Review of the geometry of screw Axes

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Contents

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1 Definitions, Important formulas, and terminology

1.1 Planar displacement

General motion of a body in 2D that includes rotation and translation. We use $[A]$ for the rotation matrix, the vector $\mathbf{d}$ for the translation and the matrix $[T]$ for displacement. Hence we write

$$\mathbf{X} = [A]\mathbf{x} + \mathbf{d}$$

1.2 Spatial displacement

General motion of a body in 2D that includes rotation and translation. Another name for this is rigid displacement. Distance between points in a body remain unchanged before and after spatial displacement.

1.3 Homogeneous Transformation

Recall that the transformation that defines spatial displacement, which is given by $\mathbf{X} = [A]\mathbf{x} + \mathbf{d}$ is non-linear due to the presence of the translation term $\mathbf{d}$. It is more convenient to be able to work with linear transformation, therefore we add a fourth component to the position vector which is always 1 and rewrite the transformation, now calling it as T which maps X to x as

$$X = T\,x$$

Or in full component form

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ 1 \end{Bmatrix} = \overbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & d_1 \\ a_{21} & a_{22} & a_{23} & d_2 \\ a_{31} & a_{32} & a_{33} & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{T} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{Bmatrix}$$

Or in short form

$$\begin{Bmatrix} \mathbf{X} \\ 1 \end{Bmatrix} = \overbrace{\begin{bmatrix} A & \mathbf{d} \\ 000 & 1 \end{bmatrix}}^{T} \begin{Bmatrix} \mathbf{x} \\ 1 \end{Bmatrix}$$

1.4   The rotation axis

The set of points that remain fixed during the rotation defined by $[A]$. We use Rodrigues vector $\mathbf{b}$ to define the rotation axis.

1.5   A Pole or a Fixed point

A pole is point that remains fixed during planar or spatial displacement. Planar displacement (2D) have a pole, but 3D spatial displacement do not have a pole in general, since the requirement for a pole is to have an inverse for $[I - A]$ which for 3D is not possible since A has an eigenvalue of 1.

The following diagram is an example of a pole under planar displacement.
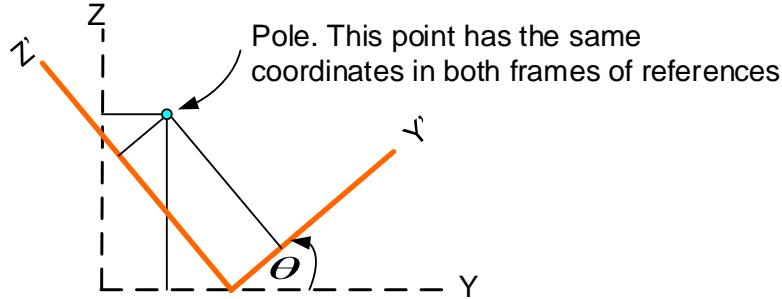
For spatial displacement (3D), one condition, when satisfied will result in a fixed point. This condition occurs when the translation vector $\mathbf{d}$ is perpendicular to the rotation vector $\mathbf{b}$. The pole in this case is given by

$$\mathbf{c} = \frac{\mathbf{b} \times (\mathbf{d} - \mathbf{b} \times \mathbf{d})}{2\mathbf{b} \cdot \mathbf{b}}$$

Not only is this point $\mathbf{c}$ fixed, by any point on the line $\mathbf{c} + t\mathbf{S}$ is also fixed. Where t is a parameter and $\mathbf{S}$ is a unit vector. Hence we can a fixed line under such a spatial displacement, and this line is called the rotation axis.

1.6 Rotational displacement

This is a special case of spatial displacement when the translation vector $\mathbf{d}$ is perpendicular to the rotation axis $\mathbf{b}$. To make the difference more clear, the following diagram is an illustration of spatial displacement which is not rotational displacement, and one which is.

Spatial displacement

Rotational displacement
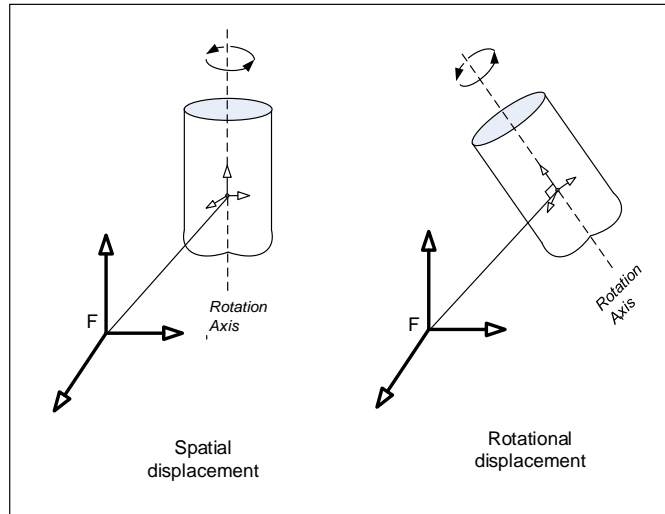
1.7 Rodrigues vector b

Defines the rotation axis under $[A]$. Using $\mathbf{S}$ as the unit vector along $\mathbf{b}$, then $\mathbf{b} = k\mathbf{S}$ where k is the length of vector $\mathbf{b}$ given by $\tan\left(\frac{\theta}{2}\right)$ where θ is the rotation angle around the rotation axis. Hence we can write

$$\mathbf{b} = \tan\left(\frac{\theta}{2}\right)\mathbf{S}$$

We see that the

$$\|\mathbf{b}\| = \tan\left(\frac{\theta}{2}\right)$$

Hence at $\theta = \pm 180^0$, $\|\mathbf{b}\| = \tan\left(\pm\frac{\pi}{2}\right)$ which goes to infinity. A plot of the function $\tan(\alpha)$ is below showing the discontinuities at $\pm\frac{\pi}{2}$
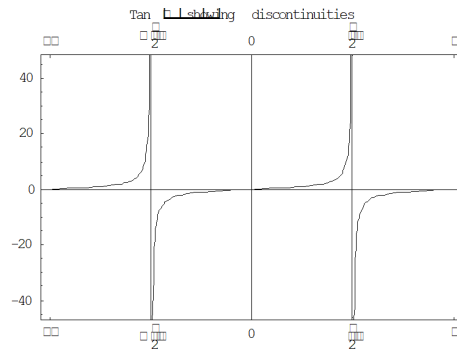
1.8 Screw

Screw is defined as a pair of vectors $(\mathbf{W}, \mathbf{V})^T$ such that $\mathbf{W} \cdot \mathbf{V} \neq 0$ and $|\mathbf{W}| \neq 1$. The pitch of the screw p_ω is defined as $\frac{\mathbf{W} \cdot \mathbf{V}}{\mathbf{W} \cdot \mathbf{W}}$

1.9 Plücker coordinates of a line

Given a line defined by a parametric equation $L(k) = \mathbf{C} + k\mathbf{S}$, where C is a reference point and S is a unit vector along the line, the Plücker coordinates of this line is given by $\left\{ \begin{array}{c} \mathbf{S} \\ \mathbf{C} \times \mathbf{S} \end{array} \right\}$ $\mathbf{C} \times \mathbf{S}$ represents the moment of the line around the origin of the reference frame. The following diagram helps to illustrates this.

Line L has Plücker coordinates $= \left\{ \begin{array}{c} \mathbf{S} \\ \mathbf{C} \times \mathbf{S} \end{array} \right\}$
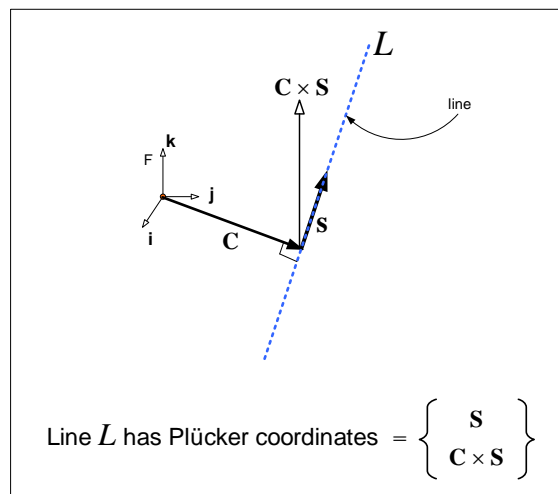
1.10 screw defined in terms of the plücker coordinates of its axis

We said above that a screw is defined as a pair of vectors $(\mathbf{W}, \mathbf{V})^T$. Given the plücker coordinates of the screw line $(\mathbf{W}, \mathbf{C} \times \mathbf{W})^T$, then we can write the pair of vectors that defines the screw associated with this screw line as follows

$$\left\{ \begin{array}{c} \omega \mathbf{S} \\ \omega \mathbf{C} \times \mathbf{S} + \omega p_\omega \mathbf{S} \end{array} \right\}$$

Where p_ω is the screw pitch and $\omega = |\mathbf{W}|$

1.11 Important relations between Cayley Matrix B

These are some important formulas put here for quick reference.

$$(\mathbf{x}_1 + \mathbf{x}_2)^T [B] (\mathbf{x}_1 + \mathbf{x}_2) = 0$$

See (4A) below for derivation of the above where

$$[B] = \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix}$$

and also

$$[B] = \tan \frac{\theta}{2} [S]$$

Where

$$[S] = \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}$$

From $[B]$ we obtain a row vector and call it

$$\mathbf{b} = (b_z, b_y, b_z)^T$$

(This is rodrigues vector) From $[S]$ we obtain a row vector and call it

$$\mathbf{s} = (s_x, s_y, s_z)^T$$

(This is unit vector along the screw axis). We also have

$$[B] = \tan \frac{\theta}{2} [S]$$

And in terms of the vectors $\mathbf{b}, \mathbf{s}$ the above becomes

$$\mathbf{b} = \tan \frac{\theta}{2} \mathbf{s}$$

2 Introduction to the rotation matrix in 3D

We seek to derive an expression for the rotation matrix in 3D. Consider a point x_1 in 3D being acted upon by a rotation matrix A. Let the final coordinates of this point be x_2. We consider the position vectors of these points, so we will designate these points by their position vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ from now on. The following diagram illustrate this.

rotation around x axis by angle θ $|\mathbf{x}_1| = |\mathbf{x}_2|$
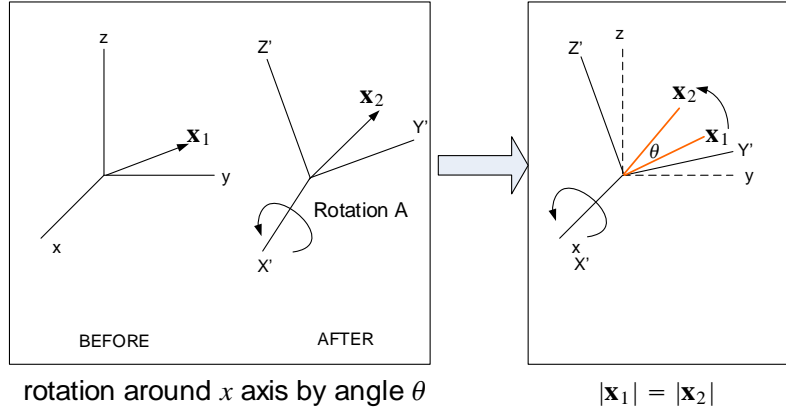
Hence we have that

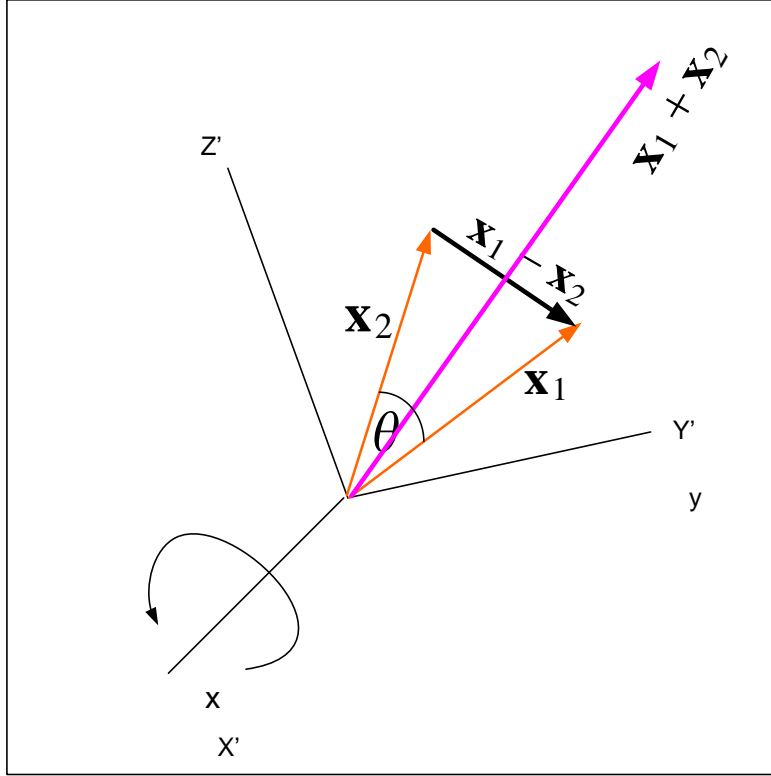$$\mathbf{x}_2 = A\mathbf{x}_1$$

Since $|\mathbf{x}_1| = |\mathbf{x}_2|$ we can write

$$\mathbf{x}_1 \cdot \mathbf{x}_1 = \mathbf{x}_2 \cdot \mathbf{x}_2$$

$$\mathbf{x}_1 \cdot \mathbf{x}_1 - \mathbf{x}_2 \cdot \mathbf{x}_2 = 0$$

$$(\mathbf{x}_1 - \mathbf{x}_2) \cdot (\mathbf{x}_1 + \mathbf{x}_2) = 0 \tag{1}$$

The geometric meaning of the above last equation is shown in the following diagram

To introduce the rotation matrix A into the equations, we can write $\mathbf{x}_1 - \mathbf{x}_2$ as $\mathbf{x}_1 - A\mathbf{x}_1$. Hence

$$\mathbf{x}_1 - \mathbf{x}_2 = [I - A]\mathbf{x}_1 \tag{2}$$

Similarly, we obtain

$$\mathbf{x}_1 + \mathbf{x}_2 = [I + A]\mathbf{x}_1 \tag{3}$$

From (3) we obtain

$$[A + I]^{-1}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{x}_1$$

Substitute the above expression for $\mathbf{x}_1$ into (2) we obtain

$$\mathbf{x}_1 - \mathbf{x}_2 = [A - I][A + I]^{-1}(\mathbf{x}_1 + \mathbf{x}_2) \tag{3A}$$

We now call the matrix $[A - I][A + I]^{-1}$ as B

$$B = [A - I][A + I]^{-1} \tag{3B}$$

We can also write from above the following

$$A = [I - B]^{-1}[I + B] \tag{3C}$$

Now rewrite (3A) as

$$\mathbf{x}_1 - \mathbf{x}_2 = [B]\,(\mathbf{x}_1 + \mathbf{x}_2) \tag{4}$$

What is the geometric meaning of $[B]$? we see that it is an operator that acts on vector $\mathbf{x}_1 + \mathbf{x}_2$ to produce the vector $\mathbf{x}_1 - \mathbf{x}_2$, however from the diagram above we see that the vector $\mathbf{x}_1 - \mathbf{x}_2$ is perpendicular to $\mathbf{x}_1 + \mathbf{x}_2$ and scaled down.

Hence

$$(\mathbf{x}_1 + \mathbf{x}_2)^T\,[B]\,(\mathbf{x}_1 + \mathbf{x}_2) = 0 \tag{4A}$$

This implied that B is skew-symmetric and have the form given by

$$B = \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix} \tag{4A}$$

Matrix B can be written as a column vector called $\mathbf{b} = [b_x, b_y, b_z]^T$ such that for any vector y we have

$$[B]\,\mathbf{y} = \mathbf{b} \times \mathbf{y}$$

The vector $\mathbf{b}$ is called Rodrigues vector.

Let the vector $\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{y}$ then we write

$$[B]\,\mathbf{y} = \mathbf{b} \times \mathbf{y}$$

Where $\mathbf{b}$ is some vector perpendicular to $\mathbf{y}$ such that its cross product with y results in $[B]\,\mathbf{y}$

The vector $\mathbf{b}$ is the vector that defines the rotation axis. The length of this vector is k and a unit vector along $\mathbf{b}$ is called $\mathbf{S}$, Hence we can write
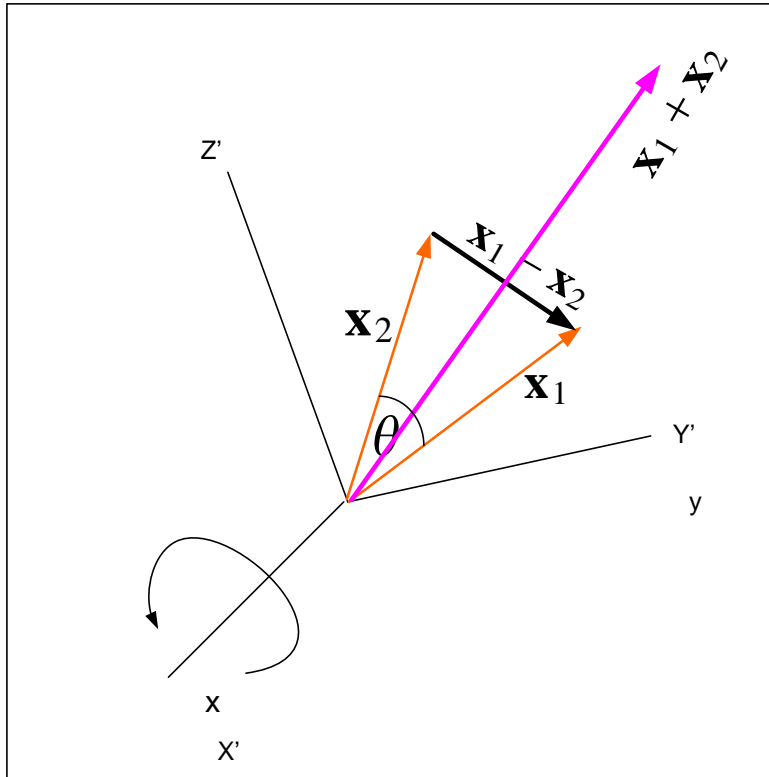
$$\mathbf{b} = k\mathbf{S}$$

Now we solve for B.

We will use equation (3B) to find the form of B for different rotations.

Consider for example the 3D rotation around the x axis given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

From (3B) we obtain

Z'

$\mathbf{x}_1 + \mathbf{x}_2$

$\mathbf{x}_1 - \mathbf{x}_2$

$\mathbf{x}_2$

$\mathbf{x}_1$

θ

Y'

y

x

X'

Out[9]//MatrixForm=

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{(-1+\cos[\theta])\,(2+2\cos[\theta])}{2+4\cos[\theta]+2\cos[\theta]^2+2\sin[\theta]^2} + \frac{2\sin[\theta]^2}{2+4\cos[\theta]+2\cos[\theta]^2+2\sin[\theta]^2} & \frac{2\,(-1+\cos[\theta])\,\sin[\theta]}{2+4\cos[\theta]+2\cos[\theta]^2+2\sin[\theta]^2} - \frac{(2+2\cos[\theta])\,\sin[\theta]}{2+4\cos[\theta]+2\cos[\theta]^2+2\sin[} \\
0 & -\frac{2\,(-1+\cos[\theta])\,\sin[\theta]}{2+4\cos[\theta]+2\cos[\theta]^2+2\sin[\theta]^2} + \frac{(2+2\cos[\theta])\,\sin[\theta]}{2+4\cos[\theta]+2\cos[\theta]^2+2\sin[\theta]^2} & \frac{(-1+\cos[\theta])\,(2+2\cos[\theta])}{2+4\cos[\theta]+2\cos[\theta]^2+2\sin[\theta]^2} + \frac{2\sin[\theta]^2}{2+4\cos[\theta]+2\cos[\theta]^2+2\sin[}
\end{pmatrix}
$$

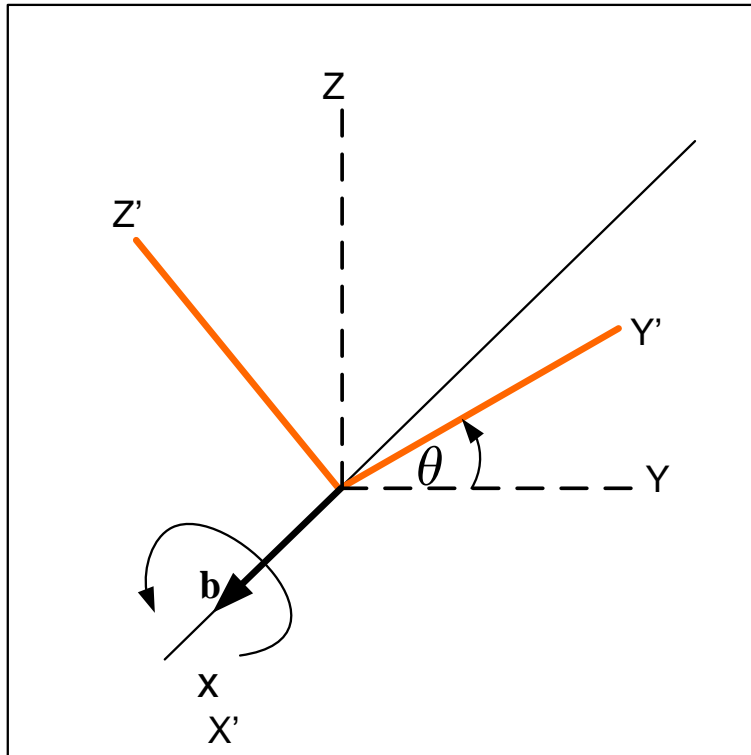For example for $\theta = 45^0$ the B matrix is

$$
B = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -0.414214 \\
0 & 0.414214 & 0
\end{bmatrix}
$$

Hence we see that $b_z = 0$, $b_y = 0$, $b_x = -0.41421$ hence

$$
\mathbf{b} = \begin{bmatrix}
0.41421 \\
0 \\
0
\end{bmatrix}
$$

Hence geometrically, Rodrigues vector is along the x axis and in the positive direction as illustrated in the following diagram.

Rodrigues vector b defines the rotation axis

The size of the **b** vector depends on the angle or rotation θ. The Rodrigues vector will be largest when θ is almost 180^0 and smallest when θ is zero.

3   Screw displacement

We now derive an expression for the invariant under spatial displacement, which is the screw axis.

Consider the following general spatial displacement

Spatial displacement showing the rotation matrix [A] and the translation vector d. The coordinates system F is the initial position and the coordinates system M is the final position.
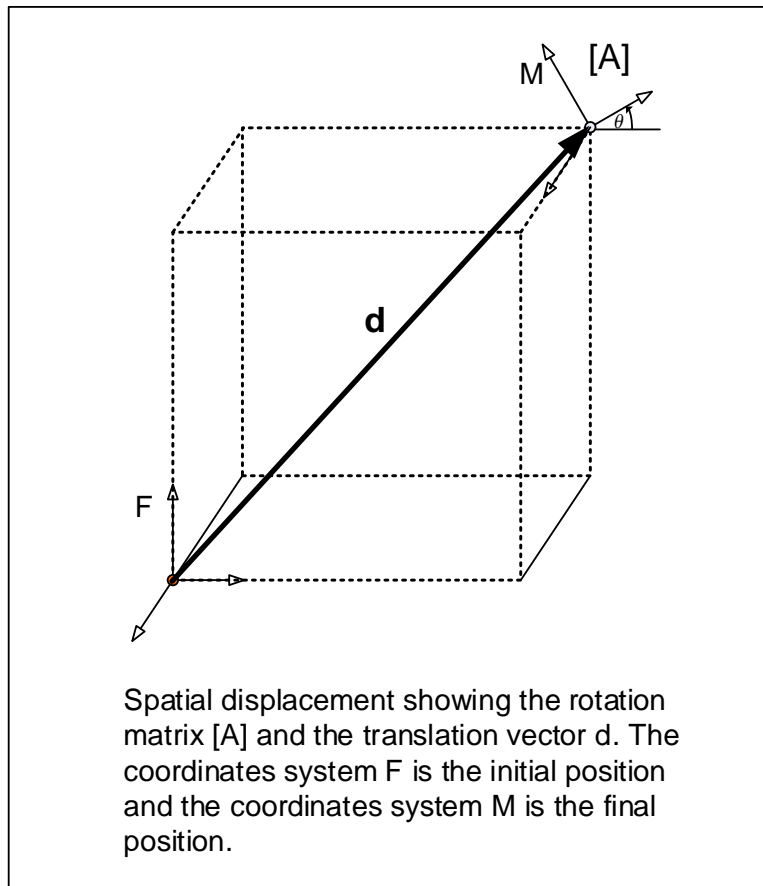
We start be decomposing the translation vector $\mathbf{d}$ into 2 components: One parallel $(\mathbf{d_2})$ and one perpendicular $(\mathbf{d^*})$ to the rotation axis of $[A]$ as follows (notice that $\mathbf{d^*}$ is perpendicular to the $x - axis$ which is where the rotation occurs around)

Decompose translation vector into its components
d=d*+d2
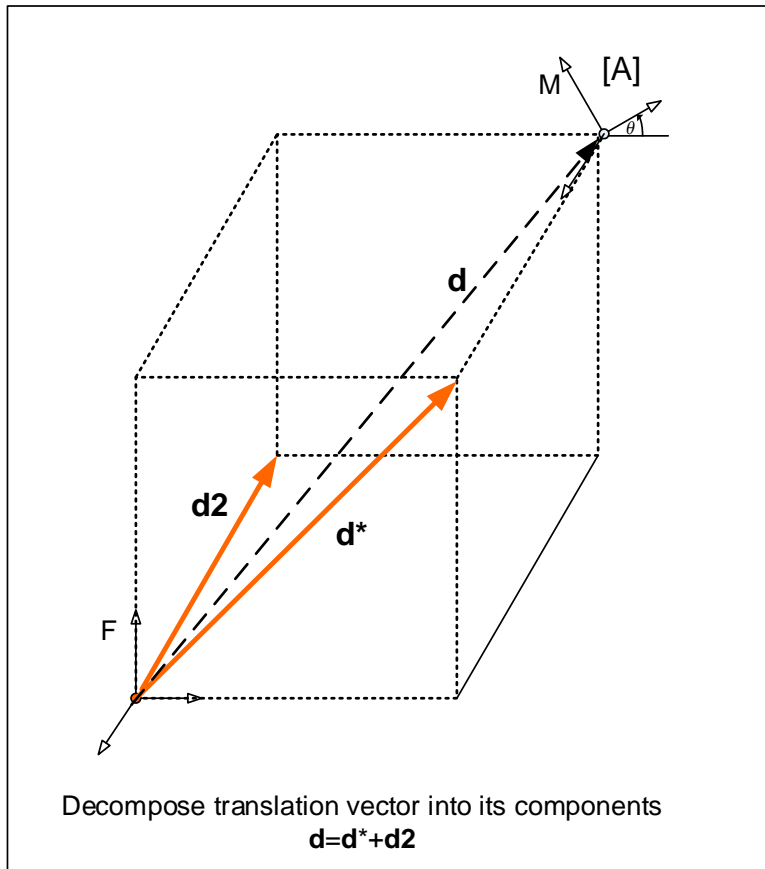
Recall from earlier that the vector $\mathbf{S}$ is a unit vector along the rotation axis of $[A]$ in the direction of $\mathbf{b}$. We found earlier that $\mathbf{b} = \tan\left(\frac{\theta}{2}\right)\mathbf{S}$. Hence we can write that $\mathbf{d}_2 = -k\mathbf{S}$ where $k = \tan\left(\frac{\theta}{2}\right)$

Let us redraw the above diagram putting all of these symbols to make the discussion more clear.

Decompose translation vector into its components
d=d*+k S. Also showing the Rodigues's vector of
rotation
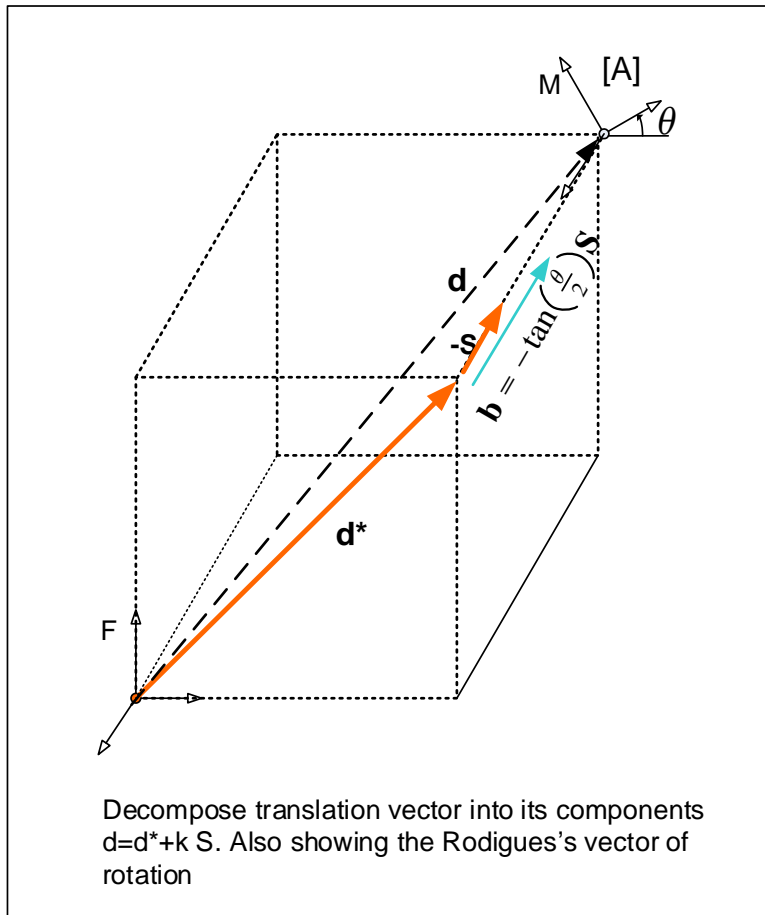
Now since

$$\mathbf{d} = \mathbf{d}^* - k\mathbf{S}$$

Then the spatial displacement operator T can be written as follows

$$T = [A, d]$$
$$= [A, \mathbf{d}^* - k\mathbf{S}]$$

Hence the spatial displacement is

$$\mathbf{X} = T\mathbf{x}$$
$$= [A, \mathbf{d}^* - k\mathbf{S}]\mathbf{x}$$
$$= [A, \mathbf{d}^*]\mathbf{x} - [I, k\mathbf{S}]\mathbf{x}$$

Hence we see that the spatial displacement can be viewed as rotational displacement followed by pure translation. Recall from above that rotational displacement is a special

type of spatial displacement where the translation part is perpendicular to the rotation axis. We can represent the above equation geometrically as follows

The rotational displacement part   The translation part along k**S**
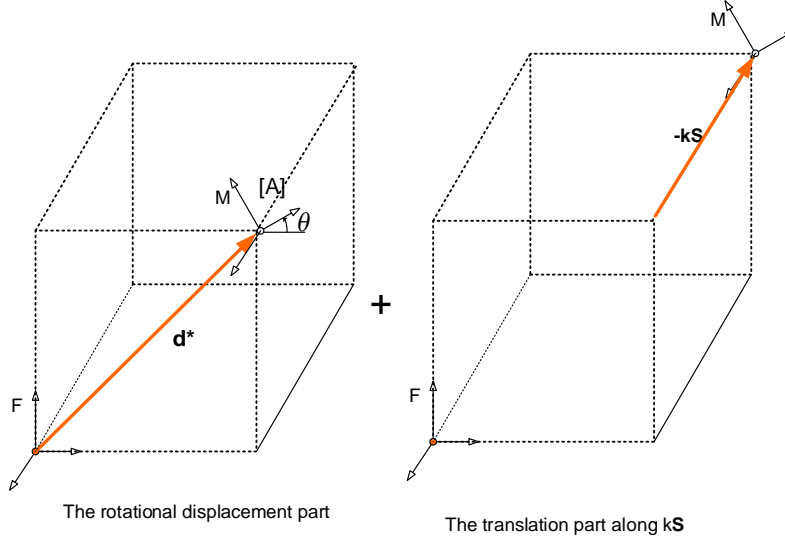
3.1 Derivation for expression for finding reference point for screw axis

Rotational displacement have a fixed point given by

$$\mathbf{C} = \frac{\mathbf{b} \times (\mathbf{d}^* - \mathbf{b} \times \mathbf{d}^*)}{2\mathbf{b} \cdot \mathbf{b}}$$

The derivation of the above equation is as follows.

Since we seek a fixed point $\mathbf{C}$, then we write

$$\begin{aligned}
\mathbf{C} &= [T]\,\mathbf{C} \\
&= [A]\,\mathbf{C} + \mathbf{d}^*
\end{aligned} \tag{4}$$

Using Cayley's formula, derived above in equation (3C), reproduced below

$$A = [I - B]^{-1}[I + B] \tag{3C}$$

and substitute for A in (4), we obtain

$$\mathbf{C} = [I - B]^{-\mathbf{1}}[I + B]\mathbf{C} + \mathbf{d}^*$$

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Multiply both sides by $[I - B]$

$$\mathbf{C} - [B]\,\mathbf{C} = [I + B]\mathbf{C} + [I - B]\mathbf{d}^*$$
$$0 = 2[B]\,\mathbf{C} + [I - B]\mathbf{d}^*$$
$$-\frac{1}{2}[I - B]\mathbf{d}^* = [B]\,\mathbf{C} \tag{4B}$$

But by definition

$$[B]\,\mathbf{C} = \mathbf{b} \times \mathbf{C}$$

Hence (4B) becomes

$$-\frac{1}{2}\mathbf{d}^* + \frac{1}{2}B\mathbf{d}^* = \mathbf{b} \times \mathbf{C}$$
$$-\frac{1}{2}\mathbf{d}^* + \frac{1}{2}\mathbf{b} \times \mathbf{d}^* = \mathbf{b} \times \mathbf{C}$$
$$\frac{1}{2}(\mathbf{b} \times \mathbf{d}^* - \mathbf{d}^*) = \mathbf{b} \times \mathbf{C}$$

Take the cross product of both sides w.r.t. $\mathbf{b}$ we obtain

$$\mathbf{b} \times \left(\frac{1}{2}(\mathbf{b} \times \mathbf{d}^* - \mathbf{d}^*)\right) = \mathbf{b} \times (\mathbf{b} \times \mathbf{C}) \tag{4C}$$

To simplify the above, use the relation

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Apply the above relation on the RHS of (4C), hence (4C) can be rewritten as

$$\mathbf{b} \times \left(\frac{1}{2}(\mathbf{b} \times \mathbf{d}^* - \mathbf{d}^*)\right) = \mathbf{b} \times (\mathbf{b} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{b} \cdot \mathbf{b})$$

But the vector $\mathbf{C}$ is perpendicular to $\mathbf{b}$ hence $\mathbf{b} \cdot \mathbf{C} = \mathbf{0}$ and the above simplifies to

$$\mathbf{C} = \frac{\mathbf{b} \times (\mathbf{d}^* - \mathbf{b} \times \mathbf{d}^*)}{2\mathbf{b} \cdot \mathbf{b}}$$

Now we continue to derive an expression for the screw axis.

We now consider a line L that passed through this point $\mathbf{C}$ and is parallel to the rotation axis of $[A]$ (in other words, along the same direction as the vector $\mathbf{S}$). Any point along this line remain fixed relative to the rotational displacement $\mathbf{X} = [A, d^*]\mathbf{x}$ part of the spatial displacement.

In addition, since the translation part of the spatial displacement, and given by $\mathbf{X} = [I, k\mathbf{S}]\mathbf{x}$, is a translation in the same direction and slides along the vector kS as k changes, then this line will also remain fixed relative to the translation part as well.

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Hence we conclude that the line L will remain fixed relative to the overall spatial displacement T.

This line is called the screw axis. And this type of decomposing the spatial displacement into rotational displacement followed by pure translation is called the screw displacement.
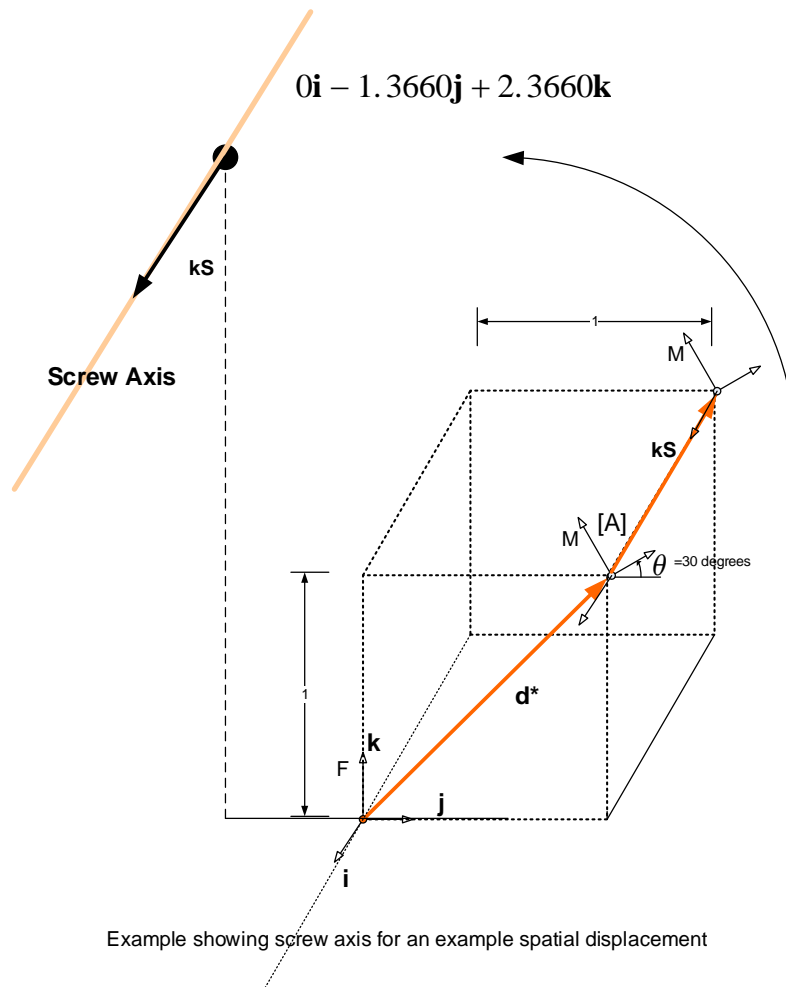
How to geometrically find the screw axis? Let us find the point $\mathbf{C}$ first. Let take an example similar to the above diagrams, where say $\theta = 30^0$, $\mathbf{d^*} = 0\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{S} = \mathbf{i}$, hence

$$\mathbf{b} = \tan \overbrace{\left(\frac{30^0}{2} \right)}^{k} \mathbf{S}$$

$$= 0.26795\mathbf{i}$$

hence

$$\mathbf{C} = \frac{\mathbf{b} \times (\mathbf{d^*} - \mathbf{b} \times \mathbf{d^*})}{2\mathbf{b} \cdot \mathbf{b}}$$

$$\mathbf{C} = \frac{0.26795\mathbf{i} \times \{(0\mathbf{i} + \mathbf{j} + \mathbf{k}) - (0.26795\mathbf{i}) \times (0\mathbf{i} + \mathbf{j} + \mathbf{k})\}}{2\,(0.26795\mathbf{i}) \cdot (0.26795\mathbf{i})}$$

$$\mathbf{C} = 0\mathbf{i} - 1.3660\mathbf{j} + 2.3660\mathbf{k}$$

On the above diagram we now can draw the screw axis using the above coordinates for the point $\mathbf{C}$

$$0\mathbf{i} - 1.3660\mathbf{j} + 2.3660\mathbf{k}$$

Screw Axis

kS

M

kS

M [A]

θ =30 degrees

d*

k

F

j

i

Example showing screw axis for an example spatial displacement

It is important to note that it is the line given by $L = \mathbf{C} + k\mathbf{S}$ (the screw axis) which is fixed under the spatial displacement, and not any one single point on this line.

4 The Screw Matrix

We now derive a new expression for spatial displacement using the screw axis line, which we denote as S, the angle of rotation θ and the amount of slide k along the screw axis.

The screw matrix is a new mathematical operator that we can use to denote spatial displacement between 2 different reference frames. Earlier we showed that we can use the homogeneous transformation operator $T(A, d) = [A, d]$ to denote spatial displacement,

and now we seek to obtain a new expression for a spatial displacement operator which is a function of the following 3 parameters

1. The screw axis line which we call S with the plucker coordinates $(\mathbf{s}, \mathbf{C} \times \mathbf{s})$

2. The angle or rotation θ

3. The amount of slide k

This is in addition to the mathematical object we examined earlier which is

$$\mathbf{d} = \mathbf{d}^* + k\mathbf{S} \tag{5}$$

Since C is a fixed point under the translation by $\mathbf{d}^*$ hence we write

$$\begin{aligned} \mathbf{C} &= [A, \mathbf{d}^*]\mathbf{C} \\ &= A\mathbf{C} + \mathbf{d}^* \end{aligned}$$

Hence

$$\mathbf{d}^* = [I - A]\mathbf{C}$$

Substitute the above into (5) we obtain

$$\mathbf{d} = [I - A]\mathbf{C} + k\mathbf{S} \tag{5A}$$

But the spatial displacement T is defined as

$$[T] = [A, \mathbf{d}]$$

Hence using (5A) the above becomes

$$[T] = [A, [I - A]\mathbf{C} + k\mathbf{S}] \tag{5B}$$

Using the notation of θ for angle of rotation and the slide k and S to denote the screw axis, we can write (5B) as

$$[T(\theta, k, S)] = [A(\theta, \mathbf{S}), [I - A(\theta, \mathbf{S})]\mathbf{C} + k\mathbf{S}] \tag{5C}$$

So now (5C) is an expression for the spatial operator T in terms of $\mathbf{S}$,k and θ. Recall that
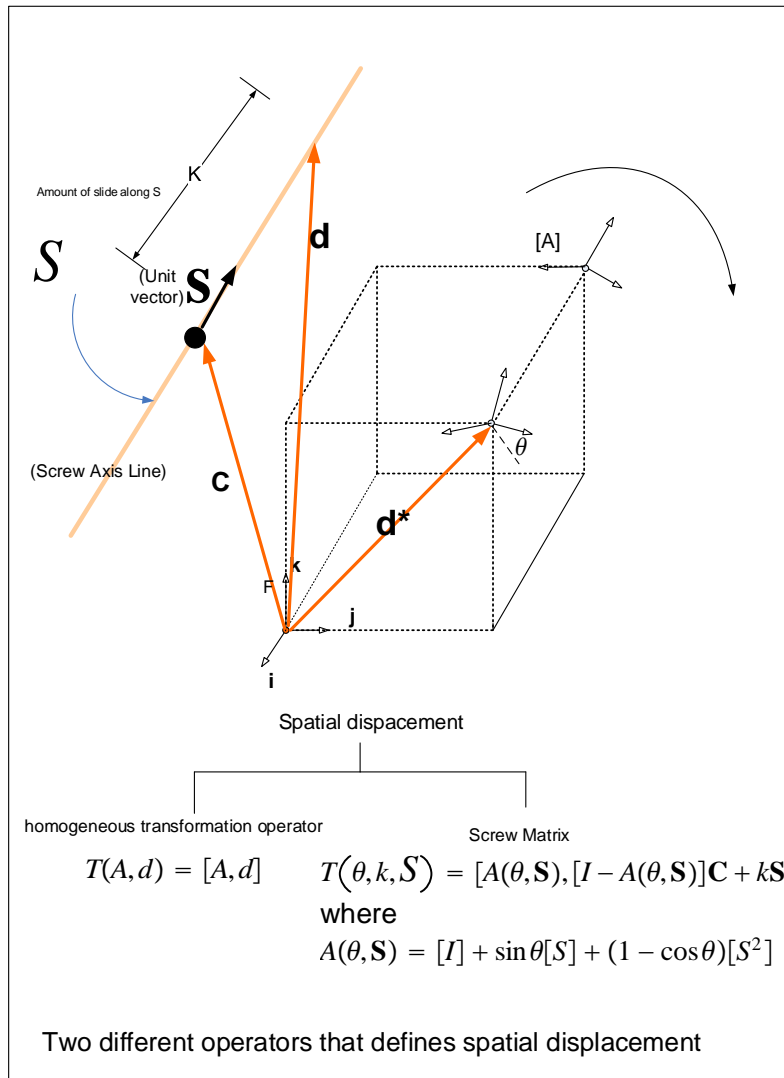
$$A(\theta, \mathbf{S}) = [I] + \sin\theta[S] + (1 - \cos\theta)\left[S^2\right]$$

Where

$$[S] = \begin{bmatrix} 0 & -s_x & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}$$

We call $[T(\theta, k, S)]$ the screw Matrix.

The following diagram helps to illustrate this.

Two different operators that defines spatial displacement

5   The spatial displacement of screws

So far we have discussed spatial displacements applied to points. We showed two Matrices can be used to accomplish this. The homogeneous transformation matrix $T(A,d) = [A,d]$ and the screw matrix $T(\theta,k,S) = [A(\theta,\mathbf{S}),[I - A(\theta,\mathbf{S})]\mathbf{C} + k\mathbf{S}]$ where $A(\theta,\mathbf{S}) = [I] + \sin\theta[S] + (1 - \cos\theta)[S^2]$.

We now show a matrix $[\hat{T}]$ which is used for the spatial displacement of a line and not just a point. This is based on using the plücker coordinates of a line to represent the line. Geometrically this is illustrated in the following diagram

Spatial displacement of lines using the new transformation matrix $[\hat{T}]$
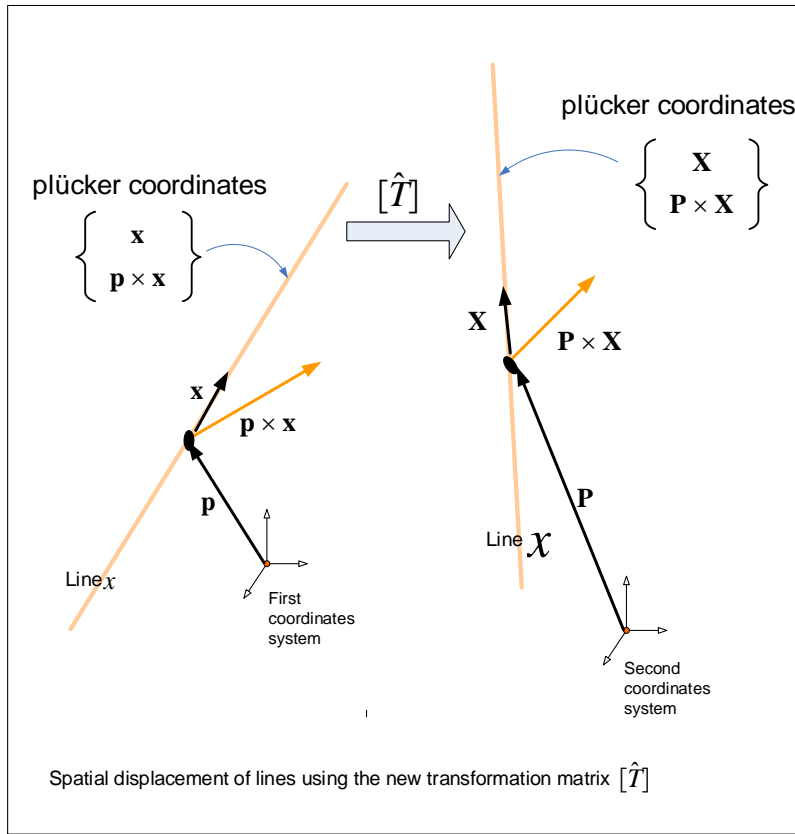
Since $\left[\hat{T}\right]$ operates on the Plücker coordinates of a line, then we write

$$X = \left[\hat{T}\right] x$$

$$\left\{ \begin{matrix} \mathbf{X} \\ \mathbf{P} \times \mathbf{X} \end{matrix} \right\} = \left[\hat{T}\right] \left\{ \begin{matrix} \mathbf{x} \\ \mathbf{p} \times \mathbf{x} \end{matrix} \right\} \tag{6A}$$

To processed further, we now assume a point $\mathbf{q}$ on the line x such that $\mathbf{x} = \mathbf{q} - \mathbf{p}$ as illustrated below
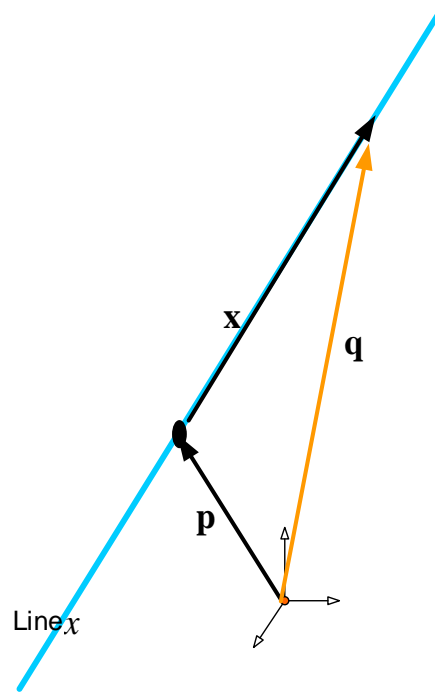
$$\mathbf{x} = \mathbf{q} - \mathbf{p}$$

Hence we can now write, in the new coordinates

$$\left\{ \begin{array}{c} \mathbf{X} \\ \mathbf{P} \times \mathbf{X} \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{Q} - \mathbf{P} \\ \mathbf{P} \times (\mathbf{Q} - \mathbf{P}) \end{array} \right\} \tag{6B}$$

But

$$\begin{aligned} \mathbf{Q} - \mathbf{P} &= [T]\,\mathbf{q} - [T]\mathbf{p} \\ &= [A]\,\mathbf{q} + \mathbf{d} - ([A]\,\mathbf{p} + \mathbf{d}) \\ &= [A]\,\mathbf{q} - [A]\,\mathbf{p} \\ &= [A]\,(\mathbf{q} - \mathbf{p}) \\ &= [A]\,\mathbf{x} \end{aligned} \tag{6C}$$

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And

$$\mathbf{P} \times (\mathbf{Q} - \mathbf{P}) = \mathbf{P} \times \mathbf{Q} - \mathbf{P} \times \mathbf{P}$$
$$= \mathbf{P} \times \mathbf{Q}$$
$$= [T]\mathbf{p} \times [T]\mathbf{q}$$
$$= ([A]\,\mathbf{p} + \mathbf{d}) \times ([A]\,\mathbf{q} + \mathbf{d})$$

$$= ([A]\,\mathbf{p} \times [A]\,\mathbf{q}) + ([A]\,\mathbf{p} \times \mathbf{d}) + (\mathbf{d} \times [A]\,\mathbf{q}) + \overbrace{\mathbf{d} \times \mathbf{d}}^{0}$$
$$= ([A]\,\mathbf{p} \times [A]\,\mathbf{q}) + ([A]\,\mathbf{p} \times \mathbf{d}) + (\mathbf{d} \times [A]\,\mathbf{q}) \qquad (6D)$$

Since $[A]$ is a rotation matrix, then

$$([\mathbf{A}]\,\mathbf{p} \times \mathbf{d}) + (\mathbf{d} \times [\mathbf{A}]\,\mathbf{q}) = [D]\,[A]\,(\mathbf{q} - \mathbf{p})$$

Where $[D]$ is a skew-symmetric matrix defined such that $[D]\mathbf{y} = \mathbf{d} \times \mathbf{y}$

Hence (6D) can be written as

$$\mathbf{P} \times (\mathbf{Q} - \mathbf{P}) = ([A]\,\mathbf{p} \times [A]\,\mathbf{q}) + [D]\,[A]\,(\mathbf{q} - \mathbf{p})$$
$$= [A]\,(\mathbf{p} \times \mathbf{q}) + [D]\,[A]\,(\mathbf{q} - \mathbf{p}) \qquad (6E)$$

By substitution of (6E) and (6C) into RHS of (6B) we obtain

$$\begin{Bmatrix} \mathbf{X} \\ \mathbf{P} \times \mathbf{X} \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q} - \mathbf{P} \\ \mathbf{P} \times (\mathbf{Q} - \mathbf{P}) \end{Bmatrix} = \begin{Bmatrix} [A]\,\mathbf{x} \\ [A]\,(\mathbf{p} \times \mathbf{q}) + [D]\,[A]\,(\mathbf{q} - \mathbf{p}) \end{Bmatrix}$$

But $\mathbf{q} - \mathbf{p} = \mathbf{x}$. Hence the above becomes

$$\begin{Bmatrix} \mathbf{X} \\ \mathbf{P} \times \mathbf{X} \end{Bmatrix} = \begin{Bmatrix} [A]\,\mathbf{x} \\ [A]\,(\mathbf{p} \times \mathbf{q}) + [D]\,[A]\,\mathbf{x} \end{Bmatrix}$$

Now substitute $\mathbf{q} = \mathbf{p} + \mathbf{x}$ in the above we obtain

$$\begin{Bmatrix} \mathbf{X} \\ \mathbf{P} \times \mathbf{X} \end{Bmatrix} = \begin{Bmatrix} [A]\,\mathbf{x} \\ [A]\,(\mathbf{p} \times (\mathbf{p} + \mathbf{x})) + [D]\,[A]\,\mathbf{x} \end{Bmatrix}$$
$$= \begin{Bmatrix} [A]\,\mathbf{x} \\ [A]\,(\mathbf{p} \times \mathbf{p} + \mathbf{p} \times \mathbf{x}) + [D]\,[A]\,\mathbf{x} \end{Bmatrix}$$
$$= \begin{Bmatrix} [A]\,\mathbf{x} \\ [A]\,(\mathbf{p} \times \mathbf{x}) + [D]\,[A]\,\mathbf{x} \end{Bmatrix}$$
$$= \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{p} \times \mathbf{x} \end{Bmatrix}$$

Hence

$$[\hat{T}] = \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}$$

The we can write

$$\left\{ \begin{matrix} \mathbf{X} \\ \mathbf{P} \times \mathbf{X} \end{matrix} \right\} = [\hat{T}] \left\{ \begin{matrix} \mathbf{x} \\ \mathbf{p} \times \mathbf{x} \end{matrix} \right\}$$
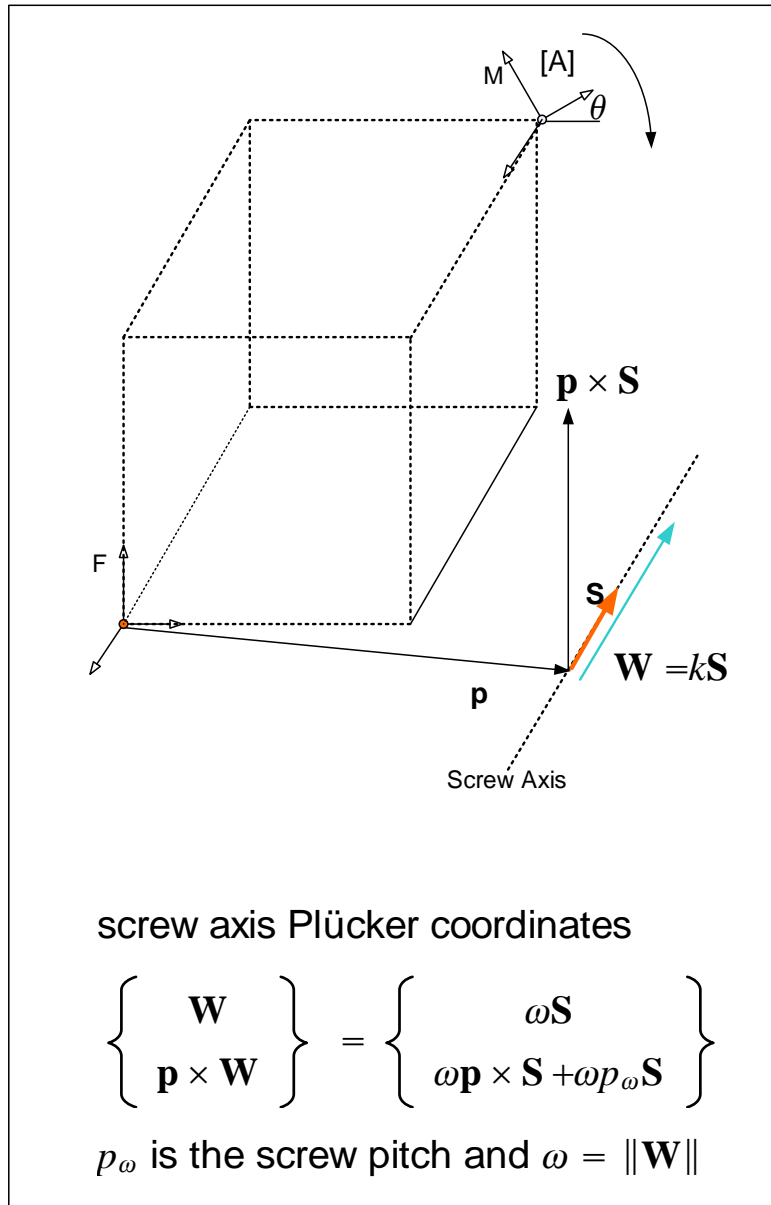
We now analyze the spatial displacement of the screw axis under $[\hat{T}]$

Recall that a screw axis Plücker coordinates are written as

$$\left\{ \begin{matrix} \mathbf{W} \\ \mathbf{p} \times \mathbf{W} \end{matrix} \right\} = \left\{ \begin{matrix} \omega \mathbf{S} \\ \omega \mathbf{p} \times \mathbf{S} + \omega p_\omega \mathbf{S} \end{matrix} \right\}$$

Where $\mathbf{S}$ is the unit vector in along the axis, $\mathbf{p}$ is the fixed reference point on the axis and p_ω is the screw pitch and $\omega = \|\mathbf{W}\|$ where $\mathbf{W}$ is the Rodrigues vector. To make things more clear, we illustrate these quantities in the following diagram

screw axis Plücker coordinates

$$\left\{ \begin{array}{c} \mathbf{W} \\ \mathbf{p} \times \mathbf{W} \end{array} \right\} = \left\{ \begin{array}{c} \omega \mathbf{S} \\ \omega \mathbf{p} \times \mathbf{S} + \omega p_\omega \mathbf{S} \end{array} \right\}$$

p_ω is the screw pitch and $\omega = \|\mathbf{W}\|$

We now perform spatial displacement on the screw axis using its Plücker coordinates

$$\left\{ \begin{array}{c} \omega \mathbf{S} \\ \omega \mathbf{P} \times \mathbf{S} + \omega p_\omega \mathbf{S} \end{array} \right\} \tag{6C}$$

Rewrite the above as general plücker coordinates $\left\{ \begin{array}{c} \mathbf{W} \\ \mathbf{V} \end{array} \right\}$ then the spatial displacement

of this general line is as seen above in (6B) becomes

$$\left\{ \begin{array}{c} [A]\,\mathbf{w} \\ [D][A]\mathbf{w} + [A]\,\mathbf{v} \end{array} \right\}$$

We need seek to evaluate the above coordinates for the screw axis given in (6C)

In other words, given $\mathbf{v} = \omega\mathbf{p} \times \mathbf{s} + \omega p_\omega\mathbf{s}$ and $\mathbf{W} = \omega\mathbf{S}$ we need to find $[A]\,\mathbf{v}, [A]\,\mathbf{w}$ and $[D][A]\mathbf{w}$

The first plücker coordinate $\omega\mathbf{S}$ transforms easily as

$$\omega\mathbf{S} = \omega[A]\,\mathbf{s}$$
$$= [\mathbf{A}]\omega\mathbf{s}$$

But $\omega\mathbf{s}$ is just the Rodrigues vector in the new coordinates system which we call lower case $\mathbf{w}$ hence

$$\omega\mathbf{S} = [\mathbf{A}]\,\mathbf{w}$$

Now we need to transform the second plücker coordinate $\omega\mathbf{p} \times \mathbf{s} + \omega p_\omega\mathbf{s}$

With the help of the $[D]$ matrix which can be used to rewrite the cross product of 2 vectors as $[D]$ times one the 2 vectors, we can write

$$[D]\,\mathbf{S} = \mathbf{d} \times \mathbf{S}$$

But $\mathbf{S} = [A]\,\mathbf{s}$ hence the above becomes

$$[D]\,[A]\,\mathbf{s} = \mathbf{d} \times \mathbf{S}$$

And since ω is a scalar, we can write the above as

$$\omega[D]\,[A]\,\mathbf{s} = \omega\mathbf{d} \times \mathbf{S}$$
$$[D]\,[A]\,(\omega\mathbf{s}) = \omega\mathbf{d} \times \mathbf{S}$$
$$[D]\,[A]\,\mathbf{w} = \omega\mathbf{d} \times \mathbf{S} \tag{6D}$$

Now we need to compute $[A]\,\mathbf{v}$ which is $[A]\,(\omega\mathbf{p} \times \mathbf{s} + \omega p_\omega\mathbf{s})$

$$[A]\,(\omega\mathbf{p} \times \mathbf{s} + \omega p_\omega\mathbf{s}) = \omega([A]\,(\mathbf{p} \times \mathbf{s}) + \omega p_\omega[A]\,\mathbf{s})$$
$$= \omega(([A]\,\mathbf{p} \times [A]\,\mathbf{s}) + \omega p_\omega[A]\,\mathbf{s})$$
$$= \omega(([A]\,\mathbf{p} \times \mathbf{S}) + \omega p_\omega\mathbf{S}) \tag{6E}$$

Hence

$$\left\{ \begin{array}{c} \mathbf{W} \\ \mathbf{V} \end{array} \right\} = \left\{ \begin{array}{c} [A]\,\mathbf{w} \\ [D][A]\mathbf{w} + [A]\,\mathbf{v} \end{array} \right\}$$

6 The screw axis of a displacement

Here we show that the screw axis is invariant of the 6×6 transformation matrix $[\hat{T}]$ derived in the last section.

Given the screw axis line S defined by its plucker coordinates $\begin{bmatrix} \mathbf{S} \\ \mathbf{V} \end{bmatrix}$ we need to show the following

$$S = [\hat{T}]S \tag{1}$$

(1) can be written as

$$S - [\hat{T}]S = 0$$
$$\left[I - \hat{T} \right] S = 0 \tag{2}$$

Now, if we can find solution to the above others than $S = 0$ then we have showed that (1) is valid. Equation (1) can be written as

$$\left[I - \hat{T} \right] \begin{bmatrix} \mathbf{S} \\ \mathbf{V} \end{bmatrix} = 0$$

But

$$[\hat{T}] = \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}$$

Hence we obtain

$$\left[\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix} \right] \begin{bmatrix} \mathbf{S} \\ \mathbf{V} \end{bmatrix} = 0$$

$$\begin{bmatrix} I - A & 0 \\ -DA & I - A \end{bmatrix} \begin{bmatrix} \mathbf{S} \\ \mathbf{V} \end{bmatrix} = 0$$

$$\begin{bmatrix} (I - A)\,\mathbf{S} \\ -[D]\,[A]\,\mathbf{S} + [I - A]\,\mathbf{V} \end{bmatrix} = 0$$

Hence we obtain 2 equations

$$\begin{cases} (I - [A])\,\mathbf{S} = \mathbf{0} \\ -[D]\,[A]\,\mathbf{S} + [I - A]\,\mathbf{V} = \mathbf{0} \end{cases}$$

From the first equation we obtain $[A]\,\mathbf{S} = \mathbf{S}$ substitute into the second equation

$$-[D]\,\mathbf{S} + [I - A]\,\mathbf{V} = \mathbf{0}$$
$$[I - A]\,\mathbf{V} = [D]\,\mathbf{S}$$

Introduce $[D]\,\mathbf{S} = -[S]\,\mathbf{d}$ hence the above becomes

$$[I - A]\,\mathbf{V} = -[S]\,\mathbf{d} \tag{3}$$

And from the cayley's formula for A

$$A = [I - B]^{-1}[I + B]$$

Then (3) becomes

$$\left[I - [I - B]^{-1}[I + B]\right]\mathbf{V} = -[S]\,\mathbf{d}$$
$$\left[I - \frac{[I + B]}{[I - B]}\right]\mathbf{V} = -[S]\,\mathbf{d}$$

Hence

$$[[I - B] - [I + B]]\,\mathbf{V} = [I - B](-[S]\,\mathbf{d})$$
$$-2[B]\,\mathbf{V} = [I - B](-[S]\,\mathbf{d})$$
$$[B]\,\mathbf{V} = \frac{1}{2}[I - B][S]\,\mathbf{d} \tag{4}$$

But we know that $[B] = \tan\left(\frac{\theta}{2}\right)[S]$ where θ is the rotation angle, and $[S] = \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}$

hence (4) becomes

$$\tan\left(\frac{\theta}{2}\right)[S]\,\mathbf{V} = \frac{1}{2}[I - B][S]\,\mathbf{d}$$
$$[S]\,\mathbf{V} = [S]\frac{1}{2\tan\left(\frac{\theta}{2}\right)}[I - B]\mathbf{d}$$

Hence

$$\mathbf{V} = \frac{1}{2\tan\left(\frac{\theta}{2}\right)}[I - B]\mathbf{d} \tag{5}$$

Hence we showed that a non zero solution for (2) exist given by $S = \begin{bmatrix} \mathbf{S} \\ \mathbf{V} \end{bmatrix}$ where V is given in (5). This shows that (1) is valid which is what we wanted to show.

Therefore the screw axis S is invariant of the 6×6 transformation matrix $[\hat{T}]$.

7 References

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