Small note on solving second order ode using eigenvalues approach

Nasser M. Abbasi

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1 Conditions for using BVP eigenvalue method

We are only here talking about second order linear ode of the form $a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$ with boundary conditions. If it is an IVP ode, then this note does not apply. The first thing to check for is that B.C. are homogeneous. Which means they are equal to zero. An example is $y(0) = 0, y(L) = 0$ or $y(0) = 0, y'(L) = 0$ and so on. The BC do not have to be in terms of unknown L. They can be $y(0) = 0, y(1) = 0$ for example. If the BC are not homogeneous then we have to do some preprocessing which is not considered here.

There are two general cases. The first is if the ode itself has an unknown in it or not. For example $y'' + y' + \lambda y = 0$ would qualify. Also $ay'' + y' + y = 0$ will qualify. The second general case is that the ode has no unknown it is. Which means all the coefficients are known. So we have then a total of 4 cases to consider:

1. The ode has at least one unknown in it such as $y'' + y' + \lambda y = 0$ but the BC have no unknown. For example $y(0) = 0, y(1) = 0$. This is a BVP eigenvalue problem.

The above gives the solution

$$y(x) = \begin{cases} 2e^{-\frac{x}{2}} c_1 \sinh\left(\frac{1}{2}\sqrt{1-4\lambda}x\right) & \sinh\left(\frac{1}{2}\sqrt{1-4\lambda}\right) = 0 \\ 0 & \text{otherwise} \end{cases}$$

2. The ode has at least one unknown in it such as $y'' + y' + \lambda y = 0$ and also BC have unknown. For example $y(0) = 0, y(L) = 0$ where here L is unknown. This is also BVP eigenvalue problem

$$y(x) = \begin{cases} 2e^{-\frac{x}{2}} c_1 \sinh\left(\frac{1}{2}\sqrt{1-4\lambda}x\right) & \sinh\left(\frac{1}{2}\sqrt{1-4\lambda}L\right) = 0 \\ 0 & \text{otherwise} \end{cases}$$

3. The ode has *NO* unknown in it such as $y'' + y' + 5y = 0$ but BC have at least one unknown. For example $y(0) = 0, y(L) = 0$ where here L is unknown. This is BVP but not an eigenvalue problem. There is one eigenfunction The solution will have the unknown L in it.

$$y(x) = \begin{cases} e^{-\frac{x}{2}} c_1 \sin\left(\frac{\sqrt{19}}{2}x\right) & L = \frac{4n\pi}{\sqrt{19}} \text{or } L = \frac{2(\pi+2n\pi)}{\sqrt{19}} \text{ where } n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

4. The ode has *NO* unknown in it such as $y'' + y' + 5y = 0$ and also BC have no unknown. For example $y(0) = 0, y(1) = 0$. This is BVP but not an eigenvalue problem. The solution is $y(x) = 0$ in this case.

In the above, only 1,2 are considered BVP eigenvalue problems. The others are just BVP problems. In (3), even though there is no eigenvalues (because the ODE has no unknown in it), we still give conditions for non-trivial solution, because depending on value of L this can happen. In (4), only trivial solution is possible.

2 Example 1

Let us solve the second order BVP

$$\begin{aligned} y'' + 3y &= 0 \\ y(0) &= 0 \\ y(L) &= 0 \end{aligned}$$

If the boundary condition do not have an arbitrary L in them, i.e. if $y(L) = 0$ happened to be for example $y(1) = 0$ then this can be solved using normal methods giving $y = 0$ as solution and there is nothing more to say.

But if we try to solve this using normal methods with $y(L) = 0$, where L is now a symbol and not a number, we will see that the result will be also be $y(0) = 0$ which is the trivial solution again.

But there is a non-trivial solution to this when L is symbol (undefined length). We have to use eigenvalue/eigenfunction method instead of normal methods to find this non-trivial solution. And this note is to show the difference.

First we will solve

$$\begin{aligned} y'' + 3y &= 0 \\ y(0) &= 0 \\ y(L) &= 0 \end{aligned}$$

Using normal methods. Since this is constant coefficient ode, then the characteristic equation is

$$\begin{aligned} r^2 + 3 &= 0 \\ r &= \pm\sqrt{3}i \end{aligned}$$

This means the basis solutions are $y_1 = e^{\sqrt{3}ix}, y_2 = e^{-\sqrt{3}ix}$ and the general solution is

$$y = c_1 e^{\sqrt{3}ix} + c_2 e^{-\sqrt{3}ix}$$

Which can be written using Euler formula as

$$y = c_1 \cos\left(\sqrt{3}x\right) + c_2 \sin\left(\sqrt{3}x\right) \tag{1}$$

Now we apply boundary conditions to find c_1, c_2. When $y(0) = 0$ the above becomes

$$0 = c_1$$

Hence (1) now becomes

$$y = c_2 \sin\left(\sqrt{3}x\right)$$

And when $y(L) = 0$ the above gives

$$0 = c_2 \sin\left(\sqrt{3}L\right) \tag{2}$$

So we conclude here that $c_2 = 0$ since L is not known and can not be zero, which results in trivial solution

$$y = 0$$

The difference between the above method and the eigenvalue/eigenfunction method to solving BVP, is that in (2), we do not choose $c_2 = 0$, but instead we find L values for L that makes $\sin(3L) = 0$ instead of the constant c_2 and thus avoiding trivial solution. We know that $\sin\left(\sqrt{3}L\right) = 0$ when $\sqrt{3}L = 0, \pm\pi, \pm 2\pi, \cdots$ but since $L > 0$ this becomes $\sqrt{3}L = \pi, 2\pi, 3\pi, \cdots$

$$\sqrt{3}L = n\pi \qquad n = \mathbb{Z}, n > 0$$

Hence

$$L = \frac{n\pi}{\sqrt{3}} \qquad n = \mathbb{Z}, n > 0$$

The L values above are called the eigenvalues of the ode. These are the values that gives non trivial solution. The general solution now becomes

$$y = \begin{cases} c_2 \sin\left(\sqrt{3}x\right) & L = \frac{n\pi}{\sqrt{3}}, n = \mathbb{Z}, n > 0 \\ 0 & \text{otherwise} \end{cases}$$

$\sin\left(\sqrt{3}x\right)$ above is the eigenfunction. There is only one eigenfuction in this example. The above says that the solution is trivial only when L does not satisfy $L = \frac{n\pi}{\sqrt{3}}$.

Mathematica DSolve command handles this automatically and gives both trivial and non-trivial and the conditions on L.

3 Example 2

Another variation of this problem is when the ode has the arbitrary value in the ode itself, instead of in the boundary condition as in the above example. Now the boundary conditions do not have an arbitrary symbol in them. An example is

$$\begin{aligned} y'' + \lambda y &= 0 \\ y(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

Let us solve this using normal method again. Since this is constant coefficient ode, then the characteristic equation is

$$\begin{aligned} r^2 + \lambda &= 0 \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Hence the general solution is

$$\begin{aligned} y &= c_1 e^{\sqrt{\lambda} i x} + c_2 e^{-\sqrt{\lambda} i x} \\ &= c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right) \end{aligned} \tag{3}$$

At $y(0) = 0$ the above gives

$$0 = c_1$$

And (3) becomes

$$y = c_2 \sin\left(\sqrt{\lambda} x\right)$$

At $y(1) = 0$ the above gives

$$0 = c_2 \sin\left(\sqrt{\lambda}\right)$$

And we conclude that $c_2 = 0$ which gives the trivial solution

$$y = 0$$

Now the ode is solved using eigenvalue/eigenvector approach. Starting from $r = \pm\sqrt{-\lambda}$ and since we do not know what λ is, we have to check each possible case. We assume λ is real value in all of this.

<u>case $\lambda < 0$</u> Hence $-\lambda > 0$ and we let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. The roots then are $r = \pm\mu$ and the general solution is

$$y = c_1 \cos(\mu x) + c_2 \sinh(\mu x)$$

At $y(0) = 0$ the above gives

$$0 = c_1$$

So the solution becomes

$$y = c_2 \sinh(\mu x)$$

At $y(1) = 0$ the above becomes

$$0 = c_2 \sinh(\mu)$$

But $\sinh(\mu)$ is only zero when $\mu = 0$ or $\lambda = 0$ which is not the case here. This means only other option is $c_2 = 0$ which leads to trivial solution. Hence $\lambda < 0$ is not valid assumption.

<u>case $\lambda = 0$</u> The roots now are $r = 0$ double root. In other words, the ode is $y'' = 0$. Hence the general solution is

$$y = c_1 + c_2 x$$

At $y(0) = 0$ this gives $c_1 = 0$. Hence $y = c_2 x$. At $y(1) = 0$ this gives $0 = c_2$. Which means the trivial solution $y = 0$. So $\lambda = 0$ is again not a valid assumption.

<u>case $\lambda > 0$</u> The roots now are $r = \pm i\sqrt{\lambda}$ which means the general solution is

$$y = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right)$$

At $y(0) = 0$ the above gives $0 = c_1$ and the solution now becomes

$$y = c_2 \sin\left(\sqrt{\lambda} x\right)$$

At $y(1) = 0$ the above gives

$$0 = c_2 \sin\left(\sqrt{\lambda}\right)$$

For non-trivial solution we want $\sin\left(\sqrt{\lambda}\right) = 0$. This means $\sqrt{\lambda} = \pi, 2\pi, 3\pi, \cdots$ or

$$\begin{aligned} \sqrt{\lambda} &= n\pi \qquad n = \mathbb{Z}, n > 0 \\ \lambda_n &= n^2\pi^2 \end{aligned}$$

Notice that $n > 0$ because we assumed $\lambda > 0$ we can't pick $-\pi, -2\pi, \cdots$ values. We also can not pick $n = 0$. The general solution hence becomes

$$y = \begin{cases} c_2 \sin\left(\sqrt{\lambda_n} x\right) & \lambda = n^2\pi^2, n = \mathbb{Z}, n > 0 \\ 0 & \text{otherwise} \end{cases}$$

In the above λ_n are called the eigenvalues and $\Phi_n = \sin\left(\sqrt{\lambda_n} x\right), n = 1, 2, \cdots$ are called the eigenfunctions. In this example, there is one eigenfunction Φ_n associated with each eigenvalue λ_n.

4 Example 3

Final example is when the ode has an arbitrary value (the eigenvalue) in the ode itself and the boundary condition also has an arbitrary value.

$$\begin{aligned} y'' + \lambda y &= 0 \\ y(0) &= 0 \\ y(L) &= 0 \end{aligned}$$

Let us solve this using normal method again. Since this is constant coefficient ode, then the characteristic equation is

$$\begin{aligned} r^2 + \lambda &= 0 \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Hence the general solution is

$$\begin{aligned} y &= c_1 e^{\sqrt{\lambda} i x} + c_2 e^{-\sqrt{\lambda} i x} \\ &= c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right) \end{aligned} \tag{3}$$

At $y(0) = 0$ the above gives

$$0 = c_1$$

And (3) becomes

$$y = c_2 \sin\left(\sqrt{\lambda} x\right)$$

At $y(L) = 0$ the above gives

$$0 = c_2 \sin\left(\sqrt{\lambda} L\right)$$

And since L can not be zero, we conclude that $c_2 = 0$ which gives the trivial solution

$$y = 0$$

Now the problem is solved using eigenvalue/eigenvector approach. Starting from $r = \pm\sqrt{-\lambda}$ and since we do not know what λ is, we have to check each possible case. We assume λ is real value in all of this.

<u>case $\lambda < 0$</u> Hence $-\lambda > 0$ and we let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. The roots then are $r = \pm\mu$ and the general solution is

$$y = c_1 \cos(\mu x) + c_2 \sinh(\mu x)$$

At $y(0) = 0$ the above gives

$$0 = c_1$$

So the solution becomes

$$y = c_2 \sinh(\mu x)$$

At $y(L) = 0$ the above becomes

$$0 = c_2 \sinh(\mu L)$$

But $\sinh(\mu)$ is only zero when $\mu L = 0$ or $\lambda = 0$ (since L can not be zero), which is not the case here. This means only other option is $c_2 = 0$ which leads to trivial solution. Hence $\lambda < 0$ is not valid assumption.

<u>case $\lambda = 0$</u> The roots now are $r = 0$ double root. In other words, the ode is $y'' = 0$. Hence the general solution is

$$y = c_1 + c_2 x$$

At $y(0) = 0$ this gives $c_1 = 0$. Hence $y = c_2 x$. At $y(L) = 0$ this gives $0 = L c_2$. Which means $c_2 = 0$ since L can not be zero. Hence the trivial solution $y = 0$. So $\lambda = 0$ is again not a valid assumption.

<u>case $\lambda > 0$</u> The roots now are $r = \pm i\sqrt{\lambda}$ which means the general solution is

$$y = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right)$$

At $y(0)=0$ the above gives $0=c_1$ and the solution now becomes

$$y=c_2\sin\left(\sqrt{\lambda}x\right)$$

At $y(L)=0$ the above gives

$$0=c_2\sin\left(\sqrt{\lambda}L\right)$$

For non-trivial solution we want $\sin\left(\sqrt{\lambda}L\right)=0$. This means $\sqrt{\lambda}L=\pi,2\pi,\cdots$ or

$$\begin{aligned}\sqrt{\lambda}&=\frac{n\pi}{L} \qquad n=\mathbb{Z},n>0\\ \lambda_n&=\frac{n^2\pi^2}{L^2}\end{aligned}$$

Hence the general solution now becomes

$$y=\begin{cases} c_2\sin\left(\sqrt{\lambda_n}x\right) & \lambda=\frac{n^2\pi^2}{L^2},n=\mathbb{Z},n>0\\ 0 & \text{otherwise}\end{cases}$$

In the above λ_n are called the eigenvalues and $\Phi_n=\sin\left(\sqrt{\lambda_n}x\right),n=1,2,3,\cdots$ are called the eigenfunctions. This is basically the same solution as second example, with the difference is that the length is now unknown L and not a specific value 1 as before. That is why the length L shows in the eigenvalues.