HW 9, ME 740, Spring 2013

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Let the left most spring stiffness be k_1, the middle spring stiffness be k_2 and the rightmost spring stiffness be k_3. Let the mass of the left cylinder be m_1 and the mass of the right cylinder be m_2.

The generalized coordinates are θ_1 and θ_2. Using Lagrangian $L = T - V$ method we find

$$\begin{aligned} T &= \frac{1}{2}I\dot{\theta}_1^2 + \frac{1}{2}m_1\left(r\dot{\theta}_1\right)^2 + \frac{1}{2}I\dot{\theta}_2^2 + \frac{1}{2}m_2\left(r\dot{\theta}_2\right)^2 \\ V &= \frac{1}{2}k_1\left(r\theta_1\right)^2 + \frac{1}{2}k_2\left(r\theta_2 - r\theta_1\right)^2 + \frac{1}{2}k_3\left(r\theta_2\right)^2 \end{aligned}$$

Therefore

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}I\dot{\theta}_1^2 + \frac{1}{2}m_1\left(r\dot{\theta}_1\right)^2 + \frac{1}{2}I\dot{\theta}_2^2 + \frac{1}{2}m_2\left(r\dot{\theta}_2\right)^2 - \frac{1}{2}k_1\left(r\theta_1\right)^2 - \frac{1}{2}k_2\left(r\theta_2 - r\theta_1\right)^2 - \frac{1}{2}k_3\left(r\theta_2\right)^2 \end{aligned}$$

and for θ_1

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_1} &= I\dot{\theta}_1 + m_1 r^2 \dot{\theta}_1 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} &= \left(I + m_1 r^2\right)\ddot{\theta}_1 \\
\frac{\partial L}{\partial \theta_1} &= -k_1 r^2 \theta_1 + r k_2 \left(r\theta_2 - r\theta_1\right) \\
&= -\left(k_1 r^2 + r^2 k_2\right)\theta_1 + r^2 k_2 \theta_2
\end{aligned}$$

and for θ_2

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_2} &= I\dot{\theta}_2 + m_2 r^2 \dot{\theta}_2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} &= \left(I + m_2 r^2\right)\ddot{\theta}_2 \\
\frac{\partial L}{\partial \theta_2} &= -r k_2 \left(r\theta_2 - r\theta_1\right) - r k_3 \left(r\theta_2\right) \\
&= \left(r^2 k_2\right)\theta_1 - \theta_2 \left(r^2 k_2 + r^2 k_3\right)
\end{aligned}$$

Hence EQM are

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} &= 0 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} &= 0
\end{aligned}$$

or

$$\begin{aligned}
\left(I + m_1 r^2\right)\ddot{\theta}_1 + \left(k_1 r^2 + r^2 k_2\right)\theta_1 - r^2 k_2 \theta_2 &= 0 \\
\left(I + m_2 r^2\right)\ddot{\theta}_2 - \left(r^2 k_2\right)\theta_1 + \left(r^2 k_2 + r^2 k_3\right)\theta_2 &= 0
\end{aligned}$$

we obtain the equations of motion

$$MX'' + kX = 0$$

$$r^2 \begin{pmatrix} \frac{3}{2}m_1 & 0 \\ 0 & \frac{3}{2}m_2 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + r^2 \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For the special case when $k_1 = k$,$k_2 = 2k$, $k_3 = 3k$ and $m_1 = m_2 = m$ we obtain

$$r^2 \begin{pmatrix} \frac{3}{2}m & 0 \\ 0 & \frac{3}{2}m \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + kr^2 \begin{pmatrix} 3 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$