2+1}+\operatorname{arcsinh}(x)\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(\frac{1}{2}-\frac{i}{2}\right) x\sqrt{2}\right) \\
&\quad + c_2\left(\left(\frac{1}{2}-\frac{i}{2}\right) x\sqrt{2}\left(i\left(-\frac{\left(x^2+1\right)^{3/2}}{x}+x\sqrt{x^2+1}+\operatorname{arcsinh}(x)\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.722.2 Maple step by step solution

1.722.3 Maple trace

```
Methods for second order ODEs:
```

1.722.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 23

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = -\sqrt{x^2+1}\, c_2 + x(c_2 \operatorname{arcsinh}(x) + c_1)$$

1.722.5 Mathematica DSolve solution

Solving time : 0.074 (sec)
Leaf size : 39

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x \operatorname{arctanh}\left(\frac{x}{\sqrt{x^2+1}}\right) - c_2\sqrt{x^2+1} + c_1 x$$

1.723 problem 740

Internal problem ID [8861]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 740
Date solved : `Monday, October 21, 2024 at 05:22:57 PM`
CAS classification : `[_Hermite]`

Solve

$$y'' - xy' + 2y = 0$$

1.723.1 Solved as second order ode using Kovacic algorithm

Time used: 0.250 (sec)

Writing the ode as

$$y'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 10$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1377: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{5}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) = 0 \\ a_1 x + 2a_0 + 2 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 1\right) e^{\int -\frac{x}{2} dx} \\
&= \left(x^2 - 1\right) e^{-\frac{x^2}{4}} \\
&= \left(x^2 - 1\right) e^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(e^{\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2 - 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2 - 1)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1(x^2-1) + c_2\left(x^2-1\left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right)\right)$$

Will add steps showing solving for IC soon.

1.723.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) - a_k(k-2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} - a_k(k-2) = 0$

- Recursion relation; series terminates at $k = 2$

  $a_{k+2} = \frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

$a_2 = -a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

  $y = A_2x^2 + A_1x - a_0$

1.723.3 Maple trace

```
Methods for second order ODEs:
```

1.723.4 Maple dsolve solution

Solving time : `0.012` (sec)
Leaf size : 39

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -2\,\mathrm{e}^{\frac{x^2}{2}} c_1 x + (x-1)(x+1)\left(\sqrt{\pi}\,\sqrt{2}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right) c_1 + c_2\right)$$

1.723.5 Mathematica DSolve solution

Solving time : `0.136` (sec)
Leaf size : 54

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}c_2\left(\sqrt{2\pi}(x^2-1)\,\mathrm{erfi}\left(\frac{x}{\sqrt{2}}\right) - 2e^{\frac{x^2}{2}}x\right) + c_1(x^2-1)$$

1.724 problem 741

Internal problem ID [8862]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 741
Date solved : `Monday, October 21, 2024 at 05:22:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(-x^2+1\right) y''-y'+y=0$$

1.724.1 Solved as second order ode using Kovacic algorithm

Time used: 0.702 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x - 3}{4\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 - 4x - 3$$
$$t = 4\left(x^2 - 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x - 3}{4\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1379: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(x-1)^2} + \frac{7}{16(x-1)} - \frac{7}{16(x+1)} + \frac{5}{16(x+1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4x^2 - 4x - 3}{4(x^2-1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=1$. Hence

$$\begin{aligned}E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\ &= \{2\}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1,2,3\}$
-1	2	$\{-1,2,5\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \; e_2 = -1, \; e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\ &= \frac{1}{2}(2-(1+(-1)))\\ &= 1\end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{1}{(x-(1))} + \frac{-1}{(x-(-1))}\right)\\ &= \frac{1}{2x-2} - \frac{1}{2(x+1)}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{4a_0 - 6}{(x+1)^2 (x-1)} = 0$$

And solving for p gives

$$p = x + \frac{3}{2}$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x+\frac{3}{2}} + \frac{1}{2x-2} - \frac{1}{2(x+1)}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{3}{2}} + \frac{1}{2x-2} - \frac{1}{2(x+1)}\right) w + \frac{-8x^3 - 4x^2 + 10x + 7}{4(x^2-1)^2 (2x+3)} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{5}\,\sqrt{(x-1)(x+1)}\,x + 2\sqrt{5}\,\sqrt{(x-1)(x+1)} + 2x^2 + 2x + 1}{2(2x+3)(x-1)(x+1)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{2\sqrt{5}\,\sqrt{(x-1)(x+1)}\,x+2\sqrt{5}\,\sqrt{(x-1)(x+1)}+2x^2+2x+1}{2(2x+3)(x-1)(x+1)}dx} \\ &= \frac{(x-1)^{1/4}\sqrt{2x+3}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} 5^{1/4}}{(x+1)^{1/4}\sqrt{\frac{5\sqrt{x^2-1}+(2+3x)\sqrt{5}}{\sqrt{x^2-1}\sqrt{-\frac{(2x+3)^2}{x^2-1}}}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{-x^2+1}\,dx} \\
&= z_1 e^{\frac{\operatorname{arctanh}(x)}{2}} \\
&= z_1 \left(\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x+3}\,(5x-5)^{1/4}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{2x+3}}\,(x+1)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\operatorname{arctanh}(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{i\sqrt{x+1}\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{(2x+3)^2\sqrt{5x-5}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x+3}\,(5x-5)^{1/4}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{2x+3}}\,(x+1)^{1/4}}\right)$$

$$+c_2 \left(\frac{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x+3}\,(5x-5)^{1/4}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}+2\sqrt{5}\right)}{2x+3}}\,(x+1)^{1/4}}\left(\int \frac{i\sqrt{x+1}\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}\left(3\sqrt{5}\,x+5\sqrt{x^2\right.}{(2x+3)^2\sqrt{5x-5}}\right.\right.$$

Will add steps showing solving for IC soon.

1.724.2 Maple step by step solution

Let's solve

$\left(-x^2+1\right)\left(\frac{d}{dx}y'\right)-y'+y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{y}{x^2-1}-\frac{y'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{y'}{x^2-1}-\frac{y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{1}{x^2-1}, P_3(x)=-\frac{1}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=-\frac{1}{2}$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

$(x^2-1)\left(\frac{d}{dx}y'\right)+y'-y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+\frac{d}{du}y(u)-y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$

  ◦ Shift index using $k->k+1$

  $\frac{d}{du}y(u)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-a_0r(-3+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(2k-1+2r)+a_k(k^2+2kr+r^2-k-r-1)\right)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(-3+2r)=0$

- Values of r that satisfy the indicial equation

$r\in\left\{0,\frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$-2(k+1+r)\left(k-\frac{1}{2}+r\right)a_{k+1}+\left(k^2+(2r-1)\,k+r^2-r-1\right)a_k=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{(k^2+2kr+r^2-k-r-1)a_k}{(k+1+r)(2k-1+2r)}$

- Recursion relation for $r=0$

$a_{k+1}=\frac{(k^2-k-1)a_k}{(k+1)(2k-1)}$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{(k^2-k-1) a_k}{(k+1)(2k-1)}\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1}=\frac{(k^2-k-1) a_k}{(k+1)(2k-1)}\right]$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+1}=\frac{(k^2+2k-\frac{1}{4}) a_k}{(k+\frac{5}{2})(2k+2)}$$

- Solution for $r=\frac{3}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1}=\frac{(k^2+2k-\frac{1}{4}) a_k}{(k+\frac{5}{2})(2k+2)}\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{3}{2}}, a_{k+1}=\frac{(k^2+2k-\frac{1}{4}) a_k}{(k+\frac{5}{2})(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x+1)^{k+\frac{3}{2}}\right), a_{k+1}=\frac{(k^2-k-1) a_k}{(k+1)(2k-1)}, b_{k+1}=\frac{(k^2+2k-\frac{1}{4}) b_k}{(k+\frac{5}{2})(2k+2)}\right]$$

1.724.3 Maple trace

```
Methods for second order ODEs:
```

1.724.4 Maple dsolve solution

Solving time : `0.026` (sec)
Leaf size : 66

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-diff(y(x),x)+y(x) = 0,
          y(x),singsol=all)
```

$$y = c_1 \,\mathrm{hypergeom}\left(\left[-\frac{1}{2}-\frac{\sqrt{5}}{2}, -\frac{1}{2}+\frac{\sqrt{5}}{2}\right], \left[-\frac{1}{2}\right], \frac{x}{2}+\frac{1}{2}\right) + 2c_2\sqrt{2x+2}\,\mathrm{hypergeom}\left(\left[1+\frac{\sqrt{5}}{2}, 1-\frac{\sqrt{5}}{2}\right], \left[\frac{5}{2}\right], \frac{x}{2}+\frac{1}{2}\right)(x+1)$$

1.724.5 Mathematica DSolve solution

Solving time : `4.595` (sec)
Leaf size : 195

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to{} & \left(\sqrt{x-1}-\sqrt{x+1}\right)^{-\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(\sqrt{x-1}+\sqrt{x+1}\right)^{\frac{1}{2}\left(\sqrt{5}-1\right)}\left(\sqrt{x-1}-\sqrt{5}\sqrt{x+1}\right)\left(c_2\int_1^x\right. \\
& -\frac{2\sqrt{K[1]+1}\left(\sqrt{K[1]-1}-\sqrt{K[1]+1}\right)^{\sqrt{5}}\left(\sqrt{K[1]-1}+\sqrt{K[1]+1}\right)^{-\sqrt{5}}}{\sqrt{1-K[1]}\left(\sqrt{K[1]-1}-\sqrt{5}\sqrt{K[1]+1}\right)^2}dK[1] \\
& \left.+c_1\right)
\end{aligned}
$$

1.725 problem 742

Internal problem ID [8863]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 742
Date solved : `Monday, October 21, 2024 at 05:23:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(x+1)^2 y'' + \left(-x^2+1\right) y' + (x-1)\, y = 0$$

1.725.1 Solved as second order ode using Kovacic algorithm

Time used: 0.186 (sec)

Writing the ode as

$$x(x+1)^2 y'' + \left(-x^2+1\right) y' + (x-1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x(x+1)^2 \\ B &= -x^2+1 \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1381: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x^2+1}{x(x+1)^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2} + \ln(x+1)} \\ &= z_1 \left(\frac{x+1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x + 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-x^2+1}{x(x+1)^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-\ln(x)+2\ln(x+1)}}{(y_1)^2}\,dx$$
$$= y_1(\ln(x))$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1(x+1) + c_2(x + 1(\ln(x)))$$

Will add steps showing solving for IC soon.

1.725.2 Maple step by step solution

Let's solve

$x(x+1)^2 \left(\frac{d}{dx}y'\right) + (-x^2+1)\,y' + (x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x-1)y}{x(x+1)^2} + \frac{y'(x-1)}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'(x-1)}{x(x+1)} + \frac{(x-1)y}{x(x+1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x-1}{(x+1)x}, P_3(x) = \frac{x-1}{x(x+1)^2}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.((x+1)\cdot P_2(x))\right|_{x=-1} = -2$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left.((x+1)^2\cdot P_3(x))\right|_{x=-1} = 2$

- ○ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$
- Multiply by denominators

  $x(x+1)^2 \left(\frac{d}{dx}y'\right) - (x+1)(x-1)y' + (x-1)y = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^3 - u^2)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u^2 + 2u)\left(\frac{d}{du}y(u)\right) + (u-2)\,y(u) = 0$
- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$
- □ Rewrite ODE with series expansions
  - ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$
  - ○ Shift index using $k- > k-m$

    $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$
  - ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$
  - ○ Shift index using $k- > k+1-m$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
  - ○ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 2..3$

    $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$
  - ○ Shift index using $k- > k+2-m$

    $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0(-1+r)(-2+r)\,u^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-1)(k+r-2) + a_{k-1}(k+r-2)^2\right) u^{k+r}\right) = 0$$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-1+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation

$r \in \{1, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r-1)(k+r-2) + a_{k-1}(k+r-2)^2 = 0$
- Shift index using $k- > k+1$

  $-a_{k+1}(k+r)(k+r-1) + a_k(k+r-1)^2 = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-1)}{k+r}$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k k}{k+1}$
- Solution for $r = 1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+1} = \frac{a_k k}{k+1}\right]$
- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+1}, a_{k+1} = \frac{a_k k}{k+1}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+1)}{k+2}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k(k+1)}{k+2}\right]$
- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+2}, a_{k+1} = \frac{a_k(k+1)}{k+2}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+2}\right), a_{k+1} = \frac{a_k k}{k+1}, b_{k+1} = \frac{b_k(k+1)}{k+2}\right]$

1.725.3 Maple trace

```
Methods for second order ODEs:
```

1.725.4 Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 14

```
dsolve(x*(x+1)^2*diff(diff(y(x),x),x)+(-x^2+1)*diff(y(x),x)+(x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = (x+1)\left(c_2 \ln\left(x\right) + c_1\right)$$

1.725.5 Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 17

```
DSolve[{x*(x+1)^2*D[y[x],{x,2}]+(1-x^2)*D[y[x],x]+(x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (x+1)(c_2 \log(x) + c_1)$$

1.726 problem 743

Internal problem ID [8864]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 743
Date solved : `Monday, October 21, 2024 at 05:23:01 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$2xy'' - y' + 2y = 0$$

1.726.1 Solved as second order ode using Kovacic algorithm

Time used: 0.228 (sec)

Writing the ode as

$$2xy'' - y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{5 - 16x}{16x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 5 - 16x \\
t &= 16x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1383: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right) \\ &= -\frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = x + \frac{1}{4}$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x+\frac{1}{4}} - \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{1}{4}} - \frac{1}{2x}\right) w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4(4x+1)x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= e^{\int \frac{16x\sqrt{-x}+4x-1}{4(4x+1)x} dx} \\ &= \frac{(2\sqrt{-x} - 1)\, e^{2\sqrt{-x}}}{(-x)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{2x} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{4}} \\ &= z_1 \left(x^{1/4}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}(2\sqrt{-x} - 1)\, e^{2\sqrt{-x}}}{(-x)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{2x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{e^{-4\sqrt{-x}}}{8} + \frac{e^{-4\sqrt{-x}}}{8\sqrt{-x} - 4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/4}\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}}\right) + c_2 \left(\frac{x^{1/4}\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}}\left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.726.2 Maple step by step solution

Let's solve

$2x\left(\frac{d}{dx}y'\right) - y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x} + \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'}{2x} + \frac{y}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ○ Define functions

    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{1}{x}\right]$

  - ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{1}{2}$

  - ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  - ○ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x\left(\frac{d}{dx}y'\right) - y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- ◦ Shift index using $k->k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$

- ◦ Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- ◦ Shift index using $k->k+1$

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k - \frac{1}{2} + r\right)(k+1+r)\, a_{k+1} + 2a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k}{(2k-1+2r)(k+1+r)}$

- Recursion relation for $r=0$

  $a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}$

- Solution for $r=0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}\right]$

- Recursion relation for $r=\frac{3}{2}$

  $a_{k+1} = -\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}$

- Solution for $r=\frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}, b_{k+1} = -\frac{2b_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

1.726.3 Maple trace

```
Methods for second order ODEs:
```

1.726.4 Maple dsolve solution

Solving time : `0.015` (sec)
Leaf size : 36

```
dsolve(2*x*diff(diff(y(x),x),x)-diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \left(2c_1\sqrt{x} + c_2\right)\cos\left(2\sqrt{x}\right) - \sin\left(2\sqrt{x}\right)\left(-2c_2\sqrt{x} + c_1\right)$$

1.726.5 Mathematica DSolve solution

Solving time : `0.13` (sec)
Leaf size : 59

```
DSolve[{2*x*D[y[x],{x,2}]-D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{2i\sqrt{x}}\left(2\sqrt{x} + i\right) + \frac{1}{8}c_2 e^{-2i\sqrt{x}}\left(1 + 2i\sqrt{x}\right)$$

1.727 problem 744

Internal problem ID [8865]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 744
Date solved : `Monday, October 21, 2024 at 05:23:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + xy' - 2y = 0$$

1.727.1 Solved as second order ode using Kovacic algorithm

Time used: 0.243 (sec)

Writing the ode as

$$xy'' + xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x+8}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$s = x + 8$$
$$t = 4x$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x+8}{4x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1385: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x} - \frac{4}{x^2} + \frac{16}{x^3} - \frac{80}{x^4} + \frac{448}{x^5} - \frac{2688}{x^6} + \frac{16896}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x+8}{4x} \\
&= Q + \frac{R}{4x} \\
&= \left(\frac{1}{4}\right) + \left(\frac{2}{x}\right) \\
&= \frac{1}{4} + \frac{2}{x}
\end{aligned}
$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 8. Dividing this by leading coefficient in t which is 4 gives 2. Now b can be found.

$$
\begin{aligned}
b &= (2) - (0) \\
&= 2
\end{aligned}
$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 0\right) &&= 2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 0\right) &&= -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x+8}{4x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	2	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{x} + \frac{1}{2} \\
&= \frac{1}{x} + \frac{1}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} + \frac{1}{2}\right)^2 - \left(\frac{x+8}{4x}\right)\right) &= 0 \\
\frac{2 - a_0}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 2 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2 + x)\, e^{\int \left(\frac{1}{x} + \frac{1}{2}\right) dx} \\
&= (2 + x)\, e^{\frac{x}{2} + \ln(x)} \\
&= (2 + x)\, x\, e^{\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (2+x)\,x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-x}}{4x} + \frac{\mathrm{Ei}_1(x)}{2} + \frac{\mathrm{e}^{-x}}{-8-4x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1((2+x)\,x) + c_2\left((2+x)\,x\left(-\frac{\mathrm{e}^{-x}}{4x} + \frac{\mathrm{Ei}_1(x)}{2} + \frac{\mathrm{e}^{-x}}{-8-4x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.727.2 Maple step by step solution

1.727.3 Maple trace

```
Methods for second order ODEs:
```

1.727.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 28

```
dsolve(x*diff(diff(y(x),x),x)+x*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{c_2(x+1)\,\mathrm{e}^{-x}}{2} + x\left(c_1 + \frac{c_2\,\mathrm{Ei}_1\left(x\right)}{2}\right)(2+x)$$

1.727.5 Mathematica DSolve solution

Solving time : 0.087 (sec)
Leaf size : 39

```
DSolve[{x*D[y[x],{x,2}]+x*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x(x+2) - \frac{1}{2}c_2 e^{-x}(e^x(x+2)x\,\mathrm{ExpIntegralEi}(-x) + x + 1)$$

1.728 problem 745

Internal problem ID [8866]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 745
Date solved : `Monday, October 21, 2024 at 05:23:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(x-1)^2 y'' - 2y = 0$$

1.728.1 Solved as second order ode using Kovacic algorithm

Time used: 0.170 (sec)

Writing the ode as

$$x(x-1)^2 y'' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x(x-1)^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{(x-1)^2 x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1386: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 0 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x-1)^2} - \frac{2}{x-1} + \frac{2}{x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{(x-1)^2 x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= 0 - (0) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x} - \frac{1}{x-1} + (0) \\ &= \frac{1}{x} - \frac{1}{x-1} \\ &= -\frac{1}{x(x-1)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{x-1}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{(x-1)^2}\right) + \left(\frac{1}{x} - \frac{1}{x-1}\right)^2 - \left(\frac{2}{(x-1)^2\, x}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{x} - \frac{1}{x-1}\right) dx} \\ &= \frac{x}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{x}{x-1} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{x-1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \frac{x}{x-1} \int \frac{1}{\frac{x^2}{(x-1)^2}}\,dx \\
&= \frac{x}{x-1}\left(x - 2\ln(x) - \frac{1}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{x-1}\right) + c_2\left(\frac{x}{x-1}\left(x - 2\ln(x) - \frac{1}{x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.728.2 Maple step by step solution

Let's solve

$x(x-1)^2\left(\frac{d}{dx}y'\right) - 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{2y}{x(x-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y}{x(x-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{2}{(x-1)^2 x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x-1)^2 \left(\frac{d}{dx}y'\right) - 2y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..3$

    $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k- > k+2-m$

    $$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-1+r)\, x^{-1+r} + \left(a_1(1+r)\, r - 2a_0(r^2-r+1)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)(k+r) - 2a_k(k^2\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term must be 0

  $a_1(1+r)\, r - 2a_0(r^2-r+1) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-2a_k + a_{k-1} + a_{k+1})\, k^2 + \left((-4a_k + 2a_{k-1} + 2a_{k+1})\, r + 2a_k - 3a_{k-1} + a_{k+1}\right) k + (-2a_k + a_{k-1}$

- Shift index using $k- > k+1$

  $(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + \left((-4a_{k+1} + 2a_k + 2a_{k+2})\, r + 2a_{k+1} - 3a_k + a_{k+2}\right)(k+1) + (-$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kra_k - 4kra_{k+1} + r^2 a_k - 2r^2 a_{k+1} - ka_k - 2ka_{k+1} - ra_k - 2ra_{k+1} - 2a_{k+1}}{k^2 + 2kr + r^2 + 3k + 3r + 2}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - ka_k - 2ka_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}$

- Solution for $r = 0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{k^2 a_k-2k^2 a_{k+1}-k a_k-2k a_{k+1}-2a_{k+1}}{k^2+3k+2}, -2a_0=0\right]$$

- Recursion relation for $r=1$

$a_{k+2}=-\frac{k^2 a_k-2k^2 a_{k+1}+k a_k-6k a_{k+1}-6a_{k+1}}{k^2+5k+6}$

- Solution for $r=1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{k^2 a_k-2k^2 a_{k+1}+k a_k-6k a_{k+1}-6a_{k+1}}{k^2+5k+6}, 2a_1-2a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2}=-\frac{k^2 a_k-2k^2 a_{k+1}-k a_k-2k a_{k+1}-2a_{k+1}}{k^2+3k+2}, -2a_0=0, b_{k+2}=-\frac{k^2 b}{}\right.$$

1.728.3 Maple trace

```
Methods for second order ODEs:
```

1.728.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 27

```
dsolve(x*(x-1)^2*diff(diff(y(x),x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{2\ln(x)\, c_2 x-c_2\, x^2+c_1 x+c_2}{x-1}$$

1.728.5 Mathematica DSolve solution

Solving time : `0.057` (sec)
Leaf size : 33

```
DSolve[{x*(x-1)^2*D[y[x],{x,2}]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{-c_2 x^2-c_1 x+2c_2 x\log(x)+c_2}{x-1}$$

1.729 problem 746

Internal problem ID [8867]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 746
Date solved : `Monday, October 21, 2024 at 05:23:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 2xy' + x^2y = 0$$

1.729.1 Solved as second order ode using Kovacic algorithm

Time used: 0.171 (sec)

Writing the ode as

$$y'' - 2xy' + x^2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x \\ C &= x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1388: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1}\,dx}\\ &= z_1 e^{\frac{x^2}{2}}\\ &= z_1\left(\mathrm{e}^{\frac{x^2}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x^2}{2}}\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{x^2}}{(y_1)^2}\,dx\\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{x^2}{2}} \cos(x)\right) + c_2\left(\mathrm{e}^{\frac{x^2}{2}} \cos(x) (\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.729.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - 2xy' + x^2 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^2 \cdot y$ to series expansion

    $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$
  - ○ Shift index using $k- > k-2$

    $x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$
  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$
  - ○ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - ○ Shift index using $k- > k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + (6a_3 - 2a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) - 2a_k k + a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 = 0, 6a_3 - 2a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = 0, a_3 = \frac{a_1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - 2a_k k + a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $\left((k + 2)^2 + 3k + 8\right) a_{k+4} - 2a_{k+2}(k + 2) + a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2k a_{k+2} - a_k + 4a_{k+2}}{k^2 + 7k + 12}, a_2 = 0, a_3 = \frac{a_1}{3}\right]$

1.729.3 Maple trace

```
Methods for second order ODEs:
```

1.729.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 20

```
dsolve(diff(diff(y(x),x),x)-2*x*diff(y(x),x)+x^2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{\frac{x^2}{2}} (c_1 \cos(x) + c_2 \sin(x))$$

1.729.5 Mathematica DSolve solution

Solving time : `0.049` (sec)
Leaf size : 39

```
DSolve[{D[y[x],{x,2}]-2*x*D[y[x],x]+x^2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{\frac{1}{2}x(x-2i)} \left(2c_1 - ic_2 e^{2ix}\right)$$

1.730 problem 747

Internal problem ID [8868]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 747
Date solved : `Monday, October 21, 2024 at 05:23:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x\left(-x^2+2\right) y''-\left(x^2+4x+2\right) \left(\left(1-x\right) y'+y\right)=0$$

1.730.1 Solved as second order ode using Kovacic algorithm

Time used: 0.518 (sec)

Writing the ode as

$$\left(-x^3+2x\right) y''+\left(x^3+3x^2-2x-2\right) y'+\left(-x^2-4x-2\right) y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+2x \\ B &= x^3+3x^2-2x-2 \\ C &= -x^2-4x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4\left(x^3 - 2x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12 \\
t &= 4\left(x^3 - 2x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4\left(x^3 - 2x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1390: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\left(x - \sqrt{2}\right)^2} + \frac{3}{4\left(x + \sqrt{2}\right)^2} + \frac{-\frac{5\sqrt{2}}{8} - \frac{1}{2}}{x - \sqrt{2}} + \frac{\frac{5\sqrt{2}}{8} - \frac{1}{2}}{x + \sqrt{2}} + \frac{3}{2x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - \sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x=-\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\ \alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty)=0$ then

$$v=\frac{-O_r(\infty)}{2}=\frac{0}{2}=0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0\leq i\leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned}\tag{8}$$

Let a be the coefficient of $x^v=x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r}\approx\frac{1}{2}+\frac{1}{2x}-\frac{1}{2x^2}-\frac{3}{2x^3}+\frac{21}{4x^4}-\frac{43}{4x^5}+\frac{135}{4x^6}-\frac{147}{4x^7}+\dots\tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a=\frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i\\ &= \frac{1}{2}\end{aligned}\tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^{-1}=\frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2=\frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^6+2x^5-5x^4-16x^3+24x^2+24x+12}{4x^6-16x^4+16x^2} \\
&= Q + \frac{R}{4x^6-16x^4+16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{2x^5-x^4-16x^3+20x^2+24x+12}{4x^6-16x^4+16x^2}\right) \\
&= \frac{1}{4} + \frac{2x^5-x^4-16x^3+20x^2+24x+12}{4x^6-16x^4+16x^2}
\end{aligned}$$

Since the degree of t is 6, then we see that the coefficient of the term x^5 in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{1}{2}\right) - (0) \\
&= \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^6+2x^5-5x^4-16x^3+24x^2+24x+12}{4\left(x^3-2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \left(\frac{1}{2}\right) \\ &= \frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2} \\ &= \frac{x^3 + x^2 - 2x - 6}{2x^3 - 4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2}\right)(0) + \left(\left(-\frac{3}{2x^2} + \frac{1}{2\left(x-\sqrt{2}\right)^2} + \frac{1}{2\left(x+\sqrt{2}\right)^2}\right) + \left(\frac{3}{2x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= e^{\int \left(\frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2}\right) dx} \\
&= \frac{x^{3/2} e^{\frac{x}{2}}}{\sqrt{x+\sqrt{2}}\,\sqrt{x-\sqrt{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3+3x^2-2x-2}{-x^3+2x}\, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2} + \frac{\ln\left(x^2-2\right)}{2}} \\
&= z_1 \left(\sqrt{x}\,\sqrt{x^2-2}\, e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2 e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3+3x^2-2x-2}{-x^3+2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x)+\ln(x^2-2)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{(x-1)\, \mathrm{e}^{x+\ln(x)+\ln(x^2-2)}\mathrm{e}^{-2x}}{x^3\,(x^2-2)} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2 \mathrm{e}^x\right) + c_2 \left(x^2 \mathrm{e}^x \left(-\frac{(x-1)\, \mathrm{e}^{x+\ln(x)+\ln(x^2-2)}\mathrm{e}^{-2x}}{x^3\,(x^2-2)} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.730.2 Maple step by step solution

Let's solve

$x(-x^2+2)\left(\frac{d}{dx}y'\right) - (x^2+4x+2)\left((1-x)\,y' + y\right) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+4x+2)y}{x(x^2-2)} + \frac{(x^2+4x+2)(x-1)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x^2+4x+2)(x-1)y'}{x(x^2-2)} + \frac{(x^2+4x+2)y}{x(x^2-2)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{(x^2+4x+2)(x-1)}{x(x^2-2)}, P_3(x) = \frac{x^2+4x+2}{x(x^2-2)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x^2-2)\left(\frac{d}{dx}y'\right) - (x^2+4x+2)(x-1)\,y' + (x^2+4x+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 1..3$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-2+r)\,x^{-1+r} + \left(-2a_1(1+r)(-1+r) + 2a_0(1+r)\right)x^r + \left(-2a_2(2+r)\,r + 2a_1(2+r) +\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 2\}$

- The coefficients of each power of x must be 0

  $[-2a_1(1+r)(-1+r)+2a_0(1+r)=0, -2a_2(2+r)r+2a_1(2+r)+a_0(-2+r)^2=0]$
- Solve for the dependent coefficient(s)

  $\left\{a_1=\frac{a_0}{-1+r}, a_2=\frac{a_0(r^2-5r+10)}{2(r^2+r-2)}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k-1}(k-3+r)^2-2a_{k+1}(k+r+1)(k+r-1)+(2a_k-a_{k-2})k+(2a_k-a_{k-2})r+2a_k+3a_{k-}$
- Shift index using $k->k+2$

  $a_{k+1}(k+r-1)^2-2a_{k+3}(k+3+r)(k+r+1)+(2a_{k+2}-a_k)(k+2)+(2a_{k+2}-a_k)r+2a_{k+2}$
- Recursion relation that defines series solution to ODE

  $a_{k+3}=\frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}-a_kr-2ra_{k+1}+2ra_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3+r)(k+r+1)}$
- Recursion relation for $r=0$

  $a_{k+3}=\frac{k^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}$
- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty}a_kx^k, a_{k+3}=\frac{k^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}, a_1=-a_0, a_2=-\frac{5a_0}{2}\right]$
- Recursion relation for $r=2$

  $a_{k+3}=\frac{k^2a_{k+1}-a_kk+2ka_{k+1}+2ka_{k+2}-a_k+a_{k+1}+10a_{k+2}}{2(k+5)(k+3)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+2}, a_{k+3}=\frac{k^2a_{k+1}-a_kk+2ka_{k+1}+2ka_{k+2}-a_k+a_{k+1}+10a_{k+2}}{2(k+5)(k+3)}, a_1=a_0, a_2=\frac{a_0}{2}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+2}\right), a_{k+3}=\frac{k^2a_{k+1}-ka_k-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}, a_1=-a_0, a_2=\right.$

1.730.3 Maple trace

```
Methods for second order ODEs:
```

1.730.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 17

```
dsolve(x*(-x^2+2)*diff(diff(y(x),x),x)-(x^2+4*x+2)*((1-x)*diff(y(x),x)+y(x)) = 0,
       y(x),singsol=all)
```

$$y = c_1(x-1) + c_2\, x^2 \mathrm{e}^x$$

1.730.5 Mathematica DSolve solution

Solving time : 0.117 (sec)
Leaf size : 21

```
DSolve[{x*(2-x^2)*D[y[x],{x,2}]-(x^2+4*x+2)*((1-x)*D[y[x],x]+y[x])==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x x^2 + c_2(x-1)$$

1.731 problem 748

Internal problem ID [8869]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 748
Date solved : `Monday, October 21, 2024 at 05:23:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(1+x)\,y''-(1+2x)\,(xy'-y)=0$$

1.731.1 Solved as second order ode using Kovacic algorithm

Time used: 0.231 (sec)

Writing the ode as

$$x^2(1+x)\,y''+\left(-2x^2-x\right)y'+(1+2x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\B&=-2x^2-x\\C&=1+2x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4x - 1}{4\left(x^2 + x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4x - 1 \\
t &= 4\left(x^2 + x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-4x - 1}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1392: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{2x} + \frac{3}{4(1+x)^2} - \frac{1}{4x^2} + \frac{1}{2+2x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-4x - 1}{4\left(x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(1 + x)} + \frac{1}{2x} + (0) \\
&= -\frac{1}{2(1 + x)} + \frac{1}{2x} \\
&= \frac{1}{2x(1 + x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\,(1+x)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2\,(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{2\,(1+x)} + \frac{1}{2x}\right)^2 - \left(\frac{-4x-1}{4\,(x^2+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(1+x)} + \frac{1}{2x}\right) dx} \\
&= \frac{\sqrt{x}}{\sqrt{1+x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-x}{x^2(1+x)}\, dx} \\
&= z_1 e^{\frac{\ln(x(1+x))}{2}} \\
&= z_1 \left(\sqrt{x\,(1+x)}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x\,(1+x)}\,\sqrt{x}}{\sqrt{1+x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-x}{x^2(1+x)} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{\ln(x(1+x))}}{(y_1)^2} dx \\ &= y_1(x + \ln(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{x(1+x)}\,\sqrt{x}}{\sqrt{1+x}}\right) + c_2 \left(\frac{\sqrt{x(1+x)}\,\sqrt{x}}{\sqrt{1+x}}(x + \ln(x))\right)\end{aligned}$$

Will add steps showing solving for IC soon.

1.731.2 Maple step by step solution

Let's solve

$x^2(1+x)\left(\frac{d}{dx}y'\right) - (1+2x)(xy' - y) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(1+2x)y}{x^2(1+x)} + \frac{(1+2x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(1+2x)y'}{x(1+x)} + \frac{(1+2x)y}{x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1+2x}{x(1+x)}, P_3(x) = \frac{1+2x}{x^2(1+x)}\right]$

  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = -1$

  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

$$\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x^2(1+x)\left(\frac{d}{dx}y'\right) - x(1+2x)\,y' + (1+2x)\,y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - 2u^2 + u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-2u^2 + 3u - 1)\left(\frac{d}{du}y(u)\right) + (-1 + 2u)\,y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r)\,u^{-1+r} + \left(a_1(1+r)\,(-1+r) - a_0(2r^2 - 5r + 1)\right)u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term must be 0
  $a_1(1+r)(-1+r) - a_0(2r^2 - 5r + 1) = 0$
- Each term in the series must be 0, giving the recursion relation
  $(-2a_k + a_{k-1} + a_{k+1}) k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1}) r + 5a_k - 5a_{k-1}) k + (-2a_k + a_{k-1} + a_{k+1})$
- Shift index using $k- > k+1$
  $(-2a_{k+1} + a_k + a_{k+2}) (k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2}) r + 5a_{k+1} - 5a_k) (k+1) + (-2a_{k+1} +$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2kra_k - 4kra_{k+1} + r^2a_k - 2r^2a_{k+1} - 3ka_k + ka_{k+1} - 3ra_k + ra_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2kr + r^2 + 2k + 2r}$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 3ka_k + ka_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 3ka_k + ka_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2k}$
- Recursion relation for $r = 2$
  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2 + 6k + 8}$
- Solution for $r = 2$
  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2 + 6k + 8}, 3a_1 + a_0 = 0\right]$
- Revert the change of variables $u = 1 + x$
  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+2}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + ka_k - 7ka_{k+1} - 4a_{k+1}}{k^2 + 6k + 8}, 3a_1 + a_0 = 0\right]$

1.731.3 Maple trace

```
Methods for second order ODEs:
```

1.731.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 15

```
dsolve(x^2*(1+x)*diff(diff(y(x),x),x)-(1+2*x)*(x*diff(y(x),x)-y(x)) = 0,
       y(x),singsol=all)
```

$$y = x(c_2 \ln(x) + c_2 x + c_1)$$

1.731.5 Mathematica DSolve solution

Solving time : 0.281 (sec)
Leaf size : 132

```
DSolve[{x^2*(1+x)*D[y[x],{x,2}]-(1+2*x)*(x*D[y[x],x]+y[x])==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x^{1+\sqrt{2}}\,\text{Hypergeometric2F1}\left(-\frac{1}{2}+\sqrt{2}-\frac{\sqrt{17}}{2},-\frac{1}{2}+\sqrt{2}+\frac{\sqrt{17}}{2},1+2\sqrt{2},-x\right) + c_1 x^{1-\sqrt{2}}\,\text{Hypergeometric2F1}\left(\frac{1}{2}\left(-1-2\sqrt{2}-\sqrt{17}\right),\frac{1}{2}\left(-1-2\sqrt{2}+\sqrt{17}\right),1-2\sqrt{2},-x\right)$$

1.732 problem 749

Internal problem ID [8870]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 749
Date solved : `Monday, October 21, 2024 at 05:23:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2(2-x)\,x^2y''-(4-x)\,xy'+(3-x)\,y=0$$

1.732.1 Solved as second order ode using Kovacic algorithm

Time used: 0.188 (sec)

Writing the ode as

$$\left(-2x^3+4x^2\right)y''+\left(x^2-4x\right)y'+(3-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+4x^2 \\ B &= x^2-4x \\ C &= 3-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{16(-2+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = 16(-2+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{16(-2+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1394: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(-2+x)^2$. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(-2+x)^2}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(-2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16(-2+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{16\left(-2+x\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
2	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{-8+4x} + (-)\left(0\right) \\ &= \frac{1}{-8+4x} \\ &= \frac{1}{-8+4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{-8+4x}\right)(0) + \left(\left(-\frac{1}{4\,(-2+x)^2}\right) + \left(\frac{1}{-8+4x}\right)^2 - \left(-\frac{3}{16\,(-2+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{-8+4x} dx} \\ &= (-2+x)^{1/4} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^2-4x}{-2x^3+4x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \frac{\ln(-2+x)}{4}} \\ &= z_1 \left(\frac{\sqrt{x}}{(-2+x)^{1/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{x^2-4x}{-2x^3+4x^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{\ln(x)-\frac{\ln(-2+x)}{2}}}{(y_1)^2}\,dx$$
$$= y_1 \left(\frac{2\,\mathrm{e}^{\ln(x)-\frac{\ln(-2+x)}{2}}(-2+x)}{x}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\left(\frac{2\,\mathrm{e}^{\ln(x)-\frac{\ln(-2+x)}{2}}(-2+x)}{x}\right)\right)$$

Will add steps showing solving for IC soon.

1.732.2 Maple step by step solution

Let's solve

$2(2-x)\,x^2\left(\frac{d}{dx}y'\right) - (4-x)\,xy' + (3-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(-3+x)y}{2(-2+x)x^2} + \frac{(x-4)y'}{2(-2+x)x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x-4)y'}{2(-2+x)x} + \frac{(-3+x)y}{2(-2+x)x^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x-4}{2x(-2+x)}, P_3(x) = \frac{-3+x}{2(-2+x)x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{3}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2(-2+x)\, x^2\left(\frac{d}{dx}y'\right) - x(x-4)\, y' + (-3+x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..3$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- > k + 2 - m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(-1+2r)\,(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(2k+2r-1)\,(2k+2r-3) + a_{k-1}(2k+2r-3)\,(k-2\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-1+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$-4\left(k+r-\frac{3}{2}\right)\left(\left(-\frac{k}{2}-\frac{r}{2}+1\right)a_{k-1}+a_k\left(k+r-\frac{1}{2}\right)\right)=0$

- Shift index using $k->k+1$

$-4\left(k+r-\frac{1}{2}\right)\left(\left(-\frac{k}{2}+\frac{1}{2}-\frac{r}{2}\right)a_k+a_{k+1}\left(k+\frac{1}{2}+r\right)\right)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{(k+r-1)a_k}{2k+1+2r}$

- Recursion relation for $r=\frac{1}{2}$

$a_{k+1}=\frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}$

- Solution for $r=\frac{1}{2}$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=\frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}\right]$

- Recursion relation for $r=\frac{3}{2}$

$a_{k+1}=\frac{\left(k+\frac{1}{2}\right)a_k}{2k+4}$

- Solution for $r=\frac{3}{2}$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1}=\frac{\left(k+\frac{1}{2}\right)a_k}{2k+4}\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1}=\frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}, b_{k+1}=\frac{\left(k+\frac{1}{2}\right)b_k}{2k+4}\right]$

1.732.3 Maple trace

```
Methods for second order ODEs:
```

1.732.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 19

```
dsolve(2*(2-x)*x^2*diff(diff(y(x),x),x)-(4-x)*x*diff(y(x),x)+(3-x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1\sqrt{x}+c_2\sqrt{(-2+x)\,x}$$

1.732.5 Mathematica DSolve solution

Solving time : 0.097 (sec)
Leaf size : 41

```
DSolve[{2*(2-x)*x^2*D[y[x],{x,2}]-(4-x)*x*D[y[x],x]+(3-x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[4]{x-2}\sqrt{x}\left(2c_2\sqrt{x-2}+c_1\right)}{\sqrt[4]{2-x}}$$

1.733 problem 750

Internal problem ID [8871]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 750
Date solved : `Monday, October 21, 2024 at 05:23:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(1-x)\,x^2y''+(5x-4)\,xy'+(6-9x)\,y=0$$

1.733.1 Solved as second order ode using Kovacic algorithm

Time used: 0.210 (sec)

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(5x^2-4x\right)y'+(6-9x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^3+x^2\\B&=5x^2-4x\\C&=6-9x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x + 4}{4x\left(-1 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x + 4$$
$$t = 4x(-1 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x + 4}{4x\left(-1 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1396: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(-1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(-1+x\right)^2} - \frac{1}{-1+x} + \frac{1}{x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+4}{4x\left(-1+x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x+4}{4x\left(-1+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} - \frac{1}{2(-1+x)} + (-)(0) \\
&= \frac{1}{x} - \frac{1}{2(-1+x)} \\
&= \frac{-2+x}{2(-1+x)x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{x} - \frac{1}{2(-1+x)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{2(-1+x)^2}\right) + \left(\frac{1}{x} - \frac{1}{2(-1+x)}\right)^2 - \left(\frac{-x+4}{4x(-1+x)^2}\right)\right) = 0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{2(-1+x)}\right) dx} \\
&= \frac{x}{\sqrt{-1+x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x^2-4x}{-x^3+x^2}\, dx} \\
&= z_1 e^{\frac{\ln(-1+x)}{2} + 2\ln(x)} \\
&= z_1 \left(\sqrt{-1+x}\, x^2\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2-4x}{-x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(-1+x)+4\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{1}{x} + \ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x^3) + c_2 \left(x^3 \left(\frac{1}{x} + \ln(x)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.733.2 Maple step by step solution

1.733.3 Maple trace

```
Methods for second order ODEs:
```

1.733.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 18

```
dsolve((1-x)*x^2*diff(diff(y(x),x),x)+(5*x-4)*x*diff(y(x),x)+(6-9*x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^2 (c_2 x \ln(x) + c_1 x + c_2)$$

1.733.5 Mathematica DSolve solution

Solving time : 0.062 (sec)
Leaf size : 24

```
DSolve[{(1-x)*x^2*D[y[x],{x,2}]+(5*x-4)*x*D[y[x],x]+(6-9*x)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x^2(c_1 x - c_2(x \log(x) + 1))$$

1.734 problem 751

Internal problem ID [8872]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 751
Date solved : `Monday, October 21, 2024 at 05:23:08 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (4x^2 + 1) y' + 4x(x^2 + 1) y = 0$$

1.734.1 Solved as second order ode using Kovacic algorithm

Time used: 0.188 (sec)

Writing the ode as

$$xy'' + (4x^2 + 1) y' + (4x^3 + 4x) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 4x^2 + 1 \\ C &= 4x^3 + 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1397: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx}\\ &= e^{\int \frac{1}{2x} dx}\\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{4x^2+1}{x} \, dx}\\ &= z_1 e^{-x^2 - \frac{\ln(x)}{2}}\\ &= z_1\left(\frac{e^{-x^2}}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{4x^2+1}{x}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-2x^2-\ln(x)}}{(y_1)^2}\,dx$$
$$= y_1(\ln(x))$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(e^{-x^2}\right) + c_2\left(e^{-x^2}(\ln(x))\right)$$

Will add steps showing solving for IC soon.

1.734.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + (4x^2+1)\,y' + 4x(x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = (-4x^2-4)\,y - \frac{(4x^2+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(4x^2+1)y'}{x} + (4x^2+4)\,y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{4x^2+1}{x}, P_3(x) = 4x^2+4\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right)+\left(4x^2+1\right)y'+4x\left(x^2+1\right)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=1..3$

    $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=0..2$

    $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

    $x\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k->k+1$

    $x\cdot\left(\frac{d}{dx}y'\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r}+a_1(1+r)^2 x^r+\left(a_2(2+r)^2+4a_0(1+r)\right)x^{1+r}+\left(a_3(3+r)^2+4a_1(2+r)\right)x^{2+r}+\Big($

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2=0$

- Values of r that satisfy the indicial equation

  $r=0$

- The coefficients of each power of x must be 0

  $\left[a_1(1+r)^2=0, a_2(2+r)^2+4a_0(1+r)=0, a_3(3+r)^2+4a_1(2+r)=0\right]$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = -\frac{4a_0(1+r)}{r^2+4r+4}, a_3 = 0\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + 4a_{k-1}k + 4a_{k-3} = 0$

- Shift index using $k- >k+3$

  $a_{k+4}(k+4)^2 + 4a_{k+2}(k+3) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}$

- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}, a_1 = 0, a_2 = -a_0, a_3 = 0\right]$

1.734.3 Maple trace

```
Methods for second order ODEs:
```

1.734.4 Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+(4*x^2+1)*diff(y(x),x)+4*x*(x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{-x^2}(c_1 + \ln(x) c_2)$$

1.734.5 Mathematica DSolve solution

Solving time : `0.049` (sec)
Leaf size : 21

```
DSolve[{x*D[y[x],{x,2}]+(4*x^2+1)*D[y[x],x]+4*x*(x^2+1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x^2}(c_2 \log(x) + c_1)$$

1.735 problem 754

Internal problem ID [8873]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 754
Date solved : `Monday, October 21, 2024 at 05:23:09 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-2xy'+12y=0$$

1.735.1 Solved as second order ode using Kovacic algorithm

Time used: 0.303 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-2xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -2x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{12x^2 - 13}{(x^2 - 1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 12x^2 - 13 \\
t &= (x^2 - 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{12x^2 - 13}{(x^2 - 1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1399: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{25}{4(x+1)} - \frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2} + \frac{25}{4(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{12x^2 - 13}{(x^2 - 1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=12$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -3
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{12x^2 - 13}{(x^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	4	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 4 - (1) \\
&= 3
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(6x + 2a_2) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(3x^2 + 2xa_2 + a_1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(x+1)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x \cdots}\right.\right.
$$

$$
\frac{-6a_2x^2 + (-10a_1 \cdots}{x^{\cdots}}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = 0, a_1 = -\frac{3}{5}, a_2 = 0\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^3 - \frac{3}{5}x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^3 - \frac{3}{5}x\right) e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= \left(x^3 - \frac{3}{5}x\right) \sqrt{(x-1)(x+1)} \\
&= \frac{(5x^3 - 3x)\sqrt{x^2-1}}{5}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(5x^3-3x)\sqrt{x^2-1}}{5\sqrt{x-1}\,\sqrt{x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{25}{9x}+\frac{125x}{36\left(x^2-\frac{3}{5}\right)}+\frac{25\ln(x-1)}{8}-\frac{25\ln(x+1)}{8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(5x^3-3x)\sqrt{x^2-1}}{5\sqrt{x-1}\,\sqrt{x+1}}\right) \\
&\quad + c_2\left(\frac{(5x^3-3x)\sqrt{x^2-1}}{5\sqrt{x-1}\,\sqrt{x+1}}\left(\frac{25}{9x}+\frac{125x}{36\left(x^2-\frac{3}{5}\right)}+\frac{25\ln(x-1)}{8}-\frac{25\ln(x+1)}{8}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.735.2 Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d}{dx}y'\right)-2xy'+12y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{12y}{x^2-1}-\frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{2xy'}{x^2-1}-\frac{12y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{12}{x^2-1}\right]$

- ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=1$

- ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$(x^2-1)\left(\frac{d}{dx}y'\right)+2xy'-12y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-12y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r^2 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+4)(k+r-3)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-2a_{k+1}(k+1)^2 + a_k(k+4)(k-3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+4)(k-3)}{2(k+1)^2}$$

- Recursion relation for $r = 0$; series terminates at $k = 3$

$$a_{k+1} = \frac{a_k(k+4)(k-3)}{2(k+1)^2}$$

- Apply recursion relation for $k = 0$

$$a_1 = -6a_0$$

- Apply recursion relation for $k = 1$

$$a_2 = -\frac{5a_1}{4}$$

- Express in terms of a_0

$$a_2 = \frac{15a_0}{2}$$

- Apply recursion relation for $k = 2$

$$a_3 = -\frac{a_2}{3}$$

- Express in terms of a_0

$$a_3 = -\frac{5a_0}{2}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot \left(1 - 6u + \frac{15}{2}u^2 - \frac{5}{2}u^3\right)$$

- Revert the change of variables $u = x + 1$

$$\left[y = a_0\left(\frac{3}{2}x - \frac{5}{2}x^3\right)\right]$$

1.735.3 Maple trace

```
Methods for second order ODEs:
```

1.735.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 55

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+12*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(5x^3 - 3x)\, c_2 \ln(x-1)}{24} + \frac{(-5x^3 + 3x)\, c_2 \ln(x+1)}{24} - \frac{5c_1\, x^3}{3} + \frac{5c_2\, x^2}{12} + c_1 x - \frac{c_2}{9}$$

1.735.5 Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 59

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+12*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}c_1 x(5x^2 - 3) + c_2\left(-\frac{5x^2}{2} - \frac{1}{4}(5x^2 - 3)\, x(\log(1-x) - \log(x+1)) + \frac{2}{3}\right)$$

1.736 problem 755

Internal problem ID [8874]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 755
Date solved : `Monday, October 21, 2024 at 05:23:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x(x+2)\, y'' + 2(x+1)\, y' - 2y = 0$$

1.736.1 Solved as second order ode using Kovacic algorithm

Time used: 0.239 (sec)

Writing the ode as

$$\left(x^2+2x\right) y'' + (2x+2)\, y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+2x \\ B &= 2x+2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{2x^2 + 4x - 1}{\left(x^2 + 2x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 2x^2 + 4x - 1 \\
t &= \left(x^2 + 2x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{2x^2 + 4x - 1}{\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1401: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4(x+2)} + \frac{5}{4x} - \frac{1}{4(x+2)^2} - \frac{1}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 + 4x - 1}{(x^2 + 2x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=2\\ \alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{2x^2+4x-1}{\left(x^2+2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$\begin{aligned}d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{+}\right)\\ &= 2-(1)\\ &= 1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x+4} + \frac{1}{2x} + (0) \\
&= \frac{1}{2x+4} + \frac{1}{2x} \\
&= \frac{x+1}{x(x+2)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x+4} + \frac{1}{2x}\right)(1) + \left(\left(-\frac{1}{2(x+2)^2} - \frac{1}{2x^2}\right) + \left(\frac{1}{2x+4} + \frac{1}{2x}\right)^2 - \left(\frac{2x^2+4x-1}{(x^2+2x)^2}\right)\right) = 0
$$

$$
\frac{2-2a_0}{x(x+2)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+1)\, e^{\int \left(\frac{1}{2x+4} + \frac{1}{2x}\right) dx} \\
&= (x+1)\sqrt{x(x+2)} \\
&= (x+1)\sqrt{x(x+2)}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x+2}{x^2+2x}\,dx} \\
&= z_1 e^{-\frac{\ln(x(x+2))}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x\,(x+2)}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x + 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x+2}{x^2+2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x(x+2))}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\ln(x+2)}{2} + \frac{\ln(x)}{2} + \frac{1}{x+1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x+1) + c_2\left(x + 1\left(-\frac{\ln(x+2)}{2} + \frac{\ln(x)}{2} + \frac{1}{x+1}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.736.2 Maple step by step solution

Let's solve

$x(x+2)\left(\frac{d}{dx}y'\right)+2(x+1)\,y'-2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{2y}{x(x+2)}-\frac{2(x+1)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{2(x+1)y'}{x(x+2)}-\frac{2y}{x(x+2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{2(x+1)}{x(x+2)},P_3(x)=-\frac{2}{x(x+2)}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=1$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $x(x+2)\left(\frac{d}{dx}y'\right)+(2x+2)\,y'-2y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+2)\,(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-2r^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2a_{k+1}(k+1)^2 + a_k(k+2)\,(k-1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$
- Apply recursion relation for $k = 0$
  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y(u) = a_0 \cdot (-u+1)$
- Revert the change of variables $u = x+2$
  $[y = a_0(-x-1)]$

1.736.3 Maple trace

```
Methods for second order ODEs:
```

1.736.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 28

```
dsolve(x*(x+2)*diff(diff(y(x),x),x)+2*(x+1)*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -\frac{c_2(x+1)\ln(x+2)}{2} + \frac{c_2(x+1)\ln(x)}{2} + c_1 x + c_1 + c_2$$

1.736.5 Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 37

```
DSolve[{x*(x+2)*D[y[x],{x,2}]+2*(x+1)*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1(x+1) - \frac{1}{2}c_2((x+1)\log(-x) - (x+1)\log(x+2) + 2)$$

1.737 problem 757

Internal problem ID [8875]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 757
Date solved : `Monday, October 21, 2024 at 05:23:11 PM`
CAS classification :
`[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]`

Solve

$$x(x+2)\,y''+(x+1)\,y'-4y=0$$

1.737.1 Solved as second order ode using Kovacic algorithm

Time used: 0.227 (sec)

Writing the ode as

$$\left(x^2+2x\right)y''+(x+1)\,y'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+2x\\B&=x+1\\C&=-4\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15x^2 + 30x - 3$$
$$t = 4\left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1403: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{33}{16\,(x+2)} + \frac{33}{16x} - \frac{3}{16x^2} - \frac{3}{16\,(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 + 30x - 3}{4\,(x^2 + 2x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4\left(x+2\right)} + \frac{3}{4x} + (0) \\
&= \frac{3}{4\left(x+2\right)} + \frac{3}{4x} \\
&= \frac{\frac{3x}{2}+\frac{3}{2}}{x\left(x+2\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4\left(x+2\right)} + \frac{3}{4x}\right)(1) + \left(\left(-\frac{3}{4\left(x+2\right)^2} - \frac{3}{4x^2}\right) + \left(\frac{3}{4\left(x+2\right)} + \frac{3}{4x}\right)^2 - \left(\frac{15x^2+30x-3}{4\left(x^2+2x\right)^2}\right)\right) = \frac{3-3a_0}{x\left(x+2\right)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+1)\, e^{\int \left(\frac{3}{4(x+2)} + \frac{3}{4x}\right) dx} \\
&= (x+1)\left(x(x+2)\right)^{3/4} \\
&= (x+1)\left(x(x+2)\right)^{3/4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x+1}{x^2+2x}\,dx} \\
&= z_1 e^{-\frac{\ln(x(x+2))}{4}} \\
&= z_1 \left(\frac{1}{(x(x+2))^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x(x+2)}\,(x+1)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x+1}{x^2+2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x(x+2))}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{2x^2+4x+1}{\sqrt{x(x+2)}\,(x+1)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x(x+2)}\,(x+1)\right) + c_2\left(\sqrt{x(x+2)}\,(x+1)\left(-\frac{2x^2+4x+1}{\sqrt{x(x+2)}\,(x+1)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.737.2 Maple step by step solution

Let's solve

$x(x+2)\left(\frac{d}{dx}y'\right)+(x+1)\,y'-4y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{4y}{x(x+2)}-\frac{(x+1)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{(x+1)y'}{x(x+2)}-\frac{4y}{x(x+2)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{x+1}{x(x+2)},P_3(x)=-\frac{4}{x(x+2)}\right]$

- ○ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=\frac{1}{2}$

- ○ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

- ○ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $x(x+2)\left(\frac{d}{dx}y'\right)+(x+1)\,y'-4y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)-4y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(k+r+2)\,(k+r-2)\right) u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + a_k(k+r+2)\,(k+r-2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r+2)(k+r-2)}{(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 2$

$$a_{k+1} = \frac{a_k(k+2)(k-2)}{(2k+1)(k+1)}$$

- Apply recursion relation for $k = 0$

$$a_1 = -4a_0$$

- Apply recursion relation for $k = 1$

$$a_2 = -\frac{a_1}{2}$$

- Express in terms of a_0

$$a_2 = 2a_0$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot (2u^2 - 4u + 1)$$

- Revert the change of variables $u = x + 2$

$$[y = a_0(2x^2 + 4x + 1)]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1}=\frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Revert the change of variables $u=x+2$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+2)^{k+\frac{1}{2}}, a_{k+1}=\frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=a_0\left(2x^2+4x+1\right)+\left(\sum_{k=0}^{\infty} b_k(x+2)^{k+\frac{1}{2}}\right), b_{k+1}=\frac{b_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

1.737.3 Maple trace

```
Methods for second order ODEs:
```

1.737.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 28

```
dsolve(x*(x+2)*diff(diff(y(x),x),x)+(x+1)*diff(y(x),x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y=c_2\sqrt{x\,(x+2)}\,(x+1)+2\left(x^2+2x+\frac{1}{2}\right)c_1$$

1.737.5 Mathematica DSolve solution

Solving time : `0.266` (sec)
Leaf size : 73

```
DSolve[{x*(x+2)*D[y[x],{x,2}]+(x+1)*D[y[x],x]-4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_1\cosh\left(8\operatorname{arctanh}\left(\frac{\sqrt{x}-1}{\sqrt{3}-\sqrt{x+2}}\right)\right)-ic_2\sinh\left(8\operatorname{arctanh}\left(\frac{\sqrt{x}-1}{\sqrt{3}-\sqrt{x+2}}\right)\right)$$

1.738 problem 758

Internal problem ID [8876]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 758
Date solved : `Monday, October 21, 2024 at 05:23:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\, y''-x y'+y = 0$$

1.738.1 Solved as second order ode using Kovacic algorithm

Time used: 0.245 (sec)

Writing the ode as

$$(x-1)\, y''-x y'+y = 0 \tag{1}$$
$$A y''+B y'+C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1405: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4\left(x-1\right)^2} - \frac{1}{2\left(x-1\right)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4(x-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{x \, e^{x+\ln(x-1)} e^{-2x}}{x-1}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2 \left(e^x \left(-\frac{x \, e^{x+\ln(x-1)} e^{-2x}}{x-1}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.738.2 Maple step by step solution

Let's solve

$(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$
- $(x-1)\cdot P_2(x)$ is analytic at $x=1$
  $$\left((x-1)\cdot P_2(x)\right)\bigg|_{x=1} = -1$$
- $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$
  $$\left((x-1)^2\cdot P_3(x)\right)\bigg|_{x=1} = 0$$
- $x=1$is a regular singular point
  Check to see if $x_0=1$ is a regular singular point
  $x_0=1$

- Multiply by denominators
  $(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$
- Change variables using $x=u+1$ so that the regular singular point is at $u=0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions
- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
- Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$
- Shift index using $k->k+1$
  $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$
  Rewrite ODE with series expansions
  $$a_0r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Revert the change of variables $u=x-1$

  $\left[y=\sum_{k=0}^{\infty} a_k(x-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Revert the change of variables $u=x-1$

  $\left[y=\sum_{k=0}^{\infty} a_k(x-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k(x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

1.738.3 Maple trace

```
Methods for second order ODEs:
```

1.738.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2 \,\mathrm{e}^{x}$$

1.738.5 Mathematica DSolve solution

Solving time : 0.052 (sec)
Leaf size : 17

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2 x$$

1.739 problem 759

Internal problem ID [8877]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 759
Date solved : `Monday, October 21, 2024 at 05:23:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+1\right)y''-2xy'+2y=0$$

1.739.1 Solved as second order ode using Kovacic algorithm

Time used: 0.286 (sec)

Writing the ode as

$$\left(x^2+1\right)y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+1\\B&=-2x\\C&=2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (x^2+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1407: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{3/2}}{(ix+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1} \, dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1 \left(\sqrt{x^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x}{(x+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2 \left(\frac{(x^2+1)^2}{(ix+1)^2} \left(-\frac{x}{(x+i)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.739.2 Maple step by step solution

1.739.3 Maple trace

```
Methods for second order ODEs:
```

1.739.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 16

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$
y = c_2\,x^2 + c_1 x - c_2
$$

1.739.5 Mathematica DSolve solution

Solving time : 0.068 (sec)
Leaf size : 21

```
DSolve[{(1+x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x - c_1 (x - i)^2$$

1.740 problem 760

Internal problem ID [8878]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 760
Date solved : `Monday, October 21, 2024 at 05:23:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x^2-2x+10)\,y''+xy'-4y=0$$

1.740.1 Solved as second order ode using Kovacic algorithm

Time used: 0.373 (sec)

Writing the ode as

$$(x^2-2x+10)\,y''+xy'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-2x+10 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 15x^2 - 32x + 180 \\
t &= 4\left(x^2 - 2x + 10\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1408: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x + 10)^2$. There is a pole at $x = 1 + 3i$ of order 2. There is a pole at $x = 1 - 3i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{7}{36} + \frac{i}{24}}{(x - 1 - 3i)^2} + \frac{-\frac{7}{36} - \frac{i}{24}}{(x - 1 + 3i)^2} - \frac{149i}{216\,(x - 1 - 3i)} + \frac{149i}{216\,(x - 1 + 3i)}$$

For the pole at $x = 1 + 3i$ let b be the coefficient of $\frac{1}{(x-1-3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} + \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} + \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} - \frac{i}{12} \end{aligned}$$

For the pole at $x = 1 - 3i$ let b be the coefficient of $\frac{1}{(x-1+3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} - \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} - \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} + \frac{i}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$1+3i$	2	0	$\frac{3}{4} + \frac{i}{12}$	$\frac{1}{4} - \frac{i}{12}$
$1-3i$	2	0	$\frac{3}{4} - \frac{i}{12}$	$\frac{1}{4} + \frac{i}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i} + (0) \\
&= \frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i} \\
&= \frac{3x-4}{2x^2-4x+20}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right)(1) + \left(\left(\frac{-\frac{3}{4}-\frac{i}{12}}{(x-1-3i)^2} + \frac{-\frac{3}{4}+\frac{i}{12}}{(x-1+3i)^2}\right) + \left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right.\right.
$$

$$
-\frac{3}{(-
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{4}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{4}{3}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{\frac{3\ln\left(x^2-2x+10\right)}{4} - \frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}} \\
&= \frac{(3x-4)\left(x^2-2x+10\right)^{3/4} e^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}}}{3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2-2x+10}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2-2x+10\right)}{4}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}}}{\left(x^2-2x+10\right)^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2-2x+10}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{9(3x^2-4x+15)\,\mathrm{e}^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}\mathrm{e}^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}\right) \\
&\quad + c_2\left(\frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}\left(-\frac{9(3x^2-4x+15)\,\mathrm{e}^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}\mathrm{e}^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.740.2 Maple step by step solution

1.740.3 Maple trace

```
Methods for second order ODEs:
```

1.740.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 31

```
dsolve((x^2-2*x+10)*diff(diff(y(x),x),x)+x*diff(y(x),x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y = 3c_2\left(x-\frac{4}{3}\right)(x-1+3i)^{\frac{1}{2}-\frac{i}{6}}(x-1-3i)^{\frac{1}{2}+\frac{i}{6}}+c_1\left(x^2-\frac{4}{3}x+5\right)$$

1.740.5 Mathematica DSolve solution

Solving time : 1.038 (sec)
Leaf size : 92

```
DSolve[{(x^2-2*x+10)*D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3}(3x - 4)\sqrt{x^2-2x+10}e^{-\frac{1}{3}\arctan\left(\frac{x-1}{3}\right)}\left(c_2\int_1^x \frac{9e^{\frac{1}{3}\arctan\left(\frac{1}{3}(K[1]-1)\right)}}{(4-3K[1])^2\left(K[1]^2-2K[1]+10\right)^{3/2}}dK[1] + c_1\right)$$

1.741 problem 761

Internal problem ID [8879]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 761
Date solved : `Monday, October 21, 2024 at 05:23:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-2x+10\right)y''+xy'-4y=0$$

1.741.1 Solved as second order ode using Kovacic algorithm

Time used: 0.391 (sec)

Writing the ode as

$$\left(x^2-2x+10\right)y''+xy'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2-2x+10\\ B&=x\\ C&=-4\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15x^2 - 32x + 180$$
$$t = 4\left(x^2 - 2x + 10\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1409: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x + 10)^2$. There is a pole at $x = 1 + 3i$ of order 2. There is a pole at $x = 1 - 3i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{7}{36} + \frac{i}{24}}{(x - 1 - 3i)^2} + \frac{-\frac{7}{36} - \frac{i}{24}}{(x - 1 + 3i)^2} - \frac{149i}{216\,(x - 1 - 3i)} + \frac{149i}{216\,(x - 1 + 3i)}$$

For the pole at $x = 1 + 3i$ let b be the coefficient of $\frac{1}{(x-1-3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} + \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} + \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} - \frac{i}{12} \end{aligned}$$

For the pole at $x = 1 - 3i$ let b be the coefficient of $\frac{1}{(x-1+3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} - \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} - \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} + \frac{i}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$1+3i$	2	0	$\frac{3}{4} + \frac{i}{12}$	$\frac{1}{4} - \frac{i}{12}$
$1-3i$	2	0	$\frac{3}{4} - \frac{i}{12}$	$\frac{1}{4} + \frac{i}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i} + (0) \\
&= \frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i} \\
&= \frac{3x-4}{2x^2-4x+20}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right)(1) + \left(\left(\frac{-\frac{3}{4}-\frac{i}{12}}{(x-1-3i)^2} + \frac{-\frac{3}{4}+\frac{i}{12}}{(x-1+3i)^2}\right) + \left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right.\right.$$

$$-\frac{3}{(-\cdots}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{4}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{4}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{\frac{3\ln\left(x^2-2x+10\right)}{4} - \frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}} \\
&= \frac{(3x-4)\left(x^2-2x+10\right)^{3/4} e^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}}}{3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2-2x+10}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2-2x+10\right)}{4}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{6}}}{\left(x^2-2x+10\right)^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2-2x+10}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{9(3x^2-4x+15)\,\mathrm{e}^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}\mathrm{e}^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}\right) \\
&\quad + c_2 \left(\frac{\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}(3x-4)}{3}\left(-\frac{9(3x^2-4x+15)\,\mathrm{e}^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}\mathrm{e}^{\frac{2\arctan\left(\frac{x}{3}-\frac{1}{3}\right)}{3}}}{410\,(3x-4)}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.741.2 Maple step by step solution

1.741.3 Maple trace

```
Methods for second order ODEs:
```

1.741.4 Maple dsolve solution

Solving time : 0.014 (sec)
Leaf size : 31

```
dsolve((x^2-2*x+10)*diff(diff(y(x),x),x)+x*diff(y(x),x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y = 3(x-1+3i)^{\frac{1}{2}-\frac{i}{6}}\left(x-\frac{4}{3}\right)c_2(x-1-3i)^{\frac{1}{2}+\frac{i}{6}}+c_1\left(x^2-\frac{4}{3}x+5\right)$$

1.741.5 Mathematica DSolve solution

Solving time : 0.843 (sec)
Leaf size : 92

```
DSolve[{(x^2-2*x+10)*D[y[x],{x,2}]+x*D[y[x],x]-4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3}(3x-4)\sqrt{x^2-2x+10}e^{-\frac{1}{3}\arctan\left(\frac{x-1}{3}\right)}\left(c_2\int_1^x \frac{9e^{\frac{1}{3}\arctan\left(\frac{1}{3}(K[1]-1)\right)}}{(4-3K[1])^2(K[1]^2-2K[1]+10)^{3/2}}dK[1]+c_1\right)$$

1.742 problem 762

Internal problem ID [8880]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 762
Date solved : `Monday, October 21, 2024 at 05:23:15 PM`
CAS classification : `[_Hermite]`

Solve

$$y'' - xy' + 2y = 0$$

1.742.1 Solved as second order ode using Kovacic algorithm

Time used: 0.252 (sec)

Writing the ode as

$$y'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1410: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) = 0$$

$$a_1 x + 2a_0 + 2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x^2 - 1) \, e^{\int -\frac{x}{2} dx} \\ &= (x^2 - 1) \, e^{-\frac{x^2}{4}} \\ &= (x^2 - 1) \, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 - 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2 - 1)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 (x^2 - 1) + c_2 \left(x^2 - 1 \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2} dx\right)\right)$$

Will add steps showing solving for IC soon.

1.742.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$

  ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$

  ○ Shift index using $k- > k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k(k-2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) \, a_{k+2} - a_k(k-2) = 0$

- Recursion relation; series terminates at $k = 2$

  $a_{k+2} = \frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

$a_2 = -a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

  $y = A_2x^2 + A_1x - a_0$

1.742.3 Maple trace

```
Methods for second order ODEs:
```

1.742.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 39

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -2\,\mathrm{e}^{\frac{x^2}{2}} c_1 x + (x-1)(x+1)\left(\sqrt{\pi}\,\sqrt{2}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right) c_1 + c_2\right)$$

1.742.5 Mathematica DSolve solution

Solving time : 0.143 (sec)
Leaf size : 54

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}c_2\left(\sqrt{2\pi}(x^2-1)\,\mathrm{erfi}\left(\frac{x}{\sqrt{2}}\right) - 2e^{\frac{x^2}{2}}x\right) + c_1(x^2-1)$$

1.743 problem 763

Internal problem ID [8881]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 763
Date solved : `Monday, October 21, 2024 at 05:23:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x+2)\,y''+xy'-y=0$$

1.743.1 Solved as second order ode using Kovacic algorithm

Time used: 0.270 (sec)

Writing the ode as

$$(x+2)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x+2 \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 12}{4\left(x+2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x + 12$$
$$t = 4(x+2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 12}{4\left(x+2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1412: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+2)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x^2} - \frac{8}{x^3} + \frac{20}{x^4} - \frac{32}{x^5} + \frac{16}{x^6} + \frac{64}{x^7} - \frac{80}{x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 12}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{8}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{8}{4x^2 + 16x + 16} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 12}{4(x+2)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x+2} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{x+2} - \frac{1}{2} \\
&= -\frac{4+x}{2\,(x+2)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x+2} - \frac{1}{2}\right)(1) + \left(\left(\frac{1}{(x+2)^2}\right) + \left(-\frac{1}{x+2} - \frac{1}{2}\right)^2 - \left(\frac{x^2+4x+12}{4\,(x+2)^2}\right)\right) &= 0 \\
\frac{a_0 - 4}{x+2} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 4\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 4 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (4+x)\, e^{\int \left(-\frac{1}{x+2} - \frac{1}{2}\right) dx} \\
&= (4+x)\, e^{-\frac{x}{2} - \ln(x+2)} \\
&= \frac{(4+x)\, e^{-\frac{x}{2}}}{x+2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x+2}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\ln(x+2)} \\
&= z_1\left((x+2)\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}(4+x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x+2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+2\ln(x+2)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x\,\mathrm{e}^{-x+2\ln(x+2)}\mathrm{e}^{2x}}{(4+x)\,(x+2)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}(4+x)\right) + c_2\left(\mathrm{e}^{-x}(4+x)\left(\frac{x\,\mathrm{e}^{-x+2\ln(x+2)}\mathrm{e}^{2x}}{(4+x)\,(x+2)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.743.2 Maple step by step solution

Let's solve

$(x+2)\left(\frac{d}{dx}y'\right)+xy'-y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{y}{x+2}-\frac{xy'}{x+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{xy'}{x+2}-\frac{y}{x+2}=0$

□ Check to see if $x_0=-2$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x}{x+2}, P_3(x)=-\frac{1}{x+2}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=-2$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

Check to see if $x_0=-2$ is a regular singular point

$x_0=-2$

- Multiply by denominators

$(x+2)\left(\frac{d}{dx}y'\right)+xy'-y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$u\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)-y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^{m} \cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty} a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u \cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)$ to series expansion

$$u \cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =0}^{\infty} a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -1}$$

- Shift index using $k->k +1$

$$u \cdot \left(\frac{d}{d u}\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1}^{\infty} a_{k +1} \left(k +1+r \right) \left(k +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$a_{0} r \left(-3+r \right) u^{-1+r}+\left(\sum_{k =0}^{\infty} \left(a_{k +1} \left(k +1+r \right) \left(k -2+r \right)+a_{k} \left(k +r -1\right)\right) u^{k +r}\right)=0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$r \left(-3+r \right)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, 3\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k +1} \left(k +1+r \right) \left(k -2+r \right)+a_{k} \left(k +r -1\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k +1}=-\frac{a_{k} \left(k +r -1\right)}{\left(k +1+r \right) \left(k -2+r \right)}$$

- Recursion relation for $r =0$; series terminates at $k =1$

$$a_{k +1}=-\frac{a_{k} \left(k -1\right)}{\left(k +1\right) \left(k -2\right)}$$

- Apply recursion relation for $k =0$

$$a_{1}=-\frac{a_{0}}{2}$$

- Terminating series solution of the ODE for $r =0$. Use reduction of order to find the second li

$$y \left(u \right)=a_{0} \cdot \left(1-\frac{u}{2}\right)$$

- Revert the change of variables $u =x +2$

$$\left[y =-\frac{a_{0} x}{2}\right]$$

- Recursion relation for $r =3$

$$a_{k +1}=-\frac{a_{k} \left(k +2\right)}{\left(k +4\right) \left(k +1\right)}$$

- Solution for $r =3$

$$\left[y \left(u \right)=\sum_{k =0}^{\infty} a_{k} u^{k +3}, a_{k +1}=-\frac{a_{k} \left(k +2\right)}{\left(k +4\right) \left(k +1\right)}\right]$$

- Revert the change of variables $u =x +2$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+2)^{k+3}, a_{k+1}=-\frac{a_k(k+2)}{(k+4)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y=-\frac{a_0 x}{2}+\left(\sum_{k=0}^{\infty} b_k (x+2)^{k+3}\right), b_{k+1}=-\frac{b_k(k+2)}{(k+4)(k+1)}\right]$$

1.743.3 Maple trace

```
Methods for second order ODEs:
```

1.743.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 17

```
dsolve((x+2)*diff(diff(y(x),x),x)+x*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y=c_1 x+c_2\,\mathrm{e}^{-x}(4+x)$$

1.743.5 Mathematica DSolve solution

Solving time : `0.171` (sec)
Leaf size : 72

```
DSolve[{(x+2)*D[y[x],{x,2}]+x*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\frac{2\sqrt{\frac{2}{\pi}}e^{-x-2}\sqrt{x+2}(c_1(e^{x+2}x+x+4)-ic_2((e^{x+2}-1)\,x-4))}{\sqrt{-i(x+2)}}$$

1.744 problem 764

Internal problem ID [8882]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 764
Date solved : `Monday, October 21, 2024 at 05:23:17 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$\left(x^2+1\right) y''-6y=0$$

1.744.1 Solved as second order ode using Kovacic algorithm

Time used: 0.250 (sec)

Writing the ode as

$$\left(x^2+1\right) y''-6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 0 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{x^2 + 1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 6 \\ t &= x^2 + 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{6}{x^2 + 1}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1414: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2 + 1$. There is a pole at $x = i$ of order 1. There is a pole at $x = -i$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = i$ of order 1 then

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= 1\\ \alpha_c^- &= 1\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6}{x^2 + 1}$$

Since the $\gcd(s, t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6}{x^2 + 1}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= 3 - (1) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x-i} + (0) \\ &= \frac{1}{x-i} \\ &= \frac{1}{x-i} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2)+2\left(\frac{1}{x-i}\right)(2x+a_1)+\left(\left(-\frac{1}{(x-i)^2}\right)+\left(\frac{1}{x-i}\right)^2-\left(\frac{6}{x^2+1}\right)\right)=0$$

$$2+\frac{-4x-2a_1}{-x+i}+\frac{-6x^2-6a_1x-6a_0}{x^2+1}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0=0, a_1=i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=x^2+ix$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= \left(x^2+ix\right) e^{\int \frac{1}{x-i} dx} \\
&= \left(x^2+ix\right) e^{\frac{\ln\left(x^2+1\right)}{2}+i\arctan(x)} \\
&= x(x+i)(ix+1)
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= x(x+i)(ix+1)
\end{aligned}$$

Which simplifies to

$$y_1 = ix^3+ix$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= ix^3 + ix \int \frac{1}{(ix^3 + ix)^2}\, dx \\
&= ix^3 + ix \left(\frac{1}{x} + \frac{x}{2x^2 + 2} + \frac{3 \arctan(x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (ix^3 + ix) + c_2 \left(ix^3 + ix \left(\frac{1}{x} + \frac{x}{2x^2 + 2} + \frac{3 \arctan(x)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.744.2 Maple step by step solution

1.744.3 Maple trace

```
Methods for second order ODEs:
```

1.744.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 31

```
dsolve((x^2+1)*diff(diff(y(x),x),x)-6*y(x) = 0,
       y(x),singsol=all)
```

$$
y = \frac{3x c_2 (x^2 + 1) \arctan(x)}{2} + c_1 x^3 + \frac{3 c_2 x^2}{2} + c_1 x + c_2
$$

1.744.5 Mathematica DSolve solution

Solving time : 0.078 (sec)
Leaf size : 36

```
DSolve[{(x^2+1)*D[y[x],{x,2}]-6*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1(x^3 + x) - \frac{1}{2}c_2(3(x^3 + x)\arctan(x) + 3x^2 + 2)$$

1.745 problem 765

Internal problem ID [8883]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 765
Date solved : `Monday, October 21, 2024 at 05:23:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2+2\right) y''+3xy'-y=0$$

1.745.1 Solved as second order ode using Kovacic algorithm

Time used: 0.381 (sec)

Writing the ode as

$$\left(x^2+2\right) y''+3xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+2 \\ B &= 3x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 20}{4(x^2 + 2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 7x^2 + 20$$
$$t = 4(x^2 + 2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 20}{4(x^2 + 2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1415: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+2)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\left(x - i\sqrt{2}\right)^2} - \frac{3}{16\left(x + i\sqrt{2}\right)^2} - \frac{17i\sqrt{2}}{32\left(x - i\sqrt{2}\right)} + \frac{17i\sqrt{2}}{32\left(x + i\sqrt{2}\right)}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7x^2 + 20}{4\left(x^2 + 2\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{7}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
$i\sqrt{2}$	2	$\{1,2,3\}$
$-i\sqrt{2}$	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2-(1+(1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(i\sqrt{2}))} + \frac{1}{(x-(-i\sqrt{2}))}\right) \\ &= \frac{1}{2x-2i\sqrt{2}} + \frac{1}{2x+2i\sqrt{2}} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x - 2i\sqrt{2}} + \frac{1}{2x + 2i\sqrt{2}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2x - 2i\sqrt{2}} + \frac{1}{2x + 2i\sqrt{2}}\right)w + \frac{7x^2 + 16}{4\left(\sqrt{2} + ix\right)^2 \left(x + i\sqrt{2}\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{x + 2\sqrt{2x^2 + 4}}{2x^2 + 4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= e^{\int \frac{x + 2\sqrt{2x^2+4}}{2x^2+4} dx} \\ &= \left(x^2 + 2\right)^{1/4} e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2+2}\, dx} \\ &= z_1 e^{-\frac{3\ln\left(x^2+2\right)}{4}} \\ &= z_1 \left(\frac{1}{\left(x^2 + 2\right)^{3/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2+2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+2\right)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-2\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}}\right) + c_2 \left(\frac{e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} \left(\int \frac{e^{-2\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.745.2 Maple step by step solution

1.745.3 Maple trace

```
Methods for second order ODEs:
```

1.745.4 Maple dsolve solution

Solving time : 0.027 (sec)
Leaf size : 45

```
dsolve((x^2+2)*diff(diff(y(x),x),x)+3*x*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2\left(\sqrt{x^2+2}+x\right)^{-\sqrt{2}} + c_1\left(\sqrt{x^2+2}+x\right)^{\sqrt{2}}}{\sqrt{x^2+2}}$$

1.745.5 Mathematica DSolve solution

Solving time : 0.158 (sec)
Leaf size : 92

```
DSolve[{(x^2+2)*D[y[x],{x,2}]+3*x*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2^{3/4}c_1 \cos\left(2\sqrt{2}\arcsin\left(\frac{1}{2}\sqrt{2-i\sqrt{2}x}\right)\right)}{\sqrt{\pi}\sqrt{x^2+2}} + \frac{c_2 Q^{\frac{1}{2}}_{-\frac{1}{2}+\sqrt{2}}\left(\frac{ix}{\sqrt{2}}\right)}{\sqrt[4]{x^2+2}}$$

1.746 problem 766

Internal problem ID [8884]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 766
Date solved : `Monday, October 21, 2024 at 05:23:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-1)\,y''-xy'+y=0$$

1.746.1 Solved as second order ode using Kovacic algorithm

Time used: 0.245 (sec)

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 4x + 6}{4\left(x - 1\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 4x + 6 \\
t &= 4(x - 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 - 4x + 6}{4\left(x - 1\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1416: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(x-1)} + \frac{3}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2(x-1)} + \frac{1}{2} \\ &= \frac{x-2}{2x-2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(x-1)^2}\right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\ &= \frac{e^{\frac{x}{2}}}{\sqrt{x-1}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1}\, dx} \\ &= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\ &= z_1 \left(\sqrt{x-1}\, e^{\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{x\, \mathrm{e}^{x+\ln(x-1)} \mathrm{e}^{-2x}}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2 \left(\mathrm{e}^{x} \left(-\frac{x\, \mathrm{e}^{x+\ln(x-1)} \mathrm{e}^{-2x}}{x-1}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.746.2 Maple step by step solution

Let's solve

$(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$
- $(x-1)\cdot P_2(x)$ is analytic at $x = 1$
  $$\left((x-1)\cdot P_2(x)\right)\bigg|_{x=1} = -1$$
- $(x-1)^2\cdot P_3(x)$ is analytic at $x = 1$
  $$\left((x-1)^2\cdot P_3(x)\right)\bigg|_{x=1} = 0$$
- $x = 1$is a regular singular point
  Check to see if $x_0 = 1$ is a regular singular point
  $x_0 = 1$

- Multiply by denominators
  $(x-1)\left(\frac{d}{dx}y'\right) - xy' + y = 0$
- Change variables using $x = u+1$ so that the regular singular point is at $u = 0$
  $u\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$
- Assume series solution for $y(u)$
  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$
    $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$
  - Shift index using $k->k+1-m$
    $$u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
  - Convert $u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion
    $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$
  - Shift index using $k->k+1$
    $$u\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

  Rewrite ODE with series expansions
  $$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

1.746.3 Maple trace

```
Methods for second order ODEs:
```

1.746.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
dsolve((x-1)*diff(diff(y(x),x),x)-x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2 \,\mathrm{e}^{x}$$

1.746.5 Mathematica DSolve solution

Solving time : 0.05 (sec)
Leaf size : 17

```
DSolve[{(x-1)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2 x$$

1.747 problem 769

Internal problem ID [8885]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 769
Date solved : `Monday, October 21, 2024 at 05:23:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0$$

1.747.1 Solved as second order ode using Kovacic algorithm

Time used: 0.365 (sec)

Writing the ode as

$$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= \frac{5}{3}x + x^2 \\ C &= -\frac{1}{3} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 + 30x + 7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9x^2 + 30x + 7$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2 + 30x + 7}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1418: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{7}{36x^2} + \frac{5}{6x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{5}{6x} - \frac{1}{2x^2} + \frac{5}{6x^3} - \frac{59}{36x^4} + \frac{385}{108x^5} - \frac{2681}{324x^6} + \frac{19525}{972x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{9x^2 + 30x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{30x + 7}{36x^2}\right) \\ &= \frac{1}{4} + \frac{30x + 7}{36x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 30. Dividing this by leading coefficient in t which is 36 gives $\frac{5}{6}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{5}{6}\right) - (0) \\
&= \frac{5}{6}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{6}}{\frac{1}{2}} - 0\right) &&= \frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{6}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{9x^2 + 30x + 7}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{5}{6}$	$-\frac{5}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= \frac{5}{6} - \left(-\frac{1}{6}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{6x} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{6x} + \frac{1}{2} \\ &= -\frac{1}{6x} + \frac{1}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6x} + \frac{1}{2}\right)(1) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} + \frac{1}{2}\right)^2 - \left(\frac{9x^2 + 30x + 7}{36x^2}\right) \right) = 0$$

$$\frac{-1 - 3a_0}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = -\frac{1}{3} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{1}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{1}{3}\right) \mathrm{e}^{\int \left(-\frac{1}{6x} + \frac{1}{2}\right) dx} \\
&= \left(x - \frac{1}{3}\right) \mathrm{e}^{\frac{x}{2} - \frac{\ln(x)}{6}} \\
&= \frac{(-1 + 3x)\, \mathrm{e}^{\frac{x}{2}}}{3x^{1/6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{5}{3}x + x^2}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \frac{5\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{x^{5/6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{-1 + 3x}{3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{5}{3}x + x^2}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x - \frac{5\ln(x)}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{9\, \mathrm{e}^{-x - \frac{5\ln(x)}{3}} x^2}{(-1 + 3x)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{-1+3x}{3x}\right) + c_2 \left(\frac{-1+3x}{3x} \left(\int \frac{9\,\mathrm{e}^{-x-\frac{5\ln(x)}{3}} x^2}{(-1+3x)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.747.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + \left(\frac{5}{3}x + x^2\right) y' - \frac{y}{3} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{y}{3x^2} - \frac{(5+3x)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{(5+3x)y'}{3x} - \frac{y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{5+3x}{3x}, P_3(x) = -\frac{1}{3x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = \frac{5}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{3}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $3x^2 \left(\frac{d}{dx} y'\right) + x(5+3x)\, y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(3k+3r-1) + 3a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-1, \frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3(k+r+1)\left(k+r-\frac{1}{3}\right) a_k + 3a_{k-1}(k+r-1) = 0$

- Shift index using $k- > k+1$

  $3(k+2+r)\left(k+\frac{2}{3}+r\right) a_{k+1} + 3a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{3a_k(k+r)}{(k+2+r)(3k+2+3r)}$

- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = -\frac{3a_k(k-1)}{(k+1)(3k-1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -3a_0$

- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot (1-3x)$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+1} = -\frac{3a_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{3a_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}\right]$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot (1-3x) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), b_{k+1} = -\frac{3b_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}\right]$$

1.747.3 Maple trace

```
Methods for second order ODEs:
```

1.747.4 Maple dsolve solution

Solving time : 0.025 (sec)
Leaf size : 29

```
dsolve(x^2*diff(diff(y(x),x),x)+(5/3*x+x^2)*diff(y(x),x)-1/3*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1\, x^{4/3}\,\text{hypergeom}\left(\left[2\right], \left[\frac{7}{3}\right], x\right) \mathrm{e}^{-x} - 3c_2 x + c_2}{x}$$

1.747.5 Mathematica DSolve solution

Solving time : 0.123 (sec)
Leaf size : 47

```
DSolve[{x^2*D[y[x],{x,2}]+(5/3*x+x^2)*D[y[x],x]-1/3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{-3c_1 x + 3c_2 e^{-x}\sqrt[3]{x} + c_2(1-3x)\Gamma\left(\frac{1}{3}, x\right) + c_1}{3x}$$

1.748 problem 770

Internal problem ID [8886]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 770
Date solved : `Monday, October 21, 2024 at 05:23:21 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$2xy'' - y' + 2y = 0$$

1.748.1 Solved as second order ode using Kovacic algorithm

Time used: 0.225 (sec)

Writing the ode as

$$2xy'' - y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5 - 16x$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1420: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\
&= \frac{1}{2}(1-(-1))\\
&= 1
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\
&= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right)\\
&= -\frac{1}{2x}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d=1$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}
$$

Since $d=1$, then letting

$$
p = x + a_0 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$
\frac{1-4a_0}{x^2} = 0
$$

And solving for p gives

$$
p = x + \frac{1}{4}
$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta + \frac{p'}{p}\\
&= \frac{1}{x+\frac{1}{4}} - \frac{1}{2x}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0
$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{1}{4}} - \frac{1}{2x}\right)w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4\,(4x+1)\,x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= e^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(4x+1)x}dx} \\
&= \frac{\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{\left(-x\right)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{2x}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{4}} \\
&= z_1\left(x^{1/4}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{1/4}\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{\left(-x\right)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{2x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{1/4} \left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}}\right) + c_2 \left(\frac{x^{1/4} \left(2\sqrt{-x}-1\right) \mathrm{e}^{2\sqrt{-x}}}{(-x)^{1/4}} \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.748.2 Maple step by step solution

Let's solve

$2x\left(\frac{d}{dx}y'\right) - y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x} + \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'}{2x} + \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x\left(\frac{d}{dx}y'\right) - y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-1+2r)+2a_k\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2\left(k-\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 2a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k}{(2k-1+2r)(k+1+r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}\right]$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = -\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}$
- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}, b_{k+1} = -\frac{2b_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

1.748.3 Maple trace

```
Methods for second order ODEs:
```

1.748.4 Maple dsolve solution

Solving time : `0.015` (sec)
Leaf size : 36

```
dsolve(2*x*diff(diff(y(x),x),x)-diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \left(2c_1\sqrt{x} + c_2\right)\cos\left(2\sqrt{x}\right) - \sin\left(2\sqrt{x}\right)\left(-2c_2\sqrt{x} + c_1\right)$$

1.748.5 Mathematica DSolve solution

Solving time : `0.124` (sec)
Leaf size : 59

```
DSolve[{2*x*D[y[x],{x,2}]-D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{2i\sqrt{x}}\left(2\sqrt{x} + i\right) + \frac{1}{8}c_2 e^{-2i\sqrt{x}}\left(1 + 2i\sqrt{x}\right)$$

1.749 problem 771

Internal problem ID [8887]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 771
Date solved : `Monday, October 21, 2024 at 05:23:22 PM`
CAS classification : `[_Laguerre]`

Solve

$$2xy'' - (3 + 2x)\, y' + y = 0$$

1.749.1 Solved as second order ode using Kovacic algorithm

Time used: 0.338 (sec)

Writing the ode as

$$2xy'' + (-3 - 2x)\, y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -3 - 2x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 + 4x + 21$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x + 21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1422: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{21}{16x^2} + \frac{1}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4x} + \frac{5}{4x^2} - \frac{5}{8x^3} - \frac{5}{4x^4} + \frac{35}{16x^5} + \frac{105}{64x^6} - \frac{1005}{128x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 4x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x + 21}{16x^2}\right) \\ &= \frac{1}{4} + \frac{4x + 21}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 4x + 21}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= \frac{1}{4} - \left(-\frac{3}{4}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{3}{4x} + \left(\frac{1}{2}\right) \\
&= -\frac{3}{4x} + \frac{1}{2} \\
&= -\frac{3}{4x} + \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4x} + \frac{1}{2}\right)(1) + \left(\left(\frac{3}{4x^2}\right) + \left(-\frac{3}{4x} + \frac{1}{2}\right)^2 - \left(\frac{4x^2 + 4x + 21}{16x^2}\right) \right) = 0$$

$$\frac{-3 - 2a_0}{2x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = -\frac{3}{2} \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{3}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{3}{2}\right) \mathrm{e}^{\int \left(-\frac{3}{4x} + \frac{1}{2}\right) dx} \\
&= \left(x - \frac{3}{2}\right) \mathrm{e}^{\frac{x}{2} - \frac{3 \ln(x)}{4}} \\
&= \frac{(-3 + 2x) \, \mathrm{e}^{\frac{x}{2}}}{2x^{3/4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3-2x}{2x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{3 \ln(x)}{4}} \\
&= z_1 \left(x^{3/4} \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^x (-3 + 2x)}{2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3-2x}{2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x + \frac{3 \ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{4 \, \mathrm{e}^{x + \frac{3 \ln(x)}{2}} \mathrm{e}^{-2x}}{(-3 + 2x)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{x}(-3+2x)}{2}\right) + c_2 \left(\frac{e^{x}(-3+2x)}{2} \left(\int \frac{4\, e^{x+\frac{3\ln(x)}{2}} e^{-2x}}{(-3+2x)^2} dx\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.749.2 Maple step by step solution

Let's solve

$2x\left(\frac{d}{dx}y'\right) - (3+2x)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{2x} + \frac{(3+2x)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(3+2x)y'}{2x} + \frac{y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{3+2x}{2x}, P_3(x) = \frac{1}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -\frac{3}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x\left(\frac{d}{dx}y'\right) + (-3-2x)\,y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^{m}\cdot y' = \sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$$

◦ Shift index using $k- >k+1-m$

$$x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

◦ Convert $x\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$$x\cdot\left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-1}$$

◦ Shift index using $k- >k+1$

$$x\cdot\left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_{0}r(-5+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-3+2r) - a_{k}(2k+2r-1)\right) x^{k+r}\right) = 0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $r(-5+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{5}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+1+r)\left(k-\frac{3}{2}+r\right) a_{k+1} - 2\left(k+r-\frac{1}{2}\right) a_{k} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(2k+2r-1)a_{k}}{(k+1+r)(2k-3+2r)}$

- Recursion relation for $r=0$

  $a_{k+1} = \frac{(2k-1)a_{k}}{(k+1)(2k-3)}$

- Solution for $r=0$

  $\left[y = \sum_{k=0}^{\infty} a_{k}x^{k}, a_{k+1} = \frac{(2k-1)a_{k}}{(k+1)(2k-3)}\right]$

- Recursion relation for $r=\frac{5}{2}$

  $a_{k+1} = \frac{(2k+4)a_{k}}{\left(k+\frac{7}{2}\right)(2k+2)}$

- Solution for $r=\frac{5}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_{k}x^{k+\frac{5}{2}}, a_{k+1} = \frac{(2k+4)a_{k}}{\left(k+\frac{7}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+1} = \frac{(2k-1)a_k}{(k+1)(2k-3)}, b_{k+1} = \frac{(2k+4)b_k}{(k+\frac{7}{2})(2k+2)}\right]$$

1.749.3 Maple trace

```
Methods for second order ODEs:
```

1.749.4 Maple dsolve solution

Solving time : `0.016` (sec)
Leaf size : 24

```
dsolve(2*x*diff(diff(y(x),x),x)-(3+2*x)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \operatorname{hypergeom}\left([2], \left[\frac{7}{2}\right], x\right) x^{5/2} - \frac{2\left(x-\frac{3}{2}\right) c_2 \,\mathrm{e}^x}{3}$$

1.749.5 Mathematica DSolve solution

Solving time : `0.126` (sec)
Leaf size : 54

```
DSolve[{2*x*D[y[x],{x,2}]-(3+2*x)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}\left(-\sqrt{\pi} c_2 e^x (2x-3) \operatorname{erf}\left(\sqrt{x}\right) + 2c_1 e^x (2x-3) - 6c_2\sqrt{x}\right)$$

1.750 problem 772

Internal problem ID [8888]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 772
Date solved : `Monday, October 21, 2024 at 05:23:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + 3xy' + (2x - 1)\,y = 0$$

1.750.1 Solved as second order ode using Kovacic algorithm

Time used: 0.232 (sec)

Writing the ode as

$$2x^2y'' + 3xy' + (2x - 1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 3x \\ C &= 2x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5 - 16x$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1424: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1, \ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{1}{2}(1-(-1)) \\
&= 1
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\
&= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right) \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d=1$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}
$$

Since $d=1$, then letting

$$
p = x + a_0 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$
\frac{1-4a_0}{x^2} = 0
$$

And solving for p gives

$$
p = x + \frac{1}{4}
$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta + \frac{p'}{p} \\
&= \frac{1}{x+\frac{1}{4}} - \frac{1}{2x}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0
$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{1}{4}} - \frac{1}{2x}\right) w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4\,(4x+1)\,x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \,dx} \\ &= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(4x+1)x}dx} \\ &= \frac{\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{\left(-x\right)^{1/4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{2x^2}\,dx} \\ &= z_1 e^{-\frac{3\ln(x)}{4}} \\ &= z_1\left(\frac{1}{x^{3/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{x^{3/4}\left(-x\right)^{1/4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{3x}{2x^2}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx$$

$$= y_1 \left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{x^{3/4}\left(-x\right)^{1/4}}\right) + c_2 \left(\frac{\left(2\sqrt{-x}-1\right)\mathrm{e}^{2\sqrt{-x}}}{x^{3/4}\left(-x\right)^{1/4}}\left(\frac{\mathrm{e}^{-4\sqrt{-x}}}{8} + \frac{\mathrm{e}^{-4\sqrt{-x}}}{8\sqrt{-x}-4}\right)\right)$$

Will add steps showing solving for IC soon.

1.750.2 Maple step by step solution

Let's solve

$2x^2\left(\frac{d}{dx}y'\right) + 3xy' + (2x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(2x-1)y}{2x^2} - \frac{3y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{3y'}{2x} + \frac{(2x-1)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3}{2x}, P_3(x) = \frac{2x-1}{2x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{3}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{2}$

- $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $2x^2\left(\frac{d}{dx}y'\right) + 3xy' + (2x-1)\,y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$
    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$
    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion
    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions
  $a_0(1+r)\,(-1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r-1) + 2a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{-1, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $2(k+r+1)\left(k+r-\frac{1}{2}\right) a_k + 2a_{k-1} = 0$
- Shift index using $k->k+1$
  $2(k+2+r)\left(k+\frac{1}{2}+r\right) a_{k+1} + 2a_k = 0$
- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{2a_k}{(k+2+r)(2k+1+2r)}$

- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}, b_{k+1} = -\frac{2b_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

1.750.3 Maple trace

```
Methods for second order ODEs:
```

1.750.4 Maple dsolve solution

Solving time : `0.097` (sec)
Leaf size : 73

```
dsolve(2*x^2*diff(diff(y(x),x),x)+3*x*diff(y(x),x)+(2*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2 \sqrt{\frac{(2\sqrt{x}-i)(4x+1)}{2\sqrt{x}+i}}\, \mathrm{e}^{-2i\sqrt{x}} + c_1 \sqrt{\frac{(2\sqrt{x}+i)(4x+1)}{2\sqrt{x}-i}}\, \mathrm{e}^{2i\sqrt{x}}}{x}$$

1.750.5 Mathematica DSolve solution

Solving time : 0.13 (sec)
Leaf size : 64

```
DSolve[{2*x^2*D[y[x],{x,2}]+3*x*D[y[x],x]+(2*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-2i\sqrt{x}}\left(8c_1e^{4i\sqrt{x}}\left(2\sqrt{x}+i\right)+c_2\left(1+2i\sqrt{x}\right)\right)}{8x}$$

1.751 problem 773

Internal problem ID [8889]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 773
Date solved : `Monday, October 21, 2024 at 05:23:24 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + 2y' - xy = 0$$

1.751.1 Solved as second order ode using Kovacic algorithm

Time used: 0.085 (sec)

Writing the ode as

$$xy'' + 2y' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1426: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-x}}{x}\right) + c_2 \left(\frac{e^{-x}}{x} \left(\frac{e^{2x}}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.751.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 2y' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = y - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2y'}{x} - y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -1\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + 2y' - xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - Shift index using $k- > k-1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$$

  - Shift index using $k- > k+1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1) x^{k+r}$$

  - Convert $x \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

    $$x \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r) (k+r-1) x^{k+r-1}$$

  - Shift index using $k- > k+1$

    $$x \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1) (k+r) x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r) x^{-1+r} + a_1 (1+r) (2+r) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1) (k+2+r) - a_{k-1}) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term must be 0

  $a_1(1+r)(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r) - a_{k-1} = 0$
- Shift index using $k- > k+1$

  $a_{k+2}(k+2+r)(k+3+r) - a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = \frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = \frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

1.751.3 Maple trace

```
Methods for second order ODEs:
```

1.751.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sinh(x) + c_2 \cosh(x)}{x}$$

1.751.5 Mathematica DSolve solution

Solving time : `0.039` (sec)
Leaf size : 28

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-x} + c_2 e^x}{2x}$$

1.752 problem 774

Internal problem ID [8890]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 774
Date solved : `Monday, October 21, 2024 at 05:23:25 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

1.752.1 Solved as second order ode using Kovacic algorithm

Time used: 0.165 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1428: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(x)}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.752.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx}y'\right) + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

$$a_1(3+2r)\,(1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$$

- Shift index using $k- >k + 2$

$$a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

1.752.3 Maple trace

```
Methods for second order ODEs:
```

1.752.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1 \sin(x)+c_2 \cos(x)}{\sqrt{x}}$$

1.752.5 Mathematica DSolve solution

Solving time : 0.051 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{e^{-ix}(2c_1-ic_2e^{2ix})}{2\sqrt{x}}$$

1.753 problem 775

Internal problem ID [8891]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 775
Date solved : `Monday, October 21, 2024 at 05:23:26 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (x - 6)\,y' - 3y = 0$$

1.753.1 Solved as second order ode using Kovacic algorithm

Time used: 0.283 (sec)

Writing the ode as

$$xy'' + (x - 6)\,y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x - 6 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 48}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 48$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 48}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1430: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{12}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{12}{x^2} - \frac{144}{x^4} + \frac{3456}{x^6} - \frac{103680}{x^8} + \frac{3483648}{x^{10}} - \frac{125411328}{x^{12}} + \frac{4729798656}{x^{14}} + \dots \quad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \quad (10)$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 48}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{12}{x^2}\right) \\ &= \frac{1}{4} + \frac{12}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 48}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= 0 - (-3) \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{x} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{3}{x} - \frac{1}{2} \\
&= -\frac{6 + x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(6x + 2a_2) + 2\left(-\frac{3}{x} - \frac{1}{2}\right)\left(3x^2 + 2a_2 x + a_1\right) + \left(\left(\frac{3}{x^2}\right) + \left(-\frac{3}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 + 48}{4x^2}\right)\right) = 0
$$

$$
\frac{(a_2 - 12)\,x^2 + 2(a_1 - 5a_2)\,x + 3a_0 - 6a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 120, a_1 = 60, a_2 = 12\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^3 + 12x^2 + 60x + 120
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^3 + 12x^2 + 60x + 120\right) e^{\int \left(-\frac{3}{x} - \frac{1}{2}\right) dx} \\
&= \left(x^3 + 12x^2 + 60x + 120\right) e^{-\frac{x}{2} - 3\ln(x)} \\
&= \frac{\left(x^3 + 12x^2 + 60x + 120\right) e^{-\frac{x}{2}}}{x^3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x-6}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}+3\ln(x)} \\
&= z_1 \left(x^3 \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}\left(x^3 + 12x^2 + 60x + 120\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x-6}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+6\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(x^3 - 12x^2 + 60x - 120)\,\mathrm{e}^{-x+6\ln(x)}\mathrm{e}^{2x}}{(x^3 + 12x^2 + 60x + 120)\,x^6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\left(x^3 + 12x^2 + 60x + 120\right)\right) \\
&\quad + c_2\left(\mathrm{e}^{-x}\left(x^3 + 12x^2 + 60x + 120\right)\left(\frac{(x^3 - 12x^2 + 60x - 120)\,\mathrm{e}^{-x+6\ln(x)}\mathrm{e}^{2x}}{(x^3 + 12x^2 + 60x + 120)\,x^6}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.753.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right)+\left(x-6\right)y'-3y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=\frac{3y}{x}-\frac{(x-6)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(x-6)y'}{x}-\frac{3y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{x-6}{x}, P_3(x)=-\frac{3}{x}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-6$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

- ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$x\left(\frac{d}{dx}y'\right)+\left(x-6\right)y'-3y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

$x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-7+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k-6+r) + a_k(k+r-3)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-7+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 7\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-6+r) + a_k(k+r-3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-3)}{(k+1+r)(k-6+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 3$

  $a_{k+1} = -\frac{a_k(k-3)}{(k+1)(k-6)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{a_0}{2}$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{5}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{10}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2}{12}$
- Express in terms of a_0

  $a_3 = -\frac{a_0}{120}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right)$
- Recursion relation for $r = 7$

  $a_{k+1} = -\frac{a_k(k+4)}{(k+8)(k+1)}$
- Solution for $r = 7$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+7}, a_{k+1} = -\frac{a_k(k+4)}{(k+8)(k+1)}\right]$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot \left(1-\frac{1}{2}x+\frac{1}{10}x^2-\frac{1}{120}x^3\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+7}\right), b_{k+1} = -\frac{b_k(k+4)}{(k+8)(k+1)}\right]$$

1.753.3 Maple trace

```
Methods for second order ODEs:
```

1.753.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 39

```
dsolve(x*diff(diff(y(x),x),x)+(x-6)*diff(y(x),x)-3*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1\left(x^3-12x^2+60x-120\right)+c_2\,\mathrm{e}^{-x}\left(x^3+12x^2+60x+120\right)$$

1.753.5 Mathematica DSolve solution

Solving time : `0.116` (sec)
Leaf size : 98

```
DSolve[{x*D[y[x],{x,2}]+(x-6)*D[y[x],x]-3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2e^{-x/2}\sqrt{x}\left((c_1x^3+12ic_2x^2+60c_1x+120ic_2)\cosh\left(\frac{x}{2}\right)-(12c_1(x^2+10)+ic_2x(x^2+60))\sinh\left(\frac{x}{2}\right)\right)}{\sqrt{\pi}\sqrt{-ix}}$$

1.754 problem 776

Internal problem ID [8892]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 776
Date solved : `Monday, October 21, 2024 at 05:23:27 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$x^4y'' + \lambda y = 0$$

1.754.1 Solved as second order ode using Kovacic algorithm

Time used: 0.256 (sec)

Writing the ode as

$$x^4y'' + \lambda y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= 0 \\ C &= \lambda \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\lambda}{x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\lambda \\ t &= x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{\lambda}{x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1432: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{\lambda}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i\sqrt{\lambda}}{x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{i\sqrt{\lambda}}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = i\sqrt{\lambda}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{i\sqrt{\lambda}}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{0}{i\sqrt{\lambda}} + 2\right) &= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{0}{i\sqrt{\lambda}} + 2\right) &= 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{\lambda}{x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{i\sqrt{\lambda}}{x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x} + (-)\,(0) \\ &= -\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x} \\ &= \frac{-i\sqrt{\lambda} + x}{x^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{2i\sqrt{\lambda}}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right)^2 - \left(-\frac{\lambda}{x^4}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right) dx} \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \int \frac{1}{x^2 \mathrm{e}^{\frac{2i\sqrt{\lambda}}{x}}}\,dx \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \left(-\frac{i\mathrm{e}^{-\frac{2i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}\right) + c_2\left(x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}\left(-\frac{i\mathrm{e}^{-\frac{2i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.754.2 Maple step by step solution

1.754.3 Maple trace

```
Methods for second order ODEs:
```

1.754.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 31

```
dsolve(x^4*diff(diff(y(x),x),x)+lambda*y(x) = 0,
       y(x),singsol=all)
```

$$y = x\left(c_1 \sinh\left(\frac{\sqrt{-\lambda}}{x}\right) + c_2 \cosh\left(\frac{\sqrt{-\lambda}}{x}\right)\right)$$

1.754.5 Mathematica DSolve solution

Solving time : 0.125 (sec)
Leaf size : 52

```
DSolve[{x^4*D[y[x],{x,2}]+\[Lambda]*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x e^{\frac{i\sqrt{\lambda}}{x}} - \frac{i c_2 x e^{-\frac{i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}$$

1.755 problem 777

Internal problem ID [8893]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 777
Date solved : `Monday, October 21, 2024 at 05:23:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + 4xy' + \left(4x^2 - 25\right) y = 0$$

1.755.1 Solved as second order ode using Kovacic algorithm

Time used: 0.316 (sec)

Writing the ode as

$$4x^2y'' + 4xy' + \left(4x^2 - 25\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 4x \\ C &= 4x^2 - 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 6$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1433: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= -1 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 + 6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + (-)(i) \\ &= -\frac{2}{x} - i \\ &= -\frac{2}{x} - i\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2 + 6}{x^2}\right)\right) &= 0 \\ \frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3, a_1 = -3i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 3ix - 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= (x^2 - 3ix - 3)\, e^{\int \left(-\frac{2}{x} - i\right) dx} \\ &= (x^2 - 3ix - 3)\, e^{-2\ln(x) - ix} \\ &= \frac{(x^2 - 3ix - 3)\, e^{-ix}}{x^2}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{4x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}\right) + c_2\left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{5/2}}\left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.755.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right)+4xy'+\left(4x^2-25\right)y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{(4x^2-25)y}{4x^2}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{y'}{x}+\frac{(4x^2-25)y}{4x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{4x^2-25}{4x^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\Big|_{x=0}=1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=-\frac{25}{4}$

- ○ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$4x^2\left(\frac{d}{dx}y'\right)+4xy'+\left(4x^2-25\right)y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ○ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x\cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(5+2r)\,(-5+2r)\, x^r + a_1(7+2r)\,(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+5)\,(2k+2r-5) + 4a_{\cdot}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+2r)\,(-5+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{5}{2}, \frac{5}{2}\right\}$
- Each term must be 0

  $a_1(7+2r)\,(-3+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r+5)\,(2k+2r-5) + 4a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(2k+9+2r)\,(2k-1+2r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$
- Recursion relation for $r = -\frac{5}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}$
- Solution for $r = -\frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0\right]$$
- Recursion relation for $r = \frac{5}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}$
- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}, a_1 = 0\right]$$
- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}, a_1=0, b_{k+2}=-\frac{4b_k}{(2k+14)(2k+4)}, b_1=0\right]$$

1.755.3 Maple trace

```
Methods for second order ODEs:
```

1.755.4 Maple dsolve solution

Solving time : `0.023` (sec)
Leaf size : 43

```
dsolve(4*x^2*diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(4*x^2-25)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{-3c_2\left(ix-\frac{1}{3}x^2+1\right)\mathrm{e}^{-ix}+3\left(ix+\frac{1}{3}x^2-1\right)\mathrm{e}^{ix}c_1}{x^{5/2}}$$

1.755.5 Mathematica DSolve solution

Solving time : `0.114` (sec)
Leaf size : 59

```
DSolve[{4*x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2-25)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\frac{\sqrt{\frac{2}{\pi}}((-c_2x^2+3c_1x+3c_2)\cos(x)+(c_1(x^2-3)+3c_2x)\sin(x))}{x^{5/2}}$$

1.756 problem 778

Internal problem ID [8894]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 778
Date solved : `Monday, October 21, 2024 at 05:23:28 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right) y = 0$$

1.756.1 Solved as second order ode using Kovacic algorithm

Time used: 0.174 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 36x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-36}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -36$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -36z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1435: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -36$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(6x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx}\\ &= z_1 e^{-\frac{\ln(x)}{2}}\\ &= z_1\left(\frac{1}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(6x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\tan(6x)}{6}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(6x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(6x)}{\sqrt{x}} \left(\frac{\tan(6x)}{6}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.756.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(144x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{x} + \frac{(144x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{144x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 4xy' + (144x^2 - 1)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- > k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

  - Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 144a\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 144a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 144a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{144a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{144a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{144a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{144a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{144a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{144a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{144b_k}{4k^2+20k+24}, b_1 = 0\right]$$

1.756.3 Maple trace

```
Methods for second order ODEs:
```

1.756.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(36*x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(6x) + c_2 \cos(6x)}{\sqrt{x}}$$

1.756.5 Mathematica DSolve solution

Solving time : `0.061` (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(36*x^2-1/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-6ix}(12c_1 - ic_2 e^{12ix})}{12\sqrt{x}}$$

1.757 problem 779

Internal problem ID [8895]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 779
Date solved : `Monday, October 21, 2024 at 05:23:29 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + (x^2 - 2)\, y = 0$$

1.757.1 Solved as second order ode using Kovacic algorithm

Time used: 0.286 (sec)

Writing the ode as

$$x^2y'' + (x^2 - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1437: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{x^2} - \frac{i}{2x^4} - \frac{i}{2x^6} - \frac{5i}{8x^8} - \frac{7i}{8x^{10}} - \frac{21i}{16x^{12}} - \frac{33i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{2}{x^2}\right) \\ &= -1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)(i) \\
&= -\frac{1}{x} - i \\
&= -\frac{1}{x} - i
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - i\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - i\right)^2 - \left(\frac{-x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{2ia_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -i\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - i
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x - i)\, \mathrm{e}^{\int \left(-\frac{1}{x} - i\right) dx} \\
&= (x - i)\, \mathrm{e}^{-\ln(x) - ix} \\
&= \frac{(x - i)\, \mathrm{e}^{-ix}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \frac{(x - i)\,\mathrm{e}^{-ix}}{x}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x - i)\,\mathrm{e}^{-ix}}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \frac{(x - i)\,\mathrm{e}^{-ix}}{x} \int \frac{1}{\frac{(x-i)^2 \mathrm{e}^{-2ix}}{x^2}}\,dx \\
&= \frac{(x - i)\,\mathrm{e}^{-ix}}{x} \left(\frac{(ix - 1)\,\mathrm{e}^{2ix}}{-2x + 2i}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x - i)\,\mathrm{e}^{-ix}}{x}\right) + c_2 \left(\frac{(x - i)\,\mathrm{e}^{-ix}}{x} \left(\frac{(ix - 1)\,\mathrm{e}^{2ix}}{-2x + 2i}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.757.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right)+\left(x^2-2\right)y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{\left(x^2-2\right)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{\left(x^2-2\right)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ○ Define functions

$\left[P_2(x)=0, P_3(x)=\frac{x^2-2}{x^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=0$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-2$

- ○ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2\left(\frac{d}{dx}y'\right)+\left(x^2-2\right)y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ○ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- ○ Convert $x^2\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+r)(-2+r)\, x^r + a_1(2+r)(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+1)(k+r-2) + a_{k-2})\, x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(1+r)(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 2\}$
- Each term must be 0
  $a_1(2+r)(-1+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+1)(k+r-2) + a_{k-2} = 0$
- Shift index using $k- > k+2$
  $a_{k+2}(k+3+r)(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{(k+3+r)(k+r)}$
- Recursion relation for $r = -1$
  $a_{k+2} = -\frac{a_k}{(k+2)(k-1)}$
- Solution for $r = -1$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+2)(k-1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$
  $a_{k+2} = -\frac{a_k}{(k+5)(k+2)}$
- Solution for $r = 2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+5)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k-1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+5)(k+2)}, b_1 = 0\right]$

1.757.3 Maple trace

```
Methods for second order ODEs:
```

1.757.4 Maple dsolve solution

Solving time : 0.011 (sec)
Leaf size : 27

```
dsolve(x^2*diff(diff(y(x),x),x)+(x^2-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(c_1 x + c_2) \cos(x) + \sin(x) (c_2 x - c_1)}{x}$$

1.757.5 Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 21

```
DSolve[{x^2*D[y[x],{x,2}]+(x^2-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x(c_1 j_1(x) - c_2 y_1(x))$$

1.758 problem 780

Internal problem ID [8896]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 780
Date solved : `Monday, October 21, 2024 at 05:23:30 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$xy'' + 3y' + x^3y = 0$$

1.758.1 Solved as second order ode using Kovacic algorithm

Time used: 0.290 (sec)

Writing the ode as

$$xy'' + 3y' + x^3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4x^4 + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1439: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 4\\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx ix - \frac{3i}{8x^3} - \frac{9i}{128x^7} - \frac{27i}{1024x^{11}} - \frac{405i}{32768x^{15}} - \frac{1701i}{262144x^{19}} - \frac{15309i}{4194304x^{23}} - \frac{72171i}{33554432x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= ix \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-4x^4 + 3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(-x^2\right) + \left(\frac{3}{4x^2}\right) \\
&= -x^2 + \frac{3}{4x^2}
\end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= ix \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-4x^4 + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	ix	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(ix) \\
&= -\frac{1}{2x} - ix \\
&= -\frac{1}{2x} - ix
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - ix\right)(0) + \left(\left(\frac{1}{2x^2} - i\right) + \left(-\frac{1}{2x} - ix\right)^2 - \left(\frac{-4x^4 + 3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - ix\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{i x^2}{2}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3}{x} \, dx} \\
&= z_1 e^{-\frac{3 \ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{i x^2}{2}}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{i e^{i x^2}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-\frac{i x^2}{2}}}{x^2}\right) + c_2 \left(\frac{e^{-\frac{i x^2}{2}}}{x^2} \left(-\frac{i e^{i x^2}}{2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.758.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 3y' + x^3 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -x^2 y - \frac{3y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{3y'}{x} + x^2 y = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{3}{x}, P_3(x) = x^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $$x\left(\frac{d}{dx}y'\right) + 3y' + x^3 y = 0$$
- Assume series solution for y
  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - Convert $x^3 \cdot y$ to series expansion
    $$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$
  - Shift index using $k- >k-3$
    $$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$
  - Convert y' to series expansion
    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$
  - Shift index using $k- >k+1$
    $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
    $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-1}$$
  - Shift index using $k- >k+1$
    $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^r + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 0\}$
- The coefficients of each power of x must be 0
  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$
- Solve for the dependent coefficient(s)
  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k+r+3) + a_{k-3} = 0$
- Shift index using $k- >k+3$
  $a_{k+4}(k+4+r)\,(k+6+r) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+4} = -\frac{a_k}{(k+4+r)(k+6+r)}$
- Recursion relation for $r = -2$
  $a_{k+4} = -\frac{a_k}{(k+2)(k+4)}$
- Solution for $r = -2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Recursion relation for $r = 0$
  $a_{k+4} = -\frac{a_k}{(k+4)(k+6)}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+4} = -\frac{a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{b_k}{(k+4)(k+6)}\right.$

1.758.3 Maple trace

```
Methods for second order ODEs:
```

1.758.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 25

```
dsolve(x*diff(diff(y(x),x),x)+3*diff(y(x),x)+x^3*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(\frac{x^2}{2}\right) + c_2 \cos\left(\frac{x^2}{2}\right)}{x^2}$$

1.758.5 Mathematica DSolve solution

Solving time : 0.075 (sec)
Leaf size : 43

```
DSolve[{x*D[y[x],{x,2}]+3*D[y[x],x]+x^3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{ix^2}{2}}\left(2c_1 - ic_2 e^{ix^2}\right)}{2x^2}$$

1.759 problem 781

Internal problem ID [8897]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 781
Date solved : `Monday, October 21, 2024 at 05:23:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 4xy' + (x^2 + 2) y = 0$$

1.759.1 Solved as second order ode using Kovacic algorithm

Time used: 0.150 (sec)

Writing the ode as

$$x^2y'' + 4xy' + (x^2 + 2) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1441: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\, dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1 \left(\frac{1}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(x)}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x^2}\right) + c_2 \left(\frac{\cos(x)}{x^2}(\tan(x))\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.759.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + 4xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+2)y}{x^2} - \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{4y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 4$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + 4xy' + (x^2+2)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - Shift index using $k- >k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(1+r)\, x^r + a_1(3+r)\,(2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(k+r+1) + a_{k-2})\, x^{k+r}\right) = 0$$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$
- Each term must be 0

  $a_1(3+r)\,(2+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)\,(k+r+1) + a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+4+r)\,(k+3+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+4+r)(k+3+r)}$
- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r = -2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$$

- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$
- Solution for $r=-1$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$$
- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$$

1.759.3 Maple trace

```
Methods for second order ODEs:
```

1.759.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1 \sin(x)+c_2 \cos(x)}{x^2}$$

1.759.5 Mathematica DSolve solution

Solving time : `0.045` (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{2c_1 e^{-ix}-ic_2 e^{ix}}{2x^2}$$

1.760 problem 782

Internal problem ID [8898]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 782
Date solved : `Monday, October 21, 2024 at 05:23:32 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$16x^2y'' + 32xy' + \left(x^4 - 12\right) y = 0$$

1.760.1 Solved as second order ode using Kovacic algorithm

Time used: 0.289 (sec)

Writing the ode as

$$16x^2y'' + 32xy' + \left(x^4 - 12\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^2 \\ B &= 32x \\ C &= x^4 - 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 + 12}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^4 + 12$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 + 12}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1443: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{x^2}{16} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{ix}{4} - \frac{3i}{2x^3} - \frac{9i}{2x^7} - \frac{27i}{x^{11}} - \frac{405i}{2x^{15}} - \frac{1701i}{x^{19}} - \frac{15309i}{x^{23}} - \frac{144342i}{x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{ix}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{-x^4 + 12}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(-\frac{x^2}{16}\right) + \left(\frac{3}{4x^2}\right) \\ &= -\frac{x^2}{16} + \frac{3}{4x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned}b &= (0) - (0) \\ &= 0\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{ix}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{i}{4}} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{i}{4}} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^4 + 12}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{ix}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)\left(\frac{ix}{4}\right) \\
&= -\frac{1}{2x} - \frac{ix}{4} \\
&= -\frac{1}{2x} - \frac{ix}{4}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - \frac{ix}{4}\right)(0) + \left(\left(\frac{1}{2x^2} - \frac{i}{4}\right) + \left(-\frac{1}{2x} - \frac{ix}{4}\right)^2 - \left(\frac{-x^4 + 12}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{ix}{4}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{ix^2}{8}}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{32x}{16x^2} \, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{ix^2}{8}}}{x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{32x}{16x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-2i e^{\frac{ix^2}{4}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-\frac{ix^2}{8}}}{x^{3/2}}\right) + c_2 \left(\frac{e^{-\frac{ix^2}{8}}}{x^{3/2}} \left(-2i e^{\frac{ix^2}{4}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.760.2 Maple step by step solution

Let's solve

$16x^2 \left(\frac{d}{dx} y'\right) + 32xy' + (x^4 - 12)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(x^4-12)y}{16x^2} - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{2y'}{x} + \frac{(x^4-12)y}{16x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = \frac{x^4-12}{16x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{3}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

• Multiply by denominators

  $16x^2\left(\frac{d}{dx}y'\right) + 32xy' + \left(x^4 - 12\right)y = 0$

• Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..4$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $4a_0(3+2r)(-1+2r)\, x^r + 4a_1(5+2r)(1+2r)\, x^{1+r} + 4a_2(7+2r)(3+2r)\, x^{2+r} + 4a_3(9+2r)\,($

• $a_0$cannot be 0 by assumption, giving the indicial equation

  $4(3+2r)(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{3}{2}, \frac{1}{2}\right\}$
- The coefficients of each power of x must be 0

  $[4a_1(5+2r)(1+2r) = 0, 4a_2(7+2r)(3+2r) = 0, 4a_3(9+2r)(5+2r) = 0]$
- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation

  $16\left(k+r-\frac{1}{2}\right)\left(k+r+\frac{3}{2}\right)a_k + a_{k-4} = 0$
- Shift index using $k- > k+4$

  $16\left(k+\frac{7}{2}+r\right)\left(k+\frac{11}{2}+r\right)a_{k+4} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{a_k}{4(2k+7+2r)(2k+11+2r)}$
- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+4} = -\frac{a_k}{4(2k+4)(2k+8)}$
- Solution for $r = -\frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+4} = -\frac{a_k}{4(2k+4)(2k+8)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+4} = -\frac{a_k}{4(2k+8)(2k+12)}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+4} = -\frac{a_k}{4(2k+8)(2k+12)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+4} = -\frac{a_k}{4(2k+4)(2k+8)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{}{4(2k-}\right.$

1.760.3 Maple trace

```
Methods for second order ODEs:
```

1.760.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 25

```
dsolve(16*x^2*diff(diff(y(x),x),x)+32*x*diff(y(x),x)+(x^4-12)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(\frac{x^2}{8}\right) + c_2 \cos\left(\frac{x^2}{8}\right)}{x^{3/2}}$$

1.760.5 Mathematica DSolve solution

Solving time : 0.089 (sec)
Leaf size : 42

```
DSolve[{16*x^2*D[y[x],{x,2}]+32*x*D[y[x],x]+(x^4-12)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{ix^2}{8}}\left(c_1 - 2ic_2 e^{\frac{ix^2}{4}}\right)}{x^{3/2}}$$

1.761 problem 783

Internal problem ID [8899]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 783
Date solved : `Monday, October 21, 2024 at 05:23:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2 y' + xy = 0$$

1.761.1 Solved as second order ode using Kovacic algorithm

Time used: 0.378 (sec)

Writing the ode as

$$y'' - x^2 y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x(x^3 - 8)}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x(x^3 - 8) \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x(x^3 - 8)}{4}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1445: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 4 \\
&= -4
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{2} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{x^2}{2} - \frac{2}{x} - \frac{4}{x^4} - \frac{16}{x^7} - \frac{80}{x^{10}} - \frac{448}{x^{13}} - \frac{2688}{x^{16}} - \frac{16896}{x^{19}} + \dots \qquad (9)
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 2$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\
&= \frac{x^2}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x(x^3-8)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^4 - 2x\right) + (0) \\
&= \frac{1}{4}x^4 - 2x
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -2. Now b can be found.

$$\begin{aligned}
b &= (-2) - (0) \\
&= -2
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^2}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 2\right) &&= -3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 2\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3-8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	-3	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^2}{2}\right) \\ &= -\frac{x^2}{2} \\ &= -\frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x^2}{2}\right)(1) + \left((-x) + \left(-\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3-8)}{4}\right)\right) = 0$$

$$x a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, \mathrm{e}^{\int -\frac{x^2}{2} dx} \\ &= (x) \, \mathrm{e}^{-\frac{x^3}{6}} \\ &= x \, \mathrm{e}^{-\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x^2}{1} \, dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^3}{6}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{3^{2/3}(-1)^{1/3} \left(-\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}} + \frac{3\, 3^{1/3}(-1)^{2/3} \mathrm{e}^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{2/3}} \right)}{9} \right) \end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1(x) + c_2\left(x\left(\frac{3^{2/3}(-1)^{1/3}\left(-\frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}\right)}{(-x^3)^{2/3}} + \frac{3\,3^{1/3}(-1)^{2/3}e^{\frac{x^3}{3}}}{x} + \frac{3x^2(-1)^{2/3}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{2/3}}\right)}{9}\right)\right)$$

Will add steps showing solving for IC soon.

1.761.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - x^2y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k - 1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

  - Shift index using $k- >k - 1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k + 2$

$\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite ODE with series expansions

$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_{k-1}(k-2))x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)a_{k+2} - a_{k-1}(k-2) = 0$

- Shift index using $k- > k+1$

  $\left((k+1)^2 + 3k + 5\right)a_{k+3} - a_k(k-1) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k(k-1)}{k^2+5k+6}, 2a_2 = 0\right]$

1.761.3 Maple trace

```
Methods for second order ODEs:
```

1.761.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x^2*diff(y(x),x)+x*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2(-x^3)^{1/3}\,3^{2/3}\Gamma\left(\frac{2}{3}\right) - c_2(-x^3)^{1/3}\,3^{2/3}\Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) + 3\,\mathrm{e}^{\frac{x^3}{3}}c_2 + c_1 x$$

1.761.5 Mathematica DSolve solution

Solving time : 0.082 (sec)
Leaf size : 41

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]+x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x - \frac{c_2 \sqrt[3]{-x^3}\,\Gamma\left(-\frac{1}{3}, -\frac{x^3}{3}\right)}{3\sqrt[3]{3}}$$

1.762 problem 784

Internal problem ID [8900]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 784
Date solved : Monday, October 21, 2024 at 05:23:34 PM
CAS classification : [_Laguerre]

Solve

$$xy'' - (x + 2) y' + 2y = 0$$

1.762.1 Solved as second order ode using Kovacic algorithm

Time used: 0.237 (sec)

Writing the ode as

$$xy'' + (-x - 2) y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x - 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1447: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -1$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - (\alpha_{c_1}^{-}) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{x} \\
&= \frac{x-2}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x-2}{x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \ln(x)} \\
&= z_1 \left(x \, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-2}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(x^2+2x+2)\,\mathrm{e}^{x+2\ln(x)}\mathrm{e}^{-2x}}{x^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-\frac{(x^2+2x+2)\,\mathrm{e}^{x+2\ln(x)}\mathrm{e}^{-2x}}{x^2}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.762.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) - (x+2)\,y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2y}{x} + \frac{(x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x+2)y'}{x} + \frac{2y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x+2}{x}, P_3(x) = \frac{2}{x}\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$
  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -2$$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$
- $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $x\left(\frac{d}{dx}y'\right) + (-x-2)\,y' + 2y = 0$
- Assume series solution for y
  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$
  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$
- Shift index using $k- >k+1-m$
  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$
- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$
- Shift index using $k- >k+1$
  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions
  $$a_0 r(-3+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-2) - a_k(k+r-2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation
  $(k+r-2)\left(a_{k+1}(k+1+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=3$
  $a_{k+1}=\frac{a_k}{k+4}$
- Solution for $r=3$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1}=\frac{a_k}{k+4}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+4}\right]$

1.762.3 Maple trace

```
Methods for second order ODEs:
```

1.762.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 19

```
dsolve(x*diff(diff(y(x),x),x)-(x+2)*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y=c_1\,\mathrm{e}^{x}+c_2\left(x^2+2x+2\right)$$

1.762.5 Mathematica DSolve solution

Solving time : 0.048 (sec)
Leaf size : 24

```
DSolve[{x*D[y[x],{x,2}]-(x+2)*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2 (x^2 + 2x + 2)$$

1.763 problem 785

Internal problem ID [8901]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 785
Date solved : `Monday, October 21, 2024 at 05:23:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + xy' + 2y = 0$$

1.763.1 Solved as second order ode using Kovacic algorithm

Time used: 0.240 (sec)

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 6}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 6 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1449: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, e^{\int -\frac{x}{2} dx} \\ &= (x) \, e^{-\frac{x^2}{4}} \\ &= x \, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(e^{-\frac{x^2}{4}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{x^2}{2}} x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}} x\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} x \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.763.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

1.763.3 Maple trace

```
Methods for second order ODEs:
```

1.763.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = -x\left(c_2\pi\ \text{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)-c_1\right)\text{e}^{-\frac{x^2}{2}}+i\sqrt{\pi}\,\sqrt{2}\,c_2$$

1.763.5 Mathematica DSolve solution

Solving time : `0.073` (sec)
Leaf size : 69

```
DSolve[{D[y[x],{x,2}]+x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to -\sqrt{\frac{\pi}{2}}c_2e^{-\frac{x^2}{2}}\sqrt{x^2}\text{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right)+\sqrt{2}c_1e^{-\frac{x^2}{2}}x+c_2$$

1.764 problem 786

Internal problem ID [8902]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 786
Date solved : `Monday, October 21, 2024 at 05:23:36 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-2 x y'+2 y=0$$

1.764.1 Solved as second order ode using Kovacic algorithm

Time used: 0.262 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-2 x y'+2 y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -2 x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2 A} d x}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{2x^2 - 3}{(x^2 - 1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 2x^2 - 3 \\
t &= (x^2 - 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{2x^2 - 3}{(x^2 - 1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1451: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{4(x-1)} - \frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2} - \frac{5}{4(x+1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2-3}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= 2-(1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(x+1)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)^2 - \left(\frac{2x^2-3}{(x^2-1)^2}\right)\right.
$$

$$
-\frac{2a_0}{x^2-1} =
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= (x) \sqrt{(x-1)(x+1)} \\
&= x\sqrt{x^2-1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\ln(x+1)}{2}+\frac{1}{x}+\frac{\ln(x-1)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}\right) + c_2 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{x+1}}\left(-\frac{\ln(x+1)}{2}+\frac{1}{x}+\frac{\ln(x-1)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.764.2 Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d}{dx}y'\right)-2xy'+2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{2y}{x^2-1}-\frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{2xy'}{x^2-1}-\frac{2y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{2}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=1$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d}{dx}y'\right)+2xy'-2y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+2)\,(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-2a_{k+1}(k+1)^2 + a_k(k+2)\,(k-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$$a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$$

- Apply recursion relation for $k = 0$

$$a_1 = -a_0$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot (-u+1)$$

- Revert the change of variables $u = x+1$

$$[y = -a_0x]$$

1.764.3 Maple trace

```
Methods for second order ODEs:
```

1.764.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 25

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\ln(x-1)\, c_2 x}{2} - \frac{\ln(x+1)\, c_2 x}{2} + c_1 x + c_2$$

1.764.5 Mathematica DSolve solution

Solving time : 0.03 (sec)
Leaf size : 33

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

1.765 problem 787

Internal problem ID [8903]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 787
Date solved : `Monday, October 21, 2024 at 05:23:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0$$

1.765.1 Solved as second order ode using Kovacic algorithm

Time used: 0.084 (sec)

Writing the ode as

$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1453: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\ &= z_1 e^{x^2} \\ &= z_1 \left(e^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{x^2}\right) + c_2\left(\mathrm{e}^{x^2}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.765.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - 4xy' + (4x^2 - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ◦ Shift index using $k- > k - m$

$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ◦ Convert $x \cdot y'$ to series expansion

$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ◦ Convert $\frac{d}{dx}y'$ to series expansion

$\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ◦ Shift index using $k- > k + 2$

$\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 - 2a_0 + (6a_3 - 6a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2})\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = a_0, a_3 = a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$
- Shift index using $k- >k+2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2ka_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$

1.765.3 Maple trace

```
Methods for second order ODEs:
```

1.765.4 Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)-4*x*diff(y(x),x)+(4*x^2-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{x^2}(c_2 x + c_1)$$

1.765.5 Mathematica DSolve solution

Solving time : `0.03` (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]-4*x*D[y[x],x]+(4*x^2-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x^2}(c_2 x + c_1)$$

1.766 problem 788

Internal problem ID [8904]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 788
Date solved : `Monday, October 21, 2024 at 05:23:37 PM`
CAS classification : `[_Gegenbauer]`

Solve

$$\left(-x^2+1\right) y''-2xy'+30y=0$$

1.766.1 Solved as second order ode using Kovacic algorithm

Time used: 0.319 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''-2xy'+30y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= -2x \\ C &= 30 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{30x^2 - 31}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 30x^2 - 31$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{30x^2 - 31}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1455: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{61}{4(x-1)} - \frac{61}{4(x+1)} - \frac{1}{4(x+1)^2} - \frac{1}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{30x^2 - 31}{(x^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=30$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=6\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-5
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{30x^2-31}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	6	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=6$ then

$$\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= 6-(1)\\
&= 5
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\left(20x^3 + 12x^2a_4 + 6xa_3 + 2a_2\right) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)\left(5x^4 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(-\frac{1}{2(x -
$$

$$
-10a_4x^4 + (-18a_3 - 20
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = 0, a_1 = \frac{5}{21}, a_2 = 0, a_3 = -\frac{10}{9}, a_4 = 0\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right) e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= \left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right) \sqrt{(x-1)(x+1)} \\
&= \frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{-x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{3087\left(-23x^3+\frac{935}{63}x\right)}{1600\left(x^4-\frac{10}{9}x^2+\frac{5}{21}\right)} + \frac{3969\ln(x-1)}{128} - \frac{3969\ln(x+1)}{128} + \frac{441}{25x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{x+1}}\right) \\
&\quad + c_2 \left(\frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{x+1}}\left(-\frac{3087\left(-23x^3+\frac{935}{63}x\right)}{1600\left(x^4-\frac{10}{9}x^2+\frac{5}{21}\right)} + \frac{3969\ln(x-1)}{128} \right.\right. \\
&\qquad \left.\left. - \frac{3969\ln(x+1)}{128} + \frac{441}{25x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.766.2 Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d}{dx}y'\right)-2xy'+30y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=\frac{30y}{x^2-1}-\frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'+\frac{2xy'}{x^2-1}-\frac{30y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{30}{x^2-1}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1}=1$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\left(\frac{d}{dx}y'\right)+2xy'-30y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-30y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+6)\,(k+r-5)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-2a_{k+1}(k+1)^2 + a_k(k+6)\,(k-5) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+6)(k-5)}{2(k+1)^2}$$

- Recursion relation for $r = 0$; series terminates at $k = 5$

$$a_{k+1} = \frac{a_k(k+6)(k-5)}{2(k+1)^2}$$

- Apply recursion relation for $k = 0$

$$a_1 = -15a_0$$

- Apply recursion relation for $k = 1$

$$a_2 = -\frac{7a_1}{2}$$

- Express in terms of a_0

$$a_2 = \frac{105a_0}{2}$$

- Apply recursion relation for $k = 2$

$$a_3 = -\frac{4a_2}{3}$$

- Express in terms of a_0

$$a_3 = -70a_0$$

- Apply recursion relation for $k = 3$

$$a_4 = -\frac{9a_3}{16}$$

- Express in terms of a_0

$$a_4 = \frac{315a_0}{8}$$

- Apply recursion relation for $k = 4$

  $a_5 = -\frac{a_4}{5}$
- Express in terms of a_0

  $a_5 = -\frac{63a_0}{8}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 15u + \frac{105}{2}u^2 - 70u^3 + \frac{315}{8}u^4 - \frac{63}{8}u^5\right)$
- Revert the change of variables $u = x + 1$

  $\left[y = a_0\left(-\frac{15}{8}x + \frac{35}{4}x^3 - \frac{63}{8}x^5\right)\right]$

1.766.3 Maple trace

```
Methods for second order ODEs:
```

1.766.4 Maple dsolve solution

Solving time : `0.064` (sec)
Leaf size : 71

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+30*y(x) = 0,
       y(x),singsol=all)
```

$$
\begin{aligned}
y = &\frac{21c_2 x\left(x^4 - \frac{10}{9}x^2 + \frac{5}{21}\right)\ln\left(x-1\right)}{640} - \frac{21c_2 x\left(x^4 - \frac{10}{9}x^2 + \frac{5}{21}\right)\ln\left(x+1\right)}{640} \\
&+ \frac{21c_1 x^5}{5} + \frac{21c_2 x^4}{320} - \frac{14c_1 x^3}{3} - \frac{49c_2 x^2}{960} + c_1 x + \frac{c_2}{225}
\end{aligned}
$$

1.766.5 Mathematica DSolve solution

Solving time : `0.039` (sec)
Leaf size : 76

```
DSolve[{(1-x^2)*D[y[x],{x,2}]-2*x*D[y[x],x]+30*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to &\frac{1}{8}c_1 x\left(63x^4 - 70x^2 + 15\right) \\
&+ c_2\left(-\frac{63x^4}{8} + \frac{49x^2}{8} - \frac{1}{16}\left(63x^4 - 70x^2 + 15\right)x(\log(1-x) - \log(x+1)) - \frac{8}{15}\right)
\end{aligned}
$$

1.767 problem 789

Internal problem ID [8905]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 789
Date solved : `Monday, October 21, 2024 at 05:23:38 PM`
CAS classification : `[_Lienard]`

Solve

$$xy'' + 2y' + xy = 0$$

1.767.1 Solved as second order ode using Kovacic algorithm

Time used: 0.144 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1457: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{x}\right) + c_2\left(\frac{\cos(x)}{x}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.767.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -y - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2y'}{x} + y = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$

- Shift index using $k- >k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1) x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r) x^{k+r}$

Rewrite ODE with series expansions

$$a_0 r(1+r) x^{-1+r} + a_1(1+r)(2+r) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)(k+2+r) + a_{k-1}) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r) + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$$

1.767.3 Maple trace

```
Methods for second order ODEs:
```

1.767.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{x}$$

1.767.5 Mathematica DSolve solution

Solving time : `0.04` (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

1.768 problem 790

Internal problem ID [8906]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 790
Date solved : `Monday, October 21, 2024 at 05:23:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (2x + 1)\,y' + (x + 1)\,y = 0$$

1.768.1 Solved as second order ode using Kovacic algorithm

Time used: 0.181 (sec)

Writing the ode as

$$xy'' + (2x + 1)\,y' + (x + 1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2x + 1 \\ C &= x + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1459: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x+1}{x}\, dx} \\ &= z_1 e^{-x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{-x}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x+1}{x} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-2x-\ln(x)}}{(y_1)^2} dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.768.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + (2x+1)\, y' + (x+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x+1)y}{x} - \frac{(2x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(2x+1)y'}{x} + \frac{(x+1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2x+1}{x}, P_3(x) = \frac{x+1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x\left(\frac{d}{dx}y'\right) + (2x+1)\, y' + (x+1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 + a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k(2k+2r+1) + a_{k-1}\right) x^k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

$a_1(1+r)^2 + a_0(1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 + 2a_k k + a_k + a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$a_{k+2}(k+2)^2 + 2a_{k+1}(k+1) + a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}, a_1 + a_0 = 0\right]$$

1.768.3 Maple trace

```
Methods for second order ODEs:
```

1.768.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 15

```
dsolve(x*diff(diff(y(x),x),x)+(2*x+1)*diff(y(x),x)+(x+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{-x}(c_1 + \ln(x)\, c_2)$$

1.768.5 Mathematica DSolve solution

Solving time : `0.041` (sec)
Leaf size : 19

```
DSolve[{x*D[y[x],{x,2}]+(2*x+1)*D[y[x],x]+(x+1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x}(c_2 \log(x) + c_1)$$

1.769 problem 791

Internal problem ID [8907]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 791
Date solved : `Monday, October 21, 2024 at 05:23:40 PM`
CAS classification : `[_Jacobi]`

Solve

$$2x(x-1)\,y''-(x+1)\,y'+y=0$$

1.769.1 Solved as second order ode using Kovacic algorithm

Time used: 0.204 (sec)

Writing the ode as

$$\left(2x^2-2x\right)y''+(-x-1)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^2-2x\\B&=-x-1\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 + 18x - 3}{16\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 + 18x - 3$$
$$t = 16\left(x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 + 18x - 3}{16\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1461: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4(x-1)} - \frac{3}{16x^2} + \frac{3}{4(x-1)^2} + \frac{3}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 18x - 3}{16(x^2 - x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-3x^2+18x-3}{16\left(x^2-x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}\right)\\
&= \frac{1}{4}-\left(\frac{1}{4}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4x} - \frac{1}{2(x-1)} + (-)(0) \\
&= \frac{3}{4x} - \frac{1}{2(x-1)} \\
&= \frac{x-3}{4x(x-1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4x} - \frac{1}{2(x-1)}\right)(0) + \left(\left(-\frac{3}{4x^2} + \frac{1}{2(x-1)^2}\right) + \left(\frac{3}{4x} - \frac{1}{2(x-1)}\right)^2 - \left(\frac{-3x^2+18x-3}{16(x^2-x)^2}\right)\right) =
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{3}{4x} - \frac{1}{2(x-1)}\right) dx} \\
&= \frac{x^{3/4}}{\sqrt{x-1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x-1}{2x^2-2x} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{4} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\frac{\sqrt{x-1}}{x^{1/4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x-1}{2x^2-2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2}+\ln(x-1)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{2(x+1)\,\mathrm{e}^{-\frac{\ln(x)}{2}+\ln(x-1)}}{x-1} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{x}\right) + c_2 \left(\sqrt{x} \left(\frac{2(x+1)\,\mathrm{e}^{-\frac{\ln(x)}{2}+\ln(x-1)}}{x-1} \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.769.2 Maple step by step solution

Let's solve

$2x(x-1)\left(\frac{d}{dx}y'\right) - (x+1)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{2x(x-1)} + \frac{(x+1)y'}{2x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x+1)y'}{2x(x-1)} + \frac{y}{2x(x-1)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{x+1}{2x(x-1)}, P_3(x)=\frac{1}{2x(x-1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x=0$

    $(x \cdot P_2(x))\bigg|_{x=0}=\frac{1}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0}=0$
  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$
- Multiply by denominators

  $2x(x-1)\left(\frac{d}{dx}y'\right)+(-x-1)\,y'+y=0$
- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m=0..1$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m=1..2$

    $x^m \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

    Rewrite ODE with series expansions

    $-a_0r(-1+2r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}(-a_{k+1}(k+1+r)\,(2k+1+2r)+a_k(2k+2r-1)\,(k+r-1))\,x^{k-}\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-1+2r)=0$

- Values of r that satisfy the indicial equation
  $r \in \{0, \frac{1}{2}\}$
- Each term in the series must be 0, giving the recursion relation
  $-2(k+1+r)\left(k+\frac{1}{2}+r\right)a_{k+1}+2\left(k+r-\frac{1}{2}\right)(k+r-1)\,a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{(2k+2r-1)(k+r-1)a_k}{(k+1+r)(2k+1+2r)}$
- Recursion relation for $r=0$; series terminates at $k=1$
  $a_{k+1}=\frac{(2k-1)(k-1)a_k}{(k+1)(2k+1)}$
- Apply recursion relation for $k=0$
  $a_1=a_0$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li
  $y=a_0\cdot(x+1)$
- Recursion relation for $r=\frac{1}{2}$
  $a_{k+1}=\frac{2k\left(k-\frac{1}{2}\right)a_k}{\left(k+\frac{3}{2}\right)(2k+2)}$
- Solution for $r=\frac{1}{2}$
  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+1}=\frac{2k\left(k-\frac{1}{2}\right)a_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters
  $\left[y=a_0\cdot(x+1)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), b_{k+1}=\frac{2k\left(k-\frac{1}{2}\right)b_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

1.769.3 Maple trace

```
Methods for second order ODEs:
```

1.769.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(2*x*(x-1)*diff(diff(y(x),x),x)-(x+1)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y=c_2\sqrt{x}+c_1 x+c_1$$

1.769.5 Mathematica DSolve solution

Solving time : 0.063 (sec)
Leaf size : 21

```
DSolve[{2*x*(x-1)*D[y[x],{x,2}]-(x+1)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1\sqrt{x} - 2c_2(x+1)$$

1.770 problem 792

Internal problem ID [8908]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 792
Date solved : `Monday, October 21, 2024 at 05:23:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + 2y' + 4xy = 0$$

1.770.1 Solved as second order ode using Kovacic algorithm

Time used: 0.155 (sec)

Writing the ode as

$$xy'' + 2y' + 4xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1463: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(2x)}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(2x)}{x}\right) + c_2\left(\frac{\cos(2x)}{x}\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.770.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 2y' + 4xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -4y - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2y'}{x} + 4y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 4\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + 2y' + 4xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + 4a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + 4a_{k-1} = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{4a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{4a_k}{(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{4a_k}{(k+2)(k+3)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{4a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{4a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{4b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$$

1.770.3 Maple trace

```
Methods for second order ODEs:
```

1.770.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 21

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+4*x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(2x) + c_2 \cos(2x)}{x}$$

1.770.5 Mathematica DSolve solution

Solving time : `0.046` (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+4*x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1 e^{-2ix} - ic_2 e^{2ix}}{4x}$$

1.771 problem 793

Internal problem ID [8909]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 793
Date solved : `Monday, October 21, 2024 at 05:23:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (2-2x)\,y' + (x-2)\,y = 0$$

1.771.1 Solved as second order ode using Kovacic algorithm

Time used: 0.092 (sec)

Writing the ode as

$$xy'' + (2-2x)\,y' + (x-2)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2-2x \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1465: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2-2x}{x}\,dx} \\
&= z_1 e^{x-\ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^{x}}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2-2x}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{x}\right) + c_2 \left(\frac{e^x}{x}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.771.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + (2-2x)\,y' + (x-2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x-2)y}{x} + \frac{2(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2(x-1)y'}{x} + \frac{(x-2)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(x-1)}{x}, P_3(x) = \frac{x-2}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + (2-2x)\,y' + (x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$
  - Shift index using $k->k+1$

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r) x^{-1+r} + (a_1(1+r)(2+r) - 2a_0(1+r)) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+2+r) - 2a_k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term must be 0

  $a_1(1+r)(2+r) - 2a_0(1+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+2+r) - 2a_k k - 2a_k r - 2a_k + a_{k-1} = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+3+r) - 2a_{k+1}(k+1) - 2r a_{k+1} - 2a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2k a_{k+1} + 2r a_{k+1} - a_k + 4a_{k+1}}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$

$a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}, 2a_1 - 2a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0, b_{k+2} = \frac{2kb_{k+1}-b_k+4b_{k+1}}{(k+2)(k+3)}, 2b_1 - 2b\right.$

1.771.3 Maple trace

```
Methods for second order ODEs:
```

1.771.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 15

```
dsolve(x*diff(diff(y(x),x),x)+(2-2*x)*diff(y(x),x)+(x-2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{x}(c_1 x + c_2)}{x}$$

1.771.5 Mathematica DSolve solution

Solving time : `0.039` (sec)
Leaf size : 19

```
DSolve[{x*D[y[x],{x,2}]+(2-2*x)*D[y[x],x]+(x-2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{x}(c_2 x + c_1)}{x}$$

1.772 problem 794

Internal problem ID [8910]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 794
Date solved : `Monday, October 21, 2024 at 05:23:42 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 6xy' + \left(4x^2 + 6\right) y = 0$$

1.772.1 Solved as second order ode using Kovacic algorithm

Time used: 0.164 (sec)

Writing the ode as

$$x^2y'' + 6xy' + \left(4x^2 + 6\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 6x \\ C &= 4x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1467: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6x}{x^2}\,dx} \\
&= z_1 e^{-3\ln(x)} \\
&= z_1 \left(\frac{1}{x^3}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(2x)}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-6\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(2x)}{x^3}\right) + c_2 \left(\frac{\cos(2x)}{x^3} \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.772.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + 6xy' + (4x^2+6)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2(2x^2+3)y}{x^2} - \frac{6y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{6y'}{x} + \frac{2(2x^2+3)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{6}{x}, P_3(x) = \frac{2(2x^2+3)}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 6$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 6$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) + 6xy' + (4x^2+6)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(2+r)\, x^r + a_1(4+r)\,(3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+3)\,(k+r+2) + 4a_{k-2}\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, -2\}$

- Each term must be 0

  $a_1(4+r)\,(3+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)\,(k+r+2) + 4a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $a_{k+2}(k+5+r)\,(k+4+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(k+5+r)(k+4+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}$

- Solution for $r = -3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}, a_1 = 0\right]$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{4a_k}{(k+3)(k+2)}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{4a_k}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right), a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(k+3)(k+2)}, b_1 = 0\right]$$

1.772.3 Maple trace

```
Methods for second order ODEs:
```

1.772.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+6*x*diff(y(x),x)+(4*x^2+6)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(2x) + c_2 \cos(2x)}{x^3}$$

1.772.5 Mathematica DSolve solution

Solving time : `0.05` (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]+6*x*D[y[x],x]+(4*x^2+6)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1 e^{-2ix} - ic_2 e^{2ix}}{4x^3}$$

1.773 problem 795

Internal problem ID [8911]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 795
Date solved : `Monday, October 21, 2024 at 05:23:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (1 - 2x)\, y' + (x - 1)\, y = 0$$

1.773.1 Solved as second order ode using Kovacic algorithm

Time used: 0.182 (sec)

Writing the ode as

$$xy'' + (1 - 2x)\, y' + (x - 1)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1 - 2x \\ C &= x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1469: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1-2x}{x}\, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{2}} \\ &= z_1\left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-2x}{x} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2} dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\ln(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.773.2 Maple step by step solution

Let's solve
$x\left(\frac{d}{dx}y'\right) + (1-2x)\,y' + (x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{(x-1)y}{x} + \frac{(2x-1)y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x\left(\frac{d}{dx}y'\right) + (1-2x)\,y' + (x-1)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ◦ Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- ◦ Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- ◦ Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$

- ◦ Shift index using $k->k+1$

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 - a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k+2r+1) + a_{k-1}\right) x^{k}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

$a_1(1+r)^2 - a_0(1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 + (-2k-1)\, a_k + a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$a_{k+2}(k+2)^2 + (-2k-3)\, a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$$

1.773.3 Maple trace

```
Methods for second order ODEs:
```

1.773.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 13

```
dsolve(x*diff(diff(y(x),x),x)+(1-2*x)*diff(y(x),x)+(x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^x (c_1 + \ln(x)\, c_2)$$

1.773.5 Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 17

```
DSolve[{x*D[y[x],{x,2}]+(1-2*x)*D[y[x],x]+(x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^x (c_2 \log(x) + c_1)$$

1.774 problem 796

Internal problem ID [8912]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 796
Date solved : `Monday, October 21, 2024 at 05:23:44 PM`
CAS classification : `[_Jacobi]`

Solve

$$x(1-x)\,y'' + \left(\frac{1}{2}+2x\right)y' - 2y = 0$$

1.774.1 Solved as second order ode using Kovacic algorithm

Time used: 0.282 (sec)

Writing the ode as

$$\left(-x^2+x\right)y'' + \left(\frac{1}{2}+2x\right)y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+x \\ B &= \frac{1}{2}+2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{48x - 3}{16(x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 48x - 3$$
$$t = 16(x^2 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{48x - 3}{16(x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1471: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{8x} - \frac{3}{16x^2} + \frac{45}{16(-1+x)^2} - \frac{21}{8(-1+x)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(-1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{48x - 3}{16\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} - \frac{5}{4(-1+x)} + (0) \\ &= \frac{1}{4x} - \frac{5}{4(-1+x)} \\ &= -\frac{4x+1}{4x(-1+x)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} - \frac{5}{4(-1+x)}\right)(1) + \left(\left(-\frac{1}{4x^2} + \frac{5}{4(-1+x)^2}\right) + \left(\frac{1}{4x} - \frac{5}{4(-1+x)}\right)^2 - \left(\frac{48x-3}{16(x^2-x)^2}\right)\right)$$
$$\frac{-1+4a_0}{2x(-1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{4}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{4}\right) e^{\int \left(\frac{1}{4x} - \frac{5}{4(-1+x)}\right) dx} \\
&= \left(x + \frac{1}{4}\right) e^{-\frac{5\ln(-1+x)}{4} + \frac{\ln(x)}{4}} \\
&= \frac{\left(x + \frac{1}{4}\right) x^{1/4}}{(-1+x)^{5/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{1}{2}+2x}{-x^2+x} \, dx} \\
&= z_1 e^{\frac{5\ln(-1+x)}{4} - \frac{\ln(x)}{4}} \\
&= z_1 \left(\frac{(-1+x)^{5/4}}{x^{1/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x + \frac{1}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{1}{2}+2x}{-x^2+x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{5\ln(-1+x)}{2} - \frac{\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\sqrt{-1+x}\,\sqrt{x}\,\left(12\ln\left(-\frac{1}{2}+x+\sqrt{x\,(-1+x)}\right)x - 4\sqrt{x\,(-1+x)}\,x + 3\ln\left(-\frac{1}{2}+x+\sqrt{x\,(-1+x)}\right)\right.}{\sqrt{x\,(-1+x)}\,(4x+1)}\right.
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x + \frac{1}{4}\right) + c_2\left(x\right. \\
&\quad + \frac{1}{4}\left(-\frac{\sqrt{-1+x}\,\sqrt{x}\,\left(12\ln\left(-\frac{1}{2}+x+\sqrt{x\,(-1+x)}\right)x - 4\sqrt{x\,(-1+x)}\,x + 3\ln\left(-\frac{1}{2}+x+\sqrt{x\,(-1\right.\right.}{\sqrt{x\,(-1+x)}\,(4x+1)}\right.
\end{aligned}$$

Will add steps showing solving for IC soon.

1.774.2 Maple step by step solution

Let's solve

$x(1-x)\left(\frac{d}{dx}y'\right)+\left(\frac{1}{2}+2x\right)y'-2y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y'=-\frac{2y}{x(-1+x)}+\frac{(4x+1)y'}{2x(-1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y'-\frac{(4x+1)y'}{2x(-1+x)}+\frac{2y}{x(-1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{4x+1}{2x(-1+x)},P_3(x)=\frac{2}{x(-1+x)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x(-1+x)\left(\frac{d}{dx}y'\right)+(-4x-1)y'+4y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

    $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m\cdot\left(\frac{d}{dx}y'\right)$ to series expansion for $m=1..2$

$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$-a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(2k+1+2r) + 2a_k(k+r-1)(k+r-2)\right) x^{k+r}$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-r(-1+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $-2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 2a_k(k+r-1)(k+r-2) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{2a_k(k+r-1)(k+r-2)}{(2k+1+2r)(k+1+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 1$
  $a_{k+1} = \frac{2a_k(k-1)(k-2)}{(2k+1)(k+1)}$
- Apply recursion relation for $k = 0$
  $a_1 = 4a_0$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li
  $y = a_0 \cdot (4x+1)$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+1} = \frac{2a_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}$
- Solution for $r = \frac{1}{2}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$
- Combine solutions and rename parameters
  $\left[y = a_0 \cdot (4x+1) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), b_{k+1} = \frac{2b_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

1.774.3 Maple trace

```
Methods for second order ODEs:
```

1.774.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 53

```
dsolve(x*(1-x)*diff(diff(y(x),x),x)+(1/2+2*x)*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = (-12x-3)\, c_2 \ln\left(-1+2x+2\sqrt{x\,(-1+x)}\right) + (4x+26)\, c_2 \sqrt{x\,(-1+x)} + 4(3\ln(2)\, c_2 + c_1)\left(x+\frac{1}{4}\right)$$

1.774.5 Mathematica DSolve solution

Solving time : 0.181 (sec)
Leaf size : 64

```
DSolve[{x*(1-x)*D[y[x],{x,2}]+(1/2+2*x)*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}c_2\left(\sqrt{-((x-1)x)}(2x+13) - 6(4x+1)\arctan\left(\frac{\sqrt{1-x}}{\sqrt{x}+1}\right)\right) + c_1\left(x+\frac{1}{4}\right)$$

1.775 problem 797

Internal problem ID [8913]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 797
Date solved : `Monday, October 21, 2024 at 05:23:45 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4\left(t^2-3t+2\right)y''-2y'+y=0$$

1.775.1 Solved as second order ode using Kovacic algorithm

Time used: 0.268 (sec)

Writing the ode as

$$\left(4t^2-12t+8\right)y''-2y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=4t^2-12t+8\\ B&=-2\\ C&=1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4t^2 + 20t - 19}{16\left(t^2 - 3t + 2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4t^2 + 20t - 19$$
$$t = 16\left(t^2 - 3t + 2\right)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{-4t^2 + 20t - 19}{16\left(t^2 - 3t + 2\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1473: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(t^2 - 3t + 2)^2$. There is a pole at $t = 2$ of order 2. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(t-1)^2} + \frac{5}{16(t-2)^2} + \frac{3}{8(t-1)} - \frac{3}{8(t-2)}$$

For the pole at $t = 2$ let b be the coefficient of $\frac{1}{(t-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4t^2 + 20t - 19}{16(t^2 - 3t + 2)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4t^2+20t-19}{16\left(t^2-3t+2\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{t-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{4(t-2)} + \frac{3}{4(t-1)} + (-)(0) \\ &= -\frac{1}{4(t-2)} + \frac{3}{4(t-1)} \\ &= \frac{2t-5}{4(t-1)(t-2)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right)(0) + \left(\left(\frac{1}{4(t-2)^2} - \frac{3}{4(t-1)^2}\right) + \left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right)^2 - \left(\frac{-4t}{16(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega\, dt} \\ &= e^{\int \left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right) dt} \\ &= \frac{(t-1)^{3/4}}{(t-2)^{1/4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{4t^2-12t+8}\, dt} \\ &= z_1 e^{-\frac{\ln(t-1)}{4} + \frac{\ln(t-2)}{4}} \\ &= z_1 \left(\frac{(t-2)^{1/4}}{(t-1)^{1/4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{t-1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{4t^2-12t+8}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-\frac{\ln(t-1)}{2}+\frac{\ln(t-2)}{2}}}{(y_1)^2}\, dt \\
&= y_1 \left(-\frac{2\sqrt{t-2}}{\sqrt{t-1}} + \frac{\ln\left(-\frac{3}{2}+t+\sqrt{t^2-3t+2}\right)\sqrt{(t-1)(t-2)}}{\sqrt{t-2}\,\sqrt{t-1}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{t-1}\right) + c_2\left(\sqrt{t-1}\left(-\frac{2\sqrt{t-2}}{\sqrt{t-1}} + \frac{\ln\left(-\frac{3}{2}+t+\sqrt{t^2-3t+2}\right)\sqrt{(t-1)(t-2)}}{\sqrt{t-2}\,\sqrt{t-1}}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.775.2 Maple step by step solution

Let's solve
$4(t^2-3t+2)\left(\frac{d}{dt}y'\right) - 2y' + y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dt}y'$
- Isolate 2nd derivative
  $\frac{d}{dt}y' = -\frac{y}{4(t^2-3t+2)} + \frac{y'}{2(t^2-3t+2)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y' - \frac{y'}{2(t^2-3t+2)} + \frac{y}{4(t^2-3t+2)} = 0$

□ Check to see if t_0 is a regular singular point

- Define functions

  $\left[P_2(t) = -\frac{1}{2(t^2-3t+2)}, P_3(t) = \frac{1}{4(t^2-3t+2)}\right]$

- $(t-1) \cdot P_2(t)$ is analytic at $t = 1$

  $\left((t-1) \cdot P_2(t)\right)\Bigg|_{t=1} = \frac{1}{2}$

- $(t-1)^2 \cdot P_3(t)$ is analytic at $t = 1$

  $\left((t-1)^2 \cdot P_3(t)\right)\Bigg|_{t=1} = 0$

- $t = 1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = 1$

- Multiply by denominators

  $(4t^2 - 12t + 8)\left(\frac{d}{dt}y'\right) - 2y' + y = 0$

- Change variables using $t = u + 1$ so that the regular singular point is at $u = 0$

  $(4u^2 - 4u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) - 2\frac{d}{du}y(u) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1}$

- Shift index using $k- >k+1$

  $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$-2a_0 r(-1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(2k+2r-1)^2\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$-2r(-1+2r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{0, \frac{1}{2}\right\}$$
- Each term in the series must be 0, giving the recursion relation
$$a_k(2k+2r-1)^2 - 4(k+1+r)\left(k+\frac{1}{2}+r\right) a_{k+1} = 0$$
- Recursion relation that defines series solution to ODE
$$a_{k+1} = \frac{a_k(2k+2r-1)^2}{2(k+1+r)(2k+1+2r)}$$
- Recursion relation for $r = 0$
$$a_{k+1} = \frac{a_k(2k-1)^2}{2(k+1)(2k+1)}$$
- Solution for $r = 0$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(2k-1)^2}{2(k+1)(2k+1)}\right]$$
- Revert the change of variables $u = t-1$
$$\left[y = \sum_{k=0}^{\infty} a_k(t-1)^k, a_{k+1} = \frac{a_k(2k-1)^2}{2(k+1)(2k+1)}\right]$$
- Recursion relation for $r = \frac{1}{2}$
$$a_{k+1} = \frac{2a_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}$$
- Solution for $r = \frac{1}{2}$
$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$
- Revert the change of variables $u = t-1$
$$\left[y = \sum_{k=0}^{\infty} a_k(t-1)^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$
- Combine solutions and rename parameters
$$\left[y = \left(\sum_{k=0}^{\infty} a_k(t-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(t-1)^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k(2k-1)^2}{2(k+1)(2k+1)}, b_{k+1} = \frac{2b_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

1.775.3 Maple trace

```
Methods for second order ODEs:
```

1.775.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 56

```
dsolve(4*(t^2-3*t+2)*diff(diff(y(t),t),t)-2*diff(y(t),t)+y(t) = 0,
       y(t),singsol=all)
```

$$y = c_1\sqrt{t-1} + \frac{c_2\left(-\frac{\left(-\ln(2)+\ln\left(-3+2t+2\sqrt{(t-1)(t-2)}\right)\right)\sqrt{t^2-3t+2}}{2} + t - 2\right)}{\sqrt{t-2}}$$

1.775.5 Mathematica DSolve solution

Solving time : 0.188 (sec)
Leaf size : 53

```
DSolve[{4*(t^2-3*t+2)*D[y[t],{t,2}]-2*D[y[t],t]+y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \sqrt{1-t}\left(-2c_2\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{t-1}{t-2}}}\right) + \frac{2c_2}{\sqrt{\frac{t-1}{t-2}}} + c_1\right)$$

1.776 problem 798

Internal problem ID [8914]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 798
Date solved : `Monday, October 21, 2024 at 05:23:46 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2\left(t^2-5t+6\right)y''+(2t-3)\,y'-8y=0$$

1.776.1 Solved as second order ode using Kovacic algorithm

Time used: 0.259 (sec)

Writing the ode as

$$\left(2t^2-10t+12\right)y''+(2t-3)\,y'-8y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2-10t+12 \\ B &= 2t-3 \\ C &= -8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{60t^2 - 308t + 381}{16\left(t^2 - 5t + 6\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 60t^2 - 308t + 381$$
$$t = 16\left(t^2 - 5t + 6\right)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{60t^2 - 308t + 381}{16\left(t^2 - 5t + 6\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1475: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(t^2 - 5t + 6)^2$. There is a pole at $t = 3$ of order 2. There is a pole at $t = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{29}{8(t-2)} + \frac{5}{16(t-2)^2} - \frac{3}{16(t-3)^2} + \frac{29}{8(t-3)}$$

For the pole at $t = 3$ let b be the coefficient of $\frac{1}{(t-3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $t = 2$ let b be the coefficient of $\frac{1}{(t-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{60t^2 - 308t + 381}{16(t^2 - 5t + 6)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{60t^2-308t+381}{16\left(t^2-5t+6\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
3	2	0	$\frac{3}{4}$	$\frac{1}{4}$
2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^+\right) \\
&= \frac{5}{2}-\left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{t-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4t-12} + \frac{5}{4(t-2)} + (0) \\
&= \frac{1}{4t-12} + \frac{5}{4(t-2)} \\
&= \frac{6t-17}{4(t-2)(t-3)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right)(1) + \left(\left(-\frac{1}{4(t-3)^2} - \frac{5}{4(t-2)^2}\right) + \left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right)^2 - \left(\frac{60t^2 - \cdots}{16(t^2 \cdots}\right.\right.$$

$$\frac{-6\cdots}{2t^2 - \cdots}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{17}{6}\right\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t - \frac{17}{6}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t - \frac{17}{6}\right) e^{\int \left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right) dt} \\
&= \left(t - \frac{17}{6}\right) e^{\frac{\ln(t-3)}{4} + \frac{5\ln(t-2)}{4}} \\
&= \left(t - \frac{17}{6}\right) (t-3)^{1/4} (t-2)^{5/4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2t-3}{2t^2-10t+12}\,dt} \\
&= z_1 e^{-\frac{3\ln(t-3)}{4}+\frac{\ln(t-2)}{4}} \\
&= z_1 \left(\frac{(t-2)^{1/4}}{(t-3)^{3/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(t-2)^{3/2}(6t-17)}{6\sqrt{t-3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2t-3}{2t^2-10t+12}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\frac{3\ln(t-3)}{2}+\frac{\ln(t-2)}{2}}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{24(t-3)^2(24t^2-104t+111)\,\mathrm{e}^{-\frac{3\ln(t-3)}{2}+\frac{\ln(t-2)}{2}}}{5(6t-17)(t-2)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(t-2)^{3/2}(6t-17)}{6\sqrt{t-3}}\right) \\
&\quad + c_2 \left(\frac{(t-2)^{3/2}(6t-17)}{6\sqrt{t-3}}\left(\frac{24(t-3)^2(24t^2-104t+111)\,\mathrm{e}^{-\frac{3\ln(t-3)}{2}+\frac{\ln(t-2)}{2}}}{5(6t-17)(t-2)^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.776.2 Maple step by step solution

Let's solve

$2(t^2-5t+6)\left(\frac{d}{dt}y'\right)+(2t-3)\,y'-8y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dt}y'$

- Isolate 2nd derivative

$\frac{d}{dt}y'=\frac{4y}{t^2-5t+6}-\frac{(2t-3)y'}{2(t^2-5t+6)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dt}y'+\frac{(2t-3)y'}{2(t^2-5t+6)}-\frac{4y}{t^2-5t+6}=0$

□ Check to see if t_0 is a regular singular point

- ◦ Define functions

$\left[P_2(t)=\frac{2t-3}{2(t^2-5t+6)},P_3(t)=-\frac{4}{t^2-5t+6}\right]$

- ◦ $(t-2)\cdot P_2(t)$ is analytic at $t=2$

$\left.((t-2)\cdot P_2(t))\right|_{t=2}=-\frac{1}{2}$

- ◦ $(t-2)^2\cdot P_3(t)$ is analytic at $t=2$

$\left.\left((t-2)^2\cdot P_3(t)\right)\right|_{t=2}=0$

- ◦ $t=2$is a regular singular point

Check to see if t_0 is a regular singular point

$t_0=2$

- Multiply by denominators

$(2t^2-10t+12)\left(\frac{d}{dt}y'\right)+(2t-3)\,y'-8y=0$

- Change variables using $t=u+2$ so that the regular singular point is at $u=0$

$(2u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(2u+1)\left(\frac{d}{du}y(u)\right)-8y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-3+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} (-a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k(k+r+2)\,(k+r-2))\, u^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2(k+1+r)\left(k+r-\frac{1}{2}\right)a_{k+1} + 2a_k(k+r+2)\,(k+r-2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k(k+r+2)(k+r-2)}{(k+1+r)(2k-1+2r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 2$

$$a_{k+1} = \frac{2a_k(k+2)(k-2)}{(k+1)(2k-1)}$$

- Apply recursion relation for $k = 0$

$$a_1 = 8a_0$$

- Apply recursion relation for $k = 1$

$$a_2 = -3a_1$$

- Express in terms of a_0

$$a_2 = -24a_0$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot (-24u^2 + 8u + 1)$$

- Revert the change of variables $u = t - 2$

$$[y = a_0(-24t^2 + 104t - 111)]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = \frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}$$

- Solution for $r = \frac{3}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1}=\frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$$

- Revert the change of variables $u=t-2$

$$\left[y=\sum_{k=0}^{\infty} a_k(t-2)^{k+\frac{3}{2}}, a_{k+1}=\frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y=a_0(-24t^2+104t-111)+\left(\sum_{k=0}^{\infty} b_k(t-2)^{k+\frac{3}{2}}\right), b_{k+1}=\frac{2b_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$$

1.776.3 Maple trace

```
Methods for second order ODEs:
```

1.776.4 Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 35

```
dsolve(2*(t^2-5*t+6)*diff(diff(y(t),t),t)+(2*t-3)*diff(y(t),t)-8*y(t) = 0,
       y(t),singsol=all)
```

$$y=\frac{c_1(24t^2-104t+111)}{24}+\frac{c_2(t-2)^{3/2}(6t-17)}{\sqrt{t-3}}$$

1.776.5 Mathematica DSolve solution

Solving time : 0.467 (sec)
Leaf size : 140

```
DSolve[{2*(t^2-5*t+6)*D[y[t],{t,2}]+(2*t-3)*D[y[t],t]-8*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{\sqrt[4]{2-t}\sqrt[4]{t-3}(t-2)^{5/4}\left(5c_1(6t-17)-\frac{24c_2(\sqrt{t-2}-1)\sqrt{t-3}(-t^2+(4\sqrt{t-2}-2)t-4\sqrt{t-2}+7)(24t^2-104t+111)}{(-t+\sqrt{t-2}+2)^3(-t+2\sqrt{t-2}+1)}\right)}{30(3-t)^{3/4}}$$

1.777 problem 799

Internal problem ID [8915]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 799
Date solved : `Monday, October 21, 2024 at 05:23:47 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$3t(1+t)\,y''+ty'-y=0$$

1.777.1 Solved as second order ode using Kovacic algorithm

Time used: 0.309 (sec)

Writing the ode as

$$\left(3t^2+3t\right)y''+ty'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3t^2+3t \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7t + 12}{36t\,(1+t)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 7t + 12$$
$$t = 36t(1+t)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{7t + 12}{36t\,(1+t)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1477: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 3 - 1 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36t(1+t)^2$. There is a pole at $t = 0$ of order 1. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $t = 0$ of order 1 then

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}
$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{5}{36\,(1+t)^2} - \frac{1}{3\,(1+t)} + \frac{1}{3t}
$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(1+t)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{7t + 12}{36t\,(1+t)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{6}\\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{6}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{7t+12}{36t\,(1+t)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
-1	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{7}{6}$ then

$$\begin{aligned}d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-\right)\\ &= \frac{7}{6}-\left(\frac{7}{6}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{t-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{t} + \frac{1}{6+6t} + (0) \\
&= \frac{1}{t} + \frac{1}{6+6t} \\
&= \frac{1}{t} + \frac{1}{6+6t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{t} + \frac{1}{6+6t}\right)(0) + \left(\left(-\frac{1}{t^2} - \frac{1}{6\,(1+t)^2}\right) + \left(\frac{1}{t} + \frac{1}{6+6t}\right)^2 - \left(\frac{7t+12}{36t\,(1+t)^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{t} + \frac{1}{6+6t}\right) dt} \\
&= t(1+t)^{1/6}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t}{3t^2+3t} \, dt} \\
&= z_1 e^{-\frac{\ln(1+t)}{6}} \\
&= z_1 \left(\frac{1}{(1+t)^{1/6}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{3t^2+3t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-\frac{\ln(1+t)}{3}}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{-2(1+t)^{1/3} - 1}{3\,(1+t)^{2/3} + 3\,(1+t)^{1/3} + 3} + \frac{\ln\left((1+t)^{2/3} + (1+t)^{1/3} + 1\right)}{6} \right. \\
&\quad \left. - \frac{\sqrt{3}\, \arctan\left(\frac{\left(1+2(1+t)^{1/3}\right)\sqrt{3}}{3}\right)}{3} - \frac{1}{3\left((1+t)^{1/3} - 1\right)} - \frac{\ln\left((1+t)^{1/3} - 1\right)}{3} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(t) + c_2 \left(t \left(\frac{-2(1+t)^{1/3} - 1}{3\,(1+t)^{2/3} + 3\,(1+t)^{1/3} + 3} + \frac{\ln\left((1+t)^{2/3} + (1+t)^{1/3} + 1\right)}{6} \right. \right. \\
&\quad \left. \left. - \frac{\sqrt{3}\, \arctan\left(\frac{\left(1+2(1+t)^{1/3}\right)\sqrt{3}}{3}\right)}{3} - \frac{1}{3\left((1+t)^{1/3} - 1\right)} - \frac{\ln\left((1+t)^{1/3} - 1\right)}{3} \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.777.2 Maple step by step solution

Let's solve

$3t(1+t)\left(\frac{d}{dt}y'\right)+ty'-y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dt}y'$

- Isolate 2nd derivative

  $\frac{d}{dt}y'=\frac{y}{3t(1+t)}-\frac{y'}{3(1+t)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dt}y'+\frac{y'}{3(1+t)}-\frac{y}{3t(1+t)}=0$

□ Check to see if t_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(t)=\frac{1}{3(1+t)}, P_3(t)=-\frac{1}{3t(1+t)}\right]$

- ◦ $(1+t)\cdot P_2(t)$ is analytic at $t=-1$

  $\left((1+t)\cdot P_2(t)\right)\Big|_{t=-1}=\frac{1}{3}$

- ◦ $(1+t)^2\cdot P_3(t)$ is analytic at $t=-1$

  $\left((1+t)^2\cdot P_3(t)\right)\Big|_{t=-1}=0$

- ◦ $t=-1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0=-1$

- Multiply by denominators

  $3t(1+t)\left(\frac{d}{dt}y'\right)+ty'-y=0$

- Change variables using $t=u-1$ so that the regular singular point is at $u=0$

  $(3u^2-3u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)-y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

$$u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m}\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-a_{0}r(-2+3r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(3k+3r+1)+a_{k}(3k+3r+1)\,(k+r-1)\right)u^{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-r(-2+3r)=0$
- Values of r that satisfy the indicial equation
  $r\in\left\{0,\frac{2}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $3\left(k+r+\frac{1}{3}\right)\left(\left(-k-r-1\right)a_{k+1}+a_{k}(k+r-1)\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_{k}(k+r-1)}{k+1+r}$
- Recursion relation for $r=0$; series terminates at $k=1$
  $a_{k+1}=\frac{a_{k}(k-1)}{k+1}$
- Apply recursion relation for $k=0$
  $a_{1}=-a_{0}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li
  $y(u)=a_{0}\cdot(-u+1)$
- Revert the change of variables $u=1+t$
  $[y=-a_{0}t]$
- Recursion relation for $r=\frac{2}{3}$
  $a_{k+1}=\frac{a_{k}\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}$
- Solution for $r=\frac{2}{3}$
  $\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+\frac{2}{3}},a_{k+1}=\frac{a_{k}\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$
- Revert the change of variables $u=1+t$

$$\left[y=\sum_{k=0}^{\infty} a_k (1+t)^{k+\frac{2}{3}}, a_{k+1}=\frac{a_k (k-\frac{1}{3})}{k+\frac{5}{3}}\right]$$

- Combine solutions and rename parameters

$$\left[y=-a_0 t+\left(\sum_{k=0}^{\infty} b_k (1+t)^{k+\frac{2}{3}}\right), b_{k+1}=\frac{b_k (k-\frac{1}{3})}{k+\frac{5}{3}}\right]$$

1.777.3 Maple trace

```
Methods for second order ODEs:
```

1.777.4 Maple dsolve solution

Solving time : 0.012 (sec)
Leaf size : 66

```
dsolve(3*t*(1+t)*diff(diff(y(t),t),t)+t*diff(y(t),t)-y(t) = 0,
       y(t),singsol=all)
```

$$y=c_1 t-2\sqrt{3}\arctan\left(\frac{\left(1+2(1+t)^{1/3}\right)\sqrt{3}}{3}\right) t c_2 + \ln\left((1+t)^{2/3}+(1+t)^{1/3}+1\right) t c_2 - 2\ln\left((1+t)^{1/3}-1\right) t c_2 - 6(1+t)^{2/3} c_2$$

1.777.5 Mathematica DSolve solution

Solving time : 0.179 (sec)
Leaf size : 93

```
DSolve[{3*t*(1+t)*D[y[t],{t,2}]+t*D[y[t],t]-y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{6c_1 t - c_2\left(2\sqrt{3}t\arctan\left(\frac{2\sqrt[3]{t+1}+1}{\sqrt{3}}\right)+6(t+1)^{2/3}+2t\log\left(\sqrt[3]{t+1}-1\right)-t\log\left((t+1)^{2/3}+\sqrt[3]{t+1}+1\right.\right.}{6\sqrt[6]{3}}$$

1.778 problem 800

Internal problem ID [8916]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 800
Date solved : `Monday, October 21, 2024 at 05:23:48 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \frac{\left(x+\frac{3}{4}\right)y}{4} = 0$$

1.778.1 Solved as second order ode using Kovacic algorithm

Time used: 0.222 (sec)

Writing the ode as

$$x^2y'' + \left(\frac{x}{4}+\frac{3}{16}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= \frac{x}{4}+\frac{3}{16} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4x - 3$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1479: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_{\infty} - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1+4x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+2\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{-x}}{4x} dx} \\ &= x^{1/4} \mathrm{e}^{\sqrt{-x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{1/4} \mathrm{e}^{\sqrt{-x}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{1/4} \mathrm{e}^{\sqrt{-x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= x^{1/4} \mathrm{e}^{\sqrt{-x}} \int \frac{1}{\sqrt{x}\, \mathrm{e}^{2\sqrt{-x}}} \, dx \\ &= x^{1/4} \mathrm{e}^{\sqrt{-x}} \left(-\frac{\sqrt{-x}\left(1 - \mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{1/4} \mathrm{e}^{\sqrt{-x}}\right) + c_2 \left(x^{1/4} \mathrm{e}^{\sqrt{-x}} \left(-\frac{\sqrt{-x}\left(1-\mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.778.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + \frac{\left(x+\frac{3}{4}\right)y}{4} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y(4x+3)}{16x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y(4x+3)}{16x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{4x+3}{16x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{16}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2\left(\frac{d}{dx}y'\right) + (4x+3)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+4r)(-3+4r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(4k+4r-1)(4k+4r-3) + 4a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+4r)(-3+4r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$16\left(k+r-\frac{3}{4}\right)\left(k+r-\frac{1}{4}\right) a_k + 4a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$16\left(k+\frac{1}{4}+r\right)\left(k+\frac{3}{4}+r\right) a_{k+1} + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{4a_k}{(4k+1+4r)(4k+3+4r)}$$

- Recursion relation for $r = \frac{1}{4}$

$$a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}$$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}\right]$$

- Recursion relation for $r = \frac{3}{4}$

$$a_{k+1} = -\frac{4a_k}{(4k+4)(4k+6)}$$

- Solution for $r = \frac{3}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = -\frac{4a_k}{(4k+4)(4k+6)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}, b_{k+1} = -\frac{4b_k}{(4k+4)(4k+6)}\right]$$

1.778.3 Maple trace

```
Methods for second order ODEs:
```

1.778.4 Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+1/4*(x+3/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x^{1/4}\left(c_1 \sin\left(\sqrt{x}\right) + c_2 \cos\left(\sqrt{x}\right)\right)$$

1.778.5 Mathematica DSolve solution

Solving time : 0.068 (sec)
Leaf size : 43

```
DSolve[{x^2*D[y[x],{x,2}]+1/4*(x+3/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-i\sqrt{x}}\sqrt[4]{x}\left(c_1 e^{2i\sqrt{x}} + ic_2\right)$$

1.779 problem 801

Internal problem ID [8917]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 801
Date solved : `Monday, October 21, 2024 at 05:23:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \frac{(x^2-1)\,y}{4} = 0$$

1.779.1 Solved as second order ode using Kovacic algorithm

Time used: 0.169 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(\frac{x^2}{4} - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= \frac{x^2}{4} - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1481: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(2\tan\left(\frac{x}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}} \left(2 \tan\left(\frac{x}{2}\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.779.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + x y' + \frac{(x^2-1) y}{4} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(x^2-1) y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{y'}{x} + \frac{(x^2-1) y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 \left(\frac{d}{dx} y'\right) + 4x y' + (x^2 - 1) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) + a_{k-2}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

$a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$a_k(4k^2 + 8kr + 4r^2 - 1) + a_{k-2} = 0$

- Shift index using $k- >k + 2$

$a_{k+2}\left(4(k + 2)^2 + 8(k + 2)\, r + 4r^2 - 1\right) + a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

$a_{k+2} = -\frac{a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{b_k}{4k^2+20k+24}, b_1 = 0\right]$$

1.779.3 Maple trace

```
Methods for second order ODEs:
```

1.779.4 Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+1/4*(x^2-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin\left(\frac{x}{2}\right) + c_2 \cos\left(\frac{x}{2}\right)}{\sqrt{x}}$$

1.779.5 Mathematica DSolve solution

Solving time : 0.053 (sec)
Leaf size : 36

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+1/4*(x^2-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{ix}{2}}\left(c_1 - ic_2 e^{ix}\right)}{\sqrt{x}}$$

1.780 problem 802

Internal problem ID [8918]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 802
Date solved : `Monday, October 21, 2024 at 05:23:49 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0$$

1.780.1 Solved as second order ode using Kovacic algorithm

Time used: 0.177 (sec)

Writing the ode as

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1-2x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1483: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1-2x}{x} \, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-2x}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\ln(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.780.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + (1-2x)\, y' + (x-1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x-1)y}{x} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x\left(\frac{d}{dx}y'\right) + (1-2x)\,y' + (x-1)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(a_1 (1+r)^2 - a_0 (1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1} (k+1+r)^2 - a_k (2k+2r+1) + a_{k-1}\right) x^{k}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

$a_1 (1+r)^2 - a_0 (1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 + (-2k-1)\, a_k + a_{k-1} = 0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2)^2 + (-2k-3)\, a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$$

1.780.3 Maple trace

```
Methods for second order ODEs:
```

1.780.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 13

```
dsolve(x*diff(diff(y(x),x),x)+(1-2*x)*diff(y(x),x)+(x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^x (c_1 + \ln(x)\, c_2)$$

1.780.5 Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 17

```
DSolve[{x*D[y[x],{x,2}]+(1-2*x)*D[y[x],x]+(x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^x (c_2 \log(x) + c_1)$$

1.781 problem 803

Internal problem ID [8919]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 803
Date solved : Monday, October 21, 2024 at 05:23:50 PM
CAS classification : [_Laguerre]

Solve

$$xy'' - (x+1)\,y' + y = 0$$

1.781.1 Solved as second order ode using Kovacic algorithm

Time used: 0.235 (sec)

Writing the ode as

$$xy'' + (-x-1)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 2x + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1485: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4x^2} - \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{2x^2} + \frac{1}{2x^3} + \frac{1}{4x^4} - \frac{1}{4x^5} - \frac{3}{4x^6} - \frac{3}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x + 3}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2x} \\
&= \frac{x-1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(\frac{1}{2} - \frac{1}{2x}\right)^2 - \left(\frac{x^2 - 2x + 3}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2x}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x-1}{x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x-1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(x+1)\,e^{x+\ln(x)}e^{-2x}}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(e^x) + c_2\left(e^x\left(-\frac{(x+1)\,e^{x+\ln(x)}e^{-2x}}{x}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.781.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) - (x+1)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y}{x} + \frac{(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(x+1)y'}{x} + \frac{y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions
  $\left[P_2(x) = -\frac{x+1}{x}, P_3(x) = \frac{1}{x}\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$
  $$(x \cdot P_2(x))\bigg|_{x=0} = -1$$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$$
- $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $$x\left(\frac{d}{dx}y'\right) + (-x-1)\,y' + y = 0$$
- Assume series solution for y
  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions
- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$
  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$
- Shift index using $k->k+1-m$
  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$
- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$
- Shift index using $k->k+1$
  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

  Rewrite ODE with series expansions
  $$a_0 r(-2+r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) - a_k(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $$r(-2+r) = 0$$
- Values of r that satisfy the indicial equation
  $$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)(a_{k+1}(k+1+r)-a_k)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k}{k+1+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=0$
  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$
  $a_{k+1}=\frac{a_k}{k+3}$
- Solution for $r=2$
  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters
  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$

1.781.3 Maple trace

```
Methods for second order ODEs:
```

1.781.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 13

```
dsolve(x*diff(diff(y(x),x),x)-(x+1)*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y=c_2\,\mathrm{e}^x+c_1x+c_1$$

1.781.5 Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 19

```
DSolve[{x*D[y[x],{x,2}]-(x+1)*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x - c_2(x+1)$$

1.782 problem 804

Internal problem ID [8920]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 804
Date solved : `Monday, October 21, 2024 at 05:23:51 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$xy'' + 3y' + 4x^3y = 0$$

1.782.1 Solved as second order ode using Kovacic algorithm

Time used: 0.297 (sec)

Writing the ode as

$$xy'' + 3y' + 4x^3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= 4x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -16x^4 + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1487: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(-4x^2\right) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 2ix \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)(2ix) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4 + 3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\
&= \frac{e^{-ix^2}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3}{x} \, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-i x^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{i e^{2 i x^2}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-i x^2}}{x^2}\right) + c_2 \left(\frac{e^{-i x^2}}{x^2} \left(-\frac{i e^{2 i x^2}}{4}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.782.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 3y' + 4x^3 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -4x^2 y - \frac{3y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{3y'}{x} + 4x^2 y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x\left(\frac{d}{dx}y'\right) + 3y' + 4x^3y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

  $$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$

- Shift index using $k- >k-3$

  $$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^r + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 0\}$
- The coefficients of each power of x must be 0
  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$
- Solve for the dependent coefficient(s)
  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$
- Shift index using $k- >k+3$
  $a_{k+4}(k+4+r)\,(k+6+r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$
- Recursion relation for $r = -2$
  $a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$
- Solution for $r = -2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Recursion relation for $r = 0$
  $a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{4b_k}{(k+4)(k+6)}\right.$

1.782.3 Maple trace

```
Methods for second order ODEs:
```

1.782.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 21

```
dsolve(x*diff(diff(y(x),x),x)+3*diff(y(x),x)+4*x^3*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x^2) + c_2 \cos(x^2)}{x^2}$$

1.782.5 Mathematica DSolve solution

Solving time : 0.075 (sec)
Leaf size : 41

```
DSolve[{x*D[y[x],{x,2}]+3*D[y[x],x]+4*x^3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1 e^{-ix^2} - ic_2 e^{ix^2}}{4x^2}$$

1.783 problem 805

Internal problem ID [8921]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 805
Date solved : `Monday, October 21, 2024 at 05:23:52 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-x^2+1\right)y''+2x\left(-x^2+1\right)y'-2y=0$$

1.783.1 Solved as second order ode using Kovacic algorithm

Time used: 0.225 (sec)

Writing the ode as

$$\left(-x^4+x^2\right)y''+\left(-2x^3+2x\right)y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^4+x^2\\B&=-2x^3+2x\\C&=-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{x^2 (x^2 - 1)} \tag{6}$$

Comparing the above to (5) shows that

$$s = -2$$
$$t = x^2 (x^2 - 1)$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{x^2 (x^2 - 1)}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1489: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 0\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2(x^2 - 1)$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 1. There is a pole at $x = -1$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 1$ of order 1 then

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= 1\\ \alpha_c^- &= 1\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x+1} - \frac{1}{x-1} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{x^2 (x^2 - 1)}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	1	0	0	1
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^- + \alpha_{c_2}^-) \\ &= 1 - (0) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{x - 1} - \frac{1}{x} + (-)(0) \\ &= \frac{1}{x - 1} - \frac{1}{x} \\ &= \frac{1}{x^2 - x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x-1} - \frac{1}{x}\right)(1) + \left(\left(-\frac{1}{(x-1)^2} + \frac{1}{x^2}\right) + \left(\frac{1}{x-1} - \frac{1}{x}\right)^2 - \left(-\frac{2}{x^2\left(x^2-1\right)}\right)\right) = 0$$

$$\frac{-2a_0 + 2}{x^3 - x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+1)\, e^{\int \left(\frac{1}{x-1} - \frac{1}{x}\right) dx} \\
&= (x+1)\, e^{\ln(x-1) - \ln(x)} \\
&= \frac{x^2 - 1}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^3+2x}{-x^4+x^2}\, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 - 1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+2x}{-x^4+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{1}{4(x-1)} + \frac{\ln(x-1)}{4} - \frac{1}{4(x+1)} - \frac{\ln(x+1)}{4} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2-1}{x^2} \right) + c_2 \left(\frac{x^2-1}{x^2} \left(-\frac{1}{4(x-1)} + \frac{\ln(x-1)}{4} - \frac{1}{4(x+1)} - \frac{\ln(x+1)}{4} \right) \right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.783.2 Maple step by step solution

1.783.3 Maple trace

```
Methods for second order ODEs:
```

1.783.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 47

```
dsolve(x^2*(-x^2+1)*diff(diff(y(x),x),x)+2*x*(-x^2+1)*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_2(x^2-1)\ln(x-1)+(-x^2+1)\,c_2\ln(x+1)+2c_1\,x^2-2c_2x-2c_1}{2x^2}$$

1.783.5 Mathematica DSolve solution

Solving time : 0.093 (sec)
Leaf size : 56

```
DSolve[{x^2*(1-x^2)*D[y[x],{x,2}]+2*x*(1-x^2)*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{-4c_1x^2-c_2(x^2-1)\log(1-x)+c_2(x^2-1)\log(x+1)+2c_2x+4c_1}{4x^2}$$

1.784 problem 806

Internal problem ID [8922]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 806
Date solved : `Monday, October 21, 2024 at 05:23:53 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2xy'' + (x-2)\,y' - y = 0$$

1.784.1 Solved as second order ode using Kovacic algorithm

Time used: 0.243 (sec)

Writing the ode as

$$2xy'' + (x-2)\,y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= x-2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 12}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x + 12$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 12}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1490: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{1}{4x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{1}{2x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{2}{x^4} + \frac{4}{x^5} - \frac{24}{x^6} + \frac{48}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 12}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{4x + 12}{16x^2}\right) \\ &= \frac{1}{16} + \frac{4x + 12}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{4}} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{4}} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 12}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - (\alpha_{c_1}^{-}) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\left(\frac{1}{4}\right) \\
&= -\frac{1}{2x} - \frac{1}{4} \\
&= -\frac{x+2}{4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - \frac{1}{4}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x} - \frac{1}{4}\right)^2 - \left(\frac{x^2 + 4x + 12}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2x} - \frac{1}{4}\right) dx} \\
&= \frac{e^{-\frac{x}{4}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x-2}{2x} \, dx} \\
&= z_1 e^{-\frac{x}{4} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, e^{-\frac{x}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x-2}{2x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x}{2}+\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{2(x-2)\,\mathrm{e}^{-\frac{x}{2}+\ln(x)}\mathrm{e}^{x}}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}}\left(\frac{2(x-2)\,\mathrm{e}^{-\frac{x}{2}+\ln(x)}\mathrm{e}^{x}}{x}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.784.2 Maple step by step solution

Let's solve

$2x\left(\frac{d}{dx}y'\right) + (x-2)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{y}{2x} - \frac{(x-2)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(x-2)y'}{2x} - \frac{y}{2x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{x-2}{2x}, P_3(x) = -\frac{1}{2x}\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\Big|_{x=0} = -1$$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\Big|_{x=0} = 0$$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

• Multiply by denominators

  $2x\left(\frac{d}{dx}y'\right) + (x-2)\,y' - y = 0$

• Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$
- Shift index using $k- > k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$
- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$
- Shift index using $k- > k+1$

  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$2a_0 r(-2+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)\,(k+r-1) + a_k(k+r-1)\right) x^{k+r}\right) = 0$$

• $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-2+r) = 0$

• Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation
$2\left(a_{k+1}(k+1+r)+\frac{a_k}{2}\right)(k+r-1)=0$
- Recursion relation that defines series solution to ODE
$a_{k+1}=-\frac{a_k}{2(k+1+r)}$
- Recursion relation for $r=0$
$a_{k+1}=-\frac{a_k}{2(k+1)}$
- Solution for $r=0$
$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=-\frac{a_k}{2(k+1)}\right]$
- Recursion relation for $r=2$
$a_{k+1}=-\frac{a_k}{2(k+3)}$
- Solution for $r=2$
$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=-\frac{a_k}{2(k+3)}\right]$
- Combine solutions and rename parameters
$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=-\frac{a_k}{2(k+1)}, b_{k+1}=-\frac{b_k}{2(k+3)}\right]$

1.784.3 Maple trace

```
Methods for second order ODEs:
```

1.784.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 16

```
dsolve(2*x*diff(diff(y(x),x),x)+(x-2)*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y=c_1(x-2)+c_2\,\mathrm{e}^{-\frac{x}{2}}$$

1.784.5 Mathematica DSolve solution

Solving time : 0.051 (sec)
Leaf size : 23

```
DSolve[{2*x*D[y[x],{x,2}]+(x-2)*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-x/2} + 2c_2(x-2)$$

1.785 problem 807

Internal problem ID [8923]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 807
Date solved : `Monday, October 21, 2024 at 05:23:54 PM`
CAS classification : `[_Lienard]`

Solve

$$xy'' + 2y' + xy = 0$$

1.785.1 Solved as second order ode using Kovacic algorithm

Time used: 0.143 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1492: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(x)}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x}\right) + c_2 \left(\frac{\cos(x)}{x} (\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.785.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -y - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- > k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$
  - Shift index using $k- > k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- > k+1$

    $x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$
- Shift index using $k- > k+1$

  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

1.785.3 Maple trace

```
Methods for second order ODEs:
```

1.785.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{x}$$

1.785.5 Mathematica DSolve solution

Solving time : 0.042 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

1.786 problem 808

Internal problem ID [8924]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 808
Date solved : `Monday, October 21, 2024 at 05:23:55 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 2x^2 y' + \left(x^4 + 2x - 1\right) y = 0$$

1.786.1 Solved as second order ode using Kovacic algorithm

Time used: 0.118 (sec)

Writing the ode as

$$y'' + 2x^2 y' + \left(x^4 + 2x - 1\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x^2 \\ C &= x^4 + 2x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1494: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2}{1}\,dx} \\ &= z_1 e^{-\frac{x^3}{3}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^3}{3}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x(x^2+3)}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{2x^3}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{-\frac{2x^3}{3}}\,\mathrm{e}^{\frac{2x\left(x^2+3\right)}{3}}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}}\right) + c_2\left(\mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}}\left(\frac{\mathrm{e}^{-\frac{2x^3}{3}}\,\mathrm{e}^{\frac{2x\left(x^2+3\right)}{3}}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.786.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 2x^2y' + (x^4 + 2x - 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..4$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x^2 \cdot y'$ to series expansion

$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$

- Shift index using $k- >k-1$

$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$

- Convert $\frac{d}{dx}y'$ to series expansion

$\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k- >k+2$

$\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 - a_0 + (6a_3 - a_1 + 2a_0)\, x + (12a_4 - a_2 + 4a_1)\, x^2 + (20a_5 - a_3 + 6a_2)\, x^3 + \left(\sum_{k=4}^{\infty} (a_{k+2}(k+2\right.$

- The coefficients of each power of x must be 0

$[2a_2 - a_0 = 0, 6a_3 - a_1 + 2a_0 = 0, 12a_4 - a_2 + 4a_1 = 0, 20a_5 - a_3 + 6a_2 = 0]$

- Solve for the dependent coefficient(s)

$\left\{a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6} - \frac{a_0}{3}, a_4 = \frac{a_0}{24} - \frac{a_1}{3}, a_5 = \frac{a_1}{120} - \frac{a_0}{6}\right\}$

- Each term in the series must be 0, giving the recursion relation

$(k^2 + 3k + 2)\, a_{k+2} + 2a_{k-1}k - a_k + a_{k-4} = 0$

- Shift index using $k- >k+4$

$\left((k+4)^2 + 3k + 14\right) a_{k+6} + 2a_{k+3}(k+4) - a_{k+4} + a_k = 0$

- Recursion relation that defines the series solution to the ODE

$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{2k a_{k+3} + a_k + 8a_{k+3} - a_{k+4}}{k^2 + 11k + 30}, a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6} - \frac{a_0}{3}, a_4 = \frac{a_0}{24} - \frac{a_1}{3}, a_5 = \frac{a_1}{120} - \frac{a_0}{6}\right]$

1.786.3 Maple trace

```
Methods for second order ODEs:
```

1.786.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)+2*x^2*diff(y(x),x)+(x^4+2*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \,\mathrm{e}^{-\frac{x\left(x^2-3\right)}{3}}+c_2 \,\mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}}$$

1.786.5 Mathematica DSolve solution

Solving time : 0.056 (sec)
Leaf size : 34

```
DSolve[{D[y[x],{x,2}]+2*x^2*D[y[x],x]+(x^4+2*x-1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{1}{2}e^{-\frac{1}{3}x\left(x^2+3\right)}\left(c_2e^{2x}+2c_1\right)$$

1.787 problem 809

Internal problem ID [8925]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 809
Date solved : `Monday, October 21, 2024 at 05:23:55 PM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$u'' + 2u' + u = 0$$

1.787.1 Solved as second order ode using Kovacic algorithm

Time used: 0.115 (sec)

Writing the ode as

$$u'' + 2u' + u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1496: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1 \left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \mathrm{e}^{-x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(u_1)^2}\,dx \\ &= u_1 \int \frac{e^{-2x}}{(u_1)^2}\,dx \\ &= u_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

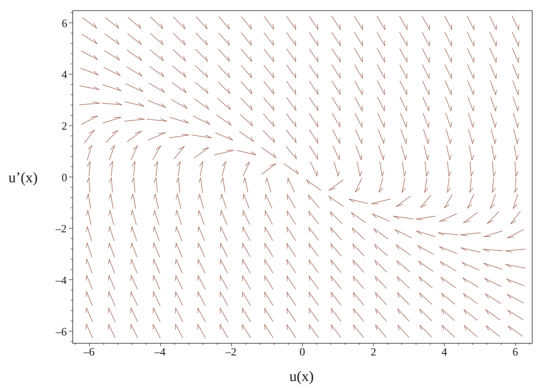

Figure 3: Slope field plot
$u'' + 2u' + u = 0$

1.787.2 Maple step by step solution

Let's solve

$\frac{d}{dx}u' + 2u' + u = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}u'$
- Characteristic polynomial of ODE

  $r^2 + 2r + 1 = 0$
- Factor the characteristic polynomial

  $(r+1)^2 = 0$
- Root of the characteristic polynomial

  $r = -1$
- 1st solution of the ODE

$u_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $u_1(x)$ by x to ensure linear independence
  $u_2(x) = x\,\mathrm{e}^{-x}$
- General solution of the ODE
  $u = C1 u_1(x) + C2 u_2(x)$
- Substitute in solutions
  $u = C1\,\mathrm{e}^{-x} + C2 x\,\mathrm{e}^{-x}$

1.787.3 Maple trace

```
Methods for second order ODEs:
```

1.787.4 Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 14

```
dsolve(diff(diff(u(x),x),x)+2*diff(u(x),x)+u(x) = 0,
       u(x),singsol=all)
```

$$u = \mathrm{e}^{-x}(c_2 x + c_1)$$

1.787.5 Mathematica DSolve solution

Solving time : 0.023 (sec)
Leaf size : 18

```
DSolve[{D[u[x],{x,2}]+2*D[u[x],x]+u[x]==0,{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to e^{-x}(c_2 x + c_1)$$

1.788 problem 810

Internal problem ID [8926]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 810
Date solved : `Monday, October 21, 2024 at 05:23:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$u'' - (2x+1)\,u' + \left(x^2+x-1\right)u = 0$$

1.788.1 Solved as second order ode using Kovacic algorithm

Time used: 0.102 (sec)

Writing the ode as

$$u'' + (-2x-1)\,u' + \left(x^2+x-1\right)u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x-1 \\ C &= x^2+x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$
u = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1498: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-1}{1}\,dx} \\ &= z_1 e^{\frac{1}{2}x^2+\frac{1}{2}x} \\ &= z_1\left(\mathrm{e}^{\frac{x(x+1)}{2}}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \mathrm{e}^{\frac{x^2}{2}}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{-2x-1}{1}\,dx}}{(u_1)^2}\,dx \\ &= u_1 \int \frac{e^{x^2+x}}{(u_1)^2}\,dx \\ &= u_1\left(\mathrm{e}^{x^2+x}\mathrm{e}^{-x^2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x^2}{2}}\right) + c_2 \left(\mathrm{e}^{\frac{x^2}{2}} \left(\mathrm{e}^{x^2+x} \mathrm{e}^{-x^2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.788.2 Maple step by step solution

Let's solve

$\frac{d}{dx}u' - (2x+1)\,u' + (x^2+x-1)\,u = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}u'$

- Isolate 2nd derivative

  $\frac{d}{dx}u' = (-x^2-x+1)\,u + (2x+1)\,u'$

- Group terms with u on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}u' + (-2x-1)\,u' + (x^2+x-1)\,u = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot u$ to series expansion for $m = 0..2$

    $x^m \cdot u = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot u = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x^m \cdot u'$ to series expansion for $m = 0..1$

    $x^m \cdot u' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot u' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

  - Convert $\frac{d}{dx}u'$ to series expansion

$$\frac{d}{dx}u' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

$$\frac{d}{dx}u' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 - a_1 - a_0 + (6a_3 - 2a_2 - 3a_1 + a_0)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - a_{k+1}(k+1) - a_k(2k+1$$

- The coefficients of each power of x must be 0

$$[2a_2 - a_1 - a_0 = 0, 6a_3 - 2a_2 - 3a_1 + a_0 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = \frac{a_1}{2} + \frac{a_0}{2}, a_3 = \frac{2a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+2} + (-2a_k - a_{k+1} + 3a_{k+2})\, k - a_k + a_{k-2} + a_{k-1} - a_{k+1} + 2a_{k+2} = 0$$

- Shift index using $k->k+2$

$$(k+2)^2\, a_{k+4} + (-2a_{k+2} - a_{k+3} + 3a_{k+4})\,(k+2) - a_{k+2} + a_k + a_{k+1} - a_{k+3} + 2a_{k+4} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2k a_{k+2} + k a_{k+3} - a_k - a_{k+1} + 5a_{k+2} + 3a_{k+3}}{k^2 + 7k + 12}, a_2 = \frac{a_1}{2} + \frac{a_0}{2}, a_3 = \frac{2a_1}{3}\right]$$

1.788.3 Maple trace

```
Methods for second order ODEs:
```

1.788.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 22

```
dsolve(diff(diff(u(x),x),x)-(2*x+1)*diff(u(x),x)+(x^2+x-1)*u(x) = 0,
       u(x),singsol=all)
```

$$u = c_1\, \mathrm{e}^{\frac{x^2}{2}} + c_2\, \mathrm{e}^{\frac{x(x+2)}{2}}$$

1.788.5 Mathematica DSolve solution

Solving time : 0.039 (sec)
Leaf size : 24

```
DSolve[{D[u[x],{x,2}]-(2*x+1)*D[u[x],x]+(x^2+x-1)*u[x]==0,{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to e^{\frac{x^2}{2}}(c_2 e^x + c_1)$$

1.789 problem 811

Internal problem ID [8927]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 811
Date solved : `Monday, October 21, 2024 at 05:23:56 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 2y' + \left(1 + \frac{2}{(1+3x)^2}\right) y = 0$$

1.789.1 Solved as second order ode using Kovacic algorithm

Time used: 0.155 (sec)

Writing the ode as

$$y'' + 2y' + \left(1 + \frac{2}{(1+3x)^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 + \frac{2}{(1+3x)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{(1+3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -2$$
$$t = (1+3x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{(1+3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1500: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+3x)^2$. There is a pole at $x = -\frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9\left(x + \frac{1}{3}\right)^2}$$

For the pole at $x = -\frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{2}{(1+3x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{(1+3x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{3}$	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{3} - \left(\frac{1}{3}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{1+3x} + (-)(0) \\ &= \frac{1}{1+3x} \\ &= \frac{1}{1+3x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{1+3x}\right)(0) + \left(\left(-\frac{1}{3\left(x+\frac{1}{3}\right)^2}\right) + \left(\frac{1}{1+3x}\right)^2 - \left(-\frac{2}{(1+3x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{1+3x} dx} \\ &= (1+3x)^{1/3} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1} \, dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}(1+3x)^{1/3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1 \left((1+3x)^{1/3}\,\mathrm{e}^{-2x}\mathrm{e}^{2x} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}(1+3x)^{1/3} \right) + c_2 \left(\mathrm{e}^{-x}(1+3x)^{1/3} \left((1+3x)^{1/3}\,\mathrm{e}^{-2x}\mathrm{e}^{2x} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.789.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 2y' + \left(1 + \frac{2}{(1+3x)^2}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{3(3x^2+2x+1)y}{(1+3x)^2} - 2y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + 2y' + \frac{3(3x^2+2x+1)y}{(1+3x)^2} = 0$

□ Check to see if $x_0 = -\frac{1}{3}$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = 2, P_3(x) = \frac{3(3x^2+2x+1)}{(1+3x)^2}\right]$

- ○ $\left(x+\frac{1}{3}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{3}$

  $\left(\left(x+\frac{1}{3}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{1}{3}} = 0$

- ○ $\left(x+\frac{1}{3}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{3}$

$$\left(\left(x+\frac{1}{3}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{3}}=\frac{2}{9}$$

- $x=-\frac{1}{3}$is a regular singular point

  Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

  $x_0=-\frac{1}{3}$

- Multiply by denominators

  $(1+3x)^2\left(\frac{d}{dx}y'\right)+2(1+3x)^2 y'+(9x^2+6x+3)\,y=0$

- Change variables using $x=u-\frac{1}{3}$ so that the regular singular point is at $u=0$

  $9u^2\left(\frac{d}{du}\frac{d}{du}y(u)\right)+18u^2\left(\frac{d}{du}y(u)\right)+(9u^2+2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- Convert $u^2\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion

  $u^2\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r+1}$

- Shift index using $k->k-1$

  $u^2\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=1}^{\infty}a_{k-1}(k-1+r)\,u^{k+r}$

- Convert $u^2\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $u^2\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k-1+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)\,(-2+3r)\,u^r+\left(a_1(2+3r)\,(1+3r)+18a_0r\right)u^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(3k+3r-1)\,(3k+3r\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)\,(-2+3r)=0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$

- Each term must be 0

  $a_1(2+3r)(1+3r)+18a_0r=0$
- Solve for the dependent coefficient(s)

  $a_1=-\frac{18a_0r}{9r^2+9r+2}$
- Each term in the series must be 0, giving the recursion relation

  $9\left(k+r-\frac{2}{3}\right)\left(k+r-\frac{1}{3}\right)a_k+18a_{k-1}k+18a_{k-1}r+9a_{k-2}-18a_{k-1}=0$
- Shift index using $k->k+2$

  $9\left(k+\frac{4}{3}+r\right)\left(k+\frac{5}{3}+r\right)a_{k+2}+18a_{k+1}(k+2)+18a_{k+1}r+9a_k-18a_{k+1}=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{9(2ka_{k+1}+2a_{k+1}r+a_k+2a_{k+1})}{(3k+4+3r)(3k+5+3r)}$
- Recursion relation for $r=\frac{1}{3}$

  $a_{k+2}=-\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}$
- Solution for $r=\frac{1}{3}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{1}{3}}, a_{k+2}=-\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1=-a_0\right]$
- Revert the change of variables $u=x+\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{3}\right)^{k+\frac{1}{3}}, a_{k+2}=-\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1=-a_0\right]$
- Recursion relation for $r=\frac{2}{3}$

  $a_{k+2}=-\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}$
- Solution for $r=\frac{2}{3}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{2}{3}}, a_{k+2}=-\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}, a_1=-a_0\right]$
- Revert the change of variables $u=x+\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{3}\right)^{k+\frac{2}{3}}, a_{k+2}=-\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}, a_1=-a_0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{3}\right)^{k+\frac{1}{3}}\right)+\left(\sum_{k=0}^{\infty}b_k\left(x+\frac{1}{3}\right)^{k+\frac{2}{3}}\right), a_{k+2}=-\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1=-a_0, b_{k+2}=\right.$

1.789.3 Maple trace

```
Methods for second order ODEs:
```

1.789.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)+2*diff(y(x),x)+(1+2/(1+3*x)^2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-x}(1 + 3x)^{1/3} \left(c_2(1 + 3x)^{1/3} + c_1\right)$$

1.789.5 Mathematica DSolve solution

Solving time : 0.077 (sec)
Leaf size : 35

```
DSolve[{D[y[x],{x,2}]+2*D[y[x],x]+(1+2/(1+3*x)^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x}\sqrt[3]{3x+1}\left(c_2\sqrt[3]{3x+1} + c_1\right)$$

1.790 problem 812

Internal problem ID [8928]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 812
Date solved : `Monday, October 21, 2024 at 05:23:57 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0$$

1.790.1 Solved as second order ode using Kovacic algorithm

Time used: 0.148 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1502: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx}\\ &= z_1 e^{\ln(x)}\\ &= z_1(x)\end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx\\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.790.2 Maple step by step solution

Let's solve
$x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d}{dx}y'$
- Isolate 2nd derivative
$\frac{d}{dx}y' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
$\frac{d}{dx}y' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
  $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$
  - $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$
- Multiply by denominators
$x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2+2)\,y = 0$
- Assume series solution for y
$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-2+r)\,x^r + a_1 r(-1+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{1, 2\}$

- Each term must be 0

$a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$a_k(k+r-1)\,(k+r-2) + a_{k-2} = 0$

- Shift index using $k- >k + 2$

$a_{k+2}(k+1+r)\,(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$$

- Recursion relation for $r = 2$

$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

1.790.3 Maple trace

```
Methods for second order ODEs:
```

1.790.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+(x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x(c_1 \sin(x) + c_2 \cos(x))$$

1.790.5 Mathematica DSolve solution

Solving time : `0.042` (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

1.791 problem 813

Internal problem ID [8929]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 813
Date solved : `Monday, October 21, 2024 at 05:23:58 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0$$

1.791.1 Solved as second order ode using Kovacic algorithm

Time used: 0.142 (sec)

Writing the ode as

$$y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= -\frac{2}{(1+x)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{(1+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= (1+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{(1+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1504: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+x)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{(1+x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{(1+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{1+x} + (-)\,(0) \\ &= -\frac{1}{1+x} \\ &= -\frac{1}{1+x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{1+x}\right)(0) + \left(\left(\frac{1}{(1+x)^2}\right) + \left(-\frac{1}{1+x}\right)^2 - \left(\frac{2}{(1+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{1+x} dx} \\
&= \frac{1}{1+x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1} \, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2 + x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{(1+x)^3}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2+x}\right) + c_2 \left(\frac{1}{x^2+x}\left(\frac{(1+x)^3}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.791.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -\frac{2}{(1+x)^2}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = 0$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = -2$

  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

$x_0 = -1$

- Multiply by denominators

  $x(1+x)^2 \left(\frac{d}{dx}y'\right) + 2(1+x)^2 y' - 2yx = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^3-u^2)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + 2u^2\left(\frac{d}{du}y(u)\right) + (-2u+2)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  ◦ Convert $u^2 \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

  $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r+1}$

  ◦ Shift index using $k->k-1$

  $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, u^{k+r}$

  ◦ Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 2..3$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(1+r)\,(-2+r)\, u^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r+1)\,(k+r-2)\right) u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$

- Each term in the series must be 0, giving the recursion relation

$-(k+r+1)(k+r-2)(a_k - a_{k-1}) = 0$

- Shift index using $k- > k+1$

  $-(k+r+2)(k-1+r)(a_{k+1} - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = a_k$

- Recursion relation for $r = -1$

  $a_{k+1} = a_k$

- Solution for $r = -1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+1} = a_k\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-1}, a_{k+1} = a_k\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = a_k$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = a_k\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+2}, a_{k+1} = a_k\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+2}\right), a_{k+1} = a_k, b_{k+1} = b_k\right]$

1.791.3 Maple trace

```
Methods for second order ODEs:
```

1.791.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 29

```
dsolve(diff(diff(y(x),x),x)+2/x*diff(y(x),x)-2/(1+x)^2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(x^3 + 3x^2 + 3x)\, c_2 + c_1}{x\,(1 + x)}$$

1.791.5 Mathematica DSolve solution

Solving time : `0.05` (sec)
Leaf size : 34

```
DSolve[{D[y[x],{x,2}]+2/x*D[y[x],x]-2/(1+x)^2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x(x^2 + 3x + 3) + 3c_1}{3x(x + 1)}$$

1.792 problem 815

Internal problem ID [8930]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 815
Date solved : `Monday, October 21, 2024 at 05:23:59 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.792.1 Solved as second order ode using Kovacic algorithm

Time used: 0.254 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 + 4x - 2}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 4x - 2 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1506: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 1 \end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0+(-)\left(\frac{x}{2}+1\right) \\ &= -1-\frac{x}{2} \\ &= -1-\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0)+2\left(-1-\frac{x}{2}\right)(1)+\left(\left(-\frac{1}{2}\right)+\left(-1-\frac{x}{2}\right)^2-\left(\frac{1}{4}x^2+x-\frac{1}{2}\right)\right) &= 0 \\ -2+a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2 + x)\, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x)\, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi}\, \mathrm{e}^{-2}\sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.792.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - ○ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - ○ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - ○ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - ○ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2+\left(\sum_{k=1}^{\infty}\left(a_{k+2}(k+2)\,(k+1)-a_k k-a_{k-1}\right)x^k\right)=0$$

- Each term must be 0

  $2a_2=0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\,a_{k+2}-a_k k-a_{k-1}=0$
- Shift index using $k->k+1$

  $\left((k+1)^2+3k+5\right)a_{k+3}-a_{k+1}(k+1)-a_k=0$
- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+3}=\frac{k a_{k+1}+a_k+a_{k+1}}{k^2+5k+6}, 2a_2=0\right]$

1.792.3 Maple trace

```
Methods for second order ODEs:
```

1.792.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y=\mathrm{e}^{-2-x}\pi c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-i\mathrm{e}^{\frac{x(2+x)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2+c_1(2+x)\,\mathrm{e}^{-x}$$

1.792.5 Mathematica DSolve solution

Solving time : `0.213` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right)+2\sqrt{2}c_1(x+2)+2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

1.793 problem 816

Internal problem ID [8931]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 816
Date solved : `Monday, October 21, 2024 at 05:24:00 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.793.1 Solved as second order ode using Kovacic algorithm

Time used: 0.254 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1508: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x)\, e^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x)\, e^{-x-\frac{1}{4}x^2} \\ &= (2 + x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x)\, e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\, e^{-2}\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.793.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ∘ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

- ∘ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

- ∘ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ∘ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ∘ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2+\left(\sum_{k=1}^{\infty}\left(a_{k+2}(k+2)\left(k+1\right)-a_k k-a_{k-1}\right)x^k\right)=0$$

- Each term must be 0

  $2a_2=0$

- Each term in the series must be 0, giving the recursion relation

  $\left(k^2+3k+2\right)a_{k+2}-a_k k-a_{k-1}=0$

- Shift index using $k->k+1$

  $\left(\left(k+1\right)^2+3k+5\right)a_{k+3}-a_{k+1}(k+1)-a_k=0$

- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+3}=\frac{k a_{k+1}+a_k+a_{k+1}}{k^2+5k+6}, 2a_2=0\right]$

1.793.3 Maple trace

```
Methods for second order ODEs:
```

1.793.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y=\mathrm{e}^{-2-x}\pi c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-i\mathrm{e}^{\frac{x(2+x)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2+c_1(2+x)\,\mathrm{e}^{-x}$$

1.793.5 Mathematica DSolve solution

Solving time : `0.166` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right)+2\sqrt{2}c_1(x+2)+2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

1.794 problem 817

Internal problem ID [8932]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 817
Date solved : `Monday, October 21, 2024 at 05:24:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.794.1 Solved as second order ode using Kovacic algorithm

Time used: 0.246 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1510: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, e^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x) \, e^{-x-\frac{1}{4}x^2} \\ &= (2 + x) \, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, e^{-2} \sqrt{2} \, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.794.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - ∘ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - ∘ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - ∘ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - ∘ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - ∘ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

1.794.3 Maple trace

```
Methods for second order ODEs:
```

1.794.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x}\pi c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i\mathrm{e}^{\frac{x(2+x)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1(2+x)\,\mathrm{e}^{-x}$$

1.794.5 Mathematica DSolve solution

Solving time : `0.168` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

1.795 problem 818

Internal problem ID [8933]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 818
Date solved : `Monday, October 21, 2024 at 05:24:01 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.795.1 Solved as second order ode using Kovacic algorithm

Time used: 0.250 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1512: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} + 1
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0$$

$$-2 + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, e^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x) \, e^{-x-\frac{1}{4}x^2} \\ &= (2 + x) \, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, e^{-2} \sqrt{2} \, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.795.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$\left(k^2 + 3k + 2\right) a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

1.795.3 Maple trace

```
Methods for second order ODEs:
```

1.795.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x}\pi c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i\mathrm{e}^{\frac{x(2+x)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1(2+x)\,\mathrm{e}^{-x}$$

1.795.5 Mathematica DSolve solution

Solving time : `0.166` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

1.796 problem 819

Internal problem ID [8934]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 819
Date solved : `Monday, October 21, 2024 at 05:24:02 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.796.1 Solved as second order ode using Kovacic algorithm

Time used: 0.254 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1514: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0 \\ -2 + a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, e^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x) \, e^{-x-\frac{1}{4}x^2} \\ &= (2 + x) \, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi}\, e^{-2}\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.796.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- > k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$

1.796.3 Maple trace

```
Methods for second order ODEs:
```

1.796.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x} \pi c_2 (2+x) \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i\mathrm{e}^{\frac{x(2+x)}{2}} \sqrt{\pi}\,\sqrt{2}\, c_2 + c_1 (2+x)\, \mathrm{e}^{-x}$$

1.796.5 Mathematica DSolve solution

Solving time : `0.168` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-x} \left(-\sqrt{2\pi} c_2 \sqrt{(x+2)^2} \operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2} c_1 (x+2) + 2 c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

1.797 problem 820

Internal problem ID [8935]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 820
Date solved : `Monday, October 21, 2024 at 05:24:03 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.797.1 Solved as second order ode using Kovacic algorithm

Time used: 0.245 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1516: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2+x)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right)dx} \\ &= (2+x)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2+x)\, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\, \mathrm{e}^{-2}\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.797.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2) a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- > k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

1.797.3 Maple trace

```
Methods for second order ODEs:
```

1.797.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x} \pi c_2 (2+x) \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i \mathrm{e}^{\frac{x(2+x)}{2}} \sqrt{\pi}\,\sqrt{2}\,c_2 + c_1 (2+x)\,\mathrm{e}^{-x}$$

1.797.5 Mathematica DSolve solution

Solving time : `0.169` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-x} \left(-\sqrt{2\pi} c_2 \sqrt{(x+2)^2} \operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2} c_1 (x+2) + 2 c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

1.798 problem 821

Internal problem ID [8936]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 821
Date solved : `Monday, October 21, 2024 at 05:24:04 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.798.1 Solved as second order ode using Kovacic algorithm

Time used: 0.249 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1518: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, \mathrm{e}^{\int (-1 - \frac{x}{2}) dx} \\ &= (2 + x) \, \mathrm{e}^{-x - \frac{1}{4}x^2} \\ &= (2 + x) \, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2 + \frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, \mathrm{e}^{-2} \sqrt{2} \, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.798.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

- ◦ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

- ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ◦ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} - a_k k - a_{k-1} = 0$
- Shift index using $k- >k + 1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

1.798.3 Maple trace

```
Methods for second order ODEs:
```

1.798.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x} \pi c_2 (2+x) \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i \mathrm{e}^{\frac{x(2+x)}{2}} \sqrt{\pi}\,\sqrt{2}\, c_2 + c_1 (2+x)\, \mathrm{e}^{-x}$$

1.798.5 Mathematica DSolve solution

Solving time : `0.169` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-x} \left(-\sqrt{2\pi} c_2 \sqrt{(x+2)^2} \operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2} c_1 (x+2) + 2 c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

1.799 problem 822

Internal problem ID [8937]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 822
Date solved : `Monday, October 21, 2024 at 05:24:05 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.799.1 Solved as second order ode using Kovacic algorithm

Time used: 0.257 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1520: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2 + x) \, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (2 + x) \, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (2 + x) \, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, \mathrm{e}^{-2} \sqrt{2} \, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.799.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

$$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2 + 3k + 2\right) a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

1.799.3 Maple trace

```
Methods for second order ODEs:
```

1.799.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x} \pi c_2 (2+x) \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i\mathrm{e}^{\frac{x(2+x)}{2}} \sqrt{\pi}\,\sqrt{2}\,c_2 + c_1 (2+x)\,\mathrm{e}^{-x}$$

1.799.5 Mathematica DSolve solution

Solving time : `0.17` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-x} \left(-\sqrt{2\pi} c_2 \sqrt{(x+2)^2} \operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2} c_1 (x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

1.800 problem 823

Internal problem ID [8938]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 823
Date solved : `Monday, October 21, 2024 at 05:24:06 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.800.1 Solved as second order ode using Kovacic algorithm

Time used: 0.250 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1522: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x) \, \mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2 + x) \, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, \mathrm{e}^{-2}\sqrt{2} \, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.800.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}) x^k\right) = 0$$

- Each term must be 0

$$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2) a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

1.800.3 Maple trace

```
Methods for second order ODEs:
```

1.800.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x} \pi c_2 (2+x) \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i\mathrm{e}^{\frac{x(2+x)}{2}} \sqrt{\pi}\,\sqrt{2}\,c_2 + c_1 (2+x)\,\mathrm{e}^{-x}$$

1.800.5 Mathematica DSolve solution

Solving time : `0.17` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2} e^{-x} \left(-\sqrt{2\pi} c_2 \sqrt{(x+2)^2} \operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2} c_1 (x+2) + 2 c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

1.801 problem 824

Internal problem ID [8939]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 824
Date solved : `Monday, October 21, 2024 at 05:24:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.801.1 Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1524: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0$$

$$-2 + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x) \, \mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2 + x) \, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi}\, \mathrm{e}^{-2}\sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)$$

Will add steps showing solving for IC soon.

1.801.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$
  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k- >k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

$$2a_2+\left(\sum_{k=1}^{\infty}\left(a_{k+2}(k+2)\left(k+1\right)-a_k k-a_{k-1}\right)x^k\right)=0$$

- Each term must be 0

  $2a_2=0$
- Each term in the series must be 0, giving the recursion relation

  $\left(k^2+3k+2\right)a_{k+2}-a_k k-a_{k-1}=0$
- Shift index using $k->k+1$

  $\left(\left(k+1\right)^2+3k+5\right)a_{k+3}-a_{k+1}(k+1)-a_k=0$
- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+3}=\frac{k a_{k+1}+a_k+a_{k+1}}{k^2+5k+6}, 2a_2=0\right]$

1.801.3 Maple trace

```
Methods for second order ODEs:
```

1.801.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y=i\sqrt{\pi}\,\mathrm{e}^{-2-x}(2+x)\,c_2\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(2+x)}{2}}c_2+c_1(2+x)\,\mathrm{e}^{-x}$$

1.801.5 Mathematica DSolve solution

Solving time : `0.17` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right)+2\sqrt{2}c_1(x+2)+2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

1.802 problem 825

Internal problem ID [8940]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 825
Date solved : `Monday, October 21, 2024 at 05:24:07 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy = 0$$

1.802.1 Solved as second order ode using Kovacic algorithm

Time used: 0.253 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1526: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, e^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x) \, e^{-x-\frac{1}{4}x^2} \\ &= (2 + x) \, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi} \, e^{-2}\sqrt{2} \, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.802.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$

1.802.3 Maple trace

```
Methods for second order ODEs:
```

1.802.4 Maple dsolve solution

Solving time : `0.006` (sec)
Leaf size : 51

```
dsolve(diff(diff(y(x),x),x)-x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-2-x}\pi c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - i\mathrm{e}^{\frac{x(2+x)}{2}}\sqrt{\pi}\,\sqrt{2}\,c_2 + c_1(2+x)\,\mathrm{e}^{-x}$$

1.802.5 Mathematica DSolve solution

Solving time : `0.169` (sec)
Leaf size : 78

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

1.803 problem 826

Internal problem ID [8941]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 826
Date solved : `Monday, October 21, 2024 at 05:24:08 PM`
CAS classification : `[_Lienard]`

Solve

$$xy'' + 2y' + xy = 0$$

1.803.1 Solved as second order ode using Kovacic algorithm

Time used: 0.145 (sec)

Writing the ode as

$$xy'' + 2y' + xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1528: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x}\right) + c_2 \left(\frac{\cos(x)}{x} (\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.803.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -y - \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{2y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x\left(\frac{d}{dx}y'\right) + 2y' + xy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$
  - Shift index using $k- > k-1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$
  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$
  - Shift index using $k- > k+1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$
  - Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$
  - Shift index using $k- > k+1$

    $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(1+r) = 0$$
- Values of r that satisfy the indicial equation

  $$r \in \{-1, 0\}$$
- Each term must be 0

  $$a_1(1+r)\,(2+r) = 0$$
- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$$
- Shift index using $k- > k+1$

  $$a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$$
- Recursion relation for $r = -1$

  $$a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

1.803.3 Maple trace

```
Methods for second order ODEs:
```

1.803.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 17

```
dsolve(x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{x}$$

1.803.5 Mathematica DSolve solution

Solving time : 0.038 (sec)
Leaf size : 37

```
DSolve[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

1.804 problem 827

Internal problem ID [8942]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 827
Date solved : `Monday, October 21, 2024 at 05:24:09 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$2x^2y'' + 3xy' - xy = 0$$

1.804.1 Solved as second order ode using Kovacic algorithm

Time used: 0.202 (sec)

Writing the ode as

$$2x^2y'' + 3xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 3x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{8x - 3}{16x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 8x - 3 \\
t &= 16x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{8x - 3}{16x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1530: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{2x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\
&= \frac{1}{2}(1-(1))\\
&= 0
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\
&= \frac{1}{2}\left(\frac{1}{(x-(0))}\right)\\
&= \frac{1}{2x}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}
$$

Since $d=0$, then letting

$$
p = 1 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$
0 = 0
$$

And solving for p gives

$$
p = 1
$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta + \frac{p'}{p}\\
&= \frac{1}{2x}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0
$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1-8x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+2\sqrt{2}\,\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= e^{\int \frac{1+2\sqrt{2}\,\sqrt{x}}{4x} dx} \\ &= x^{1/4} e^{\sqrt{2}\,\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{2x^2} \, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{4}} \\ &= z_1 \left(\frac{1}{x^{3/4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{2x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{-2\sqrt{2}\,\sqrt{x}}\sqrt{2}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}\right) + c_2 \left(\frac{e^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}} \left(-\frac{e^{-2\sqrt{2}\,\sqrt{x}}\sqrt{2}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.804.2 Maple step by step solution

Let's solve

$2x^2\left(\frac{d}{dx}y'\right) + 3xy' - xy = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{y}{2x} - \frac{3y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{3y'}{2x} - \frac{y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3}{2x}, P_3(x) = -\frac{1}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{3}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2\left(\frac{d}{dx}y'\right)x + 3y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+3+2r) - a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+\frac{3}{2}+r\right)(k+1+r)\, a_{k+1} - a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{(2k+3+2r)(k+1+r)}$$

- Recursion relation for $r=0$

$$a_{k+1} = \frac{a_k}{(2k+3)(k+1)}$$

- Solution for $r=0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{(2k+3)(k+1)}\right]$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+1} = \frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1} = \frac{a_k}{(2k+3)(k+1)}, b_{k+1} = \frac{b_k}{(2k+2)(k+\frac{1}{2})}\right]$$

1.804.3 Maple trace

```
Methods for second order ODEs:
```

1.804.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 29

```
dsolve(2*x^2*diff(diff(y(x),x),x)+3*x*diff(y(x),x)-x*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sinh\left(\sqrt{2}\,\sqrt{x}\right) + c_2 \cosh\left(\sqrt{2}\,\sqrt{x}\right)}{\sqrt{x}}$$

1.804.5 Mathematica DSolve solution

Solving time : 0.09 (sec)
Leaf size : 56

```
DSolve[{2*x^2*D[y[x],{x,2}]+3*x*D[y[x],x]-x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\sqrt{2}\sqrt{x}}\left(2c_1 e^{2\sqrt{2}\sqrt{x}} - \sqrt{2}c_2\right)}{2\sqrt{x}}$$

1.805 problem 828

Internal problem ID [8943]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 828
Date solved : `Monday, October 21, 2024 at 05:24:10 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + (3x^2 + 2x)\, y' - 2y = 0$$

1.805.1 Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$x^2y'' + (3x^2 + 2x)\, y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x^2 + 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 + 12x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9x^2 + 12x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2 + 12x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1532: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{2}{x^2} + \frac{3}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} + \frac{1}{x} + \frac{1}{3x^2} - \frac{2}{9x^3} + \frac{1}{9x^4} - \frac{2}{81x^5} - \frac{2}{81x^6} + \frac{28}{729x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^2 + 12x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{9}{4}\right) + \left(\frac{12x + 8}{4x^2}\right) \\ &= \frac{9}{4} + \frac{12x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 12. Dividing this by leading coefficient in t which is 4 gives 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{3}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{3}{\frac{3}{2}} - 0\right) = 1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{3}{\frac{3}{2}} - 0\right) = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^2 + 12x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -1$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - (\alpha_{c_1}^{-}) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{3}{2}\right) \\
&= -\frac{1}{x} - \frac{3}{2} \\
&= -\frac{1}{x} - \frac{3}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{3}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{3}{2}\right)^2 - \left(\frac{9x^2+12x+8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x} - \frac{3}{2}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{3x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x^2+2x}{x^2}\, dx} \\
&= z_1 e^{-\frac{3x}{2} - \ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^{-3x}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2+2x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3x-2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{(9x^2-6x+2)\, x^2 e^{-3x-2\ln(x)} e^{6x}}{27} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-3x}}{x^2} \right) + c_2 \left(\frac{e^{-3x}}{x^2} \left(\frac{(9x^2-6x+2)\, x^2 e^{-3x-2\ln(x)} e^{6x}}{27} \right) \right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.805.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y' \right) + (3x^2+2x)\, y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = \frac{2y}{x^2} - \frac{(3x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{(3x+2)y'}{x} - \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions
  $\left[P_2(x) = \frac{3x+2}{x}, P_3(x) = -\frac{2}{x^2}\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$
  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$
  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$
- $x = 0$is a regular singular point
  Check to see if $x_0 = 0$ is a regular singular point
  $x_0 = 0$

- Multiply by denominators
  $x^2\left(\frac{d}{dx}y'\right) + x(3x+2)\, y' - 2y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$
  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
- Shift index using $k- >k+1-m$
  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion
  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$
  Rewrite ODE with series expansions
  $a_0(2+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-1) + 3a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(2+r)\,(-1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 1\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_k(k+r+2) + 3a_{k-1}\right) = 0$
- Shift index using $k- >k+1$

$(k+r)\left(a_{k+1}(k+3+r)+3a_k\right)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=-\frac{3a_k}{k+3+r}$

- Recursion relation for $r=-2$

$a_{k+1}=-\frac{3a_k}{k+1}$

- Solution for $r=-2$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1}=-\frac{3a_k}{k+1}\right]$

- Recursion relation for $r=1$

$a_{k+1}=-\frac{3a_k}{k+4}$

- Solution for $r=1$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1}=-\frac{3a_k}{k+4}\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1}=-\frac{3a_k}{k+1}, b_{k+1}=-\frac{3b_k}{k+4}\right]$

1.805.3 Maple trace

```
Methods for second order ODEs:
```

1.805.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 27

```
dsolve(x^2*diff(diff(y(x),x),x)+(3*x^2+2*x)*diff(y(x),x)-2*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1\,\mathrm{e}^{-3x}+c_2(9x^2-6x+2)}{x^2}$$

1.805.5 Mathematica DSolve solution

Solving time : 0.021 (sec)
Leaf size : 35

```
DSolve[{x^2*D[y[x],{x,2}]+(2*x+3*x^2)*D[y[x],x]-2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1(9x^2 - 6x + 2) + 27c_2e^{-3x}}{27x^2}$$

1.806 problem 829

Internal problem ID [8944]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 829
Date solved : `Monday, October 21, 2024 at 05:24:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(x^2+x+1)\,y''+x(11x^2+11x+9)\,y'+(7x^2+10x+6)\,y=0$$

1.806.1 Solved as second order ode using Kovacic algorithm

Time used: 1.049 (sec)

Writing the ode as

$$(2x^4+2x^3+2x^2)\,y''+(11x^3+11x^2+9x)\,y'+(7x^2+10x+6)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+2x^3+2x^2 \\ B &= 11x^3+11x^2+9x \\ C &= 7x^2+10x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 21x^4 + 18x^3 + 27x^2 - 2x - 3$$
$$t = 16\left(x^3 + x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1534: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{1}{4x} + \frac{-\frac{5}{24} + \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{24} - \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{1}{8} - \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{1}{8} + \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} + \frac{i\sqrt{3}}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{6 + 6i\sqrt{3}}}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{6 + 6i\sqrt{3}}}{12} \end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} - \frac{i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6+6i\sqrt{3}}}{12}$
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{7}{4} - \left(\frac{7}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} + (0) \\
&= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\
&= \frac{7x^2+3x+1}{4x\left(x^2+x+1\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}\right) + \left(\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}\right) dx} \\
&= 2\left(x^2 + x + 1\right)^{3/4} \sqrt{2}\, x^{1/4} \mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{11x^3 + 11x^2 + 9x}{2x^4 + 2x^3 + 2x^2} \, dx} \\
&= z_1 e^{-\frac{\ln\left(x^2 + x + 1\right)}{4} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6} - \frac{9\ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{\left(x^2 + x + 1\right)^{1/4} x^{9/4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2\sqrt{x^2 + x + 1}\, \mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \sqrt{2}}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3 + 11x^2 + 9x}{2x^4 + 2x^3 + 2x^2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{\ln\left(x^2 + x + 1\right)}{2} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3} - \frac{9\ln(x)}{2}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{\ln\left(x^2 + x + 1\right)}{2} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3} - \frac{9\ln(x)}{2}} x^4 \mathrm{e}^{\frac{2\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2 + 8x + 8} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\right) \\
&\quad + c_2\left(\frac{2\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\sqrt{2}}{x^2}\left(\int \frac{\mathrm{e}^{-\frac{\ln\left(x^2+x+1\right)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}-\frac{9\ln(x)}{2}}x^4\mathrm{e}^{\frac{2\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8x^2+8x+8}\right.\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.806.2 Maple step by step solution

Let's solve

$2x^2(x^2+x+1)\left(\frac{d}{dx}y'\right)+x(11x^2+11x+9)\,y'+(7x^2+10x+6)\,y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(7x^2+10x+6)y}{2x^2(x^2+x+1)} - \frac{(11x^2+11x+9)y'}{2x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(11x^2+11x+9)y'}{2x(x^2+x+1)} + \frac{(7x^2+10x+6)y}{2x^2(x^2+x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{11x^2+11x+9}{2x(x^2+x+1)}, P_3(x)=\frac{7x^2+10x+6}{2x^2(x^2+x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=\frac{9}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=3$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(x^2+x+1)\left(\frac{d}{dx}y'\right)+x(11x^2+11x+9)\,y'+(7x^2+10x+6)\,y=0$
- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..2$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m=1..3$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m=2..4$

    $x^m \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m \cdot \left(\frac{d}{dx}y'\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(3+2r)\,x^r+\left(a_1(3+r)\,(5+2r)+a_0(5+2r)\,(2+r)\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r+2)\,(2k+\right.\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(3+2r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{-2,-\frac{3}{2}\right\}$
- Each term must be 0

  $a_1(3+r)\,(5+2r)+a_0(5+2r)\,(2+r)=0$
- Solve for the dependent coefficient(s)

  $a_1=-\frac{(2+r)a_0}{3+r}$

- Each term in the series must be 0, giving the recursion relation

$$2((a_k + a_{k-2} + a_{k-1})\, k + (a_k + a_{k-2} + a_{k-1})\, r + 2a_k - a_{k-2} + a_{k-1}) \left(k + r + \frac{3}{2}\right) = 0$$

- Shift index using $k- >k+2$

$$2((a_{k+2} + a_k + a_{k+1})\,(k+2) + (a_{k+2} + a_k + a_{k+1})\, r + 2a_{k+2} - a_k + a_{k+1}) \left(k + \frac{7}{2} + r\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{ka_k + ka_{k+1} + ra_k + ra_{k+1} + a_k + 3a_{k+1}}{k+4+r}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}, a_1 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{ka_k + ka_{k+1} - \frac{1}{2}a_k + \frac{3}{2}a_{k+1}}{k+\frac{5}{2}}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{ka_k + ka_{k+1} - \frac{1}{2}a_k + \frac{3}{2}a_{k+1}}{k+\frac{5}{2}}, a_1 = -\frac{a_0}{3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}, a_1 = 0, b_{k+2} = -\frac{kb_k + kb_{k+1} - \frac{1}{2}b_k +}{k+\frac{5}{2}}\right.$$

1.806.3 Maple trace

```
Methods for second order ODEs:
```

1.806.4 Maple dsolve solution

Solving time : `0.115` (sec)
Leaf size : 231

```
dsolve(2*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+x*(11*x^2+11*x+9)*diff(y(x),x)+(7*x^2+10*x
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}} \left(2x+1+i\sqrt{3}\right)^{\frac{5\sqrt{3}+3i}{6\sqrt{3}+6i}} \left(i\sqrt{3}-2x-1\right)^{\frac{64i\sqrt{3}+2368}{\left(\sqrt{3}+i\right)^3\left(i-\sqrt{3}\right)^4\left(13\sqrt{3}+9i\right)}} \left(\operatorname{HeunG}\left(\frac{\sqrt{3}+i}{i-\sqrt{3}}, 0, 0, \frac{5}{2}, \frac{1}{2}, \right.\right.}{x^{5/2}\left(x^2+x+1\right)}$$

1.806.5 Mathematica DSolve solution

Solving time : 1.056 (sec)
Leaf size : 93

```
DSolve[{2*x^2*(1+x+x^2)*D[y[x],{x,2}] + x*(9+11*x+11*x^2)*D[y[x],x] + (6+10*x+7*x^2)*y[x] ==
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{\sqrt{x^2+x+1}e^{-\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}}\left(c_2\int_1^x \frac{e^{\frac{\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}{\sqrt{3}}}}{\sqrt{K[1]}(K[1]^2+K[1]+1)^{3/2}}dK[1]+c_1\right)}{x^2}$$

1.807 problem 830

Internal problem ID [8945]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 830
Date solved : `Monday, October 21, 2024 at 05:24:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (1 + x)\, y' + 2y = 0$$

1.807.1 Solved as second order ode using Kovacic algorithm

Time used: 0.285 (sec)

Writing the ode as

$$xy'' + (1 + x)\, y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1 + x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1536: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2x} - \frac{5}{2x^2} - \frac{15}{2x^3} - \frac{115}{4x^4} - \frac{495}{4x^5} - \frac{2285}{4x^6} - \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-6x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{-1 + x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 6x - 1}{4x^2}\right) \right) = 0$$

$$\frac{1 + a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (-1 + x)\, e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= (-1 + x)\, e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= (-1 + x)\sqrt{x}\, e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1+x}{x}\,dx} \\ &= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}(-1+x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1+x}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{\mathrm{e}^{x}}{-1+x} - \mathrm{Ei}_1\left(-x\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}(-1+x)\right) + c_2\left(\mathrm{e}^{-x}(-1+x)\left(-\frac{\mathrm{e}^{x}}{-1+x} - \mathrm{Ei}_1\left(-x\right)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

1.807.2 Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y'\right)+(1+x)\,y'+2y=0$

- Highest derivative means the order of the ODE is 2

$\frac{d}{dx}y'$

- Isolate 2nd derivative

$\frac{d}{dx}y'=-\frac{2y}{x}-\frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$\frac{d}{dx}y'+\frac{(1+x)y'}{x}+\frac{2y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{1+x}{x},P_3(x)=\frac{2}{x}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

- ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$x\left(\frac{d}{dx}y'\right)+(1+x)\,y'+2y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ◦ Convert $x\cdot\left(\frac{d}{dx}y'\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k(k+r+2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 + a_k(k+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}\right]$$

1.807.3 Maple trace

```
Methods for second order ODEs:
```

1.807.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 29

```
dsolve(x*diff(diff(y(x),x),x)+(1+x)*diff(y(x),x)+2*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_2\,\mathrm{e}^{-x}(-1+x)\,\mathrm{Ei}_1\left(-x\right) + c_1\,\mathrm{e}^{-x}(-1+x) + c_2$$

1.807.5 Mathematica DSolve solution

Solving time : 0.091 (sec)
Leaf size : 33

```
DSolve[{x*D[y[x],{x,2}] +(1+x)*D[y[x],x]+2*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x}(c_2(x-1)\,\text{ExpIntegralEi}(x) + c_1(x-1) - c_2e^x)$$

1.808 problem 831

Internal problem ID [8946]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 831
Date solved : `Monday, October 21, 2024 at 05:24:14 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2(x^2-2x+1)\,y''-x(3+x)\,y'+(4+x)\,y=0$$

1.808.1 Solved as second order ode using Kovacic algorithm

Time used: 0.327 (sec)

Writing the ode as

$$x^2(x-1)^2\,y''+(-x^2-3x)\,y'+(4+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2(x-1)^2\\ B&=-x^2-3x\\ C&=4+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 10x - 1}{4x^2 (x - 1)^4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 7x^2 + 10x - 1$$
$$t = 4x^2(x - 1)^4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 10x - 1}{4x^2 (x - 1)^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1538: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2(x-1)^4$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{(x-1)^3} + \frac{3}{2x} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4} + \frac{7}{4(x-1)^2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{2}{(x-1)^3} + \frac{3}{2x} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4} + \frac{7}{4(x-1)^2} - \frac{1}{4x^2}$$

There is pole in r at $x = 1$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 1$ gives

$$[\sqrt{r}]_c \approx \frac{2}{(x-1)^2} - \frac{1}{2(x-1)} + \frac{21}{32} - \frac{9x}{32} + \frac{53(x-1)^2}{256} - \frac{149(x-1)^3}{1024} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{2}{(x-1)^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-1)^2}$ is

$$a = 2$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 1$. This term becomes $\frac{1}{(x-1)^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -2. Therefore

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{2}{(x-1)^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{-2}{2} + 2\right) &= \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{-2}{2} + 2\right) &= \frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2+10x-1}{4x^2(x-1)^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	4	$\frac{2}{(x-1)^2}$	$\frac{1}{2}$	$\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} + (-)(0) \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} \\ &= \frac{2x^2+x+1}{2x(x-1)^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{4}{(x-1)^3} - \frac{1}{2(x-1)^2}\right) + \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-\cdots}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right) dx} \\
&= \sqrt{x}\,\sqrt{x-1}\, e^{-\frac{2}{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2(x-1)^2}\, dx} \\
&= z_1 e^{\frac{3\ln(x)}{2} - \frac{2}{x-1} - \frac{3\ln(x-1)}{2}} \\
&= z_1 \left(\frac{x^{3/2} e^{-\frac{2}{x-1}}}{(x-1)^{3/2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{3/2} e^{-\frac{4}{x-1}} \sqrt{x(x-1)}}{(x-1)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2(x-1)^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{3\ln(x)-\frac{4}{x-1}-3\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\mathrm{e}^{-4} \operatorname{Ei}_1\left(-\frac{4}{x-1}-4\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}}{(x-1)^{3/2}}\right) + c_2 \left(\frac{x^{3/2}\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}}{(x-1)^{3/2}}\left(\mathrm{e}^{-4} \operatorname{Ei}_1\left(-\frac{4}{x-1}-4\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.808.2 Maple step by step solution

Let's solve

$x^2(x^2-2x+1)\left(\frac{d}{dx}y'\right) - x(3+x)\,y' + (4+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4+x)y}{x^2(x^2-2x+1)} + \frac{(3+x)y'}{x(x^2-2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(3+x)y'}{x(x^2-2x+1)} + \frac{(4+x)y}{x^2(x^2-2x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{3+x}{x(x^2-2x+1)}, P_3(x) = \frac{4+x}{x^2(x^2-2x+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2 - 2x + 1)\left(\frac{d}{dx}y'\right) - x(3 + x)\,y' + (4 + x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\,x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\,x^{k+r}$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\,x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2 + r)^2\,x^r + \left(a_1(-1 + r)^2 - a_0(1 + 2r)\,(-1 + r)\right)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k + r - 2)^2 - a_{k-1}(2k -\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2 + r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term must be 0

  $a_1(-1+r)^2 - a_0(1+2r)(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0(1+2r)}{-1+r}$
- Each term in the series must be 0, giving the recursion relation

  $((a_k + a_{k-2} - 2a_{k-1})k + (a_k + a_{k-2} - 2a_{k-1})r - 2a_k - 3a_{k-2} + a_{k-1})(k+r-2) = 0$
- Shift index using $k->k+2$

  $((a_{k+2} + a_k - 2a_{k+1})(k+2) + (a_{k+2} + a_k - 2a_{k+1})r - 2a_{k+2} - 3a_k + a_{k+1})(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{ka_k - 2ka_{k+1} + ra_k - 2ra_{k+1} - a_k - 3a_{k+1}}{k+r}$
- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{ka_k - 2ka_{k+1} + a_k - 7a_{k+1}}{k+2}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{ka_k - 2ka_{k+1} + a_k - 7a_{k+1}}{k+2}, a_1 = 5a_0\right]$

1.808.3 Maple trace

```
Methods for second order ODEs:
```

1.808.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 45

```
dsolve(x^2*(x^2-2*x+1)*diff(diff(y(x),x),x)-x*(3+x)*diff(y(x),x)+(4+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{x^2\left(c_2\,\mathrm{e}^{-\frac{4x}{x-1}}\,\mathrm{Ei}_1\left(-\frac{4x}{x-1}\right) + \mathrm{e}^{-\frac{4}{x-1}}c_1\right)}{x-1}$$

1.808.5 Mathematica DSolve solution

Solving time : 0.29 (sec)
Leaf size : 54

```
DSolve[{x^2*(1-2*x+x^2)*D[y[x],{x,2}] -x*(3+x)*D[y[x],x]+(4+x)*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-\frac{4x}{x-1}}\sqrt{1-x}x^2\left(c_2 \operatorname{ExpIntegralEi}\left(\frac{4x}{x-1}\right)+e^4c_1\right)}{(x-1)^{3/2}}$$

1.809 problem 832

Internal problem ID [8947]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 832
Date solved : `Monday, October 21, 2024 at 05:24:15 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y'' + 5x^2y' + (1+x)\,y = 0$$

1.809.1 Solved as second order ode using Kovacic algorithm

Time used: 0.229 (sec)

Writing the ode as

$$\left(2x^3+4x^2\right)y'' + 5x^2y' + (1+x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2 \\ B &= 5x^2 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 - 24x - 16$$
$$t = 16\left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1540: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{1}{16 + 8x} - \frac{1}{8x} + \frac{5}{16\,(2 + x)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 24x - 16}{16\,(x^2 + 2x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-3x^2-24x-16}{16\left(x^2+2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\
&= \frac{1}{4}-\left(\frac{1}{4}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(2+x\right)} + \frac{1}{2x} + (-)\left(0\right) \\
&= -\frac{1}{4\left(2+x\right)} + \frac{1}{2x} \\
&= \frac{x+4}{4x\left(2+x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4\left(2+x\right)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{4\left(2+x\right)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{4\left(2+x\right)} + \frac{1}{2x}\right)^2 - \left(\frac{-3x^2-24x-16}{16\left(x^2+2x\right)^2}\right)\right.$$

$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{4(2+x)} + \frac{1}{2x}\right)dx} \\
&= \frac{\sqrt{x}}{\left(2+x\right)^{1/4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x^2}{2x^3+4x^2}\, dx} \\
&= z_1 e^{-\frac{5\ln(2+x)}{4}} \\
&= z_1 \left(\frac{1}{\left(2+x\right)^{5/4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(2+x)^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2}{2x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(2+x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(2\sqrt{2+x} - 2\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{(2+x)^{3/2}}\right) + c_2 \left(\frac{\sqrt{x}}{(2+x)^{3/2}} \left(2\sqrt{2+x} - 2\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.809.2 Maple step by step solution

Let's solve

$2x^2(2+x)\left(\frac{d}{dx}y'\right) + 5x^2y' + (1+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(1+x)y}{2x^2(2+x)} - \frac{5y'}{2(2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + \frac{5y'}{2(2+x)} + \frac{(1+x)y}{2x^2(2+x)} = 0$

☐ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{5}{2(2+x)}, P_3(x) = \frac{1+x}{2x^2(2+x)}\right]$
  - $(2+x) \cdot P_2(x)$ is analytic at $x = -2$
    $\left((2+x) \cdot P_2(x)\right)\bigg|_{x=-2} = \frac{5}{2}$
  - $(2+x)^2 \cdot P_3(x)$ is analytic at $x = -2$
    $\left((2+x)^2 \cdot P_3(x)\right)\bigg|_{x=-2} = 0$
  - $x = -2$is a regular singular point
    Check to see if x_0 is a regular singular point
    $x_0 = -2$

- Multiply by denominators
  $2x^2(2+x)\left(\frac{d}{dx}y'\right) + 5x^2y' + (1+x)\,y = 0$
- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$
  $(2u^3 - 8u^2 + 8u)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (5u^2 - 20u + 20)\left(\frac{d}{du}y(u)\right) + (-1+u)\,y(u) = 0$
- Assume series solution for $y(u)$
  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$
    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k- >k - m$
    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$
    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $4a_0r(3+2r)\, u^{-1+r} + (4a_1(1+r)(5+2r) - a_0(8r^2+12r+1))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)(2k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r(3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{3}{2}\right\}$
- Each term must be 0

  $4a_1(1+r)(5+2r) - a_0(8r^2+12r+1) = 0$
- Each term in the series must be 0, giving the recursion relation

  $2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\, r - 12a_k - a_{k-1} + 28a_{k+1})\, k + 2(-4a_k +$
- Shift index using $k- >k+1$

  $2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\, r - 12a_{k+1} - a_k + 28a_{k+2})\,(k+1)$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} + 3ka_k - 28ka_{k+1} + 3ra_k - 28ra_{k+1} + a_k - 21a_{k+1}}{4(2k^2+4kr+2r^2+11k+11r+14)}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$
- Revert the change of variables $u = 2 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(2+x)^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$
- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 3ka_k - 4ka_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}$
- Solution for $r = -\frac{3}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - 3ka_k - 4ka_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$

- Revert the change of variables $u = 2 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (2 + x)^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (2 + x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (2 + x)^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a$$

1.809.3 Maple trace

```
Methods for second order ODEs:
```

1.809.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 39

```
dsolve(2*x^2*(2+x)*diff(diff(y(x),x),x)+5*x^2*diff(y(x),x)+(1+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\left(\sqrt{2}\,\sqrt{2+x}\,c_2 - 2\,\operatorname{arctanh}\left(\frac{\sqrt{2+x}\,\sqrt{2}}{2}\right) c_2 + c_1\right)\sqrt{x}}{(2+x)^{3/2}}$$

1.809.5 Mathematica DSolve solution

Solving time : `0.107` (sec)
Leaf size : 55

```
DSolve[{2*x^2*(2+x)*D[y[x],{x,2}] +5*x^2*D[y[x],x]+(1+x)*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{x}\left(-2\sqrt{2}c_2 \operatorname{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right) + 2c_2\sqrt{x+2} + c_1\right)}{(x+2)^{3/2}}$$

1.810 problem 833

Internal problem ID [8948]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 833
Date solved : `Monday, October 21, 2024 at 05:24:16 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 4xy' + \left(x^2+2\right)y = 0$$

1.810.1 Solved as second order ode using Kovacic algorithm

Time used: 0.168 (sec)

Writing the ode as

$$x^2y'' + 4xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1542: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\,dx} \\ &= z_1 e^{-2\ln(x)} \\ &= z_1\left(\frac{1}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x^2}\right) + c_2 \left(\frac{\cos(x)}{x^2} (\tan(x))\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.810.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + 4xy' + (x^2 + 2) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(x^2+2) y}{x^2} - \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{4y'}{x} + \frac{(x^2+2) y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{4}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 \left(\frac{d}{dx} y'\right) + 4xy' + (x^2 + 2) y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (2+r)(1+r) x^r + a_1 (3+r)(2+r) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k+r+2)(k+r+1) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$

- Each term must be 0

  $a_1 (3+r)(2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k (k+r+2)(k+r+1) + a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $a_{k+2} (k+4+r)(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+4+r)(k+3+r)}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$
- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$$

1.810.3 Maple trace

```
Methods for second order ODEs:
```

1.810.4 Maple dsolve solution

Solving time : `0.005` (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{x^2}$$

1.810.5 Mathematica DSolve solution

Solving time : `0.048` (sec)
Leaf size : 37

```
DSolve[{x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x^2}$$

1.811 problem 834

Internal problem ID [8949]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 834
Date solved : `Monday, October 21, 2024 at 05:24:17 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

1.811.1 Solved as second order ode using Kovacic algorithm

Time used: 0.178 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1544: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.811.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) + 4xy' + (4x^2 - 1)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)(-1+2r)x^r + a_1(3+2r)(1+2r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)(2k+2r-1) + 4a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

1.811.3 Maple trace

```
Methods for second order ODEs:
```

1.811.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1\sin\left(x\right)+c_2\cos\left(x\right)}{\sqrt{x}}$$

1.811.5 Mathematica DSolve solution

Solving time : `0.048` (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{e^{-ix}\left(2c_1-ic_2e^{2ix}\right)}{2\sqrt{x}}$$

1.812 problem 835

Internal problem ID [8950]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 835
Date solved : `Monday, October 21, 2024 at 05:24:18 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - xy' - \left(x^2 + \frac{5}{4}\right) y = 0$$

1.812.1 Solved as second order ode using Kovacic algorithm

Time used: 0.286 (sec)

Writing the ode as

$$x^2y'' - xy' + \left(-x^2 - \frac{5}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= -x^2 - \frac{5}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1546: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{x^2} - \frac{1}{2x^4} + \frac{1}{2x^6} - \frac{5}{8x^8} + \frac{7}{8x^{10}} - \frac{21}{16x^{12}} + \frac{33}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (1) + \left(\frac{2}{x^2}\right) \\ &= 1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{1} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{1} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(1) \\
&= -\frac{1}{x} - 1 \\
&= -\frac{1+x}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{x} - 1\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - 1\right)^2 - \left(\frac{x^2+2}{x^2}\right)\right) &= 0 \\
\frac{-2+2a_0}{x} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1+x)\, e^{\int \left(-\frac{1}{x} - 1\right) dx} \\
&= (1+x)\, e^{-x-\ln(x)} \\
&= \frac{(1+x)\, e^{-x}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(-1+x)\,\mathrm{e}^{2x}}{2+2x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}}\right) + c_2\left(\frac{(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}}\left(\frac{(-1+x)\,\mathrm{e}^{2x}}{2+2x}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.812.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - xy' - \left(x^2 + \frac{5}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{(4x^2+5)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'}{x} - \frac{(4x^2+5)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{4x^2+5}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{5}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4xy' + (-4x^2 - 5)y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-5+2r)\, x^r + a_1(3+2r)\,(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-5) - 4a$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)\,(-5+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{1}{2}, \frac{5}{2}\right\}$$

- Each term must be 0

  $$a_1(3+2r)\,(-3+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$4\left(k+r+\frac{1}{2}\right)\left(k+r-\frac{5}{2}\right) a_k - 4a_{k-2} = 0$$

- Shift index using $k- > k+2$

  $$4\left(k+\frac{5}{2}+r\right)\left(k-\frac{1}{2}+r\right) a_{k+2} - 4a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{4a_k}{(2k+5+2r)(2k-1+2r)}$$

- Recursion relation for $r = -\frac{1}{2}$

  $$a_{k+2} = \frac{4a_k}{(2k+4)(2k-2)}$$

- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4a_k}{(2k+4)(2k-2)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{5}{2}$

  $$a_{k+2} = \frac{4a_k}{(2k+10)(2k+4)}$$

- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = \frac{4a_k}{(2k+10)(2k+4)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = \frac{4a_k}{(2k+4)(2k-2)}, a_1 = 0, b_{k+2} = \frac{4b_k}{(2k+10)(2k+4)}, b_1 = 0\right]$$

1.812.3 Maple trace

```
Methods for second order ODEs:
```

1.812.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 25

```
dsolve(x^2*diff(diff(y(x),x),x)-x*diff(y(x),x)-(x^2+5/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{(1+x) c_2 e^{-x} + c_1 e^{x}(-1+x)}{\sqrt{x}}$$

1.812.5 Mathematica DSolve solution

Solving time : 0.118 (sec)
Leaf size : 53

```
DSolve[{x^2*D[y[x],{x,2}]-x*D[y[x],x]-(x^2+5/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{\frac{2}{\pi}}((ic_2x + c_1)\sinh(x) - (c_1x + ic_2)\cosh(x))}{\sqrt{-ix}}$$

1.813 problem 836

Internal problem ID [8951]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 836
Date solved : `Monday, October 21, 2024 at 05:24:19 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

1.813.1 Solved as second order ode using Kovacic algorithm

Time used: 0.167 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1548: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.813.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx}y'\right) + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 \left(\frac{d}{dx}y'\right) + 4xy' + (4x^2 - 1) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k + 2)^2 + 8(k + 2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

1.813.3 Maple trace

```
Methods for second order ODEs:
```

1.813.4 Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1\sin(x)+c_2\cos(x)}{\sqrt{x}}$$

1.813.5 Mathematica DSolve solution

Solving time : 0.042 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{e^{-ix}(2c_1-ic_2e^{2ix})}{2\sqrt{x}}$$

1.814 problem 837

Internal problem ID [8952]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 837
Date solved : `Monday, October 21, 2024 at 05:24:20 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$x^2y'' + 3xy' + 4x^4y = 0$$

1.814.1 Solved as second order ode using Kovacic algorithm

Time used: 0.309 (sec)

Writing the ode as

$$x^2y'' + 3xy' + 4x^4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x \\ C &= 4x^4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1550: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(-4x^2\right) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 2ix\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-16x^4 + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right)\\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)\,(2ix) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4 + 3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\
&= \frac{\mathrm{e}^{-ix^2}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x}{x^2} \, dx} \\
&= z_1 e^{-\frac{3 \ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{3/2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^{-i x^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{i e^{2 i x^2}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-i x^2}}{x^2}\right) + c_2 \left(\frac{e^{-i x^2}}{x^2} \left(-\frac{i e^{2 i x^2}}{4}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

1.814.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + 3 x y' + 4 x^4 y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -4 x^2 y - \frac{3 y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{3 y'}{x} + 4 x^2 y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4yx^3 + \left(\frac{d}{dx}y'\right)x + 3y' = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

  $$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$

- Shift index using $k- >k-3$

  $$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^{r} + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 0\}$
- The coefficients of each power of x must be 0
  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$
- Solve for the dependent coefficient(s)
  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$
- Shift index using $k- > k+3$
  $a_{k+4}(k+4+r)\,(k+6+r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$
- Recursion relation for $r = -2$
  $a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$
- Solution for $r = -2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Recursion relation for $r = 0$
  $a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{4b_k}{(k+4)(k+6)}\right.$

1.814.3 Maple trace

```
Methods for second order ODEs:
```

1.814.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 21

```
dsolve(x^2*diff(diff(y(x),x),x)+3*x*diff(y(x),x)+4*x^4*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x^2) + c_2 \cos(x^2)}{x^2}$$

1.814.5 Mathematica DSolve solution

Solving time : `0.074` (sec)
Leaf size : 41

```
DSolve[{x^2*D[y[x],{x,2}]+3*x*D[y[x],x]+4*x^4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4c_1 e^{-ix^2} - ic_2 e^{ix^2}}{4x^2}$$

1.815 problem 838

Internal problem ID [8953]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 838
Date solved : `Monday, October 21, 2024 at 05:24:21 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = \left(x^2+3\right) y$$

1.815.1 Solved as second order ode using Kovacic algorithm

Time used: 0.242 (sec)

Writing the ode as

$$y''+\left(-x^2-3\right) y = 0 \tag{1}$$
$$Ay''+By'+Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -x^2-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 3$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 + 3)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1552: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{2x} - \frac{9}{8x^3} + \frac{27}{16x^5} - \frac{405}{128x^7} + \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} + \frac{72171}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+3}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2+3) + (0) \\ &= x^2+3 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{3}{1} - 1\right) &= 1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{3}{1} - 1\right) &= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 + 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$
\begin{aligned}
d &= \alpha_\infty^+ \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= (+)[\sqrt{r}]_\infty \\
&= 0 + (x) \\
&= x \\
&= x
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2(x)(1) + \left((1) + (x)^2 - \left(x^2 + 3\right)\right) &= 0 \\
-2a_0 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, \mathrm{e}^{\int x dx} \\
&= (x) \, \mathrm{e}^{\frac{x^2}{2}} \\
&= x \, \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= x \, \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
$$

Which simplifies to

$$
y_1 = x \, \mathrm{e}^{\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= x \, \mathrm{e}^{\frac{x^2}{2}} \int \frac{1}{x^2 \mathrm{e}^{x^2}} \, dx \\
&= x \, \mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi} \, \mathrm{erf}\,(x) \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x \, \mathrm{e}^{\frac{x^2}{2}} \right) + c_2 \left(x \, \mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi} \, \mathrm{erf}\,(x) \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.815.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' = (x^2+3)\,y$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + (-x^2-3)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\,x^{k-2}$

- ○ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\,x^k$

Rewrite ODE with series expansions

$2a_2 - 3a_0 + (6a_3 - 3a_1)\,x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - 3a_k - a_{k-2})\,x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 3a_0 = 0, 6a_3 - 3a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\,a_{k+2} - 3a_k - a_{k-2} = 0$

- Shift index using $k- >k+2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 3a_{k+2} - a_k = 0$

- Recursion relation that defines the series solution to the ODE
$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{3a_{k+2}+a_k}{k^2+7k+12}, a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right]$$

1.815.3 Maple trace

```
Methods for second order ODEs:
```

1.815.4 Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x) = (x^2+3)*y(x),
       y(x),singsol=all)
```

$$y = x\left(c_2\sqrt{\pi}\, \operatorname{erf}\left(x\right) + c_1\right) \mathrm{e}^{\frac{x^2}{2}} + \mathrm{e}^{-\frac{x^2}{2}} c_2$$

1.815.5 Mathematica DSolve solution

Solving time : 0.092 (sec)
Leaf size : 46

```
DSolve[{D[y[x],{x,2}]==(x^2+3)*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{x^2}{2}}\left(-\sqrt{\pi}c_2e^{x^2}x\text{erf}(x) + c_1e^{x^2}x - c_2\right)$$

1.816 problem 839

Internal problem ID [8954]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 839
Date solved : `Monday, October 21, 2024 at 05:24:22 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 2xy' + \left(x^2+1\right) y = 0$$

1.816.1 Solved as second order ode using Kovacic algorithm

Time used: 0.115 (sec)

Writing the ode as

$$y'' + 2xy' + \left(x^2+1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x \\ C &= x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1554: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{1}\,dx} \\
&= z_1 e^{-\frac{x^2}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{2}}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.816.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + 2xy' + (x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k+2$

  $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + a_0 + (6a_3 + 3a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(2k+1) + a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + a_0 = 0, 6a_3 + 3a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} + 2a_k k + a_k + a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $\left((k + 2)^2 + 3k + 8\right) a_{k+4} + 2a_{k+2}(k + 2) + a_{k+2} + a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2ka_{k+2}+a_k+5a_{k+2}}{k^2+7k+12}, a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{2}\right]$

1.816.3 Maple trace

```
Methods for second order ODEs:
```

1.816.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+2*x*diff(y(x),x)+(x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-\frac{x^2}{2}}(c_2 x + c_1)$$

1.816.5 Mathematica DSolve solution

Solving time : `0.037` (sec)
Leaf size : 22

```
DSolve[{D[y[x],{x,2}]+2*x*D[y[x],x]+(x^2+1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{x^2}{2}}(c_2 x + c_1)$$

1.817 problem 840

Internal problem ID [8955]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 840
Date solved : `Monday, October 21, 2024 at 05:24:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

1.817.1 Solved as second order ode using Kovacic algorithm

Time used: 0.165 (sec)

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1556: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.817.2 Maple step by step solution

Let's solve

$x^2 \left(\frac{d}{dx} y'\right) + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Isolate 2nd derivative

  $\frac{d}{dx} y' = -\frac{(4x^2-1) y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx} y' + \frac{y'}{x} + \frac{(4x^2-1) y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 \left(\frac{d}{dx} y'\right) + 4x y' + (4x^2 - 1) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

  - Convert $x^2 \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term must be 0

  $$a_1(3+2r)\,(1+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$$

- Shift index using $k- >k + 2$

  $$a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$$

- Recursion relation for $r = -\frac{1}{2}$

  $$a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$$

1.817.3 Maple trace

```
Methods for second order ODEs:
```

1.817.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{\sqrt{x}}$$

1.817.5 Mathematica DSolve solution

Solving time : `0.046` (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

1.818 problem 841

Internal problem ID [8956]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 841
Date solved : `Monday, October 21, 2024 at 05:24:23 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$4x^2y'' + \left(-8x^2 + 4x\right) y' + \left(4x^2 - 4x - 1\right) y = 0$$

1.818.1 Solved as second order ode using Kovacic algorithm

Time used: 0.119 (sec)

Writing the ode as

$$4x^2y'' + \left(-8x^2 + 4x\right) y' + \left(4x^2 - 4x - 1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2 + 4x \\ C &= 4x^2 - 4x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1558: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\,dx} \\ &= z_1 e^{x-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^x}{\sqrt{x}}\right) + c_2 \left(\frac{\mathrm{e}^x}{\sqrt{x}}(x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.818.2 Maple step by step solution

Let's solve

$4x^2\left(\frac{d}{dx}y'\right) + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(4x^2-4x-1)y}{4x^2} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{(2x-1)y'}{x} + \frac{(4x^2-4x-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2-4x-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2\left(\frac{d}{dx}y'\right) - 4x(2x-1)y' + \left(4x^2-4x-1\right)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$
  - Shift index using $k- >k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-1+2r)\, x^r + \left(a_1(3+2r)\,(1+2r) - 4a_0(1+2r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)\,(-1+2r) = 0$$
- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$
- Each term must be 0

  $$a_1(3+2r)\,(1+2r) - 4a_0(1+2r) = 0$$
- Solve for the dependent coefficient(s)

  $$a_1 = \frac{4a_0}{3+2r}$$
- Each term in the series must be 0, giving the recursion relation

  $$a_k(4k^2 + 8kr + 4r^2 - 1) + (-8k - 8r + 4)\, a_{k-1} + 4a_{k-2} = 0$$
- Shift index using $k- >k + 2$

  $$a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + (-8k - 12 - 8r)\, a_{k+1} + 4a_k = 0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{4(2ka_{k+1} + 2ra_{k+1} - a_k + 3a_{k+1})}{4k^2 + 8kr + 4r^2 + 16k + 16r + 15}$$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1 = 2a_0\right]$$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}, a_1 = a_0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1 = 2a_0, b_{k+2} = \frac{4(2kb_{k+1}-b_k+4b_{k+1}}{4k^2+20k+24}\right.$$

1.818.3 Maple trace

```
Methods for second order ODEs:
```

1.818.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 15

```
dsolve(4*x^2*diff(diff(y(x),x),x)+(-8*x^2+4*x)*diff(y(x),x)+(4*x^2-4*x-1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \frac{e^x (c_2 x + c_1)}{\sqrt{x}}$$

1.818.5 Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 21

```
DSolve[{4*x^2*D[y[x],{x,2}]+(-8*x^2+4*x)*D[y[x],x]+(4*x^2-4*x-1)*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^x (c_2 x + c_1)}{\sqrt{x}}$$

1.819 problem 843

Internal problem ID [8957]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 843
Date solved : `Monday, October 21, 2024 at 05:24:24 PM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y'' = 0$$

1.819.1 Solved as second order ode using Kovacic algorithm

Time used: 0.049 (sec)

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1560: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(1) + c_2(1(x))
\end{aligned}
$$

Will add steps showing solving for IC soon.

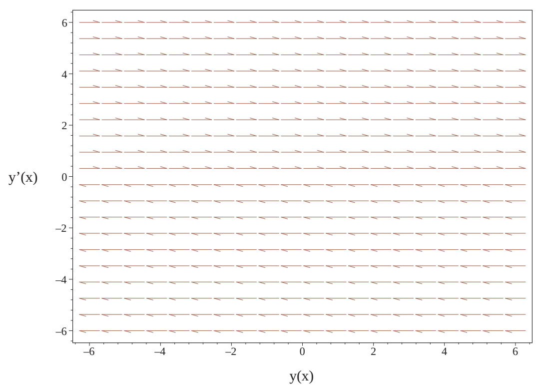

Figure 4: Slope field plot
$y'' = 0$

1.819.2 Maple step by step solution

Let's solve
$\frac{d}{dx}y' = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d}{dx}y'$
- Characteristic polynomial of ODE
$r^2 = 0$
- Use quadratic formula to solve for r
$r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$
- Roots of the characteristic polynomial
$r = 0$
- 1st solution of the ODE

$y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence
  $y_2(x) = x$
- General solution of the ODE
  $y = C1y_1(x) + C2y_2(x)$
- Substitute in solutions
  $y = C2x + C1$

1.819.3 Maple trace

```
Methods for second order ODEs:
```

1.819.4 Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 9

```
dsolve(diff(diff(y(x),x),x) = 0,
       y(x),singsol=all)
```

$$y = c_1 x + c_2$$

1.819.5 Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
DSolve[{D[y[x],{x,2}]==((4*(1/2)^2-1)/(4*x^2))*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x + c_1$$

1.820 problem 844

Internal problem ID [8958]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 844
Date solved : `Monday, October 21, 2024 at 05:24:25 PM`
CAS classification :
`[[_Emden, _Fowler], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']]`

Solve

$$y'' = \frac{2y}{x^2}$$

1.820.1 Solved as second order ode using Kovacic algorithm

Time used: 0.159 (sec)

Writing the ode as

$$y'' - \frac{2y}{x^2} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{2}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1562: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(0) \\ &= -\frac{1}{x} \\ &= -\frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \frac{1}{x}
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= \frac{1}{x} \int \frac{1}{\frac{1}{x^2}} \, dx \\
&= \frac{1}{x}\left(\frac{x^3}{3}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

1.820.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' = \frac{2y}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y}{x^2} = 0$

- Multiply by denominators of the ODE

  $\left(\frac{d}{dx}y'\right)x^2 - 2y = 0$

- Make a change of variables

  $t = \ln(x)$

- ☐ Substitute the change of variables back into the ODE
  - ○ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - ○ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - ○ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y' = \left(\frac{d}{dt}\frac{d}{dt}y(t)\right)t'(x)^2 + \left(\frac{d}{dx}t'(x)\right)\left(\frac{d}{dt}y(t)\right)$

  - ○ Compute derivative

    $\frac{d}{dx}y' = \frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - 2y(t) = 0$

- Simplify

  $\frac{d}{dt}\frac{d}{dt}y(t) - \frac{d}{dt}y(t) - 2y(t) = 0$

- Characteristic polynomial of ODE

$r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r + 1)(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the ODE

  $y_1(t) = e^{-t}$

- 2nd solution of the ODE

  $y_2(t) = e^{2t}$

- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t)$

- Substitute in solutions

  $y(t) = C1\, e^{-t} + C2\, e^{2t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{C1}{x} + C2\, x^2$

- Simplify

  $y = \frac{C1}{x} + C2\, x^2$

1.820.3 Maple trace

```
Methods for second order ODEs:
```

1.820.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(diff(diff(y(x),x),x) = 2/x^2*y(x),
       y(x),singsol=all)
```

$$y = \frac{c_2\, x^3 + c_1}{x}$$

1.820.5 Mathematica DSolve solution

Solving time : 0.016 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]==((4*(3/2)^2-1)/(4*x^2))*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x^3 + c_1}{x}$$

1.821 problem 845

Internal problem ID [8959]
Book : Collection of Kovacic problems
Section : section 1
Problem number : 845
Date solved : `Monday, October 21, 2024 at 05:24:26 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$y'' = \frac{6y}{x^2}$$

1.821.1 Solved as second order ode using Kovacic algorithm

Time used: 0.165 (sec)

Writing the ode as

$$y'' - \frac{6y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 6$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1564: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -2$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -2 - (-2) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + (-)(0) \\ &= -\frac{2}{x} \\ &= -\frac{2}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x}\right)(0) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x}\right)^2 - \left(\frac{6}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int -\frac{2}{x} dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x^2} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \frac{1}{x^2} \int \frac{1}{\frac{1}{x^4}}\, dx \\ &= \frac{1}{x^2}\left(\frac{x^5}{5}\right) \end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1\left(\frac{1}{x^2}\right) + c_2\left(\frac{1}{x^2}\left(\frac{x^5}{5}\right)\right)$$

Will add steps showing solving for IC soon.

1.821.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' = \frac{6y}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{6y}{x^2} = 0$
- Multiply by denominators of the ODE

  $\left(\frac{d}{dx}y'\right)x^2 - 6y = 0$
- Make a change of variables

  $t = \ln(x)$
- □ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$
  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y' = \left(\frac{d}{dt}\frac{d}{dt}y(t)\right)t'(x)^2 + \left(\frac{d}{dx}t'(x)\right)\left(\frac{d}{dt}y(t)\right)$
  - Compute derivative

    $\frac{d}{dx}y' = \frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - 6y(t) = 0$
- Simplify

  $\frac{d}{dt}\frac{d}{dt}y(t) - \frac{d}{dt}y(t) - 6y(t) = 0$
- Characteristic polynomial of ODE

$r^2 - r - 6 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-3) = 0$
- Roots of the characteristic polynomial

  $r = (-2, 3)$
- 1st solution of the ODE

  $y_1(t) = e^{-2t}$
- 2nd solution of the ODE

  $y_2(t) = e^{3t}$
- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t)$
- Substitute in solutions

  $y(t) = C1\, e^{-2t} + C2\, e^{3t}$
- Change variables back using $t = \ln(x)$

  $y = \frac{C1}{x^2} + C2\, x^3$
- Simplify

  $y = \frac{C1}{x^2} + C2\, x^3$

1.821.3 Maple trace

```
Methods for second order ODEs:
```

1.821.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 15

```
dsolve(diff(diff(y(x),x),x) = 6/x^2*y(x),
       y(x),singsol=all)
```

$$y = \frac{c_2\, x^5 + c_1}{x^2}$$

1.821.5 Mathematica DSolve solution

Solving time : 0.016 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]==((4*(5/2)^2-1)/(4*x^2))*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2x^5 + c_1}{x^2}$$

2 section 2. Solution found using all possible Kovacic cases

2.1 problem 1

Internal problem ID [8960]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 1
Date solved : `Monday, October 21, 2024 at 05:24:27 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = \left(-\frac{3}{16x^2} - \frac{2}{9(x-1)^2} + \frac{3}{16x(x-1)}\right) y$$

2.1.1 Solved as second order ode using Kovacic algorithm

Time used: 1.162 (sec)

Writing the ode as

$$y'' + \frac{(32x^2 - 27x + 27)\, y}{144x^2 (x-1)^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{32x^2 - 27x + 27}{144x^2 (x-1)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-32x^2 + 27x - 27}{144 (x^2 - x)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -32x^2 + 27x - 27 \\
t &= 144(x^2 - x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-32x^2 + 27x - 27}{144 (x^2 - x)^2} \right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1566: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Unable to find solution using case two.

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{3}{16x} - \frac{2}{9(x-1)^2} - \frac{3}{16x^2} + \frac{3}{16(x-1)}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{3}{16}$. Hence

$$
E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} \middle| k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}
$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$
E_c = \{3, 6, 9\}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{2}{9}$. Hence

$$
E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} \middle| k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}
$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{4, 5, 6, 7, 8\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx -\frac{2}{9x^2} - \frac{37}{144x^3} - \frac{23}{48x^4} - \frac{101}{144x^5} - \frac{133}{144x^6} - \frac{55}{48x^7} + \cdots$$

The above shows that

$$b = -\frac{2}{9}$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{4, 5, 6, 7, 8\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{3, 6, 9\}$
1	2	$\{4, 5, 6, 7, 8\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{4, 5, 6, 7, 8\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \ldots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 3,\ e_2 = 4,\ e_\infty = 7$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}
d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{4}{12}(7-(3+(4))) \\
&= 0
\end{aligned}$$

The following rational function is

$$\begin{aligned}
\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\
&= \frac{4}{12}\left(\frac{3}{(x-(0))}+\frac{4}{(x-(1))}\right) \\
&= \frac{1}{x}+\frac{4}{3x-3}
\end{aligned}$$

And

$$\begin{aligned}
S &= \prod_{c\in\Gamma}(x-c) \\
&= x(x-1)
\end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d=0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}
P_n &= -p(x) \\
&= -1 \\
P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \ldots, 0
\end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n=4$ and $r=$

$\frac{-32x^2+27x-27}{144(x^2-x)^2}$).

$$\begin{aligned}
P_4 &= -p \\
&= -1 \\
P_3 &= \frac{7x}{3} - 1 \\
P_2 &= -4x^2 + \frac{41}{12}x - \frac{3}{4} \\
P_1 &= \frac{40}{9}x^3 - \frac{409}{72}x^2 + \frac{5}{2}x - \frac{3}{8} \\
P_0 &= -\frac{64}{27}x^4 + \frac{871}{216}x^3 - \frac{257}{96}x^2 + \frac{13}{16}x - \frac{3}{32} \\
P_{-1} &= 0
\end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!}\omega^i = 0$$
$$\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!}\omega^i = 0$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$-\frac{8x^4}{81} + \frac{871x^3}{5184} - \frac{257x^2}{2304} + \frac{13x}{384} - \frac{1}{256} + \frac{x(x-1)\left(320x^3 - 409x^2 + 180x - 27\right)\omega}{432} - \frac{x^2(x-1)^2\left(48x^2 - 41x + 9\right)\omega^2}{24} + x^3(x-1)^3\left(\frac{7x}{3} - 1\right)\omega^3 - x^4(x-1)^4\omega^4 = 0$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{12x(x-1)}\left(7x-3+\sqrt{x^2+\left((x-1)^2x^3\right)^{1/3}-x} + \sqrt{-\frac{2\left(\left(-x^2+x+\frac{\left((x-1)^2x^3\right)^{1/3}}{2}\right)\sqrt{x^2+\left((x-1)^2x^3\right)^{1/3}-x}+x^2(x-1)\right)}{\sqrt{x^2+\left((x-1)^2x^3\right)^{1/3}-x}}}\right) \tag{4A}$$

This ω is used to find a solution to $z''=rz$.

$$z_1(x) = e^{\int \omega\, dx} \tag{5A}$$

Unable to integrate $\int \omega dx$. Leaving the integral unevaluated. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{\int \omega\, dx} \end{aligned}$$

Where ω given above. The second solution y_2 to the original ode is found using reduction of order

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx \\ &= e^{\int \omega\, dx} \int \frac{e^{\int -\frac{B}{A}\, dx}}{\left(e^{\int \omega\, dx}\right)^2}\, dx \end{aligned}$$

Since $B=0$ then the above reduces to

$$y_2 = e^{\int \omega\, dx} \int \left(e^{\int \omega\, dx}\right)^{-2}\, dx$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{\int \omega \, dx}\right) + c_2\left(e^{\int \omega \, dx} \int \left(e^{\int \omega \, dx}\right)^{-2} dx\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

2.1.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' = \left(-\frac{3}{16x^2} - \frac{2}{9(x-1)^2} + \frac{3}{16x(x-1)}\right) y$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(32x^2-27x+27)y}{144x^2(x-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(32x^2-27x+27)y}{144x^2(x-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{32x^2-27x+27}{144x^2(x-1)^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{3}{16}$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $144x^2(x-1)^2\left(\frac{d}{dx}y'\right) + (32x^2 - 27x + 27)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx} y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$9a_0(-1+4r)(-3+4r)\, x^r + (9a_1(3+4r)(1+4r) - 9a_0(32r^2 - 32r + 3))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (9a_k(4k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$9(-1+4r)(-3+4r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$$

- Each term must be 0

$$9a_1(3+4r)(1+4r) - 9a_0(32r^2 - 32r + 3) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = \frac{a_0(32r^2 - 32r + 3)}{16r^2 + 16r + 3}$$

- Each term in the series must be 0, giving the recursion relation

$$144(a_k + a_{k-2} - 2a_{k-1})\, k^2 + 144(2(a_k + a_{k-2} - 2a_{k-1})\, r - a_k - 5a_{k-2} + 6a_{k-1})\, k + 144(a_k + a_{k-}$$

- Shift index using $k- >k + 2$

$$144(a_{k+2} + a_k - 2a_{k+1})\,(k+2)^2 + 144(2(a_{k+2} + a_k - 2a_{k+1})\, r - a_{k+2} - 5a_k + 6a_{k+1})\,(k+2) + 1$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} + 288kra_k - 576kra_{k+1} + 144r^2a_k - 288r^2a_{k+1} - 144ka_k - 288ka_{k+1} - 144ra_k - 288ra_{k+1} + 32a_k - 27}{9(16k^2 + 32kr + 16r^2 + 48k + 48r + 35)}$$

- Recursion relation for $r = \frac{1}{4}$

$$a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} - 72ka_k - 432ka_{k+1} + 5a_k - 117a_{k+1}}{9(16k^2 + 56k + 48)}$$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{144k^2 a_k - 288k^2 a_{k+1} - 72k a_k - 432k a_{k+1} + 5a_k - 117a_{k+1}}{9(16k^2+56k+48)}, a_1 = -\frac{3a_0}{8}\right]$$

- Recursion relation for $r = \frac{3}{4}$

$$a_{k+2} = -\frac{144k^2 a_k - 288k^2 a_{k+1} + 72k a_k - 720k a_{k+1} + 5a_k - 405a_{k+1}}{9(16k^2+72k+80)}$$

- Solution for $r = \frac{3}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+2} = -\frac{144k^2 a_k - 288k^2 a_{k+1} + 72k a_k - 720k a_{k+1} + 5a_k - 405a_{k+1}}{9(16k^2+72k+80)}, a_1 = -\frac{a_0}{8}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+2} = -\frac{144k^2 a_k - 288k^2 a_{k+1} - 72k a_k - 432k a_{k+1} + 5a_k - 117a_{k+1}}{9(16k^2+56k+48)}, a_1 = -\right.$$

2.1.3 Maple trace

```
Methods for second order ODEs:
```

2.1.4 Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x) = (-3/16/x^2-2/9/(x-1)^2+3/16/x/(x-1))*y(x),
       y(x),singsol=all)
```

$$y = \sqrt{x-1}\, x^{1/4}\left(c_1 \operatorname{LegendreP}\left(-\frac{1}{6}, \frac{1}{3}, \sqrt{x}\right) + c_2 \operatorname{LegendreQ}\left(-\frac{1}{6}, \frac{1}{3}, \sqrt{x}\right)\right)$$

2.1.5 Mathematica DSolve solution

Solving time : `0.375` (sec)
Leaf size : 550

```
DSolve[{D[y[x],{x,2}]== ( -3/(16*x^2)- 2/(9*(x-1)^2) + 3/(16*x*(x-1))) *y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to{} & c_1 \exp\left(\int_1^x \mathrm{Root}\left[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2 - 702K[1]\right.\right.\\
& + \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4\\
& + \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right)\#1^3\\
& + \left(41472K[1]^6 - 118368K[1]^5 + 120096K[1]^4 - 50976K[1]^3 + 7776K[1]^2\right)\#1^2\\
& + \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right)\#1\\
& \left.\left. + 81\&, 1\right] dK[1]\right) + c_2 \exp\left(\int_1^x \mathrm{Root}\left[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2 - 702K[1]\right.\right.\\
& + \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4\\
& + \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right)\#1^3\\
& + \left(41472K[1]^6 - 118368K[1]^5 + 120096K[1]^4 - 50976K[1]^3 + 7776K[1]^2\right)\#1^2\\
& + \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right)\#1\\
& \left.\left. + 81\&, 1\right] dK[1]\right) \int_1^x \exp\left(-2\int_1^{K[2]} \mathrm{Root}\left[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2\right.\right.\\
& -702K[1] + \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4\\
& + \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right)\#1^3 + \left(41472K[1]^6 - 11836\right.\\
& \left.\left. + \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right)\#1 + 81\&, 1\right] dK[1]\right) dK[2]
\end{aligned}
$$

2.2 problem 2

Internal problem ID [8961]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 2
Date solved : `Monday, October 21, 2024 at 05:24:29 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$y'' = \frac{20y}{x^2}$$

2.2.1 Solved as second order ode using Kovacic algorithm

Time used: 0.138 (sec)

Writing the ode as

$$y'' - \frac{20y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{20}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{20}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 20$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{20}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1568: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{20}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 5 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -4
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{20}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 20$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 5 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -4
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{20}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	5	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -4$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -4 - (-4) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{4}{x} + (-)(0) \\
&= -\frac{4}{x} \\
&= -\frac{4}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{4}{x}\right)(0) + \left(\left(\frac{4}{x^2}\right) + \left(-\frac{4}{x}\right)^2 - \left(\frac{20}{x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{4}{x} dx} \\
&= \frac{1}{x^4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \frac{1}{x^4}
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= \frac{1}{x^4} \int \frac{1}{\frac{1}{x^8}} \, dx \\
&= \frac{1}{x^4}\left(\frac{x^9}{9}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^4}\right) + c_2 \left(\frac{1}{x^4}\left(\frac{x^9}{9}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

2.2.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' = \frac{20y}{x^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{20y}{x^2} = 0$

- Multiply by denominators of the ODE

  $\left(\frac{d}{dx}y'\right)x^2 - 20y = 0$

- Make a change of variables

  $t = \ln(x)$

- □ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y' = \left(\frac{d}{dt}\frac{d}{dt}y(t)\right)t'(x)^2 + \left(\frac{d}{dx}t'(x)\right)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $\frac{d}{dx}y' = \frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - 20y(t) = 0$

- Simplify

  $\frac{d}{dt}\frac{d}{dt}y(t) - \frac{d}{dt}y(t) - 20y(t) = 0$

- Characteristic polynomial of ODE

$r^2 - r - 20 = 0$

- Factor the characteristic polynomial

  $(r + 4)(r - 5) = 0$
- Roots of the characteristic polynomial

  $r = (-4, 5)$
- 1st solution of the ODE

  $y_1(t) = e^{-4t}$
- 2nd solution of the ODE

  $y_2(t) = e^{5t}$
- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t)$
- Substitute in solutions

  $y(t) = C1\, e^{-4t} + C2\, e^{5t}$
- Change variables back using $t = \ln(x)$

  $y = \frac{C1}{x^4} + C2\, x^5$
- Simplify

  $y = \frac{C1}{x^4} + C2\, x^5$

2.2.3 Maple trace

```
Methods for second order ODEs:
```

2.2.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 15

```
dsolve(diff(diff(y(x),x),x) = 20/x^2*y(x),
       y(x),singsol=all)
```

$$y = \frac{c_1\, x^9 + c_2}{x^4}$$

2.2.5 Mathematica DSolve solution

Solving time : `0.017` (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]==((4*(9/2)^2-1)/(4*x^2))*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x^9 + c_1}{x^4}$$

2.3 problem 3

Internal problem ID [8962]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 3
Date solved : `Monday, October 21, 2024 at 05:24:29 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$y'' = \frac{12y}{x^2}$$

2.3.1 Solved as second order ode using Kovacic algorithm

Time used: 0.140 (sec)

Writing the ode as

$$y'' - \frac{12y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{12}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{12}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 12 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{12}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1570: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{12}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 4 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -3
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{12}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 12$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 4 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{12}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	4	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -3$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -3 - (-3) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{x} + (-)(0) \\
&= -\frac{3}{x} \\
&= -\frac{3}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{3}{x}\right)(0) + \left(\left(\frac{3}{x^2}\right) + \left(-\frac{3}{x}\right)^2 - \left(\frac{12}{x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx}\\ &= \mathrm{e}^{\int -\frac{3}{x} dx}\\ &= \frac{1}{x^3}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \frac{1}{x^3}\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2} \, dx\\ &= \frac{1}{x^3} \int \frac{1}{\frac{1}{x^6}} \, dx\\ &= \frac{1}{x^3}\left(\frac{x^7}{7}\right)\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{1}{x^3}\right) + c_2\left(\frac{1}{x^3}\left(\frac{x^7}{7}\right)\right)$$

Will add steps showing solving for IC soon.

2.3.2 Maple step by step solution

Let's solve
$\frac{d}{dx}y' = \frac{12y}{x^2}$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{12y}{x^2} = 0$
- Multiply by denominators of the ODE
  $\left(\frac{d}{dx}y'\right)x^2 - 12y = 0$
- Make a change of variables
  $t = \ln(x)$
- ☐ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule
    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$
  - Compute derivative
    $y' = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule
    $\frac{d}{dx}y' = \left(\frac{d}{dt}\frac{d}{dt}y(t)\right)t'(x)^2 + \left(\frac{d}{dx}t'(x)\right)\left(\frac{d}{dt}y(t)\right)$
  - Compute derivative
    $\frac{d}{dx}y' = \frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE
  $\left(\frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - 12y(t) = 0$
- Simplify
  $\frac{d}{dt}\frac{d}{dt}y(t) - \frac{d}{dt}y(t) - 12y(t) = 0$
- Characteristic polynomial of ODE

$r^2 - r - 12 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-4) = 0$
- Roots of the characteristic polynomial

  $r = (-3, 4)$
- 1st solution of the ODE

  $y_1(t) = e^{-3t}$
- 2nd solution of the ODE

  $y_2(t) = e^{4t}$
- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t)$
- Substitute in solutions

  $y(t) = C1\, e^{-3t} + C2\, e^{4t}$
- Change variables back using $t = \ln(x)$

  $y = \frac{C1}{x^3} + C2\, x^4$
- Simplify

  $y = \frac{C1}{x^3} + C2\, x^4$

2.3.3 Maple trace

```
Methods for second order ODEs:
```

2.3.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 15

```
dsolve(diff(diff(y(x),x),x) = 12/x^2*y(x),
       y(x),singsol=all)
```

$$y = \frac{c_2\, x^7 + c_1}{x^3}$$

2.3.5 Mathematica DSolve solution

Solving time : `0.017` (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]==((4*(7/2)^2-1)/(4*x^2))*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x^7 + c_1}{x^3}$$

2.4 problem 4

Internal problem ID [8963]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 4
Date solved : `Monday, October 21, 2024 at 05:24:30 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$y'' - \frac{y}{4x^2} = 0$$

2.4.1 Solved as second order ode using Kovacic algorithm

Time used: 0.249 (sec)

Writing the ode as

$$y'' - \frac{y}{4x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{1}{4x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1572: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{2}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{2}}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{2}}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{2}}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}+\frac{\sqrt{2}}{2}$	$\frac{1}{2}-\frac{\sqrt{2}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+\frac{\sqrt{2}}{2}$	$\frac{1}{2}-\frac{\sqrt{2}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - \frac{\sqrt{2}}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \frac{\sqrt{2}}{2} - \left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2}-\frac{\sqrt{2}}{2}}{x} + (-)(0) \\
&= \frac{\frac{1}{2}-\frac{\sqrt{2}}{2}}{x} \\
&= -\frac{\sqrt{2}-1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{2}}{2}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{2}}{2}}{x^2}\right) + \left(\frac{\frac{1}{2} - \frac{\sqrt{2}}{2}}{x}\right)^2 - \left(\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{\frac{1}{2} - \frac{\sqrt{2}}{2}}{x} dx} \\ &= x^{\frac{1}{2} - \frac{\sqrt{2}}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{\frac{1}{2} - \frac{\sqrt{2}}{2}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{2} - \frac{\sqrt{2}}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= x^{\frac{1}{2}-\frac{\sqrt{2}}{2}} \int \frac{1}{x^{1-\sqrt{2}}}\, dx \\
&= x^{\frac{1}{2}-\frac{\sqrt{2}}{2}} \left(\frac{x\sqrt{2}\, x^{\sqrt{2}-1}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{\frac{1}{2}-\frac{\sqrt{2}}{2}}\right) + c_2 \left(x^{\frac{1}{2}-\frac{\sqrt{2}}{2}} \left(\frac{x\sqrt{2}\, x^{\sqrt{2}-1}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

2.4.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' - \frac{y}{4x^2} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Multiply by denominators of the ODE

  $4\left(\frac{d}{dx}y'\right)x^2 - y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

- ○ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

- ○ Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- ○ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $\frac{d}{dx}y' = \left(\frac{d}{dt}\frac{d}{dt}y(t)\right)t'(x)^2 + \left(\frac{d}{dx}t'(x)\right)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $\frac{d}{dx}y' = \frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $4\left(\frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - y(t) = 0$

- Simplify

  $4\frac{d}{dt}\frac{d}{dt}y(t) - 4\frac{d}{dt}y(t) - y(t) = 0$
- Isolate 2nd derivative

  $\frac{d}{dt}\frac{d}{dt}y(t) = \frac{d}{dt}y(t) + \frac{y(t)}{4}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d}{dt}\frac{d}{dt}y(t) - \frac{d}{dt}y(t) - \frac{y(t)}{4} = 0$
- Characteristic polynomial of ODE

  $r^2 - r - \frac{1}{4} = 0$
- Use quadratic formula to solve for r

  $r = \frac{1 \pm (\sqrt{2})}{2}$
- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{\sqrt{2}}{2}, \frac{1}{2} + \frac{\sqrt{2}}{2}\right)$
- 1st solution of the ODE

  $y_1(t) = e^{\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right)t}$
- 2nd solution of the ODE

  $y_2(t) = e^{\left(\frac{1}{2} + \frac{\sqrt{2}}{2}\right)t}$
- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t)$
- Substitute in solutions

  $y(t) = C1\, e^{\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right)t} + C2\, e^{\left(\frac{1}{2} + \frac{\sqrt{2}}{2}\right)t}$
- Change variables back using $t = \ln(x)$

  $y = C1\, e^{\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right)\ln(x)} + C2\, e^{\left(\frac{1}{2} + \frac{\sqrt{2}}{2}\right)\ln(x)}$
- Simplify

  $y = \sqrt{x}\left(x^{\frac{\sqrt{2}}{2}} C2 + x^{-\frac{\sqrt{2}}{2}} C1\right)$

2.4.3 Maple trace

```
Methods for second order ODEs:
```

2.4.4 Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)-1/4/x^2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\left(x^{\frac{\sqrt{2}}{2}} c_1 + x^{-\frac{\sqrt{2}}{2}} c_2\right)$$

2.4.5 Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 32

```
DSolve[{D[y[x],{x,2}]-1/(4*x^2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x^{\frac{1}{2}-\frac{1}{\sqrt{2}}}\left(c_2 x^{\sqrt{2}} + c_1\right)$$

2.5 problem 5

Internal problem ID [8964]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 5
Date solved : `Monday, October 21, 2024 at 05:24:31 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (2x + 2)\, y' + (2 + x)\, y = 0$$

2.5.1 Solved as second order ode using Kovacic algorithm

Time used: 0.164 (sec)

Writing the ode as

$$xy'' + (-2x - 2)\, y' + (2 + x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x - 2 \\ C &= 2 + x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1574: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(0) \\ &= -\frac{1}{x} \\ &= -\frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0 \\ 0 = 0\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x \, e^x)\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x \, e^{2x+2\ln(x)} e^{-2x}}{3}\right)\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x\,\mathrm{e}^{2x+2\ln(x)}\mathrm{e}^{-2x}}{3}\right)\right)$$

Will add steps showing solving for IC soon.

2.5.2 Maple step by step solution

Let's solve
$x\left(\frac{d}{dx}y'\right) - (2x+2)\,y' + (2+x)\,y = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = -\frac{(2+x)y}{x} + \frac{2(x+1)y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{2(x+1)y'}{x} + \frac{(2+x)y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{2+x}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$

- Multiply by denominators
  $x\left(\frac{d}{dx}y'\right) + (-2x-2)\,y' + (2+x)\,y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx} y'\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- > k + 1$

  $x \cdot \left(\frac{d}{dx} y'\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$$a_0 r(-3+r)\, x^{-1+r} + (a_1(1+r)\,(-2+r) - 2a_0(-1+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-2+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term must be 0

  $a_1(1+r)\,(-2+r) - 2a_0(-1+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k-2+r) - 2a_k k - 2a_k r + 2a_k + a_{k-1} = 0$

- Shift index using $k- > k + 1$

  $a_{k+2}(k+2+r)\,(k+r-1) - 2a_{k+1}(k+1) - 2r a_{k+1} + 2a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2k a_{k+1} + 2r a_{k+1} - a_k}{(k+2+r)(k+r-1)}$

- Recursion relation for $r = 0$

$a_{k+2} = \frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Recursion relation for $r = 3$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}, 4a_1 - 4a_0 = 0\right]$

2.5.3 Maple trace

```
Methods for second order ODEs:
```

2.5.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 14

```
dsolve(x*diff(diff(y(x),x),x)-(2*x+2)*diff(y(x),x)+(2+x)*y(x) = 0,
       y(x),singsol=all)
```

$$y = e^{x} \left(c_2\, x^3 + c_1\right)$$

2.5.5 Mathematica DSolve solution

Solving time : `0.04` (sec)
Leaf size : 23

```
DSolve[{x*D[y[x],{x,2}]-(2*x+2)*D[y[x],x]+(2+x)*y[x] ==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3} e^{x} \left(c_2 x^3 + 3c_1\right)$$

2.6 problem 6

Internal problem ID [8965]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 6
Date solved : `Monday, October 21, 2024 at 05:24:32 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$y'' + \frac{y}{x^2} = 0$$

2.6.1 Solved as second order ode using Kovacic algorithm

Time used: 0.247 (sec)

Writing the ode as

$$y'' + \frac{y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{1}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1576: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -1$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -1$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}+\frac{i\sqrt{3}}{2}$	$\frac{1}{2}-\frac{i\sqrt{3}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}+\frac{i\sqrt{3}}{2}$	$\frac{1}{2}-\frac{i\sqrt{3}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \frac{i\sqrt{3}}{2} - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} + (-)(0) \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} \\
&= \frac{1 - i\sqrt{3}}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x}\right)(0)+\left(\left(-\frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x^2}\right)+\left(\frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x}\right)^2-\left(-\frac{1}{x^2}\right)\right)=0$$

$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int \omega\,dx}\\
&=\mathrm{e}^{\int \frac{\frac{1}{2}-\frac{i\sqrt{3}}{2}}{x}dx}\\
&=x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned}
y_1&=z_1\\
&=x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}
\end{aligned}$$

Which simplifies to

$$y_1=x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \int \frac{1}{x^{1-i\sqrt{3}}}\, dx \\
&= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{ix\sqrt{3}\, x^{i\sqrt{3}-1}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right) + c_2 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{ix\sqrt{3}\, x^{i\sqrt{3}-1}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

2.6.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' + \frac{y}{x^2} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Multiply by denominators of the ODE

  $\left(\frac{d}{dx}y'\right) x^2 + y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y' = \left(\frac{d}{dt}\frac{d}{dt}y(t)\right) t'(x)^2 + \left(\frac{d}{dx}t'(x)\right) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $\frac{d}{dx}y' = \frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 + y(t) = 0$

- Simplify

  $\frac{d}{dt}\frac{d}{dt}y(t) - \frac{d}{dt}y(t) + y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1 \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t)$

- Substitute in solutions

  $y(t) = C1\,\mathrm{e}^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + C2\,\mathrm{e}^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- Change variables back using $t = \ln(x)$

  $y = C1\sqrt{x}\,\cos\left(\frac{\ln(x)\sqrt{3}}{2}\right) + C2\sqrt{x}\,\sin\left(\frac{\ln(x)\sqrt{3}}{2}\right)$

- Simplify

  $y = \sqrt{x}\left(C1 \cos\left(\frac{\ln(x)\sqrt{3}}{2}\right) + C2 \sin\left(\frac{\ln(x)\sqrt{3}}{2}\right)\right)$

2.6.3 Maple trace

```
Methods for second order ODEs:
```

2.6.4 Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 29

```
dsolve(diff(diff(y(x),x),x)+1/x^2*y(x) = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\left(c_1 \sin\left(\frac{\ln\left(x\right)\sqrt{3}}{2}\right) + c_2 \cos\left(\frac{\ln\left(x\right)\sqrt{3}}{2}\right)\right)$$

2.6.5 Mathematica DSolve solution

Solving time : `0.04` (sec)
Leaf size : 42

```
DSolve[{D[y[x],{x,2}]+1/x^2*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \sqrt{x}\left(c_1 \cos\left(\frac{1}{2}\sqrt{3}\log(x)\right) + c_2 \sin\left(\frac{1}{2}\sqrt{3}\log(x)\right)\right)$$

2.7 problem 7

Internal problem ID [8966]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 7
Date solved : `Monday, October 21, 2024 at 05:24:33 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(-x^2+1\right) y''+y'+y=0$$

2.7.1 Solved as second order ode using Kovacic algorithm

Time used: 0.659 (sec)

Writing the ode as

$$\left(-x^2+1\right) y''+y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x - 3}{4\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 4x - 3 \\ t &= 4\left(x^2 - 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x - 3}{4\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1578: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{7}{16(x-1)} + \frac{5}{16(x-1)^2} - \frac{7}{16(x+1)} - \frac{3}{16(x+1)^2}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\
&= \{-1, 2, 5\}
\end{aligned}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\
&= \{1, 2, 3\}
\end{aligned}
$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{4x^2 + 4x - 3}{4(x^2-1)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=1$. Hence

$$
\begin{aligned}
E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\
&= \{2\}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{-1,2,5\}$
-1	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$
e_1 = -1,\ e_2 = 1,\ e_\infty = 2
$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{1}{2}(2-(-1+(1))) \\
&= 1
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\
&= \frac{1}{2}\left(\frac{-1}{(x-(1))}+\frac{1}{(x-(-1))}\right) \\
&= -\frac{1}{2\,(x-1)}+\frac{1}{2x+2}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d=1$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}
$$

Since $d=1$, then letting

$$
p = x + a_0 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$\frac{4a_0+6}{(x-1)^2(x+1)} = 0$$

And solving for p gives

$$p = x - \frac{3}{2}$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x-\frac{3}{2}} - \frac{1}{2(x-1)} + \frac{1}{2x+2}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x-\frac{3}{2}} - \frac{1}{2(x-1)} + \frac{1}{2x+2}\right)w + \frac{-8x^3+4x^2+10x-7}{4(x^2-1)^2(2x-3)} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{5}\,\sqrt{(x-1)(x+1)}\,x - 2\sqrt{5}\,\sqrt{(x-1)(x+1)} + 2x^2 - 2x + 1}{2(2x-3)(x-1)(x+1)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \frac{2\sqrt{5}\,\sqrt{(x-1)(x+1)}\,x-2\sqrt{5}\,\sqrt{(x-1)(x+1)}+2x^2-2x+1}{2(2x-3)(x-1)(x+1)}dx} \\ &= \frac{\sqrt{2x-3}\,(x+1)^{1/4}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}5^{1/4}}{(x-1)^{1/4}\sqrt{\frac{5\sqrt{x^2-1}+(-2+3x)\sqrt{5}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{-x^2+1}\,dx} \\
&= z_1 e^{-\frac{\operatorname{arctanh}(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}}\sqrt{2x-3}\,(5x+5)^{1/4}}{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}-2\sqrt{5}\right)}{2x-3}}\,(x-1)^{1/4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{-x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\operatorname{arctanh}(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{i\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}\left(3\sqrt{5}\,x+5\sqrt{x^2-1}-2\sqrt{5}\right)\sqrt{x-1}}{(2x-3)^2\sqrt{5x+5}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{2x-3}\,(5x+5)^{1/4}}{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}-2\sqrt{5}\right)}{2x-3}}\,(x-1)^{1/4}} \right) \\
&\quad + c_2 \left(\frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{2x-3}\,(5x+5)^{1/4}}{\sqrt{\frac{x+1}{\sqrt{-x^2+1}}}\sqrt{\frac{i\left(3\sqrt{5}\,x+5\sqrt{x^2-1}-2\sqrt{5}\right)}{2x-3}}\,(x-1)^{1/4}} \left(\int \frac{i\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}\left(3\sqrt{5}\,x+5\sqrt{x^2-1}-2\sqrt{5}\right)}{(2x-3)^2\sqrt{5x+5}} \right.\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

2.7.2 Maple step by step solution

Let's solve

$(-x^2+1)\left(\frac{d}{dx}y'\right)+y'+y=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{y}{x^2-1}+\frac{y'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'}{x^2-1} - \frac{y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1}{x^2-1}, P_3(x) = -\frac{1}{x^2-1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = \frac{1}{2}$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

$(x^2-1)\left(\frac{d}{dx}y'\right)-y'-y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^2-2u)\left(\frac{d}{du}\frac{d}{du}y(u)\right)-\frac{d}{du}y(u)-y(u)=0$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

  ◦ Convert $\frac{d}{du}y(u)$ to series expansion

$$\frac{d}{du}y(u)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$$

  ◦ Shift index using $k->k+1$

$$\frac{d}{du}y(u)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$$

  ◦ Convert $u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m=1..2$

$$u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

  ◦ Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d}{du}\frac{d}{du}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(2r-1)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(2k+1+2r)+a_k(k^2+2kr+r^2-k-r-1)\right)u^k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(2r-1)=0$

- Values of r that satisfy the indicial equation

$r\in\left\{0,\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$-2\left(k+\frac{1}{2}+r\right)(k+1+r)\,a_{k+1}+\left(k^2+(2r-1)\,k+r^2-r-1\right)a_k=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{(k^2+2kr+r^2-k-r-1)a_k}{(2k+1+2r)(k+1+r)}$

- Recursion relation for $r=0$

$a_{k+1}=\frac{(k^2-k-1)a_k}{(2k+1)(k+1)}$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=\frac{(k^2-k-1)a_k}{(2k+1)(k+1)}\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1}=\frac{(k^2-k-1)a_k}{(2k+1)(k+1)}\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1}=\frac{\left(k^2-\frac{5}{4}\right)a_k}{(2k+2)\left(k+\frac{3}{2}\right)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1}=\frac{\left(k^2-\frac{5}{4}\right)a_k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{1}{2}}, a_{k+1}=\frac{\left(k^2-\frac{5}{4}\right)a_k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x+1)^{k+\frac{1}{2}}\right), a_{k+1}=\frac{(k^2-k-1)a_k}{(2k+1)(k+1)}, b_{k+1}=\frac{\left(k^2-\frac{5}{4}\right)b_k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

2.7.3 Maple trace

```
Methods for second order ODEs:
```

2.7.4 Maple dsolve solution

Solving time : `0.025` (sec)
Leaf size : 58

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y=c_1 \operatorname{hypergeom}\left(\left[-\frac{1}{2}-\frac{\sqrt{5}}{2}, -\frac{1}{2}+\frac{\sqrt{5}}{2}\right], \left[\frac{1}{2}\right], \frac{x}{2}+\frac{1}{2}\right) + c_2\sqrt{2x+2}\, \operatorname{hypergeom}\left(\left[-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right], \left[\frac{3}{2}\right], \frac{x}{2}+\frac{1}{2}\right)$$

2.7.5 Mathematica DSolve solution

Solving time : `92.212` (sec)
Leaf size : 198

```
DSolve[{(1-x^2)*D[y[x],{x,2}]+D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \left(\sqrt{x-1}-\sqrt{x+1}\right)^{-\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(\sqrt{x-1}+\sqrt{x+1}\right)^{\frac{1}{2}\left(\sqrt{5}-1\right)}\left(5\sqrt{x-1} - \sqrt{5}\sqrt{x+1}\right)\left(c_2\int_1^x -\frac{2\sqrt{1-K[1]}\left(\sqrt{K[1]-1}-\sqrt{K[1]+1}\right)^{\sqrt{5}}\left(\sqrt{K[1]-1}+\sqrt{K[1]+1}\right)^{-\sqrt{5}}}{\sqrt{K[1]+1}\left(\sqrt{5}\sqrt{K[1]+1}-5\sqrt{K[1]-1}\right)^2}dK[1] + c_1\right)$$

2.8 problem 8

Internal problem ID [8967]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 8
Date solved : `Monday, October 21, 2024 at 05:24:34 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-x\right)y''-xy'+y=0$$

2.8.1 Solved as second order ode using Kovacic algorithm

Time used: 0.195 (sec)

Writing the ode as

$$\left(x^2-x\right)y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-x \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x + 4}{4x\,(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x + 4 \\ t &= 4x(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x+4}{4x\,(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1580: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} + \frac{3}{4(x-1)^2} - \frac{1}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+4}{4x(x-1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x+4}{4x\left(x-1\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} - \frac{1}{2\left(x-1\right)} + (-)\left(0\right) \\
&= \frac{1}{x} - \frac{1}{2\left(x-1\right)} \\
&= \frac{x-2}{2x\left(x-1\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{x} - \frac{1}{2\left(x-1\right)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{2\left(x-1\right)^2}\right) + \left(\frac{1}{x} - \frac{1}{2\left(x-1\right)}\right)^2 - \left(\frac{-x+4}{4x\left(x-1\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{2(x-1)}\right) dx} \\
&= \frac{x}{\sqrt{x-1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2-x}\, dx} \\
&= z_1 e^{\frac{\ln(x-1)}{2}} \\
&= z_1\left(\sqrt{x-1}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2-x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{1}{x} + \ln(x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2\left(x\left(\frac{1}{x} + \ln(x)\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

2.8.2 Maple step by step solution

2.8.3 Maple trace

```
Methods for second order ODEs:
```

2.8.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 14

```
dsolve((x^2-x)*diff(diff(y(x),x),x)-x*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = \ln(x)\, c_2 x + c_1 x + c_2$$

2.8.5 Mathematica DSolve solution

Solving time : `0.048` (sec)
Leaf size : 20

```
DSolve[{(x^2-x)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x - c_2(x \log(x) + 1)$$

2.9 problem 9

Internal problem ID [8968]
Book : Collection of Kovacic problems
Section : section 2. Solution found using all possible Kovacic cases
Problem number : 9
Date solved : `Monday, October 21, 2024 at 05:24:35 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-x^2+2\right)y''-x\left(4x^2+3\right)y'+\left(-2x^2+2\right)y=0$$

2.9.1 Solved as second order ode using Kovacic algorithm

Time used: 0.563 (sec)

Writing the ode as

$$\left(-x^4+2x^2\right)y''+\left(-4x^3-3x\right)y'+\left(-2x^2+2\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+2x^2 \\ B &= -4x^3-3x \\ C &= -2x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{14x^2 + 5}{4\left(x^3 - 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 14x^2 + 5 \\ t &= 4\left(x^3 - 2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{14x^2 + 5}{4\left(x^3 - 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1581: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16x^2} + \frac{33}{64\left(x - \sqrt{2}\right)^2} + \frac{33}{64\left(x + \sqrt{2}\right)^2} - \frac{43\sqrt{2}}{128\left(x - \sqrt{2}\right)} + \frac{43\sqrt{2}}{128\left(x + \sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - \sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{11}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{8} \end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{14x^2 + 5}{4\left(x^3 - 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$\sqrt{2}$	2	0	$\frac{11}{8}$	$-\frac{3}{8}$
$-\sqrt{2}$	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 1 - (-1) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4x} - \frac{3}{8\left(x-\sqrt{2}\right)} - \frac{3}{8\left(x+\sqrt{2}\right)} + (-)\,(0) \\
&= -\frac{1}{4x} - \frac{3}{8\left(x-\sqrt{2}\right)} - \frac{3}{8\left(x+\sqrt{2}\right)} \\
&= \frac{-2x^2+1}{2x^3-4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{1}{4x} - \frac{3}{8\left(x-\sqrt{2}\right)} - \frac{3}{8\left(x+\sqrt{2}\right)}\right)(2x+a_1) + \left(\left(\frac{1}{4x^2} + \frac{3}{8\left(x-\sqrt{2}\right)^2} + \frac{3}{8\left(x+\sqrt{2}\right)^2}\right) + \left(-\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^2+1)\, e^{\int \left(-\frac{1}{4x}-\frac{3}{8\left(x-\sqrt{2}\right)}-\frac{3}{8\left(x+\sqrt{2}\right)}\right)dx} \\
&= (x^2+1)\, e^{-\frac{3\ln\left(x-\sqrt{2}\right)}{8}-\frac{\ln(x)}{4}-\frac{3\ln\left(x+\sqrt{2}\right)}{8}} \\
&= \frac{x^2+1}{\left(x-\sqrt{2}\right)^{3/8} x^{1/4} \left(x+\sqrt{2}\right)^{3/8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^3-3x}{-x^4+2x^2}\, dx} \\
&= z_1 e^{\frac{3\ln(x)}{4}-\frac{11\ln\left(x^2-2\right)}{8}} \\
&= z_1 \left(\frac{x^{3/4}}{\left(x^2-2\right)^{11/8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{5/2}+\sqrt{x}}{\left(x^2-2\right)^{11/8}\left(x-\sqrt{2}\right)^{3/8}\left(x+\sqrt{2}\right)^{3/8}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3-3x}{-x^4+2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{3\ln(x)}{2}-\frac{11\ln\left(x^2-2\right)}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{\frac{3\ln(x)}{2}-\frac{11\ln\left(x^2-2\right)}{4}} \left(x^2-2\right)^{11/4}\left(x-\sqrt{2}\right)^{3/4}\left(x+\sqrt{2}\right)^{3/4}}{\left(x^{5/2}+\sqrt{x}\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{x^{5/2}+\sqrt{x}}{\left(x^2-2\right)^{11/8} \left(x-\sqrt{2}\right)^{3/8} \left(x+\sqrt{2}\right)^{3/8}}\right) + c_2 \left(\frac{x^{5/2}+\sqrt{x}}{\left(x^2-2\right)^{11/8} \left(x-\sqrt{2}\right)^{3/8} \left(x+\sqrt{2}\right)^{3/8}} \left(\int \frac{\mathrm{e}^{\frac{3\ln(x)}{2}-\frac{11 \ln}{}}}{}\right.\right.$$

Will add steps showing solving for IC soon.

2.9.2 Maple step by step solution

Let's solve

$x^2(-x^2+2)\left(\frac{d}{dx}y'\right) - x(4x^2+3)\,y' + (-2x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{2(x^2-1)y}{x^2(x^2-2)} - \frac{(4x^2+3)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(4x^2+3)y'}{x(x^2-2)} + \frac{2(x^2-1)y}{x^2(x^2-2)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{4x^2+3}{x(x^2-2)}, P_3(x) = \frac{2(x^2-1)}{x^2(x^2-2)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -\frac{3}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(x^2-2)\left(\frac{d}{dx}y'\right) + x(4x^2+3)\,y' + (2x^2-2)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+2r)\,(-2+r)\, x^r - a_1(1+2r)\,(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (-a_k(2k+2r-1)\,(k+r-2) + a_{k}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-(-1+2r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{2, \frac{1}{2}\right\}$
- Each term must be 0
  $-a_1(1+2r)\,(-1+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2\left(k+r-\frac{1}{2}\right)(k+r-2)\, a_k + a_{k-2}(k+r)\,(k+r-1) = 0$
- Shift index using $k- >k + 2$
  $-2\left(k+\frac{3}{2}+r\right)(k+r)\, a_{k+2} + a_k(k+r+2)\,(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a_k(k+r+2)(k+r+1)}{(2k+3+2r)(k+r)}$$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{a_k(k+4)(k+3)}{(2k+7)(k+2)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{a_k(k+4)(k+3)}{(2k+7)(k+2)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k+\frac{3}{2}\right)}{(2k+4)\left(k+\frac{1}{2}\right)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k+\frac{3}{2}\right)}{(2k+4)\left(k+\frac{1}{2}\right)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{a_k(k+4)(k+3)}{(2k+7)(k+2)}, a_1 = 0, b_{k+2} = \frac{b_k\left(k+\frac{5}{2}\right)\left(k+\frac{3}{2}\right)}{(2k+4)\left(k+\frac{1}{2}\right)}, b_1 = 0\right]$$

2.9.3 Maple trace

```
Methods for second order ODEs:
```

2.9.4 Maple dsolve solution

Solving time : `0.019` (sec)
Leaf size : 47

```
dsolve(x^2*(-x^2+2)*diff(diff(y(x),x),x)-x*(4*x^2+3)*diff(y(x),x)+(-2*x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1\, x^2 \,\mathrm{hypergeom}\left(\left[\frac{3}{2}, 2\right], \left[\frac{7}{4}\right], \frac{x^2}{2}\right) + \frac{c_2\sqrt{x}\,(x^2+1)}{(x^2-2)\,(-2x^2+4)^{3/4}}$$

2.9.5 Mathematica DSolve solution

Solving time : 20.316 (sec)
Leaf size : 86

```
DSolve[{x^2*(2-x^2)*D[y[x],{x,2}] - x*(3+4*x^2)*D[y[x],x] + (2-2*x^2)*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2^{3/4}c_2(x^2+1)\,x^2\,\text{Hypergeometric2F1}\left(\frac{1}{4},\frac{3}{4},\frac{7}{4},\frac{x^2}{2}\right)+3c_2(2-x^2)^{3/4}\,x^2+6c_1(x^2+1)\,\sqrt{x}}{6\,(2-x^2)^{7/4}}$$

3 section 3. Problems from Kovacic related papers

3.1 problem Kovacic 1985 paper. page 13. section 3.2, example 1

Internal problem ID [8969]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Kovacic 1985 paper. page 13. section 3.2, example 1
Date solved : `Monday, October 21, 2024 at 05:24:36 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = \frac{(4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4)\,y}{4x^4}$$

3.1.1 Solved as second order ode using Kovacic algorithm

Time used: 0.552 (sec)

Writing the ode as

$$y'' + \left(-x^2 + 2x - 3 - \frac{1}{x} - \frac{7}{4x^2} + \frac{5}{x^3} - \frac{1}{x^4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -x^2 + 2x - 3 - \frac{1}{x} - \frac{7}{4x^2} + \frac{5}{x^3} - \frac{1}{x^4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4 \\ t &= 4x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1583: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 6 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = 3 + x^2 - 2x + \frac{1}{x} + \frac{7}{4x^2} - \frac{5}{x^3} + \frac{1}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{x^2} - \frac{5}{2x} - \frac{9}{4} - \frac{41x}{8} - \frac{443x^2}{32} - \frac{3017x^3}{64} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = 1$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -5. Therefore

$$\begin{aligned} b &= (-5) - (0) \\ &= -5 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{-5}{1} + 2\right) &= -\frac{3}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{-5}{1} + 2\right) &= \frac{7}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - 1 + \frac{1}{x} + \frac{3}{2x^2} + \frac{15}{8x^3} - \frac{17}{8x^4} - \frac{37}{8x^5} - \frac{85}{16x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= -1 + x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2 - 2x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4} \\ &= Q + \frac{R}{4x^4} \\ &= \left(x^2 - 2x + 3\right) + \left(\frac{4x^3 + 7x^2 - 20x + 4}{4x^4}\right) \\ &= x^2 - 2x + 3 + \frac{4x^3 + 7x^2 - 20x + 4}{4x^4} \end{aligned}$$

We see that the coefficient of the term x^3 in the quotient is 3. Now b can be found.

$$\begin{aligned} b &= (3) - (1) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= -1 + x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{1} - 1\right) &= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{1} - 1\right) &= -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{1}{x^2}$	$-\frac{3}{2}$	$\frac{7}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-1 + x$	$\frac{1}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(-\frac{3}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x^2} - \frac{3}{2x} + (-1 + x) \\ &= \frac{1}{x^2} - \frac{3}{2x} - 1 + x \\ &= \frac{1}{x^2} - \frac{3}{2x} - 1 + x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{x^2} - \frac{3}{2x} - 1 + x\right)(2x + a_1) + \left(\left(-\frac{2}{x^3} + \frac{3}{2x^2} + 1\right) + \left(\frac{1}{x^2} - \frac{3}{2x} - 1 + x\right)^2 - \left(\frac{4x^6 - 8x^5 + 12\right.\right.$$

$$\frac{-2x^3 a_1 + (-4a_0 + 2a_1 - 4)\, x^2 -}{x^2}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x^2 - 1)\, e^{\int \left(\frac{1}{x^2} - \frac{3}{2x} - 1 + x\right) dx} \\
&= (x^2 - 1)\, e^{\frac{x^2}{2} - x - \frac{1}{x} - \frac{3\ln(x)}{2}} \\
&= \frac{(x^2 - 1)\, e^{\frac{x^3 - 2x^2 - 2}{2x}}}{x^{3/2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \frac{(x^2 - 1)\, e^{\frac{x^3 - 2x^2 - 2}{2x}}}{x^{3/2}}
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{3/2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{3/2}} \int \frac{1}{\frac{(x^2-1)^2\mathrm{e}^{\frac{x^3-2x^2-2}{x}}}{x^3}}\,dx \\
&= \frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{3/2}} \left(\int \frac{x^3\mathrm{e}^{-\frac{x^3-2x^2-2}{x}}}{(x^2-1)^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{3/2}}\right) + c_2\left(\frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{3/2}}\left(\int \frac{x^3\mathrm{e}^{-\frac{x^3-2x^2-2}{x}}}{(x^2-1)^2}dx\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

3.1.2 Maple step by step solution

3.1.3 Maple trace

```
Methods for second order ODEs:
```

3.1.4 Maple dsolve solution

Solving time : 0.131 (sec)
Leaf size : 66

```
dsolve(diff(diff(y(x),x),x) = 1/4*(4*x^6-8*x^5+12*x^4+4*x^3+7*x^2-20*x+4)/x^4*y(x),
       y(x),singsol=all)
```

$$y = \frac{(x^2-1)\, e^{\frac{x^3-2x^2-2}{2x}} \left(c_2 \left(\int \frac{x^3 e^{\frac{-x^3+2x^2+2}{x}}}{(-1+x)^2 (x+1)^2} dx\right) + c_1\right)}{x^{3/2}}$$

3.1.5 Mathematica DSolve solution

Solving time : 0.936 (sec)
Leaf size : 79

```
DSolve[{D[y[x],{x,2}]== (4*x^6-8*x^5+12*x^4+4*x^3+7*x^2-20*x+4)/(4*x^4)*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{\frac{x^2}{2}-x-\frac{1}{x}}(x^2-1)\left(c_2 \int_1^x \frac{e^{-K[1]^2+2K[1]+\frac{2}{K[1]}} K[1]^3}{(K[1]^2-1)^2} dK[1] + c_1\right)}{x^{3/2}}$$

3.2 problem Kovacic 1985 paper. page 14. section 3.2, example 2

Internal problem ID [8970]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Kovacic 1985 paper. page 14. section 3.2, example 2
Date solved : `Monday, October 21, 2024 at 05:24:38 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = \left(\frac{6}{x^2} - 1\right) y$$

3.2.1 Solved as second order ode using Kovacic algorithm

Time used: 0.310 (sec)

Writing the ode as

$$y'' + \left(-\frac{6}{x^2} + 1\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{6}{x^2} + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \qquad (5)$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \qquad (6)$$

Comparing the above to (5) shows that

$$s = -x^2 + 6$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{x^2}\right) z(x) \qquad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1584: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2} - 1$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= \frac{6}{x^2} - 1 \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{i} - 0\right) &= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 + 6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + (-)\,(i) \\
&= -\frac{2}{x} - i \\
&= -\frac{2}{x} - i
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2 + 6}{x^2}\right)\right) &= 0 \\
\frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -3, a_1 = -3i\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 3ix - 3
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 3ix - 3\right) \mathrm{e}^{\int \left(-\frac{2}{x} - i\right) dx} \\
&= \left(x^2 - 3ix - 3\right) \mathrm{e}^{-2\ln(x) - ix} \\
&= \frac{\left(x^2 - 3ix - 3\right) \mathrm{e}^{-ix}}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^2}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^2} \int \frac{1}{\frac{(x^2 - 3ix - 3)^2 \mathrm{e}^{-2ix}}{x^4}}\,dx \\
&= \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^2} \left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^2} \right) + c_2 \left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^2} \left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

3.2.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' = \left(\frac{6}{x^2} - 1\right) y$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{y(x^2-6)}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{y(x^2-6)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{x^2-6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -6$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d}{dx}y'\right) x^2 + (x^2 - 6)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

$$x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)(-3+r)\, x^r + a_1(3+r)(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)(k+r-3) + a_{k-2}\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(2+r)(-3+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 3\}$
- Each term must be 0
  $a_1(3+r)(-2+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k(k+r+2)(k+r-3) + a_{k-2} = 0$
- Shift index using $k- >k+2$
  $a_{k+2}(k+4+r)(k+r-1) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{(k+4+r)(k+r-1)}$
- Recursion relation for $r = -2$
  $a_{k+2} = -\frac{a_k}{(k+2)(k-3)}$
- Solution for $r = -2$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k}{(k+2)(k-3)}, a_1 = 0\right]$
- Recursion relation for $r = 3$
  $a_{k+2} = -\frac{a_k}{(k+7)(k+2)}$
- Solution for $r = 3$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k}{(k+7)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{a_k}{(k+2)(k-3)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+7)(k+2)}, b_1 = 0\right]$

3.2.3 Maple trace

```
Methods for second order ODEs:
```

3.2.4 Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 41

```
dsolve(diff(diff(y(x),x),x) = (6/x^2-1)*y(x),
       y(x),singsol=all)
```

$$y = \frac{(c_1 x^2 + 3c_2 x - 3c_1) \cos(x) + \sin(x) (c_2 x^2 - 3c_1 x - 3c_2)}{x^2}$$

3.2.5 Mathematica DSolve solution

Solving time : `0.031` (sec)
Leaf size : 21

```
DSolve[{D[y[x],{x,2}]== ( (4*(5/2)^2-1)/(4*x^2)-1)*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x(c_1 j_2(x) - c_2 y_2(x))$$

3.3 problem Kovacic 1985 paper. page 15. Weber equation

Internal problem ID [8971]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Kovacic 1985 paper. page 15. Weber equation
Date solved : `Monday, October 21, 2024 at 05:24:39 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = \left(\frac{x^2}{4} - \frac{11}{2}\right) y$$

3.3.1 Solved as second order ode using Kovacic algorithm

Time used: 0.275 (sec)

Writing the ode as

$$y'' + \left(-\frac{x^2}{4} + \frac{11}{2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{x^2}{4} + \frac{11}{2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 22}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 22 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2}{4} - \frac{11}{2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1586: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{11}{2x} - \frac{121}{4x^3} - \frac{1331}{4x^5} - \frac{73205}{16x^7} - \frac{1127357}{16x^9} - \frac{37202781}{32x^{11}} - \frac{643076643}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 22}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{11}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{11}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{11}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{11}{2}\right) - (0) \\ &= -\frac{11}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{11}{2}}{\frac{1}{2}} - 1\right) &&= -6 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{11}{2}}{\frac{1}{2}} - 1\right) &&= 5 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{11}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-6	5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 5$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 5 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(20x^3 + 12x^2 a_4 + 6x a_3 + 2a_2\right) + 2\left(-\frac{x}{2}\right)\left(5x^4 + 4x^3 a_4 + 3x^2 a_3 + 2x a_2 + a_1\right) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4}\right.\right.$$

$$a_4 x^4 + 2(10 + a_3)\, x^3 + 3(a_2 + 4a_4)\, x^2 + 2(2a_1 + 3a_3)\, x + 5a_0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = 15, a_2 = 0, a_3 = -10, a_4 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^5 - 10x^3 + 15x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^5 - 10x^3 + 15x\right) e^{\int -\frac{x}{2} dx} \\ &= \left(x^5 - 10x^3 + 15x\right) e^{-\frac{x^2}{4}} \\ &= x\left(x^4 - 10x^2 + 15\right) e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x\left(x^4 - 10x^2 + 15\right) e^{-\frac{x^2}{4}} \end{aligned}$$

Which simplifies to

$$y_1 = x\left(x^4 - 10x^2 + 15\right) e^{-\frac{x^2}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= x\left(x^4 - 10x^2 + 15\right) e^{-\frac{x^2}{4}} \int \frac{1}{x^2 \left(x^4 - 10x^2 + 15\right)^2 e^{-\frac{x^2}{2}}} \, dx \\ &= x\left(x^4 - 10x^2 + 15\right) e^{-\frac{x^2}{4}} \left(\int \frac{e^{\frac{x^2}{2}}}{x^2 \left(x^4 - 10x^2 + 15\right)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x(x^4 - 10x^2 + 15)\, \mathrm{e}^{-\frac{x^2}{4}} \right) \\
&\quad + c_2 \left(x(x^4 - 10x^2 + 15)\, \mathrm{e}^{-\frac{x^2}{4}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{x^2 (x^4 - 10x^2 + 15)^2} dx \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

3.3.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' = \left(\frac{x^2}{4} - \frac{11}{2}\right) y$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \left(-\frac{x^2}{4} + \frac{11}{2}\right) y = 0$

- Multiply by denominators

  $4\frac{d}{dx}y' + (-x^2 + 22)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

  $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k + 2$

$\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

Rewrite ODE with series expansions

$8a_2 + 22a_0 + (24a_3 + 22a_1)x + \left(\sum_{k=2}^{\infty} (4a_{k+2}(k+2)(k+1) + 22a_k - a_{k-2})x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[8a_2 + 22a_0 = 0, 24a_3 + 22a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\left\{a_2 = -\frac{11a_0}{4}, a_3 = -\frac{11a_1}{12}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $4(k^2 + 3k + 2)a_{k+2} + 22a_k - a_{k-2} = 0$
- Shift index using $k- >k+2$

  $4\left((k+2)^2 + 3k + 8\right)a_{k+4} + 22a_{k+2} - a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{-22a_{k+2}+a_k}{4(k^2+7k+12)}, a_2 = -\frac{11a_0}{4}, a_3 = -\frac{11a_1}{12}\right]$

3.3.3 Maple trace

```
Methods for second order ODEs:
```

3.3.4 Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 39

```
dsolve(diff(diff(y(x),x),x) = (1/4*x^2-11/2)*y(x),
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{x^2}{4}}\left(15\,\mathrm{hypergeom}\left(\left[-\frac{5}{2}\right], \left[\frac{1}{2}\right], \frac{x^2}{2}\right)c_2 + x(x^4 - 10x^2 + 15)c_1\right)}{15}$$

3.3.5 Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 22

```
DSolve[{D[y[x],{x,2}]== (1/4*x^2-1/2-5)*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 \operatorname{ParabolicCylinderD}(-6, ix) + c_1 \operatorname{ParabolicCylinderD}(5, x)$$

3.4 problem Kovacic 1985 paper. page 19. section 4.2. Example 1

Internal problem ID [8972]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Kovacic 1985 paper. page 19. section 4.2. Example 1
Date solved : `Monday, October 21, 2024 at 05:24:40 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = \left(\frac{1}{x} - \frac{3}{16x^2}\right) y$$

3.4.1 Solved as second order ode using Kovacic algorithm

Time used: 0.188 (sec)

Writing the ode as

$$y'' + \frac{(-16x+3)\,y}{16x^2} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{-16x+3}{16x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 16x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{16x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1588: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1 - 16x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 4\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+4\sqrt{x}}{4x} dx} \\ &= x^{1/4} \mathrm{e}^{2\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{1/4} \mathrm{e}^{2\sqrt{x}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{1/4} \mathrm{e}^{2\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= x^{1/4} \mathrm{e}^{2\sqrt{x}} \int \frac{1}{\sqrt{x}\, \mathrm{e}^{4\sqrt{x}}} \, dx \\ &= x^{1/4} \mathrm{e}^{2\sqrt{x}} \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{1/4} e^{2\sqrt{x}}\right) + c_2 \left(x^{1/4} e^{2\sqrt{x}} \left(-\frac{e^{-4\sqrt{x}}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

3.4.2 Maple step by step solution

Let's solve
$\frac{d}{dx}y' = \left(\frac{1}{x} - \frac{3}{16x^2}\right) y$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Isolate 2nd derivative
  $\frac{d}{dx}y' = \frac{y(16x-3)}{16x^2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' - \frac{y(16x-3)}{16x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = 0, P_3(x) = -\frac{16x-3}{16x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{16}$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$

- Multiply by denominators
  $16\left(\frac{d}{dx}y'\right) x^2 + (-16x+3)\, y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

  $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+4r)(-3+4r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(4k+4r-1)(4k+4r-3) - 16a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+4r)(-3+4r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $16\left(k+r-\frac{1}{4}\right)\left(k+r-\frac{3}{4}\right) a_k - 16a_{k-1} = 0$

- Shift index using $k- > k + 1$

  $16\left(k+\frac{3}{4}+r\right)\left(k+\frac{1}{4}+r\right) a_{k+1} - 16a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{16a_k}{(4k+3+4r)(4k+1+4r)}$

- Recursion relation for $r = \frac{1}{4}$

  $a_{k+1} = \frac{16a_k}{(4k+4)(4k+2)}$

- Solution for $r = \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = \frac{16a_k}{(4k+4)(4k+2)}\right]$

- Recursion relation for $r = \frac{3}{4}$

  $a_{k+1} = \frac{16a_k}{(4k+6)(4k+4)}$

- Solution for $r = \frac{3}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = \frac{16a_k}{(4k+6)(4k+4)}\right]$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1}=\frac{16a_k}{(4k+4)(4k+2)}, b_{k+1}=\frac{16b_k}{(4k+6)(4k+4)}\right]$$

3.4.3 Maple trace

```
Methods for second order ODEs:
```

3.4.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 25

```
dsolve(diff(diff(y(x),x),x) = (1/x-3/16/x^2)*y(x),
       y(x),singsol=all)
```

$$y = x^{1/4}\left(c_1 \sinh\left(2\sqrt{x}\right)+c_2 \cosh\left(2\sqrt{x}\right)\right)$$

3.4.5 Mathematica DSolve solution

Solving time : `0.072` (sec)
Leaf size : 41

```
DSolve[{D[y[x],{x,2}]== (1/x-3/(16*x^2))*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to \frac{1}{2}e^{-2\sqrt{x}}\sqrt[4]{x}\left(2c_1e^{4\sqrt{x}}-c_2\right)$$

3.5 problem Kovacic 1985 paper. page 23. section 5.2. Example 1

Internal problem ID [8973]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Kovacic 1985 paper. page 23. section 5.2. Example 1
Date solved : `Monday, October 21, 2024 at 05:24:41 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = \left(-\frac{3}{16x^2} - \frac{2}{9\left(x-1\right)^2} + \frac{3}{16x\left(x-1\right)}\right)y$$

3.5.1 Solved as second order ode using Kovacic algorithm

Time used: 1.020 (sec)

Writing the ode as

$$y'' + \frac{\left(32x^2 - 27x + 27\right)y}{144x^2\left(x-1\right)^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{32x^2 - 27x + 27}{144x^2\left(x-1\right)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-32x^2 + 27x - 27}{144\,(x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -32x^2 + 27x - 27 \\ t &= 144(x^2 - x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-32x^2 + 27x - 27}{144\,(x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1590: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Unable to find solution using case two.

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{16(x-1)} - \frac{3}{16x^2} - \frac{3}{16x} - \frac{2}{9(x-1)^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{3}{16}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{3, 6, 9\}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{2}{9}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{4, 5, 6, 7, 8\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx -\frac{2}{9x^2} - \frac{37}{144x^3} - \frac{23}{48x^4} - \frac{101}{144x^5} - \frac{133}{144x^6} - \frac{55}{48x^7} + \cdots$$

The above shows that

$$b = -\frac{2}{9}$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{4, 5, 6, 7, 8\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{3, 6, 9\}$
1	2	$\{4, 5, 6, 7, 8\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{4, 5, 6, 7, 8\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \dots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 3, \ e_2 = 4, \ e_\infty = 7$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}
d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{4}{12}(7 - (3 + (4))) \\
&= 0
\end{aligned}$$

The following rational function is

$$\begin{aligned}
\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\
&= \frac{4}{12}\left(\frac{3}{(x-(0))} + \frac{4}{(x-(1))}\right) \\
&= \frac{1}{x} + \frac{4}{3x-3}
\end{aligned}$$

And

$$\begin{aligned}
S &= \prod_{c\in\Gamma}(x-c) \\
&= x(x-1)
\end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}
P_n &= -p(x) \\
&= -1 \\
P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \dots, 0
\end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 4$ and $r =$

$\frac{-32x^2+27x-27}{144(x^2-x)^2}$).

$$
\begin{aligned}
P_4 &= -p \\
&= -1 \\
P_3 &= \frac{7x}{3} - 1 \\
P_2 &= -4x^2 + \frac{41}{12}x - \frac{3}{4} \\
P_1 &= \frac{40}{9}x^3 - \frac{409}{72}x^2 + \frac{5}{2}x - \frac{3}{8} \\
P_0 &= -\frac{64}{27}x^4 + \frac{871}{216}x^3 - \frac{257}{96}x^2 + \frac{13}{16}x - \frac{3}{32} \\
P_{-1} &= 0
\end{aligned}
$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$
\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!} \omega^i = 0
$$

$$
\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!} \omega^i = 0
$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$
-\frac{8x^4}{81} + \frac{871x^3}{5184} - \frac{257x^2}{2304} + \frac{13x}{384} - \frac{1}{256} + \frac{x(x-1)\left(320x^3 - 409x^2 + 180x - 27\right)\omega}{432} - \frac{x^2(x-1)^2\left(48x^2 - 41x + 9\right)\omega^2}{24} + x^3(x-1)^3\left(\frac{7x}{3} - 1\right)\omega^3 - x^4(x-1)^4\omega^4 = 0
$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{12x(x-1)}\left(7x-3+\sqrt{x^2+\left((x-1)^2x^3\right)^{1/3}-x} + \sqrt{-\frac{2\left(\left(-x^2+x+\frac{\left((x-1)^2x^3\right)^{1/3}}{2}\right)\sqrt{x^2+\left((x-1)^2x^3\right)^{1/3}-x}+x^2(x-1)\right)}{\sqrt{x^2+\left((x-1)^2x^3\right)^{1/3}-x}}}\right) \tag{4A}$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Unable to integrate $\int \omega dx$. Leaving the integral unevaluated. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{\int \omega \, dx} \end{aligned}$$

Where ω given above. The second solution y_2 to the original ode is found using reduction of order

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx \\ &= e^{\int \omega \, dx} \int \frac{e^{\int -\frac{B}{A}\,dx}}{\left(e^{\int \omega \, dx}\right)^2}\,dx \end{aligned}$$

Since $B = 0$ then the above reduces to

$$y_2 = e^{\int \omega \, dx} \int \left(e^{\int \omega \, dx}\right)^{-2} dx$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1\left(e^{\int \omega\,dx}\right) + c_2\left(e^{\int \omega\,dx} \int \left(e^{\int \omega\,dx}\right)^{-2}\,dx\right)$$

Will add steps showing solving for IC soon.

3.5.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' = \left(-\frac{3}{16x^2} - \frac{2}{9(x-1)^2} + \frac{3}{16x(x-1)}\right)y$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(32x^2-27x+27)y}{144x^2(x-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(32x^2-27x+27)y}{144x^2(x-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{32x^2-27x+27}{144x^2(x-1)^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{16}$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $144x^2(x-1)^2\left(\frac{d}{dx}y'\right) + (32x^2 - 27x + 27)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y'\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$9a_0(-1+4r)(-3+4r)\, x^r + (9a_1(3+4r)(1+4r) - 9a_0(32r^2-32r+3))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (9a_k(4k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$9(-1+4r)(-3+4r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$$

- Each term must be 0

$$9a_1(3+4r)(1+4r) - 9a_0(32r^2-32r+3) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = \frac{a_0(32r^2-32r+3)}{16r^2+16r+3}$$

- Each term in the series must be 0, giving the recursion relation

$$144(a_k + a_{k-2} - 2a_{k-1})\, k^2 + 144(2(a_k + a_{k-2} - 2a_{k-1})\, r - a_k - 5a_{k-2} + 6a_{k-1})\, k + 144(a_k + a_{k-}$$

- Shift index using $k->k+2$

$$144(a_{k+2} + a_k - 2a_{k+1})\,(k+2)^2 + 144(2(a_{k+2} + a_k - 2a_{k+1})\, r - a_{k+2} - 5a_k + 6a_{k+1})\,(k+2) + 1$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} + 288kra_k - 576kra_{k+1} + 144r^2a_k - 288r^2a_{k+1} - 144ka_k - 288ka_{k+1} - 144ra_k - 288ra_{k+1} + 32a_k - 27}{9(16k^2+32kr+16r^2+48k+48r+35)}$$

- Recursion relation for $r = \frac{1}{4}$

$$a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} - 72ka_k - 432ka_{k+1} + 5a_k - 117a_{k+1}}{9(16k^2+56k+48)}$$

- Solution for $r = \frac{1}{4}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} - 72ka_k - 432ka_{k+1} + 5a_k - 117a_{k+1}}{9(16k^2 + 56k + 48)}, a_1 = -\frac{3a_0}{8}\right]$$

- Recursion relation for $r = \frac{3}{4}$

  $$a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} + 72ka_k - 720ka_{k+1} + 5a_k - 405a_{k+1}}{9(16k^2 + 72k + 80)}$$

- Solution for $r = \frac{3}{4}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} + 72ka_k - 720ka_{k+1} + 5a_k - 405a_{k+1}}{9(16k^2 + 72k + 80)}, a_1 = -\frac{a_0}{8}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} - 72ka_k - 432ka_{k+1} + 5a_k - 117a_{k+1}}{9(16k^2 + 56k + 48)}, a_1 = -\right.$$

3.5.3 Maple trace

```
Methods for second order ODEs:
```

3.5.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x) = (-3/16/x^2-2/9/(x-1)^2+3/16/x/(x-1))*y(x),
       y(x),singsol=all)
```

$$y = x^{1/4}\sqrt{x-1}\left(c_1 \operatorname{LegendreP}\left(-\frac{1}{6}, \frac{1}{3}, \sqrt{x}\right) + c_2 \operatorname{LegendreQ}\left(-\frac{1}{6}, \frac{1}{3}, \sqrt{x}\right)\right)$$

3.5.5 Mathematica DSolve solution

Solving time : `0.369` (sec)
Leaf size : 550

```
DSolve[{D[y[x],{x,2}]== ( -3/(16*x^2) - 2/(9*(x-1)^2) + 3/(16*x*(x-1))  )*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to{}& c_1 \exp\left(\int_1^x \mathrm{Root}\big[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2 - 702K[1]\right. \\
&+ \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4 \\
&+ \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right)\#1^3 \\
&+ \left(41472K[1]^6 - 118368K[1]^5 + 120096K[1]^4 - 50976K[1]^3 + 7776K[1]^2\right)\#1^2 \\
&+ \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right)\#1 \\
&\left.+ 81\&, 1\big]\, dK[1]\right) + c_2 \exp\left(\int_1^x \mathrm{Root}\big[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2 - 702K[1]\right. \\
&+ \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4 \\
&+ \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right)\#1^3 \\
&+ \left(41472K[1]^6 - 118368K[1]^5 + 120096K[1]^4 - 50976K[1]^3 + 7776K[1]^2\right)\#1^2 \\
&+ \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right)\#1 \\
&\left.+ 81\&, 1\big]\, dK[1]\right) \int_1^x \exp\left(-2\int_1^{K[2]} \mathrm{Root}\big[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2\right. \\
&-702K[1] + \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4 \\
&+ \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right)\#1^3 + \left(41472K[1]^6 - 118368\right. \\
&\left.+ \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right)\#1 + 81\&, 1\big]\, dK[1]\right) dK[2]
\end{aligned}
$$

3.6 problem Kovacic 1985 paper. page 25. section 5.2. Example 2

Internal problem ID [8974]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Kovacic 1985 paper. page 25. section 5.2. Example 2
Date solved : `Monday, October 21, 2024 at 05:24:43 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = -\frac{(5x^2+27)\,y}{36\,(x^2-1)^2}$$

3.6.1 Solved as second order ode using Kovacic algorithm

Time used: 86.168 (sec)

Writing the ode as

$$y'' + \frac{(5x^2+27)\,y}{36\,(x^2-1)^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{5x^2+27}{36\,(x^2-1)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5x^2 - 27}{36\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5x^2 - 27 \\ t &= 36\left(x^2 - 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-5x^2 - 27}{36\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1592: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Unable to find solution using case two.

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9(x+1)^2} - \frac{11}{72(x+1)} - \frac{2}{9(x-1)^2} + \frac{11}{72(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{2}{9}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{4, 5, 6, 7, 8\}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{2}{9}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{4, 5, 6, 7, 8\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx -\frac{5}{36x^2} - \frac{37}{36x^4} - \frac{23}{12x^6} - \frac{101}{36x^8} - \frac{133}{36x^{10}} - \frac{55}{12x^{12}} + \cdots$$

The above shows that

$$b = -\frac{5}{36}$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{2, 4, 6, 8, 10\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
1	2	$\{4, 5, 6, 7, 8\}$
-1	2	$\{4, 5, 6, 7, 8\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{2, 4, 6, 8, 10\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \ldots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 4, \; e_2 = 4, \; e_\infty = 8$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{n}{12}\left(e_{\infty} - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{4}{12}(8 - (4 + (4))) \\
&= 0
\end{aligned}
$$

The following rational function is

$$
\begin{aligned}
\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\
&= \frac{4}{12}\left(\frac{4}{(x-(1))} + \frac{4}{(x-(-1))}\right) \\
&= \frac{8x}{3x^2-3}
\end{aligned}
$$

And

$$
\begin{aligned}
S &= \prod_{c\in\Gamma}(x-c) \\
&= (x-1)(x+1)
\end{aligned}
$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$
p(x) = 1
$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$
\begin{aligned}
P_n &= -p(x) \\
&= -1 \\
P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \ldots, 0
\end{aligned} \tag{1A}
$$

The coefficients a_i are solved for from

$$
P_{-1} = 0 \tag{2A}
$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 4$ and $r =$

$\frac{-5x^2-27}{36(x^2-1)^2}$).

$$
\begin{aligned}
P_4 &= -p \\
&= -1 \\
P_3 &= \frac{8x}{3} \\
P_2 &= -5x^2 - \frac{1}{3} \\
P_1 &= \frac{50}{9}x^3 + \frac{14}{9}x \\
P_0 &= -\frac{125}{54}x^4 - \frac{67}{27}x^2 + \frac{1}{18} \\
P_{-1} &= 0
\end{aligned}
$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$
\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!}\omega^i = 0
$$

$$
\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!}\omega^i = 0
$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$
\frac{(x-1)^2 (x+1)^2 \left(-5x^2 - \frac{1}{3}\right)\omega^2}{2} + \frac{8(x-1)^3 (x+1)^3 x\,\omega^3}{3} - (x-1)^4 (x+1)^4 \omega^4 - \frac{125x^4}{1296} - \frac{67x^2}{648} + \frac{1}{432} + \frac{25\omega\, x^5}{27} - \frac{2\omega\, x^3}{3} - \frac{7\omega x}{27} = 0
$$

The solution ω of eq. 3A is found as

$$
\omega = \frac{1}{6x^2-6}\left(4x + \sqrt{x^2 - 1 + (x^2-1)^{2/3}} + \sqrt{-\frac{2\left(\left(-x^2 + \frac{(x^2-1)^{2/3}}{2} + 1\right)\sqrt{x^2 - 1 + (x^2-1)^{2/3}} + x^3 - x\right)}{\sqrt{x^2 - 1 + (x^2-1)^{2/3}}}}\right) \tag{4A}
$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Unable to integrate $\int \omega dx$. Leaving the integral unevaluated. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{\int \omega \, dx} \end{aligned}$$

Where ω given above. The second solution y_2 to the original ode is found using reduction of order

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx \\ &= e^{\int \omega \, dx} \int \frac{e^{\int -\frac{B}{A} \, dx}}{\left(e^{\int \omega \, dx}\right)^2} \, dx \end{aligned}$$

Since $B = 0$ then the above reduces to

$$y_2 = e^{\int \omega \, dx} \int \left(e^{\int \omega \, dx}\right)^{-2} \, dx$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{\int \omega \, dx}\right) + c_2 \left(e^{\int \omega \, dx} \int \left(e^{\int \omega \, dx}\right)^{-2} \, dx\right) \end{aligned}$$

Will add steps showing solving for IC soon.

3.6.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' = -\frac{(5x^2+27)y}{36(x^2-1)^2}$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + \frac{(5x^2+27)y}{36(x^2-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{5x^2+27}{36(x^2-1)^2}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x = -1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = 0$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x = -1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = \frac{2}{9}$

  ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $36(x^2-1)^2\left(\frac{d}{dx}y'\right) + (5x^2+27)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(36u^4 - 144u^3 + 144u^2)\left(\frac{d}{du}\frac{d}{du}y(u)\right) + (5u^2 - 10u + 32)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m = 0..2$

  $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  ◦ Shift index using $k- > k-m$

  $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion for $m = 2..4$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $16a_0(-1+3r)(-2+3r)\, u^r + (16a_1(2+3r)(1+3r) - 2a_0(72r^2-72r+5))\, u^{1+r} + \left(\sum_{k=2}^{\infty} (16a_k(\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $16(-1+3r)(-2+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$

- Each term must be 0

  $16a_1(2+3r)(1+3r) - 2a_0(72r^2-72r+5) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0(72r^2-72r+5)}{8(9r^2+9r+2)}$

- Each term in the series must be 0, giving the recursion relation

  $36(4a_k + a_{k-2} - 4a_{k-1})\, k^2 + 36(2(4a_k + a_{k-2} - 4a_{k-1})\, r - 4a_k - 5a_{k-2} + 12a_{k-1})\, k + 36(4a_k + a$

- Shift index using $k->k+2$

  $36(4a_{k+2} + a_k - 4a_{k+1})(k+2)^2 + 36(2(4a_{k+2} + a_k - 4a_{k+1})\, r - 4a_{k+2} - 5a_k + 12a_{k+1})(k+2)$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{36k^2a_k - 144k^2a_{k+1} + 72kra_k - 288kra_{k+1} + 36r^2a_k - 144r^2a_{k+1} - 36ka_k - 144ka_{k+1} - 36ra_k - 144ra_{k+1} + 5a_k - 10a_{k+1}}{16(9k^2+18kr+9r^2+27k+27r+20)}$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{36k^2a_k - 144k^2a_{k+1} - 12ka_k - 240ka_{k+1} - 3a_k - 74a_{k+1}}{16(9k^2+33k+30)}$

- Solution for $r = \frac{1}{3}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{3}}, a_{k+2} = -\frac{36k^2a_k - 144k^2a_{k+1} - 12ka_k - 240ka_{k+1} - 3a_k - 74a_{k+1}}{16(9k^2+33k+30)}, a_1 = -\frac{11a_0}{48}\right]$

- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^{k+\frac{1}{3}}, a_{k+2} = -\frac{36k^2a_k - 144k^2a_{k+1} - 12ka_k - 240ka_{k+1} - 3a_k - 74a_{k+1}}{16(9k^2+33k+30)}, a_1 = -\frac{11a_0}{48}\right]$

- Recursion relation for $r = \frac{2}{3}$

  $a_{k+2} = -\frac{36k^2a_k - 144k^2a_{k+1} + 12ka_k - 336ka_{k+1} - 3a_k - 170a_{k+1}}{16(9k^2+39k+42)}$

- Solution for $r = \frac{2}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{2}{3}}, a_{k+2} = -\frac{36k^2 a_k - 144k^2 a_{k+1} + 12k a_k - 336k a_{k+1} - 3a_k - 170a_{k+1}}{16(9k^2+39k+42)}, a_1 = -\frac{11a_0}{96}\right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{2}{3}}, a_{k+2} = -\frac{36k^2 a_k - 144k^2 a_{k+1} + 12k a_k - 336k a_{k+1} - 3a_k - 170a_{k+1}}{16(9k^2+39k+42)}, a_1 = -\frac{11a_0}{96}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+1)^{k+\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+\frac{2}{3}}\right), a_{k+2} = -\frac{36k^2 a_k - 144k^2 a_{k+1} - 12k a_k - 240k a_{k+1} - 3a_k - 74a}{16(9k^2+33k+30)}\right.$$

3.6.3 Maple trace

```
Methods for second order ODEs:
```

3.6.4 Maple dsolve solution

Solving time : `0.007` (sec)
Leaf size : 25

```
dsolve(diff(diff(y(x),x),x) = -1/36*(5*x^2+27)/(x^2-1)^2*y(x),
       y(x),singsol=all)
```

$$y = \sqrt{x^2-1}\left(\text{LegendreP}\left(-\frac{1}{6}, \frac{1}{3}, x\right) c_1 + \text{LegendreQ}\left(-\frac{1}{6}, \frac{1}{3}, x\right) c_2\right)$$

3.6.5 Mathematica DSolve solution

Solving time : `0.061` (sec)
Leaf size : 38

```
DSolve[{D[y[x],{x,2}]== -(5*x^2+27)/(36*(x^2-1)^2)*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{x^2-1}\left(c_1 P_{-\frac{1}{6}}^{\frac{1}{3}}(x) + c_2 Q_{-\frac{1}{6}}^{\frac{1}{3}}(x)\right)$$

3.7 problem Kovacic 2005 paper. Example 2

Internal problem ID [8975]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Kovacic 2005 paper. Example 2
Date solved : `Monday, October 21, 2024 at 05:26:10 PM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$y'' = -\frac{y}{4x^2}$$

3.7.1 Solved as second order ode using Kovacic algorithm

Time used: 0.168 (sec)

Writing the ode as

$$y'' + \frac{y}{4x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{1}{4x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1594: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0\\ 0 = 0\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx}\\ &= e^{\int \frac{1}{2x} dx}\\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \sqrt{x}\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \sqrt{x} \int \frac{1}{x}\, dx \\
&= \sqrt{x}(\ln (x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}(\ln (x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

3.7.2 Maple step by step solution

Let's solve
$\frac{d}{dx}y' = -\frac{y}{4x^2}$

- Highest derivative means the order of the ODE is 2
  $\frac{d}{dx}y'$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $\frac{d}{dx}y' + \frac{y}{4x^2} = 0$
- Multiply by denominators of the ODE
  $4\left(\frac{d}{dx}y'\right)x^2 + y = 0$
- Make a change of variables
  $t = \ln (x)$

□ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule
    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$
  - Compute derivative
    $y' = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule
    $\frac{d}{dx}y' = \left(\frac{d}{dt}\frac{d}{dt}y(t)\right)t'(x)^2 + \left(\frac{d}{dx}t'(x)\right)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $\frac{d}{dx}y' = \frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $4\left(\frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 + y(t) = 0$

- Simplify

  $4\frac{d}{dt}\frac{d}{dt}y(t) - 4\frac{d}{dt}y(t) + y(t) = 0$

- Isolate 2nd derivative

  $\frac{d}{dt}\frac{d}{dt}y(t) = \frac{d}{dt}y(t) - \frac{y(t)}{4}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d}{dt}\frac{d}{dt}y(t) - \frac{d}{dt}y(t) + \frac{y(t)}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - r + \frac{1}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = \frac{1}{2}$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{\frac{t}{2}}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{\frac{t}{2}}$

- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t)$

- Substitute in solutions

  $y(t) = C1\,\mathrm{e}^{\frac{t}{2}} + C2 t\,\mathrm{e}^{\frac{t}{2}}$

- Change variables back using $t = \ln(x)$

  $y = C1\sqrt{x} + C2\ln(x)\sqrt{x}$

- Simplify

  $y = \sqrt{x}\,(C2\ln(x) + C1)$

3.7.3 Maple trace

```
Methods for second order ODEs:
```

3.7.4 Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x) = -1/4/x^2*y(x),
       y(x),singsol=all)
```

$$y = \sqrt{x}\,(c_2 \ln(x) + c_1)$$

3.7.5 Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 24

```
DSolve[{D[y[x],{x,2}]== -1/(4*x^2)*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}\sqrt{x}(c_2 \log(x) + 2c_1)$$

3.8 problem David Saunders 1981 paper. Example 1

Internal problem ID [8976]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : David Saunders 1981 paper. Example 1
Date solved : `Monday, October 21, 2024 at 05:26:11 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = (x^2 + 3)\, y$$

3.8.1 Solved as second order ode using Kovacic algorithm

Time used: 0.252 (sec)

Writing the ode as

$$y'' + (-x^2 - 3)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -x^2 - 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 3$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 + 3)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1596: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{2x} - \frac{9}{8x^3} + \frac{27}{16x^5} - \frac{405}{128x^7} + \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} + \frac{72171}{2048x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= x \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+3}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2+3) + (0) \\ &= x^2+3 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{3}{1} - 1\right) &= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{3}{1} - 1\right) &= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 + 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^{+} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + (x) \\ &= x \\ &= x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2(x)\,(1) + \left((1) + (x)^2 - \left(x^2 + 3\right)\right) &= 0 \\ -2a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int x dx} \\
&= (x)\, \mathrm{e}^{\frac{x^2}{2}} \\
&= x\, \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= x\, \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\, \mathrm{e}^{\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= x\, \mathrm{e}^{\frac{x^2}{2}} \int \frac{1}{x^2 \mathrm{e}^{x^2}}\, dx \\
&= x\, \mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi}\, \mathrm{erf}(x) \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\, \mathrm{e}^{\frac{x^2}{2}} \right) + c_2 \left(x\, \mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi}\, \mathrm{erf}(x) \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

3.8.2 Maple step by step solution

Let's solve

$\frac{d}{dx}y' = (x^2+3)\, y$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' + (-x^2-3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $\frac{d}{dx}y'$ to series expansion

    $\frac{d}{dx}y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

    $\frac{d}{dx}y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 - 3a_0 + (6a_3 - 3a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - 3a_k - a_{k-2})\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 3a_0 = 0, 6a_3 - 3a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} - 3a_k - a_{k-2} = 0$

- Shift index using $k->k+2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 3a_{k+2} - a_k = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{3a_{k+2}+a_k}{k^2+7k+12}, a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right]$$

3.8.3 Maple trace

```
Methods for second order ODEs:
```

3.8.4 Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x) = (x^2+3)*y(x),
       y(x),singsol=all)
```

$$y = x\left(c_2\sqrt{\pi}\,\operatorname{erf}(x) + c_1\right) \mathrm{e}^{\frac{x^2}{2}} + \mathrm{e}^{-\frac{x^2}{2}} c_2$$

3.8.5 Mathematica DSolve solution

Solving time : 0.091 (sec)
Leaf size : 46

```
DSolve[{D[y[x],{x,2}]== (x^2+3)*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{x^2}{2}}\left(-\sqrt{\pi}c_2 e^{x^2} x\text{erf}(x) + c_1 e^{x^2} x - c_2\right)$$

3.9 problem David Saunders 1981 paper. Example 3

Internal problem ID [8977]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : David Saunders 1981 paper. Example 3
Date solved : `Monday, October 21, 2024 at 05:26:12 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x^2y'' = 2y$$

3.9.1 Solved as second order ode using Kovacic algorithm

Time used: 0.143 (sec)

Writing the ode as

$$x^2y'' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1598: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)\,(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \frac{1}{x} \int \frac{1}{\frac{1}{x^2}} \, dx \\ &= \frac{1}{x} \left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{x^3}{3}\right)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

3.9.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) = 2y$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2\left(\frac{d}{dx}y'\right) - 2y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y' = \left(\frac{d}{dt}\frac{d}{dt}y(t)\right)t'(x)^2 + \left(\frac{d}{dx}t'(x)\right)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $\frac{d}{dx}y' = \frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d}{dt}\frac{d}{dt}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2y(t) = 0$

- Simplify

$\frac{d}{dt}\frac{d}{dt}y(t) - \frac{d}{dt}y(t) - 2y(t) = 0$

- Characteristic polynomial of ODE
  $r^2 - r - 2 = 0$
- Factor the characteristic polynomial
  $(r+1)(r-2) = 0$
- Roots of the characteristic polynomial
  $r = (-1, 2)$
- 1st solution of the ODE
  $y_1(t) = e^{-t}$
- 2nd solution of the ODE
  $y_2(t) = e^{2t}$
- General solution of the ODE
  $y(t) = C1 y_1(t) + C2 y_2(t)$
- Substitute in solutions
  $y(t) = C1\, e^{-t} + C2\, e^{2t}$
- Change variables back using $t = \ln(x)$
  $y = \frac{C1}{x} + C2\, x^2$
- Simplify
  $y = \frac{C1}{x} + C2\, x^2$

3.9.3 Maple trace

```
Methods for second order ODEs:
```

3.9.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x) = 2*y(x),
       y(x),singsol=all)
```

$$y = \frac{c_2\, x^3 + c_1}{x}$$

3.9.5 Mathematica DSolve solution

Solving time : `0.016` (sec)
Leaf size : 18

```
DSolve[{x^2*D[y[x],{x,2}]== 2*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_2 x^3 + c_1}{x}$$

3.10 problem Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 1

Internal problem ID [8978]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 1
Date solved : `Monday, October 21, 2024 at 05:26:12 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0$$

3.10.1 Solved as second order ode using Kovacic algorithm

Time used: 0.102 (sec)

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1600: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\
&= z_1 e^{-x^2} \\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2}(x)\right)$$

Will add steps showing solving for IC soon.

3.10.2 Maple step by step solution

Let's solve

$\frac{d}{dx} y' + 4xy' + (4x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx} y'$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert $\frac{d}{dx} y'$ to series expansion

  $\frac{d}{dx} y' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- > k + 2$

  $\frac{d}{dx} y' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$
- Solve for the dependent coefficient(s)

  $\{a_2 = -a_0, a_3 = -a_1\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $\left((k + 2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k + 2) + 2a_{k+2} + 4a_k = 0$
- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2}+2a_k+5a_{k+2})}{k^2+7k+12}, a_2 = -a_0, a_3 = -a_1\right]$

3.10.3 Maple trace

```
Methods for second order ODEs:
```

3.10.4 Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+4*x*diff(y(x),x)+(4*x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-x^2}(c_2 x + c_1)$$

3.10.5 Mathematica DSolve solution

Solving time : `0.039` (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+4*x*D[y[x],x]+(4*x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-x^2}(c_2 x + c_1)$$

3.11 problem Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 2

Internal problem ID [8979]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 2
Date solved : `Monday, October 21, 2024 at 05:26:13 PM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0$$

3.11.1 Solved as second order ode using Kovacic algorithm

Time used: 0.165 (sec)

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1602: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x \cos(x)) + c_2(x \cos(x) (\tan(x)))
\end{aligned}
$$

Will add steps showing solving for IC soon.

3.11.2 Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2\left(\frac{d}{dx}y'\right) - 2xy' + (x^2 + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d}{dx}y'\right)$ to series expansion

    $x^2 \cdot \left(\frac{d}{dx}y'\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$
- Each term must be 0

  $a_1 r(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)\,(k+r-2) + a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+1+r)\,(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$

$a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

3.11.3 Maple trace

```
Methods for second order ODEs:
```

3.11.4 Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 15

```
dsolve(x^2*diff(diff(y(x),x),x)-2*x*diff(y(x),x)+(x^2+2)*y(x) = 0,
       y(x),singsol=all)
```

$$y = x(c_1 \sin(x) + c_2 \cos(x))$$

3.11.5 Mathematica DSolve solution

Solving time : `0.045` (sec)
Leaf size : 33

```
DSolve[{x^2*D[y[x],{x,2}]-2*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

3.12 problem Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 3

Internal problem ID [8980]
Book : Collection of Kovacic problems
Section : section 3. Problems from Kovacic related papers
Problem number : Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 3
Date solved : `Monday, October 21, 2024 at 05:26:14 PM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$(x-2)^2 y'' - (x-2) y' - 3y = 0$$

3.12.1 Solved as second order ode using Kovacic algorithm

Time used: 0.169 (sec)

Writing the ode as

$$(x-2)^2 y'' + (-x+2) y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x-2)^2 \\ B &= -x+2 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{15}{4(x-2)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 15 \\
t &= 4(x-2)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{15}{4(x-2)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1604: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-2)^2$. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4(x-2)^2}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{4(x-2)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{4 (x-2)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
2	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2 (x-2)} + (-)(0) \\
&= -\frac{3}{2 (x-2)} \\
&= -\frac{3}{2 (x-2)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2(x-2)}\right)(0) + \left(\left(\frac{3}{2(x-2)^2}\right) + \left(-\frac{3}{2(x-2)}\right)^2 - \left(\frac{15}{4(x-2)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int -\frac{3}{2(x-2)} dx} \\ &= \frac{1}{(x-2)^{3/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x+2}{(x-2)^2}\, dx} \\ &= z_1 e^{\frac{\ln(x-2)}{2}} \\ &= z_1\left(\sqrt{x-2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x-2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x+2}{(x-2)^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{\ln(x-2)}}{(y_1)^2} dx \\
&= y_1 \left(\frac{(x-2)^4}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x-2}\right) + c_2 \left(\frac{1}{x-2}\left(\frac{(x-2)^4}{4}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

3.12.2 Maple step by step solution

Let's solve

$(x-2)^2 \left(\frac{d}{dx}y'\right) - (x-2)\, y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d}{dx}y'$

- Isolate 2nd derivative

  $\frac{d}{dx}y' = \frac{3y}{(x-2)^2} + \frac{y'}{x-2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $\frac{d}{dx}y' - \frac{y'}{x-2} - \frac{3y}{(x-2)^2} = 0$

□ Check to see if $x_0 = 2$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{1}{x-2}, P_3(x) = -\frac{3}{(x-2)^2}\right]$

  ◦ $(x-2) \cdot P_2(x)$ is analytic at $x = 2$

  $\left.((x-2) \cdot P_2(x))\right|_{x=2} = -1$

- $(x-2)^2 \cdot P_3(x)$ is analytic at $x=2$

  $\left((x-2)^2 \cdot P_3(x)\right)\bigg|_{x=2} = -3$
- $x=2$is a regular singular point

  Check to see if $x_0 = 2$ is a regular singular point

  $x_0 = 2$

- Multiply by denominators

  $(x-2)^2 \left(\frac{d}{dx}y'\right) + (-x+2)\,y' - 3y = 0$
- Change variables using $x = u+2$ so that the regular singular point is at $u = 0$

  $u^2\left(\frac{d}{du}\frac{d}{du}y(u)\right) - u\left(\frac{d}{du}y(u)\right) - 3y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite DE with series expansions

- Convert $u \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r}$
- Convert $u^2 \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right)$ to series expansion

  $u^2 \cdot \left(\frac{d}{du}\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r}$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} a_k (k+r+1)\,(k+r-3)\, u^{k+r} = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+1)\,(k-3) = 0$
- Recursion relation that defines series solution to ODE

  $a_k = 0$
- Recursion relation for $r = 0$

  $a_k = 0$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_k = 0\right]$
- Revert the change of variables $u = x - 2$

$$\left[y=\sum_{k=0}^{\infty} a_k (x-2)^k , a_k = 0\right]$$

3.12.3 Maple trace

```
Methods for second order ODEs:
```

3.12.4 Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 19

```
dsolve((x-2)^2*diff(diff(y(x),x),x)-(x-2)*diff(y(x),x)-3*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{c_1 (x-2)^4 + c_2}{x-2}$$

3.12.5 Mathematica DSolve solution

Solving time : `0.056` (sec)
Leaf size : 22

```
DSolve[{(x-2)^2*D[y[x],{x,2}]-(x-2)*D[y[x],x]-3*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_1 (x-2)^3 + \frac{c_2}{x-2}$$