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Floquet's Theorem

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Abstract

For a homogeneous system of differential equations with a constant coefficient matrix, the fundamental matrix can be computed by using the eigenpairs of the coefficient matrix. However, for a homogeneous system of differential equations with a periodic coefficient matrix, another approach is needed to obtain the fundamental matrix. Floquet's theorem offers a canonical form for each fundamental matrix of these periodic systems. Moreover, Floquet's theorem provides a way to transform a system with periodic coefficients to a system with constant coefficients. The monodromy matrix is very useful for stability analyses of periodic differential systems, in particular for Hill's differential equation. In this thesis, Floquet's theorem will be proven, and the aforementioned transformation and stability analyses will be discussed.

Contents

1 Introduction

Differential equations are equations that describe the relation between a function and its derivatives. They are often used in physics, where the derivative represents the rate of change of a certain physical quantity. Examples of differential equations are Newton's law in classical mechanics, the heat equation in thermodynamics and the wave equation in fluid dynamics.

An *nth order differential equation* can be written in the following general form

$$f(t, y, y', \dots, y^{(n)}) = 0. \tag{1}$$

A solution $y = y(t)$ to (1) is defined to be an n-times differential function such that (1) is satisfied when substituting y and its derivatives into f [13]. The unknown function y depends on a single independent variable t, so (1) is called an *ordinary differential equation.* If the unknown function depends on two or more independent variables, the equation is called a *partial differential equation.*

A first order linear differential system is of the form

$$x'(t) = A(t)x(t) + b(t) \tag{2}$$

where the matrix $A \in \mathbb{C}^{n\times n}$ is called the *coefficient matrix.* The system (2) is called *homogeneous* if $b = 0$ [8].
A set of n linearly independent solutions $x_1, \dots, x_n$ of (2) is called a *fundamental system* of solutions. We write

$$X(t) = (x_1, \dots, x_n)$$

and call this the *fundamental matrix* [13].

If (λ_i, v_i), with $i = 1, \dots n$, is an eigenpair for the constant coefficient matrix A, then

$$x_i(t) = v_i e^{\lambda_i t} \tag{3}$$

defines a solution of the first order linear differential system

$$x'(t) = Ax(t) \tag{4}$$

as proved by [1]. The fundamental matrix of (4) is given by

$$X(t) = (x_1, \dots, x_n)$$

provided that $\{v_1, \dots, v_n\}$ are linearly independent.

In this thesis, we are interested in differential equations with periodic coefficients. Periodic coefficients arise in problems in the fields of technology, natural and social sciences [5]. For example in mathematical biology, we encounter problems that deal with periodic factors such as seasonal effects of weather and mating habits of birds. In the field of social sciences, periodic factors appear for instance in problems like scheduling of public transport and regulating traffic lights. In physics, one can think of periodic problems as a pendulum or a body in uniform circular motion. Consider the homogeneous system

$$x'(t) = A(t)x(t) \tag{5}$$

with the periodic coefficient matrix $A(t)$, that is, $A(t) = A(t+T)$ for all $t \in \mathbb{R}$, for some period $T > 0$. We want to find an expression for the fundamental matrix in this case. Unlike the case of a constant coefficient matrix, the fundamental matrix cannot be expressed as $X(t) = (x_1, \ldots, x_n)$ where the x_i are given by (3). This will be made clear by the following counterexample.

Example 1.1. Consider the system

$$x'(t) = \begin{pmatrix} \sin(t) & 0 \\ 0 & 2 \end{pmatrix} x(t).$$

The coefficient matrix is periodic with period 2π. If we use the eigenpairs $(\sin(t), \begin{pmatrix} 1 \\ 0 \end{pmatrix})$ and $(2, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$ and the expression in (3), the fundamental matrix is of the form

$$\tilde{X}(t) = \begin{pmatrix} e^{t\sin(t)} & 0 \\ 0 & e^{2t} \end{pmatrix}.$$

Denote $\tilde{x}_1(t) = (e^{t\sin(t)}, 0)^T$. Then

$$\tilde{x}_1'(t) = \begin{pmatrix} e^{t\sin(t)}(\sin(t) + t\cos(t)) \\ 0 \end{pmatrix}$$

but

$$\begin{pmatrix} \sin(t) & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} e^{t\sin(t)} \\ 0 \end{pmatrix} = \begin{pmatrix} \sin(t)e^{t\sin(t)} \\ 0 \end{pmatrix} \neq \tilde{x}_1'(t).$$

So the first column of $\tilde{X}(t)$ is not a solution of the problem, which means that $\tilde{X}(t)$ is not a fundamental matrix. Hence, we cannot use the eigenpairs to obtain the fundamental matrix for the periodic differential system.

Therefore, we need another approach to find the fundamental matrix. In his paper of 1883, Gaston Floquet introduces a canonical decomposition of the fundamental matrix of (5) [4]. It is given by the following result.

Theorem 1.2. (Floquet) The fundamental matrix $X(t)$ of (5) with $X(0) = I$ has a *Floquet normal form*

$$X(t) = Q(t)e^{Bt}$$

where $Q \in C^1(\mathbb{R})$ is T-periodic and the matrix $B \in \mathbb{C}^{n\times n}$ satisfies the equation $C = X(T) = e^{BT}$. We have $Q(0) = I$ and $Q(t)$ is an invertible matrix for all t.

Proving Floquet's theorem will be part of this thesis. To be able to do this, we need three assertions regarding the fundamental matrix and one result on the logarithm of a nonsingular matrix. These four statements will be discussed in Sections 2.2 and 2.3. The proof of Floquet's theorem will be given in Section 2.4.

Section 3 will be devoted to applications of Floquet's theorem. The Floquet normal form is used to transform the periodic differential equation into a system with a constant coefficient matrix. This is called the Lyapunov-Floquet transformation and will be the subject of Section 3.1. In the case of a one-dimensional coefficient matrix, this transformation can make it easier to find the solution. Limitations of the Lyapunov-Floquet transformation for higher dimensions of the coefficient matrix will be discussed shortly.

The last part of this thesis is dedicated to a stability analysis of periodic differential systems. We will make use of the eigenvalues of the monodromy matrix, the so-called Floquet multipliers. The solution to the system is stable if all its Floquet multipliers lie within the unit circle. An example of a periodic differential system is Hill's equation and will be discussed in Section 3.4. We will develop another stability criterion for Hill's equation based on the trace of the monodromy matrix. This knowledge will be used to analyze the stability of a particular example of Hill's equation: the inverted pendulum. By making use of Matlab, a stability region can be drawn for the corresponding periodic differential equation.

2 Floquet Theory

2.1 Definitions and preliminaries

Before we dive into Floquet Theory, first some basic concepts from ordinary differential equations and linear algebra are described.

Consider the following homogeneous system of n differential equations:

$$\begin{aligned} x_1'(t) &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \ldots + a_{1n}(t)x_n(t), \\ &\vdots \\ x_n'(t) &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \ldots + a_{nn}(t)x_n(t). \end{aligned}$$

We can write this system as

$$x'(t) = A(t)x(t), \tag{6}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \text{and} \quad A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}.$$

In [13], the following definition of a fundamental matrix is given:

Definition 2.1. A set of n linear independent solutions $x_1, \ldots, x_n$ to (6) is called a *fundamental system* of solutions. We write

$$X(t) = (x_1, \ldots, x_n)$$

and call this the *fundamental matrix*.

2.2 The fundamental matrix for periodic coefficients

For a homogeneous system of differential equations with a constant coefficient matrix, the fundamental matrix can be computed by using the eigenpairs for the coefficient matrix. However, for a homogeneous system of differential equations with a periodic coefficient matrix, another approach is needed to obtain the fundamental matrix, as we have seen in Example 1.1. Floquet's theorem offers a canonical form for each fundamental matrix of these periodic systems.

In this section, we will prove three statements about the fundamental system of a periodic homogeneous system. These assertions will be used to prove Floquet's theorem.

Definition 2.2. A matrix A is a *periodic matrix* with period $T > 0$ if $A(t+T) = A(t)$ for every t.

From now on, let $A(t)$ always be a periodic matrix with period T. Consider the *Floquet system*

$$x' = A(t)x. \tag{7}$$

In [15], the following is stated.

Lemma 2.3. If $X(t)$ is a fundamental matrix of (7), then so is $Y(t) = X(t)B$ for any nonsingular constant matrix B.

Proof. Write $Y(t) = X(t)B$, where B is any nonsingular constant matrix. By definition of a fundamental matrix, $X(t)$ is nonsingular. Hence, $Y(t)$ is nonsingular. We have

$$Y' = (XB)' = X'B = AXB = AY$$

so that $Y'(t) = AY(t)$. Therefore, $Y(t)$ is a fundamental matrix of (7). □

Not only multiplying a fundamental matrix by a nonsingular constant matrix results in a fundamental matrix, also shifting a fundamental matrix by a period gives this result [2].

Lemma 2.4. If $X(t)$ is a fundamental matrix of (7), then so is $X(t+T)$.

Proof. Let $Z(t) = X(t+T)$. Note that

$$Z'(t) = X'(t+T) = A(t+T)X(t+T) = A(t)Z(t).$$

Also, $\det Z(t) = \det X(t+T) \neq 0$ for all t, because $X(t)$ is a fundamental matrix. Hence, $Z(t)$ is a fundamental matrix of (7). □

The shifted fundamental matrix can be written in a particular form.

Lemma 2.5. If $X(t)$ is a fundamental matrix of (7), then there exists a nonsingular constant matrix C with $X(t+T) = X(t)C$.

Proof. Assume that $X(t)$ is a fundamental matrix of (7). By Lemma 2.4, $Y(t) := X(t+T)$ is a fundamental matrix of (7). Define the matrix

$$C(t) = X^{-1}(t)Y(t)$$

for all t. Then

$$Y(t) = X(t)C(t). \tag{8}$$

Fix t_0 and let $C_0 = C(t_0)$. By Lemma 2.3,

$$Y_0(t) = X(t)C_0 \tag{9}$$

is a fundamental solution of (7). So we have two fundamental solutions of (7), namely $Y(t)$ and $Y_0(t)$ with $Y(t_0) = Y_0(t_0)$. By uniqueness of solutions, the matrices in (8) and (9) must be equal. This means that $C_0 = C(t)$ for all t. In other words, C is a constant matrix. Hence, there exists a nonsingular constant matrix C with $X(t+T) = X(t)C$. □

Remark. Since C is a constant matrix, we can compute it by taking $t = 0$. We then have

$$C = C(0) = X^{-1}(0)Y(0) = X^{-1}(0)X(T). \tag{10}$$

If we take the initial condition $X(0) = I$, then $C = X(T)$. In conclusion, we can write

$$X(t+T) = X(t)X(T) \quad \text{if} \quad X(0) = I. \tag{11}$$

2.3 The logarithm of a nonsingular matrix

Every nonsingular matrix can be written as the exponential of one other matrix [12, 14].

Lemma 2.6. If C is an $n \times n$ nonsingular matrix, then there exists a $n \times n$ (complex) matrix B such that $e^B = C$.

Proof. Write C in the Jordan canonical form

$$J = P^{-1}CP$$

where P is a nonsingular matrix consisting of (generalized) eigenvalues of C. If $e^B = C$ were true, then

$$e^{P^{-1}BP} = P^{-1}e^BP = P^{-1}CP = J.$$

Therefore, it is sufficient to prove the statement for C having the form of a Jordan block. Let λ_j, with $j = 1, \ldots, r$ and $r \leq n$, denote the eigenvalues of C. Since C is nonsingular, $\lambda_j \neq 0$ for all $j = 1, \ldots, r$. Suppose that $C = diag(C_1, C_2, \ldots, C_r)$ where C_j is of the form

$$C_j = \begin{pmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix}, \quad j = 1, \ldots, r.$$

Let $s_j \times s_j$ be the size of C_j. We have

$$C_j = \lambda_j I_j + N_j, \quad j = 1, \ldots, r$$

with the $s_j \times s_j$ identity matrix I_j and the $s_j \times s_j$ matrix

$$N_j = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

having the property $N_j^{s_j} = O$. We have $\sum_{k=1}^{r} s_k = n$, so we constructed a matrix C of size $n \times n$. If we can prove that for every C_j, there exists a matrix B_j such that $e^{B_j} = C_j$, then $e^B = C$. Write

$$C_j = \lambda_j (I_j + \frac{N_j}{\lambda_j})$$

so that we can use the expression

$$\log(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}, \quad |x| < 1$$

for the logarithm of C_j. We can use this expansion for the case $x = \frac{N_j}{\lambda_j}$, since N_j is nilpotent, so that $(\frac{N_j}{\lambda_j})^k = 0$ for k large enough. For every $j = 1, \dots, r$, we can define

$$\begin{aligned}\log C_j &= I_j \log(\lambda_j) + \log(I_j + \frac{N_j}{\lambda_j}) \\ &= I_j \log(\lambda_j) + \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \Big(\frac{N_j}{\lambda_j}\Big)^{k+1}.\end{aligned}$$

Having $N_j^{s_j} = O$, yields

$$\begin{aligned}\log C_j &= I_j \log(\lambda_j) + \sum_{k=0}^{s_j-2} \frac{(-1)^k}{k+1} \Big(\frac{N_j}{\lambda_j}\Big)^{k+1} \\ &:= I_j \log(\lambda_j) + M_j.\end{aligned}$$

We need to verify that $C_j = \exp(\log(C_j))$, in order to get $C_j = e^{B_j}$. We can write

$$\begin{aligned}\exp(\log(C_j)) &= \exp(I_j \log(\lambda_j) + M_j) \\ &= \exp(I_j \log(\lambda_j)) \cdot \exp(M_j) \\ &= \begin{pmatrix} \lambda_j & & \\ & \ddots & \\ & & \lambda_j \end{pmatrix} \cdot \exp(M_j).\end{aligned}$$

The matrix M_j is nilpotent, since every upper triangular matrix with zeros on the diagonal is nilpotent. Hence, we can use the expression

$$\exp(M_j) = \sum_{k=0}^{s_j-1} \frac{1}{k!} M_j^k.$$

We will compute $\exp(M_j)$ for the case that M_j is a 4×4 matrix. For larger matrices, the proof is analogous.

$$\exp(M_j) = \sum_{k=0}^{s_j-1} \frac{1}{k!} M_j^k = \begin{pmatrix} 1 & 0 & & \\ & 1 & & \\ & & \ddots & 0 \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{\lambda_j} & -\frac{1}{2\lambda_j^2} & \frac{1}{3\lambda_j^3} \\ 0 & 0 & \frac{1}{\lambda_j} & -\frac{1}{2\lambda_j^2} \\ 0 & 0 & 0 & \frac{1}{\lambda_j} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & \frac{1}{2\lambda_j^2} & -\frac{1}{2\lambda_j^3} \\ 0 & 0 & 0 & \frac{1}{2\lambda_j^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \frac{1}{6\lambda_j^3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{1}{\lambda_j} & 0 & 0 \\ 0 & 1 & \frac{1}{\lambda_j} & 0 \\ 0 & 0 & 1 & \frac{1}{\lambda_j} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Computing $\exp(M_j)$ with dimension higher than 4, gives the same structure. Therefore,

$$\exp(\log(C_j)) = \begin{pmatrix} \lambda_j & & \\ & \ddots & \\ & & \lambda_j \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{1}{\lambda_j} & & \\ & 1 & \ddots & \\ & & \ddots & \frac{1}{\lambda_j} \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix} = C_j$$

Letting $B_j = I_j \log(\lambda_j) + M_j$ yields

$$\begin{aligned} C_j &= \exp(\log(C_j)) \\ &= \exp(I_j \log(\lambda_j) + M_j) \\ &= e^{B_j}. \end{aligned}$$

Hence, if we define $B = diag(B_1, \ldots, B_r) \in \mathbb{C}^{n \times n}$ where $B_j = I_j \log(\lambda_j) + M_j$, then

$$\begin{aligned} e^B &= diag(e^{B_1}, \ldots, e^{B_r}) \\ &= diag(C_1, C_2, \ldots, C_r) \\ &= C \end{aligned}$$

which is what we wanted to prove. $\square$

Remark. The matrix B in Lemma 2.6 is not uniquely determined. For example, let

$$e^B = C.$$

Consider $\hat{B} = B + 2\pi mi \cdot I$, with $m \in \mathbb{Z}$. Since $e^{2\pi mi} = 1$,

$$e^{\hat{B}} = e^{B+2\pi mi\cdot I} = e^B \cdot I = C$$

Example 2.7. We want to find the logarithm of the rotation matrix

$$R = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Since R is diagonalizable, we can find a nonsingular matrix S and a diagonal matrix D such that $R = SDS^{-1}$. Then the logarithm of R is given by

$$\log R = S(\log D)S^{-1}.$$

The eigenvalues of R are $\lambda_1 = \cos(t) - i\sin(t)$ and $\lambda_2 = \cos(t) + i\sin(t)$ and the corresponding eigenvectors are $v_1 = (-i, 1)^T$ and $v_2 = (i, 1)^T$. Hence,

$$R = SDS^{-1} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos(t) - i\sin(t) & 0 \\ 0 & \cos(t) + i\sin(t) \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}.$$

Then

$$\log R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix},$$

with

$$\begin{aligned} r_{11} = r_{22} &= \frac{1}{2}\log[\cos(t) - i\sin(t)] + \frac{1}{2}\log[\cos(t) + i\sin(t)] \\ &= \frac{1}{2}\log[(\cos(t) - i\sin(t))(\cos(t) + i\sin(t))] \\ &= \frac{1}{2}\log[\cos^2(t) + \sin^2(t)] \\ &= \frac{1}{2}\log[1] = 0 \end{aligned}$$

and

$$\begin{aligned} r_{12} &= -\frac{i}{2}\log[\cos(t) - i\sin(t)] + \frac{i}{2}\log[\cos(t) + i\sin(t)], \\ r_{21} &= \frac{i}{2}\log[\cos(t) - i\sin(t)] - \frac{i}{2}\log[\cos(t) + i\sin(t)]. \end{aligned}$$

In Figure 1, the graphs of r_{12} and r_{21} are given on the interval $[-6.3, 6.3]$. The lines in these figures intersect the t-axis at $2\pi n$ for every $n \in \mathbb{Z}$ and the slope of the graph of r_{12} and r_{21} is -1 and 1 respectively. We can conclude that

$$\log R = \begin{pmatrix} 0 & -t - 2\pi n \\ t + 2\pi n & 0 \end{pmatrix}.$$

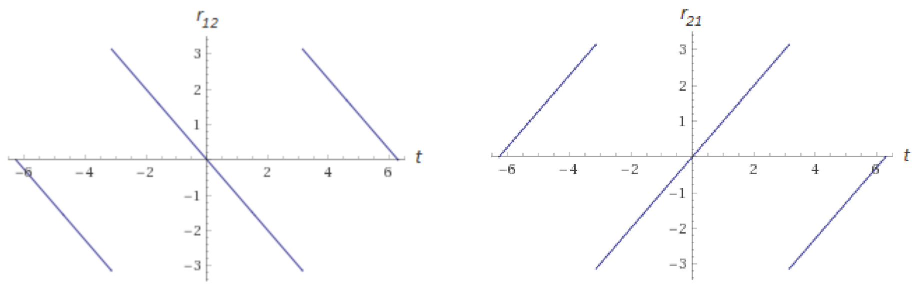


Figure 1: Plots of r_{12} and r_{21}.

2.4 Floquet's Theorem

Using the previous results, the following can be proven.

Theorem 2.8. (Floquet) The fundamental matrix $X(t)$ of (7) with $X(0) = I$ has a *Floquet normal form*

$$X(t) = Q(t)e^{Bt}$$

where $Q \in C^1(\mathbb{R})$ is T-periodic and the matrix $B \in \mathbb{C}^{n\times n}$ satisfies the equation $C = X(T) = e^{BT}$. We have $Q(0) = I$ and $Q(t)$ is an invertible matrix for all t.

Proof. By Lemma 2.5, there exists a nonsingular constant matrix C with

$$X(t+T) = X(t)C.$$

Using (11) and Lemma 2.6 gives

$$C = X(T) = e^{BT}$$

for some matrix B. If $Q(t) = X(t)e^{-Bt}$, then for all t,

$$\begin{aligned} Q(t+T) &= X(t+T)e^{-B(t+T)} \\ &= X(t)Ce^{-Bt}e^{-BT} \\ &= X(t)e^{BT}e^{-Bt}e^{-BT} \\ &= X(t)e^{-Bt} \\ &= Q(t). \end{aligned}$$

This means that

$$X(t) = Q(t)e^{Bt}$$

where $Q \in C^1(\mathbb{R})$ is T-periodic and $Q(0) = X(0)e^0 = I$. The matrix e^{-Bt} is invertible for all t, because exponentials of square matrices are invertible, and $X(t)$ is invertible. Hence, $Q(t)$ is invertible. $\square$

3 Applications

3.1 The Lyapunov-Floquet transformation

The fundamental matrix $X(t)$ of (7) satisfies $X(t)' = A(t)X(t)$. Using its Floquet normal form $X(t) = Q(t)e^{Bt}$ with the given conditions in Theorem 2.8, we can rewrite it as

$$Q'(t)e^{Bt} + Q(t)Be^{Bt} = A(t)Q(t)e^{Bt}$$

Dividing e^{Bt} on both sides and leaving out the independence on t gives

$$Q' + QB = AQ.$$

Next, we multiply this equation on both sides by the $n \times 1$ vector y. This yields

$$Q'y + QBy = AQy. \tag{12}$$

Making the substitution $x = Qy$ in $x' = Ax$ gives

$$Q'y + Qy' = AQy \tag{13}$$

Combining (12) and (13) yields

$$Q'y + Qy' = Q'y + QBy$$

implying that

$$y' = By.$$

Hence, the substitution $x = Q(t)y$ transforms the system $x' = A(t)x$ with a periodic coefficient matrix A, to the system $y' = By$ with the constant coefficient matrix B. In short, once we have obtained $Q(t)$ and solved the (easier) system $y' = By$ for y, we know the solution x for $x' = A(t)x$ by computing the *Lyapunov-Floquet transformation* $x = Q(t)y$. This can be done easily when the system of differential equations is one-dimensional. To make it more clear, consider the following example.

Example 3.1. Let us solve the one-dimensional differential equation

$$x' = \sin(t)x \tag{14}$$

by finding a Lyapunov-Floquet transformation

$$x = q(t)y \tag{15}$$

where q is 2π-periodic, so that (14) reduces to

$$y' = by \tag{16}$$

where b is a constant. Differentiating equation (15) and setting this equal to (14) gives

$$q'(t)y + q(t)y' = \sin(t)q(t)y$$

This implies

$$q'(t)y = q(t)\big(\sin(t)y - y'\big).$$

Dividing by y on both sides and using that $b = \frac{y'}{y}$ results in the following differential equation

$$\frac{dq(t)}{dt} = \big(\sin(t) - b\big)q(t). \tag{17}$$

We want to solve this for $q(t)$. We do this by separating variables and integrating both sides.

$$\begin{aligned}
(17) \implies & \int \frac{dq(t)}{q(t)} = \int \big(\sin(t) - b\big)dt \\
\implies & \log|q(t)| = -\cos(t) - bt + c_1 \\
\implies & q(t) = c_2 e^{-\cos(t) - bt}
\end{aligned}$$

for some real constant c_2. Using $q(0) = q(2\pi)$ yields $c_2e^{-1} = c_2e^{-1-2b\pi}$, implying that $b = 0$. Hence,

$$q(t) = c_2 e^{-\cos(t)}$$

is the solution to (17). We can solve (16) by the same procedure. The solution is $y = c_3e^{bt} = c_3$ (since $b = 0$), for some real constant c_3. Therefore, the solution to (14) is

$$x = q(t)y = c_2c_3e^{-\cos(t)} := ce^{-\cos(t)}.$$

Of course, we could have solved (14) by also separating variables and integrating both sides, but this was to illustrate how one could use a Lyapunov-Floquet transformation to solve a periodic differential equation.

For a system of differential equations $x' = A(t)x$ with dimension higher than 1, the computation of the unknown x is not so easy. If we want to use the Lyapunov-Floquet transformation, we have to compute B and Q. These matrices cannot be found without knowing the fundamental matrix solution for $x' = A(t)x$, because it follows from Theorem 2.8 that $B = \frac{1}{T}\log(X(T))$ and $Q(t) = X(t)e^{-Bt}$. In the following example, $X(t)$ and B will be computed so that the matrix Q can be found by using the Lyapunov-Floquet transformation.

Example 3.2. We want to compute the Floquet-normal form for the two-dimensional differential equation

$$x' = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} x. \tag{18}$$

Step 1. We wish to find a fundamental matrix $X(t)$ satisfying $X(0) = I$. Take $x = (x_1, x_2)^T$ and write $z = x_1 + ix_2$. Then

$$\begin{aligned} z' &= x_1' + ix_2' \\ &= \cos(t)x_1 - \sin(t)x_2 + i(\sin(t)x_1 + \cos(t)x_2) \\ &= \cos(t)(x_1 + ix_2) + i\ \sin(t)(x_1 + ix_2) \\ &= (\cos(t) + i\ \sin(t))(x_1 + ix_2) \\ &= e^{it}z. \end{aligned}$$

Hence, we are left with a one dimensional differential equation. We can solve this by separating variables and integrating both sides.

$$\begin{aligned} \frac{dz}{dt} = e^{it}z &\implies \int \frac{dz}{z} = \int e^{it}dt \\ &\implies \log|z| = -ie^{it} + c_1 \\ &\implies z(t) = c_2 e^{-ie^{it}} \end{aligned}$$

for some complex number c_2. Writing $c_2 = a + bi$ yields

$$\begin{aligned} z(t) &= (a+bi)e^{-i(\cos(t)+i\sin(t))} \\ &= (a+bi)e^{\sin(t)}e^{-i\cos(t)} \\ &= (a+bi)e^{\sin(t)}(\cos(\cos(t)) - i\ \sin(\cos(t))) \\ &= x_1(t) + ix_2(t) \end{aligned}$$

implying that

$$\begin{cases} x_1(t) = ae^{\sin(t)}\cos(\cos(t)) + be^{\sin(t)}\sin(\cos(t)) \\ x_2(t) = -ae^{\sin(t)}\sin(\cos(t)) + be^{\sin(t)}\cos(\cos(t)). \end{cases}$$

Hence, a fundamental matrix for (18) is given by

$$\tilde{X}(t) = \begin{pmatrix} e^{\sin(t)}\cos(\cos(t)) & e^{\sin(t)}\sin(\cos(t)) \\ -e^{\sin(t)}\sin(\cos(t)) & e^{\sin(t)}\cos(\cos(t)) \end{pmatrix}.$$

Let $X(t) := \tilde{X}(t)\tilde{X}(0)^{-1}$ to assure that $X(0) = I$. We have

$$\tilde{X}(0) = \begin{pmatrix} \cos(1) & \sin(1) \\ -\sin(1) & \cos(1) \end{pmatrix}$$

so that

$$\tilde{X}^{-1}(0) = \begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}.$$

Therefore, a fundamental matrix satisfying $X(0) = I$ is given by

$$\begin{aligned} X(t) &= \begin{pmatrix} e^{\sin(t)}\cos(1-\cos(t)) & -e^{\sin(t)}\sin(1-\cos(t)) \\ e^{\sin(t)}\sin(1-\cos(t)) & e^{\sin(t)}\cos(1-\cos(t)) \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} e^{\sin(t)} & 0 \\ 0 & e^{\sin(t)} \end{pmatrix}}_{:=S(t)} \underbrace{\begin{pmatrix} \cos(1-\cos(t)) & -\sin(1-\cos(t)) \\ \sin(1-\cos(t)) & \cos(1-\cos(t)) \end{pmatrix}}_{:=R(t)}. \end{aligned}$$

Step 2. Now we want to find the constant matrix B. By Theorem 2.8, it satisfies $X(T) = e^{BT}$ with period T. From this, it follows that $B = \frac{1}{T}\log(X(T))$. By example 2.7, the logarithm of the rotation matrix $R(t)$ is given by

$$\log\, R(t) = (1-\cos(t)+2\pi n)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for any integer n. Since S and R commute

$$\begin{aligned} \log(X(t)) &= \log(S(t)R(t)) \\ &= \log\, S(t) + \log\, R(t) \\ &= \begin{pmatrix} \sin(t) & 0 \\ 0 & \sin(t) \end{pmatrix} + (1-\cos(t)+2\pi n)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sin(t) & -1+\cos(t)-2\pi n \\ 1-\cos(t)+2\pi n & \sin(t) \end{pmatrix}. \end{aligned}$$

Hence,

$$B = \frac{1}{2\pi}\log(X(2\pi)) = \begin{pmatrix} 0 & -n \\ n & 0 \end{pmatrix}$$

for any integer n. We observe that B is a constant matrix, as was derived at the beginning of this section.

Step 3. The next step is to find the fundamental matrix $Y(t)$ for the system $y(t)' = By(t)$. Taking $y = (y_1, y_2)^T$ results in the following system of equations:

$$\begin{cases} y_1' = -ny_2 \\ y_2' = ny_1. \end{cases} \tag{19}$$

The solution to this system is given by

$$\begin{aligned} y_1(t) &= c_1\cos(nt) + c_2 i\sin(nt) \\ y_2(t) &= c_1\sin(nt) - c_2 i\cos(nt) \end{aligned}$$

If we let $c_1 = 1$ and $c_2 = i$, a fundamental matrix solution for the problem $y(t)' = By(t)$ is given by

$$Y(t) = \begin{pmatrix} \cos(nt) & -\sin(nt) \\ \sin(nt) & \cos(nt) \end{pmatrix}.$$

Step 4. The periodic matrix $Q(t)$ can be computed using the fundamental matrix form of the Lyapunov-Floquet transformation, $X(t) = Q(t)Y(t)$. Multiplying on the left by the inverse of $Y(t)$ yields

$$\begin{aligned} Q(t) &= X(t)Y^{-1}(t) \\ &= \begin{pmatrix} e^{\sin(t)}\cos(1-\cos(t)) & -e^{\sin(t)}\sin(1-\cos(t)) \\ e^{\sin(t)}\sin(1-\cos(t)) & e^{\sin(t)}\cos(1-\cos(t)) \end{pmatrix} \begin{pmatrix} \cos(nt) & \sin(nt) \\ -\sin(nt) & \cos(nt) \end{pmatrix} \\ &:= \begin{pmatrix} q_{11}(t) & q_{12}(t) \\ q_{21}(t) & q_{22}(t) \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} q_{11}(t) &= e^{\sin(t)}\cos(1-\cos(t))\cos(nt) + e^{\sin(t)}\sin(1-\cos(t))\sin(nt) \\ q_{12}(t) &= e^{\sin(t)}\cos(1-\cos(t))\sin(nt) - e^{\sin(t)}\sin(1-\cos(t))\cos(nt) \\ q_{21}(t) &= e^{\sin(t)}\sin(1-\cos(t))\cos(nt) - e^{\sin(t)}\cos(1-\cos(t))\sin(nt) \\ q_{22}(t) &= e^{\sin(t)}\sin(1-\cos(t))\sin(nt) + e^{\sin(t)}\cos(1-\cos(t))\cos(nt). \end{aligned}$$

In this example, Q was computed by using the Lyapunov-Floquet transformation. This was done in step 3 and step 4. If finding the fundamental matrix for $y'(t) = By(t)$ is difficult, one can try to find $Q(t)$ in a different way, namely, by using the relation $Q(t) = X(t)e^{-Bt}$.

In conclusion, if the periodic system $x'(t) = A(t)x(t)$ is one-dimensional, the Lyapunov-Floquet transformation is useful for finding the solution $x(t)$. In this case, the 1×1 matrices B and $Q(t)$ can be found without knowing $x(t)$. If the dimension of the system is higher than 1, the Lyapunov-Floquet transformation is useful when one wants to compute the Floquet normal form $X(t) = Q(t)e^{Bt}$ explicitly. Unfortunately, in this case, the fundamental matrix solution is needed in order to find the matrices B and $Q(t)$. The constant matrix B can be found by the formula $B = \frac{1}{T}\log(X(T))$ and the periodic matrix $Q(t)$ can be computed by applying the Lyapunov-Floquet transformation.

3.2 Floquet multipliers

We continue using the fundamental matrix $X(t)$ for (7). In Lemma 2.5, we proved that

$$X(t+T) = X(t)C$$

where C is a nonsingular constant matrix. In (10), it was mentioned that we can write $C = X^{-1}(0)X(T)$. This matrix C is known as the *monodromy matrix*.

Definition 3.3. The eigenvalues of the monodromy matrix are called the *Floquet multipliers* of (7).

Definition 3.4. The eigenvalues of the matrix B of the Floquet normal form $X(t) = Q(t)e^{Bt}$, are called the *Floquet exponents* of (7).

Since the monodromy matrix is nonsingular, its eigenvalues are nonzero. Therefore, we can state the following.

Corollary 3.5. Let $\lambda_1, \dots, \lambda_n$ be the Floquet multipliers and $\mu_1, \dots, \mu_n$ be Floquet exponents for (7). We can write

$$\lambda_j = e^{\mu_j T}$$

for all $j = 1, \dots, n$.

Proof. Write the matrix $B \in \mathbb{C}^{n \times n}$ of the Floquet normal form in the Jordan canonical form

$$J = P^{-1}BP,$$

where P is some nonsingular matrix. We have $J = diag(J_1, \dots, J_r)$, with $r \leq n$, where

$$J_k = \begin{pmatrix} \mu_k & 1 & & \\ & \mu_k & \ddots & \\ & & \ddots & 1 \\ & & & \mu_k \end{pmatrix}, \quad k = 1, \dots, r.$$

Then

$$C = X(T) = e^{BT} = e^{PJP^{-1}T} = Pe^{JT}P^{-1} = P\ diag(e^{J_1 T}, \dots, e^{J_r T})P^{-1}.$$

This means that the eigenvalues of C, given by λ_j, are the same as the eigenvalues of e^{JT}, which are given by $e^{\mu_j T}$. Hence,

$$\lambda_j = e^{\mu_j T}$$

for all $j = 1, \dots, n$. □

Remark. The Floquet exponents are not uniquely determined by (7). To see this, assume $e^{\mu_j T} = \lambda_j$ and let $\hat{\mu}_j = \mu_j + \frac{2\pi m i}{T}$ where $m \in \mathbb{Z}$. Since $e^{2\pi m i} = 1$,

$$e^{\hat{\mu}_j T} = e^{(\mu_j + \frac{2\pi m i}{T})T} = e^{\mu_j T} e^{2\pi m i} = \lambda_j$$

3.3 Stability of the Floquet system

Floquet multipliers are very useful in stability analyses of periodic systems. Recall the following definitions [7].

Definition 3.6. An eigenvalue λ of A is *simple* if its algebraic multiplicity equals 1.

Definition 3.7. Let λ be an eigenvalue of a matrix A. The *geometric multiplicity* of λ is $\dim(\mathrm{Null}(A-\lambda I))$, in other words, the number of linearly independent eigenvectors associated with λ.

Definition 3.8. An eigenvalue λ of A is *semisimple* if its geometric multiplicity equals its algebraic multiplicity.

A simple eigenvalue is always semisimple, but the converse is not true. Recall the following definition [13].

Definition 3.9. Consider the system

$$x' = A(t)x \quad \text{in} \quad V = [t_0, \infty) \tag{20}$$

and assume $A(t)$ is T-periodic and continuous in V. The solution $\psi(t)$ to the system (20) is

1. *stable* on V if for every $\epsilon > 0$, there exist a $\delta > 0$, such that

$$|\psi(t_0) - x(t_0)| < \delta \implies |\psi(t) - x(t)| < \epsilon, \quad \forall t \geq t_0$$

   and the solution $x(t)$ is defined for all $t \in V$.

2. *asymptotically stable* on V if it is stable and if in addition

$$\lim_{t\to\infty} |\psi(t) - x(t)| \to 0.$$

3. *unstable* if it is not stable on V.

It can be proven that the following stability conditions hold for the Floquet system [1, 12].

Theorem 3.10. Assume $\lambda_1, \ldots, \lambda_n$ are Floquet multipliers of system (7). Then the zero solution of (7) is

1. asymptotically stable on $[0, \infty)$ if and only if $|\lambda_j| < 1$ for all $i = 1, \ldots, n$.

2. stable on $[0, \infty)$ if $|\lambda_j| \leq 1$ for all $i = 1, \ldots, n$ and whenever $|\lambda_j| = 1$, λ_j is a semisimple eigenvalue.

3. unstable in all other cases.

Note that for the Floquet exponents, the condition $|\lambda_j| < 1$, $|\lambda_j| \leq 1$ and $|\lambda_j| > 1$ is equivalent to Re $\mu_j < 0$, Re $\mu_j \leq 0$ and Re $\mu_j > 0$.

Example 3.11. We want to find the Floquet multipliers and Floquet exponents for the following system

$$x'(t) = \begin{pmatrix} -1 & 1 \\ 0 & 1 + \cos(t) - \frac{\sin(t)}{2+\cos(t)} \end{pmatrix} x. \tag{21}$$

Let $x = (x_1, x_2)^T$. We want to find the fundamental matrix for (21) so that we can compute the monodromy matrix. Start with solving the second ODE,

$$\frac{dx_2}{dt} = \Big(1 + \cos(t) - \frac{\sin(t)}{2 + \cos(t)}\Big) x_2. \tag{22}$$

Separating variables and integrating both sides yields

$$\int \frac{dx_2}{x_2} = \int 1 + \cos(t) - \frac{\sin(t)}{2 + \cos(t)} dt.$$

We obtain

$$\log(x_2) = t + \sin(t) + \log(2 + \cos(t)) + c_1,$$

implying that

$$x_2 = e^{t+\sin(t)+c_1}(2 + \cos(t)) := c_2 e^{t+\sin(t)}(2 + \cos(t))$$

is the solution to (22). Now solve the first ODE,

$$\frac{dx_1}{dt} = -x_1 + c_2 e^{t+\sin(t)}(2 + \cos(t)). \tag{23}$$

It is of the form $x_1' + P(t)x_1 = R(t)$, where $R(t) = c_2 e^{t+\sin(t)}(2 + \cos(t))$ and $P(t) = 1$. Therefore, we can use the integrating factor $M(t) = e^{\int_0^t P(s)ds} = e^{\int_0^t ds} = e^t$. Then $\frac{d}{dt}[x_1 M(t)] = R(t)M(t)$ gives

$$\frac{d}{dt}[x_1 e^t] = c_2 e^{2t+\sin(t)}(2 + \cos(t)).$$

Integrating both sides with respect to t yields

$$x_1 e^t = c_2 e^{2t+\sin(t)} + c_3.$$

We obtain the solution to (23) by multiplying both sides with e^{-t},

$$x_1 = c_2 e^{t+\sin(t)} + c_3 e^{-t}.$$

If we let $c_2 = c_3 = 1$, a fundamental matrix for the system in (21) is given by

$$X(t) = \begin{pmatrix} e^{t+\sin(t)} & e^{-t} \\ e^{t+\sin(t)}(2+\cos(t)) & 0 \end{pmatrix}.$$

We get

$$X(T) = X(2\pi) = \begin{pmatrix} e^{2\pi} & e^{-2\pi} \\ 3e^{2\pi} & 0 \end{pmatrix}$$

and the inverse of $X(t)$ is

$$X^{-1}(t) = \begin{pmatrix} 0 & \frac{e^{-t-\sin(t)}}{2+\cos(t)} \\ e^{t} & -\frac{e^{t}}{2+\cos(t)} \end{pmatrix}$$

so that

$$X^{-1}(0) = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{1}{3} \end{pmatrix}.$$

Hence, the monodromy matrix is

$$C = X^{-1}(0)X(T) = \begin{pmatrix} e^{2\pi} & 0 \\ 0 & e^{-2\pi} \end{pmatrix}.$$

Thus the Floquet multipliers are $\lambda_1 = e^{2\pi}$ and $\lambda_2 = e^{-2\pi}$ and the Floquet exponents are $\mu_1 = 1$ and $\mu_2 = -1$. We have $|\lambda_1| > 1$, so by Theorem 3.10, the solution is unstable.

3.4 Hill's differential equation

A widely used periodic differential equation is the second order homogeneous equation

$$y'' + g_1(t)y' + g_0(t)y = 0. \tag{24}$$

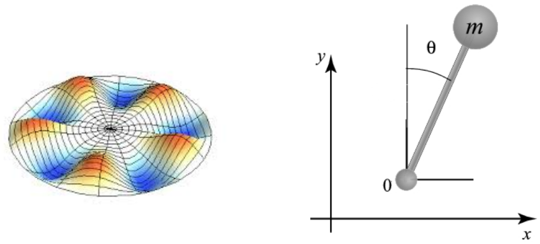
where $g_i(t) = g_i(t+T)$, for $i = 0, 1$ with $T > 0$ [9]. We want to write this system in the form of a Floquet system so that we can use Floquet multipliers to investigate its stability. The two-dimensional first order system associated with (24) is given by

$$x' = A(t)x \quad \text{with} \quad A(t) = \begin{pmatrix} 0 & 1 \\ -g_0(t) & -g_1(t) \end{pmatrix}$$

where $x = (y, y')^T$.

The first detailed theory about time-dependent periodic systems was developed by the French mathematician Émile Léonard Mathieu in 1868 [9]. He introduced the *Mathieu equation*,

$$y'' + (a - 2q\cos(2t))y = 0,$$



(a) Vibration of a homogeneous drumhead (b) The inverted pendulum

Figure 2: Applications of Mathieu's equation

where a is a constant parameter and $2q$ a parameter which represents the magnitude of the time variation. It is commonly used in vibration problems such as [9, 11]

- the vibration of a homogeneous elliptic drumhead (see Figure 2a);
- the inverted pendulum, such as the Segway (see Figure 2b);
- the stability of a floating body, for instance a vessel;
- quadrupole mass analyzers and quadrupole ion traps for mass spectrometry.

In particular, we are interested in a generalization of Mathieu's equation, called *Hill's equation*:

$$\begin{aligned} &y'' + f(t)y = 0, \\ &f(t) = f(t+T) \quad \forall t \in \mathbb{R}, \end{aligned} \tag{25}$$

where $f(t)$ is piecewise continuous and $T > 0$. It is named after the American astronomer and mathematician George William Hill (1838-1914). His most important work was the study of the 4-body problem to analyze the motion of the moon around the earth (1886).

The equation in (25) is equivalent to the two-dimensional first order system

$$x' = A(t)x \quad \text{with} \quad A(t) = \begin{pmatrix} 0 & 1 \\ -f(t) & 0 \end{pmatrix}. \tag{26}$$

where $x = (y, y')^T$. We want to investigate the stability of Hill's equation. As discussed in Section 3.3, we can do this by looking at the Floquet multipliers of system (26). Assume that a fundamental matrix for (26) is given by

$$X(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix}.$$

To satisfy $X(0) = I$, let $x_{11}(t) = x_{22}(t) = 1$ and $x_{12}(t) = x_{21}(t) = 0$. Since $X^{-1}(0) = I$, the monodromy matrix is of the form

$$C = X^{-1}(0)X(T) = \begin{pmatrix} x_{11}(T) & x_{12}(T) \\ x_{21}(T) & x_{22}(T) \end{pmatrix}.$$

The Floquet multipliers are given by the solutions of the equation

$$\det(C - \lambda I) = \lambda^2 - (\operatorname{tr} C)\lambda + \det C = 0.$$

Lemma 3.12. The monodromy matrix C of Hill's differential equation satisfies $\det C = 1$.

Proof. Let $X(t)$ be the fundamental matrix and define $W(t) = \det X(t)$. Then

$$W(0) = x_{11}(0)x_{22}(0) - x_{12}(0)x_{21}(0) = 1. \tag{27}$$

Using (26) and the fundamental matrix, we get the following system of equations:

$$\begin{cases} x'_{11} = x_{21} \\ x'_{21} = -f(t)x_{11} \end{cases} \qquad \begin{cases} x'_{12} = x_{22} \\ x'_{22} = -f(t)x_{12}. \end{cases} \tag{28}$$

Therefore,

$$\begin{aligned} \frac{dW}{dt} &= x'_{11}(t)x_{22}(t) + x_{11}(t)x'_{22}(t) - x'_{21}(t)x_{12}(t) - x_{21}(t)x'_{12}(t) \\ &= x_{21}(t)x_{22}(t) - x_{11}(t)f(t)x_{12}(t) + f(t)x_{11}(t)x_{12}(t) - x_{21}(t)x_{22}(t) \\ &= 0. \end{aligned}$$

This implies that $W(t)$ is constant, namely $W(t) = 1$ for all t, as computed in (27). Hence,

$$\det C = \det X(T) = W(T) = 1.$$

□

Consequently, we get

$$\det(C - \lambda I) = \lambda^2 - (\operatorname{tr} C)\lambda + 1$$

so that the Floquet multipliers are given by

$$\lambda_{\pm} = \frac{\operatorname{tr} C \pm \sqrt{\operatorname{tr} C^2 - 4}}{2}.$$

Using Theorem 3.10, the following conclusions can be drawn [6].

Case 1: $|\operatorname{tr} C| > 2$. If $\operatorname{tr} C > 2$, then $\lambda_+ > 1$, and if $\operatorname{tr} C < -2$, then $\lambda_- < -1$. In either case, the magnitude of the eigenvalue is larger than 1, so the zero solution is unstable.

Case 2: $|\operatorname{tr} C| < 2$. Then $\lambda_\pm = \frac{\operatorname{tr} C}{2} \pm i\beta$ with $\beta > 0$. Because the determinant of C equals the product of its eigenvalues, that is, $\lambda_+\lambda_- = 1$, it follows that $|\lambda_+| = |\lambda_-| = 1$. The algebraic multiplicity of both λ_+ and λ_- is 1, so the eigenvalues are simple, and hence, semisimple. We conclude that the zero solution is stable, but not asymptotically stable.

Case 3: $|\operatorname{tr} C| = 2$. If $\operatorname{tr} C = 2$, then $\lambda_\pm = 1$, and if $\operatorname{tr} C = -2$, then $\lambda_\pm = -1$. If the eigenvalue is semisimple, then the zero solution is stable. Otherwise, the zero solution is unstable.

The theory discussed in this section can be used for stability analyses of a Hill equation. The ODEs with the initial conditions satisfying $X(0) = I$ can be solved numerically. Setting $t = T$, one can compute the value for $\operatorname{tr} C = x_{11}(T) + x_{22}(T)$. If $|\operatorname{tr} C| < 2$, the solution is stable and if $|\operatorname{tr} C| > 2$, the solution is unstable. By only solving the system for one forcing period, conclusions can be drawn about larger time behavior of the solution. Without using Floquet theory, one would have to numerically solve the system for a larger time t to investigate the behavior of the solution [10]. A particular example of a Hill equation will be treated now [3].

Example 3.13. The stability of the inverted pendulum, illustrated in Figure 3, is determined from the Hill equation (25). Let us find an expression for the periodic function $f(t)$ assuming that the pendulum is frictionless and has a massless rod. By Newton's second law of linear motion and circular motion, we have

$$F_{pivot} = ma = m\frac{d^2y}{dt^2}, \tag{29}$$

$$\tau_{net} = I\frac{d^2\theta}{dt^2} = ml^2\frac{d^2\theta}{dt^2}, \tag{30}$$

where F_{pivot} is the force acting on the pivot, τ_{net} the netto external torque and I the moment of inertia. Without any help, the mass above the pivot point will fall over. In order to remain upright, a torque can be applied at the pivot point. The formula of the torque τ is given by $\tau = rF\sin(\theta)$, where F is the force acting on the particle and r is the distance from the axis of rotation to the particle. So in our case, the gravitational torque is

$$\tau_{grav} = rF_{grav}\sin(\theta) = mgl\sin(\theta).$$

The harmonic motion of the pendulum can be expressed as $y(t) = A\cos(\omega t)$ so that the force F_{pivot} in (29) is given by

$$F_{pivot} = -m\omega^2 A\cos(\omega t).$$

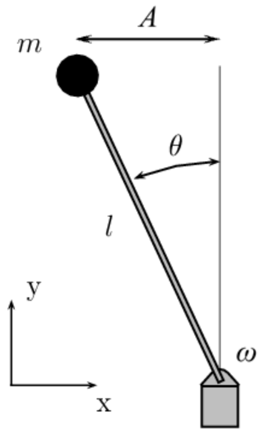


Figure 3: Inverted pendulum

The torque exerted by the pivot point is given by

$$\tau_{pivot} = rF_{pivot}\sin(\theta) = -ml\omega^2 A\cos(\omega t)\sin(\theta).$$

Hence, the total torque $\tau_{net} = \tau_{grav} + \tau_{pivot}$ yields the equation

$$ml^2\frac{d^2\theta}{dt^2} = mgl\sin(\theta) - ml\omega^2 A\cos(\omega t)\sin(\theta)$$

implying that

$$\frac{d^2\theta}{dt^2} + (\frac{g}{l} - \frac{\omega^2 A}{l}\cos(t))\sin(\theta) = 0. \tag{31}$$

Taking $\sin(\theta) \approx \theta$ for small oscillations, (31) takes the form of a Hill equation

$$\theta'' + f(t)\theta = 0, \tag{32}$$

where $f(t) = a + b\cos(t)$ is 2π-periodic and $a = \frac{g}{l}$ and $b = -\frac{\omega^2 A}{l}$.

We can use the stability analysis of Hill's equation to check for which values of a and b the pendulum equation (32) is stable. We do this by numerically solving the monodromy matrix M and storing the values for a and b for which $|\operatorname{tr} M| < 2$. The computations are done by the program `stability_diagram.m` which makes use of the function file `hill_equation.m`. The codes can be found in Appendix A. In Figure 4, the results are given, where a dot represents a stable solution for the corresponding values for a and b.

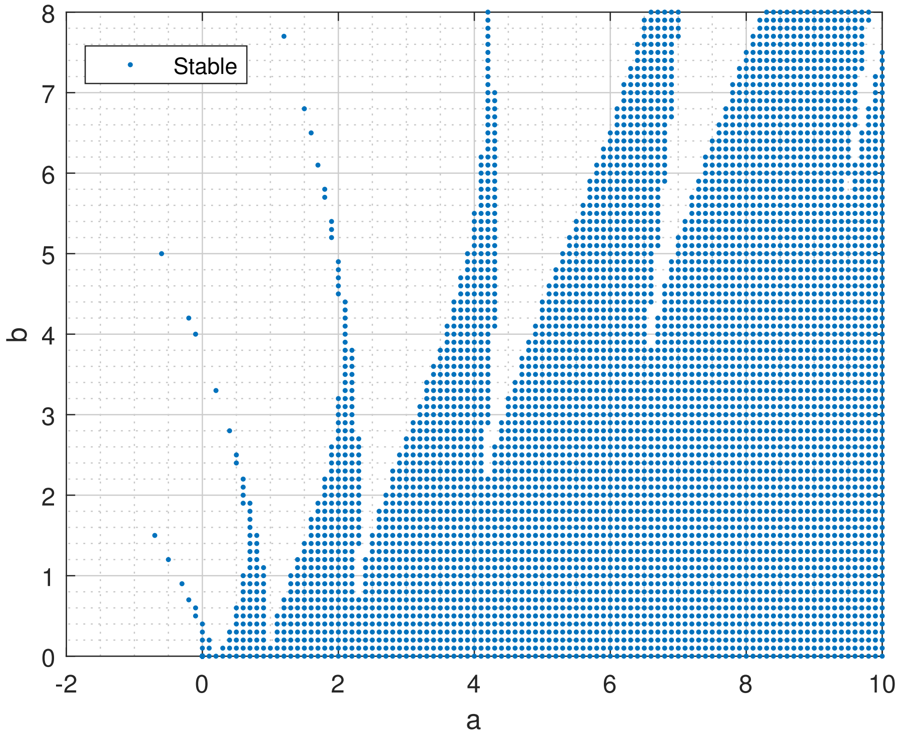
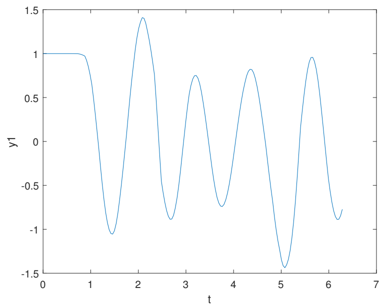


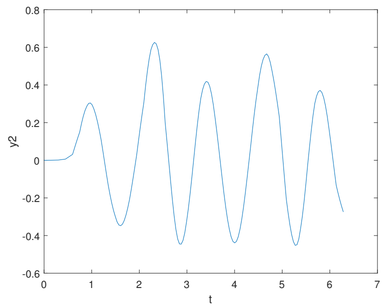
Figure 4: Stability diagram for $\theta'' + (a + b\cos(t))\theta = 0$

The sign of b does not influence the stability of the motion of the pendulum. That is because a change of sign in $b = -\frac{\omega^2 A}{l}$ corresponds to a change of sign in A. Therefore, the stability diagram is symmetric around the a-axis.

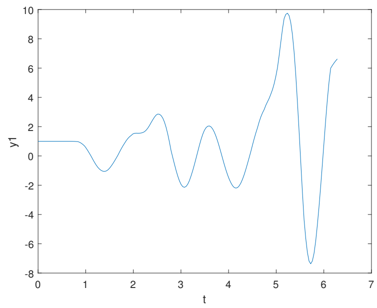
Consider the inverted pendulum with length $l = 1.64\,\mathrm{m}$, rotational speed $\omega = 3\,\mathrm{rad/sec}$ and amplitude $A = -0.9\,\mathrm{m}$. Then $a \approx 5.98$ and $b \approx 4.94$, so according to Figure 4 we expect a stable solution $\theta = (\theta_1, \theta_2)^T$. Indeed, when we look at Figure 5a and 5b, the solution is stable.
When we change the amplitude to $A = -1.3\,\mathrm{m}$, we get $b \approx 7.13$, and expect an unstable solution. Indeed, when we look at Figure 5c and 5d, the solution is unstable.



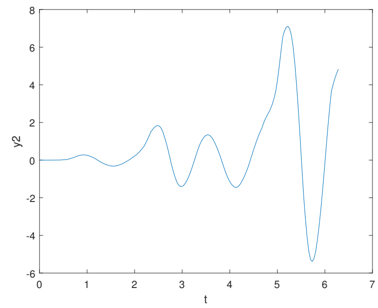
(a) Plot of θ_1 when $A = -0.9$



(b) Plot of θ_2 when $A = -0.9$



(c) Plot of θ_1 when $A = -1.3$



(d) Plot of θ_2 when $A = -1.3$

Figure 5: Graphs of solutions θ for varying amplitude

In conclusion, the stability of a Hill equation can be analyzed by looking at the trace of the corresponding monodromy matrix. In this case, the Floquet multipliers need not be computed, which saves quite some computational cost.

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References

[1] Albasrawi, F. H. (2013). Floquet theory on banach space.

[2] Coddington, E. A. and Carlson, R. (1997). *Linear ordinary differential equations*. Philadelphia: Society for Industrial and Applied Mathematics.

[3] Ewenczyk, A. et al. (2007). The inverted pendulum. `http://phy-page-imac.princeton.edu/~page/phy210/2007_projects/InvertedPendulum.pdf`.

[4] Floquet, G. (1883). Sur les equations differentielles lineaires. *Ann. ENS [2]*, 12(1883):47–88.

[5] Huang, Z., Zhang, T., and Liu, J. (2013). Theory and applications of periodic solutions and almost periodic solutions.

[6] Logemann, H. and Ryan, E. P. (2014). *Ordinary Differential Equations: Analysis, Qualitative Theory and Control*. London: Springer.

[7] Meyer, C. D. (2000). *Matrix analysis and applied linear algebra*, volume 71. Siam.

[8] Nagy, G. (2017). *Ordinary differential equations*. East Lansing: Michigan State University.

[9] Panardo, I. (2014). Stability of periodic systems and floquet theory.

[10] Rand, R. H. (2012). *Lecture notes on nonlinear vibrations*. Ithaca: Cornell University.

[11] Ruby, L. (1996). Applications of the mathieu equation. *American Journal of Physics*, 64(1):39–44.

[12] Sideris, T. C. (2013). *Ordinary differential equations and dynamical systems*. Paris: Atlantis Press.

[13] Walter, W. (1998). *Ordinary differential equations*. (Graduate texts in mathematics; readings in mathematics, 182). New York: Springer.

[14] Wang, Z. (2013). Floquet Theory. `math.ecnu.edu.cn/~zmwang/teaching/AssignmentsandSomeSupplementalMaterials/Floquet%20Theory.pdf`.

[15] Ward, M. J. (2010). Basic floquet theory. `www.math.ubc.ca/~ward/teaching/m605/every2_floquet1.pdf`.

A Matlab code

Hill equation `hill_equation.m`

```
% Hill equation y''+f(t)y=0 where f(t)=a+bcos(t)
% Written into two first order differential equations
% y'(1)=y(2)
% y'(2)=-(a+b*cos(t))*y(1)
function [dy_dt] = hill_equation(t, y, a, b)
        dy_dt = [y(2); -(a+b*cos(t))*y(1)];
```

Making a stability diagram for Hill's equation `stability_diagram.m`

```
% To plot the stability diagram for the Mathieu Equation
% of the form y''+(a+b*cos(t))y = 0

%clear all
close all
format long

a=[-1:0.5:10];
b=[0:0.5:8];

for i=1:length(a)
    for j=1:length(b)
        % solve Hill's equation with initial conditions
        % (1,0)^T and (0,1)^T for 0 <= t <= 2 pi
        tspan = [0.0 2*pi];
        [t1, y1] = ode45(@(t,y) hill_equation(t, y, a(i),
                         b(j)), tspan, [1.0; 0.0]);
        [t2, y2] = ode45(@(t,y) hill_equation(t, y, a(i),
                         b(j)), tspan, [0.0; 1.0]);

        [nrows1, ncols1] = size(y1);
        [nrows2, ncols2] = size(y2);

        % computation of monodromy matrix
        M = [y1(nrows1,:)', y2(nrows2,:)'];
        trM(j)=trace(M);

        % computation of floquet multipliers
        eigv(j,:)=eig(M);
    end

```

```
    % create list of trace(M)
    TrM(1+(i-1)*length(b):i*length(b))=trM;

    % create list of floquet multipliers
    Eigv(1+(i-1)*length(b):i*length(b),:)=eigv;

    % create list of all possible combinations of a and b
    [A,B] = meshgrid(a(i),b);
    c=cat(2,A',B');
    d=reshape(c,[],2);
    abcol(1+(i-1)*length(b):i*length(b),:)=d;
end
acol=abcol(:,1);
bcol=abcol(:,2);

% overview of a and b with corresponding floquet multipliers
T=table(acol, bcol, Eigv);
T.Properties.VariableNames = {'a' 'b' 'floquet_multipliers'};

% list the values for a and b when TrM<2
for k=1:length(Eigv)
    if abs(TrM(k))<2
        Stab(k,1:2)=[acol(k) bcol(k)];
    end
end

% make stability diagram
plot(Stab(:,1),Stab(:,2),'.')
grid on;
grid minor;
ax = gca;
ax.GridColor = [0 0 0];
ax.GridAlpha = 0.2;
hold on;
title('Stability diagram for x''''+(a+bcos(t))x=0')
axis on
xlabel('a');
ylabel('b');
legend({'Stable'})
```