Project, EGME 511 (Advanced Mechanical Vibration)
Analysis of Van Der Pol differential equation

Nasser M. Abbasi

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Contents

1 Introduction

Van der Pol differential equation is given by

$$x''(t) - c\left(1 - x^2\right) x'(t) + kx(t) = 0$$

In this analysis, we will consider the case only for positive c, k. We will analyze the stability of this equation and generate a phase diagram.

2 Stability

The first step in examining stability of a non-linear differential equation is to convert it to state space by introducing 2 state variables.

$$\left.\begin{matrix} x_1 = x \\ x_2 = x' \end{matrix}\right\} \rightarrow \left.\begin{matrix} x_1' = x' \\ x_2' = x'' = c\left(1 - x^2\right) x' - kx \end{matrix}\right\} \rightarrow \left.\begin{matrix} x_1' = x_2 \\ x_2' = c\left(1 - x_1^2\right) x_2 - kx_1 \end{matrix}\right\}$$

Therefore

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_2 \\ c\left(1 - x_1^2\right) x_2 - kx_1 \end{pmatrix} = \begin{pmatrix} g(x_1, x_2) \\ f(x_1, x_2) \end{pmatrix}$$

Equilibrium points are found by solving $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, hence from the above, we see that $x_2 = 0$ and from $c\left(1 - x_1^2\right) x_2 - kx_1 = 0$ we conclude that $x_1 = 0$ as well. Hence

$$x_{eq} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The system matrix is now found. First we note that $\frac{\partial g}{\partial x_1} = 0, \frac{\partial g}{\partial x_2} = 1, \frac{\partial f}{\partial x_1} = -2cx_2x_1 - k$, and $\frac{\partial f}{\partial x_2} = -cx_1^2 + c$, hence

$$A = \begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2cx_2x_1 - k & -cx_1^2 + c \end{pmatrix}$$

Hence A at x_{eq} becomes

$$A = \begin{pmatrix} 0 & 1 \\ -k & c \end{pmatrix}$$

Now we find the characteristic equation

$$\begin{vmatrix} -\lambda & 1 \\ -k & c - \lambda \end{vmatrix} = 0$$
$$-\lambda (c - \lambda) + k = 0$$
$$\lambda^2 - \lambda c + k = 0$$

Hence $\lambda_{1,2} = \frac{-b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4ac} = c \pm \frac{1}{2}\sqrt{c^2 - 4k}$, therefore

$$\lambda_{1,2} = c \pm \frac{1}{2}\sqrt{c^2 - k}$$

If $c^2 > k$ then both roots are on the RHS, hence system is unstable (equilibrium point is a repelling point).

If $c^2 < k$ then we have $\lambda_{1,2} = c \pm j\beta$, and we have spiral out equilibrium point, unstable.

3 Phase diagram

We need to obtain a relation between x_2 and x_1. From the differential equation

$$x''(t) - c\left(1 - x^2\right) x'(t) + kx(t) = 0$$

rewrite in state space variables, we obtain

$$\frac{dx_2}{dt} - c\left(1 - x_1^2\right) x_2 + kx_1 = 0$$
$$\frac{dx_2}{dx_1}\frac{dx_1}{dt} - c\left(1 - x_1^2\right) x_2 + kx_1 = 0$$
$$\frac{dx_2}{dx_1}x_2 - c\left(1 - x_1^2\right) x_2 + kx_1 = 0$$
$$\frac{dx_2}{dx_1} = \frac{c\left(1 - x_1^2\right) x_2 - kx_1}{x_2}$$

Hence the above is in the form $\frac{dx_2}{dx_1} = f(x_1, x_2)$, therefore the isoclines lines can be found by setting

$$f(x_1, x_2) = \xi$$

Where ξ is a constant. Hence we obtain the parameterize equation to use to plot the gradient lines as

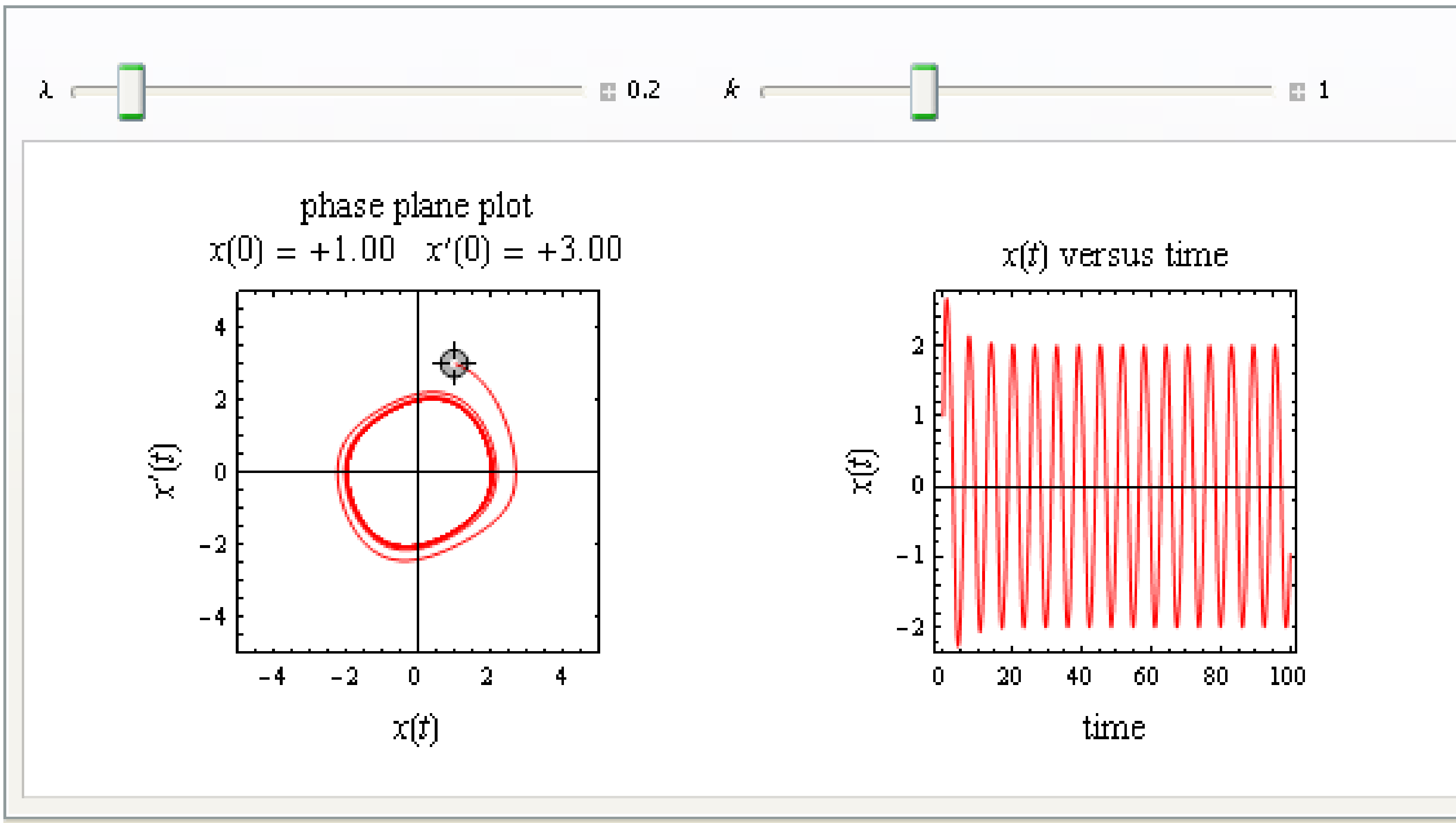
$$\xi = \frac{c\left(1 - x_1^2\right) x_2 - kx_1}{x_2}$$

4 Phase diagram

To generate the phase diagram[1], a program was written which allows one to adjust the initial conditions and the parameters k and c and observe the effect on the shape of the limit cycle. We see that starting from different initial conditions, the solution trajectory always ends up in a limit cycle.

The following is a screen shot of the program written for this project.

Phase Plane Plot of the Van der Pol Differential Equation



[1] A plot of $x'(t)$ vs. $x(t)$