Liapunov-Floquet transformation with worked examples

Nasser M. Abbasi

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1 Introduction

Given vector system of linear time variant differential equations

$$x'(t) = A(t)x(t)$$

where $x, x'(t)$ are each $n \times 1$ vectors and $A(t)$ is an $n \times n$ which is periodic in T, which means $A(t) = A(t+T)$ and given that the matrix $A(t)$ is commutative, meaning there exists a matrix $C(t)$ such that $A(t)C(t) = C(t)A(t)$ then it is possible to find closed form Matrix $P(t)$ which is $n \times n$ and periodic in T and a constant matrix B such that

$$\Phi(t,0) = P(t)e^{Bt}$$

where $\Phi(t,\tau)$ is the state transition matrix (STM) of $x' = A(t)x$

Finding $P(t)$ and B allows one to convert the time varying system $x'(t) = A(t)x(t)$ to non-time varying system using the so called Liapunov-Floquet transformation

$$x(t) = P(t)y(t)$$

And obtain the system

$$y'(t) = By(t)$$

To solve. Since this is now no longer time varying, it easier to solve. Then $x(t)$ is found by using the above Liapunov-Floquet transformation.

The method is best illustrated by worked examples. In each example, the solution found using Liapunov-Floquet transformation is then compared to the solution found by solving $x'(t) = A(t)x(t)$ using computer algebra software to verify the result.

2 Examples

2.1 Example 1

Given $x'(t) = A(t)x(t)$ as

$$x'(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} x(t)$$

Let the period of $A(t)$ be T (which is 2π in this case). The first step is to find the state transition matrix. Since the system is time varying, then

$$\Phi(t, t_0) = e^{\int_{t_0}^{t} A(s)ds} \tag{1}$$

Calculating the matrix exponential above gives

$$\Phi(t, t_0) = e^{\sin t - \sin t_0} \begin{pmatrix} \cos(\cos t - \cos t_0) & -\sin(\cos t - \cos t_0) \\ \sin(\cos t - \cos t_0) & \cos(\cos t - \cos t_0) \end{pmatrix} \tag{2}$$

Since the period is 2π then replacing t_0 by 2π in the above gives (this is Eq. 16 in first reference below but not normalized)

$$\Phi(t, 2\pi) = e^{\sin t} \begin{pmatrix} \cos(\cos t - 1) & -\sin(\cos t - 1) \\ \sin(\cos t - 1) & \cos(\cos t - 1) \end{pmatrix}$$

Now $\Phi(t, 2\pi) = e^{Bt}$. Which is valid for any t. Letting $t = 2\pi$ gives

$$\begin{aligned} \Phi(2\pi, 2\pi) &= e^{2\pi B} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= e^{2\pi B} \end{aligned}$$

Hence

$$\begin{aligned} B &= \frac{1}{2\pi} \begin{pmatrix} \ln 1 & 0 \\ 0 & \ln 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Therefore the transformed system becomes

$$
\begin{aligned}
y'(t) &= By(t) \\
y'(t) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

Which has the solution

$$
\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}
$$

Now we need to find $P(t)$ to go back to x space. Since $\Phi(t,0) = P(t)e^{Bt}$ then

$$
\begin{aligned}
P(t) &= \Phi(t,0)e^{-Bt} \\
&= \Phi(t,0)
\end{aligned}
$$

In this case. This is because $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence, from (2)

$$
P(t) = e^{\sin t} \begin{pmatrix} \cos(\cos t - 1) & -\sin(\cos t - 1) \\ \sin(\cos t - 1) & \cos(\cos t - 1) \end{pmatrix}
$$

Therefore

$$
\begin{aligned}
x(t) &= P(t)\,y(t) \\
&= e^{\sin t} \begin{pmatrix} \cos(\cos t - 1) & -\sin(\cos t - 1) \\ \sin(\cos t - 1) & \cos(\cos t - 1) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
&= e^{\sin t} \begin{pmatrix} c_1 \cos(\cos t - 1) - c_2 \sin(\cos t - 1) \\ c_1 \sin(\cos t - 1) + c_2 \cos(\cos t - 1) \end{pmatrix}
\end{aligned}
$$

Hence

$$
\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{\sin t}(c_1 \cos(\cos t - 1) - c_2 \sin(\cos t - 1)) \\ e^{\sin t}(c_1 \sin(\cos t - 1) + c_2 \cos(\cos t - 1)) \end{pmatrix}
$$

There is something wrong. The correct solution should be

$$
\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{\sin t}(c_1 \cos(\cos t) - c_2 \sin(\cos t)) \\ e^{\sin t}(c_1 \sin(\cos t) + c_2 \cos(\cos t)) \end{pmatrix}
$$

I need to find out what is wrong.

3 References

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