Small note on recursive formula for integral of trigonometric functions

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After stuggling in deriving this, I found similar one on wikpedia. References below. May be I will add Mathematica implementation for this later....

The goal is to find recusive formula for $\int \cos(x)^n \, dx$. Starting by rewriting it as

$$\int \cos(x)^n \, dx = \int \cos(x)^{n-1} \cos(x) \, dx \tag{1}$$

Integrating by parts $\int u \, dv = (uv) - \int v \, du$ and letting $u = \cos(x)^{n-1}, dv = \cos(x)$, hence $du = -(n-1)\cos(x)^{n-2}\sin(x)$ and $v = \sin(x)$ the above becomes

$$\begin{aligned}
\int \cos(x)^n \, dx &= \cos(x)^{n-1}\sin(x) + \int \sin(x)(n-1)\cos(x)^{n-2}\sin(x)\, dx \\
&= \cos(x)^{n-1}\sin(x) + \int (n-1)\cos(x)^{n-2}\sin^2(x)\, dx \\
&= \cos(x)^{n-1}\sin(x) + \int (n-1)\cos(x)^{n-2}(1-\cos(x)^2)\, dx \\
&= \cos(x)^{n-1}\sin(x) + (n-1)\int \cos(x)^{n-2} - \cos(x)^n \, dx \\
&= \cos(x)^{n-1}\sin(x) + (n-1)\int \cos(x)^{n-2}\, dx - (n-1)\int \cos(x)^n \, dx
\end{aligned}$$

The $\int \cos(x)^n \, dx$ in the RHS above is what is being solved for. Moving it to the LHS gives

$$\begin{aligned}
\int \cos(x)^n \, dx + (n-1)\int \cos(x)^n \, dx &= \cos(x)^{n-1}\sin(x) + (n-1)\int \cos(x)^{n-2}\, dx \\
n\int \cos(x)^n \, dx &= \cos(x)^{n-1}\sin(x) + (n-1)\int \cos(x)^{n-2}\, dx
\end{aligned}$$

Therefore the recusrive formula is

$$\int \cos(x)^n \, dx = \frac{\cos(x)^{n-1}\sin(x)}{n} + \frac{(n-1)}{n}\int \cos(x)^{n-2}\, dx$$

References:

1. `http://www.integraltec.com/math/math.php?f=cosPower.html#cos`
2. `http://en.wikipedia.org/wiki/Integration_by_reduction_formulae#Examples`